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The uniqueness of unconditional basis of the 2-convexified Tsirelson space, revisited

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Abstract

One of the hallmarks in the study of the classification of Banach spaces with a unique (normalized) unconditional basis was the unexpected result by Bourgain, Casazza, Lindenstrauss, and Tzafriri from their 1985 *Memoir* that the 2-convexified Tsirelson space $\mathcal{T}^{(2)}$ had that property (up to equivalence and permutation). Indeed, on one hand, finding a "pathological" space (i.e., not built out as a direct sum of the only three classical sequence spaces with a unique unconditional basis) shattered the hopeful optimism of attaining a satisfactory description of all Banach spaces which enjoy that important structural feature. On the other hand it encouraged furthering a research topic that had received relatively little attention until then. After forty years, the advances on the subject have shed light onto the underlying patterns shared by those spaces with a unique unconditional bases belonging to the same class, which has led to reproving the original theorems with fewer technicalities. Our motivation in this note is to revisit the aforementioned result on the uniqueness of unconditional basis of $\mathcal{T}^{(2)}$ from the current state-of-art of the subject and to fill in some details that we missed from the original proof.

Keywords Uniqueness of unconditional basis · Banach lattices · Tsirelson space · Sequence spaces

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1 Introduction

Let $\mathbb{X}$ be a separable Banach space over the real or complex field $\mathbb{F}$. One of the most important problems in the isomorphic theory dating back to Banach's school is the study of the existence and uniqueness of Schauder bases for $\mathbb{X}$. The question of uniqueness is formulated in a meaningful way through the notion of equivalence. Recall that two normalized bases $(\boldsymbol{x}_n)_{n=1}^\infty$ and $(\boldsymbol{y}_n)_{n=1}^\infty$ of $\mathbb{X}$ are called *equivalent* provided a series $\sum_{n=1}^\infty a_n\, \boldsymbol{x}_n$ converges if and only if $\sum_{n=1}^\infty a_n\, \boldsymbol{y}_n$ converges.

As it happens, in every infinite-dimensional Banach space with a basis there are uncountably many non-equivalent normalized bases [24]. Thus in order to get a more accurate structural information on a given space using bases as a tool, one needs to restrict the discussion on their existence and uniqueness to bases with certain special properties. The most useful and extensively studied class of special bases is that of *unconditional bases*. A basis $(\boldsymbol{x}_n)_{n=1}^\infty$ of $\mathbb{X}$ is *unconditional* if $(\boldsymbol{x}_{\pi(n)})_{n=1}^\infty$ is a basis of $\mathbb{X}$ for any permutation π of the indices.

If a Banach space has a unique normalized unconditional basis it has to be equivalent to all its permutations, i.e., it has to be *symmetric*. For a Banach space with a symmetric basis it is rather unusual to have a unique unconditional basis, to the extent that c_0, ℓ_1 and ℓ_2 are the only Banach spaces in which all normalized unconditional bases are equivalent (see [17, 20, 22]).

The fact that an unconditional basis by its very definition does not have to come with a prescribed order, shows that for unconditional bases which are not symmetric, the more natural equivalence property is that of equivalence up to permutation, (UTAP) for short. In other words, for which spaces $\mathbb{X}$ is it true that whenever $(\boldsymbol{x}_n)_{n\in\mathcal{N}}$ and $(\boldsymbol{y}_m)_{m\in\mathcal{M}}$ are two normalized unconditional bases of $\mathbb{X}$ there is a bijection $\pi: \mathcal{N} \to \mathcal{M}$ so that the map T defined by $T(\boldsymbol{x}_n) = \boldsymbol{y}_{\pi(n)}$ extends to an automorphism of $\mathbb{X}$. The first movers in this direction were Edelstein and Wojtaszczyk, who proved that finite direct sums of c_0, ℓ_1 and ℓ_2 have a (UTAP) unique unconditional basis [13]. Bourgain et al. embarked on a comprehensive study aimed at classifying those Banach spaces with a (UTAP) unique unconditional basis that culminated in 1985 with their *Memoir* [9]. They showed that the spaces $c_0(\ell_1)$, $c_0(\ell_2)$, $\ell_1(c_0)$, $\ell_1(\ell_2)$ and their complemented subspaces with unconditional basis all have a (UTAP) unique unconditional basis, while $\ell_2(\ell_1)$ and $\ell_2(c_0)$ do not. They also found a nonclassical Banach space, namely the 2-convexification $\mathcal{T}^{(2)}$ of Tsirelson space, having a (UTAP) unique unconditional basis. Let us record this result for reference.

Theorem A ([9, Theorem 7.9]) *The 2-convexified Tsirelson space $\mathcal{T}^{(2)}$ has a (UTAP) unique unconditional basis.*

The topic of uniqueness of unconditional basis has evolved considerably since the publication of [9] thanks to the impetus given by the specialists. For instance, using completely different techniques, Casazza and Kalton solved in [10] some of the open questions left open in [9] by showing that the original Tsirelson space $\mathcal{T}$ has a (UTAP) unique unconditional basis but that, surprisingly, $c_0(\mathcal{T})$ does not (solving thus another problem raised in [9], namely whether the uniqueness of unconditional basis is preserved by infinite direct sums in the sense of c_0).

The relevance of Tsirelson's space in classical Banach space theory, which we briefly recall in the next section, and the significance of Theorem A within the topic of uniqueness of lattice structure were the motivation to devote this note to revisiting the proof of Theorem A as we understand it, using the tools we can rely on nowadays, which had not been invented when the *Memoir* was published.

We will use standard notation and terminology in Banach spaces throughout, as can be found in [4]. For the definition, an extensive background, and the properties of Tsirelson's space and its derivates we refer to the monograph [11] by Casazza and Shura. We next highlight the most heavily used terminology.

Given a subset A of a Banach space $\mathbb{X}$ we will denote by $[A]$ its closed linear span. Given $A, B \subseteq \mathbb{Z}$, we put $A \le B$ (resp., $A < B$) if $a \le b$ (resp., $a < b$) for all $a \in A$ and $b \in B$. We say that a sequence $(A_j)_{j=1}^\infty$ of nonempty subsets of $\mathbb{N}$ is *increasing* if $A_j < A_{j+1}$ for all $j \in \mathbb{N}$.

The equivalence of bases can be restated using the notion of dominance. We say that a family $\mathcal{G} = (g_j)_{j\in J}$ in a Banach space $\mathbb{Y}$ *C-dominates* a family $\mathcal{F} = (f_j)_{j\in J}$ in a Banach space $\mathbb{X}$ if there is a linear map $T\colon \left[f_j \colon j \in A\right] \to \mathbb{Y}$ such that $T(f_j) = g_j$ for all $j \in J$, and $\|T\| \le C$. If $\mathcal{F}$ C-dominates $\mathcal{G}$ and $\mathcal{G}$ C-dominates $\mathcal{F}$ then $\mathcal{F}$ and $\mathcal{G}$ are C-equivalent. If $\mathcal{H} = (h_k)_{k\in K}$ is a basic sequence in $\mathbb{Y}$ and $\pi\colon J \to K$ is a bijection such that $(h_{\pi(j)})_{j\in J}$ and $\mathcal{F}$ are C-equivalent, we say that $\mathcal{H}$ is equivalent to $\mathcal{F}$ via π.

Given a constant C and families $(a_j)_{j\in J}$ and $(b_j)_{j\in J}$ of nonnegative scalars, we will use the symbol $a_j \approx_C b_j$ for $j \in J$ to mean that $a_j \le C b_j$ and $b_j \le C a_j$ for all $j \in J$. In all situations, if the constant C appearing in the equivalences is irrelevant, we drop it.

Given an unconditional basis $\mathcal{X} = (\boldsymbol{x}_n)_{n\in\mathcal{N}}$ and a family of nonzero scalars $\lambda = (\lambda_n)_{n\in\mathcal{N}}$, the unconditional basis $(\lambda_n \boldsymbol{x}_n)_{n\in\mathcal{N}}$ is equivalent to $\mathcal{X}$ if and only λ is bounded away from zero and infinity. So, a result about uniqueness of unconditional basis can be stated in terms of semi-normalized instead of normalized bases. This flexibility will be convenient in some arguments.

A (normalized) unconditional basis $\mathcal{X} = (\boldsymbol{x}_n)_{n\in\mathcal{N}}$ of a Banach space $\mathbb{X}$ induces a lattice structure on $\mathbb{X}$. Indeed, if $\tau\colon \mathbb{X} \to \mathbb{F}^{\mathcal{N}}$ denotes the *coefficient transform* relative to the basis, given by

$$\sum_{n\in\mathcal{N}} a_n \boldsymbol{x}_n \mapsto (a_n)_{n\in\mathcal{N}}$$

and we define on $L = \tau(X)$ the norm

$$\|(a_n)_{n\in\mathcal{N}}\| = \sup\left\{\left\|\sum_{n\in\mathcal{N}} b_n \boldsymbol{x}_n\right\| : b_n \in \mathbb{F},\ |b_n| \le |a_n|\right\},$$

then $\mathcal{L}$ becomes a discrete Banach lattice on $\mathcal{N}$, and $\tau\colon \mathbb{X} \to \mathcal{L}$ becomes an isomorphism. Conversely, the unit vector system of a discrete Banach lattice is an unconditional basis of its separable part. Adopting the language of lattices has some advantages. For instance, it allows the construction of new Banach spaces.

Given a discrete Banach lattice $\mathcal{L}$ over $\mathcal{N}$ Banach space $\mathbb{X}$, we can consider the Banach space $\mathcal{L}(\mathbb{X})$ consisting of all sequences $(f_n)_{n\in\mathcal{N}}$ such that $f_n \in \mathbb{X}$ and $(\|f_n\|)_{n\in\mathcal{N}} \in \mathcal{L}$. Also, lattices can be convexified. Given $1 \le p < \infty$ and a Banach lattice on a measure space (Ω, μ), its p-convexification $\mathcal{L}^{(p)}$ is the Banach lattice consisting of all measurable functions $f : \Omega \to \mathbb{F}$ such that $|f|^p \in \mathcal{L}$. This somewhat naive procedure yields spaces whose geometry can be quite different from the original one of the lattice $\mathcal{L}$. Other more specific terminology will be introduced in context when needed.

2 Preliminaries

Tsirelson's space made its appearance in the Banach space scene in the early 1970s soon to become one of its most distinguished constituents.

A historical concern of the structure theory of Banach spaces going back to Banach's book [8] had been whether there were any fundamental spaces which embedded isomorphically in every infinite-dimensional Banach space. From the point of view of the classical theory, the nicest subspace one could possibly hope to find in a general Banach space would be either c_0 or ℓ_p $(1 \le p < \infty)$. These spaces (which are prime) were the natural candidates to be the potential building blocks of any Banach space, and the feeling that every Banach space could contain a copy of c_0 or ℓ_p was supported by the fact that all classical spaces, including the Orlicz spaces, certainly do. There were other results hinting in the same direction. For instance, if a Banach space with a normalized Schauder basis $(\boldsymbol{e}_n)_{n=1}^\infty$ has the property that $(\boldsymbol{e}_n)_{n=1}^\infty$ is equivalent to all its normalized block basic sequences then $(\boldsymbol{e}_n)_{n=1}^\infty$ must be equivalent to the unit vector basis of ℓ_p or c_0 [31]. Or take Rosenthal's theorem, which states that every bounded sequence in a Banach space has a subsequence that is either weakly Cauchy or equivalent to the canonical basis of ℓ_1 [26]. Local theory also seemed to support the conjecture since an infinite-dimensional Banach space with an unconditional basis contains uniformly complemented copies of ℓ_1^n, ℓ_2^n, or ℓ_∞^n, $n \in \mathbb{N}$ [28].

The question then arose as to whether every Banach space must contain a copy of one of these spaces (see [19, 23]). This was solved in the negative in 1974 by Tsirelson who constructed an elegant counterexample of a reflexive Banach space with an unconditional basis not containing some ℓ_p $(1 \le p < \infty)$ or c_0, nor some superreflexive infinite-dimensional subspace.

In his original paper [27], Tsirelson had constructed the dual space $\mathcal{T}^*$ of the space $\mathcal{T}$ introduced by Figiel and Johnson later on in [14], which is the one nowadays known as Tsirelson's space. The authors of [14] went on to convexify the space $\mathcal{T}$ to produce other Tsirelson-like spaces with interesting properties. Let us mention, for instance, that $\mathcal{T}^{(p)}$, $1 < p < \infty$, is superreflexive and, as well as $\mathcal{T}$ and $\mathcal{T}^*$, does not contain any copy of c_0 or ℓ_q, $1 \le q < \infty$ [14]. The space $\mathcal{T}^{(2)}$ is of particular interest because it has type 2 and weak cotype 2 but it is not isomorphic to ℓ_2 (see [25, p. 205]).

The unit vector system $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^\infty$ is a normalized, 1-unconditional basis of $\mathcal{T}^{(p)}$, $1 \le p < \infty$. For the sake of readability we next quote the properties of $\mathcal{X}$ that were used in the original proof of Theorem A.

(T.a) There is a constant C such that

$$\left\| \sum_{j\in J} f_j \right\| \le C \left\| \sum_{j\in J} g_j \right\|$$

for all disjointly supported families $(f_j)_{j\in J}$ and $(g_j)_{j\in J}$ such that

$$\max(\operatorname{supp}(f_j)) \le \min(\operatorname{supp}(g_j))$$

and $\|f_j\| \le \|g_j\|$ for all $j \in J$ (see [9, Proposition 7.3]).

(T.b) For each $d \in \mathbb{N}$, $d \ge 2$, the subsequence $(\boldsymbol{x}_{d^n})_{n=1}^\infty$ of $(\boldsymbol{x}_n)_{n=1}^\infty$ is equivalent to $(\boldsymbol{x}_n)_{n=1}^\infty$ (see [9, Proposition 7.5]).

(T.c) There is a constant C such that $(\boldsymbol{x}_n)_{n\in A}$ is C-equivalent to the canonical basis of $\ell_p(A)$ for all $A \subseteq \mathbb{N}$ satisfying $|A| \le \min(A)$. (cf. [9, Proposition 7.4]).

Bourgain et al. used properties (T.a), (T.b) and (T.c) to prove that any semi-normalized unconditional basis $\mathcal{Y} = (\boldsymbol{y}_n)_{n=1}^\infty$ of $\mathcal{T}^{(2)}$ is permutatively equivalent to a subbasis of $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^\infty$ (see [9, Proposition 7.8]). Then they claimed that swapping to roles of $\mathcal{Y}$ and $\mathcal{X}$, one obtains that $\mathcal{X}$ is permutatively equivalent to a subbasis of $\mathcal{Y}$. Finally, they used (T.a) to infer from these equivalences that $\mathcal{X}$ and $\mathcal{Y}$ are permutatively equivalent.

Our objection to this proof is that the authors omitted the argument that permits to safely swap the roles of the two unconditional bases $\mathcal{Y}$ and $\mathcal{X}$. In this regard, we point out that, while properties (T.a) and (T.c) pass to subbases, it is by no means clear whether property (T.b) does. Notwithstanding, it should be conceded that, as we will show below, all subbases of the canonical basis $(\boldsymbol{x}_n)_{n=1}^\infty$ of $\mathcal{T}^{(p)}$ have property (T.b). Besides, the feature of $(\boldsymbol{x}_n)_{n=1}^\infty$ that allows us to prove this is mentioned in the *Memoir*. Let us record this property, which goes back to [12].

(T.d) There is a constant $C \in [1, \infty)$ such that

$$\frac{1}{C}\left\| \sum_{j=1}^\infty \|f_j\|\, \boldsymbol{x}_{\alpha(j)} \right\| \le \left\| \sum_{j=1}^\infty f_j \right\| \le C \left\| \sum_{j=1}^\infty \|f_j\|\, \boldsymbol{x}_{\alpha(j)} \right\|$$

for all functions $\alpha\colon \mathbb{N} \to \mathbb{N}$ and all sequences $(f_j)_{j=1}^\infty$ such that the sequence $(A_j)_{j=1}^\infty$ given by

$$A_j = \{\alpha(j)\} \cup \operatorname{supp}(f_j), \quad j \in \mathbb{N},$$

is increasing (see [9, Proposition 7.1]).

3 The proof of Theorem A, revisited

Our proof of Theorem A will be a consequence of a (formally) more general result, namely, that any Banach space with an unconditional basis $(\boldsymbol{x}_n)_{n=1}^\infty$ satisfying (T.a), (T.c), and (T.d) with $p = 2$ has a (UTAP) unique unconditional basis. To show this we will not a priori assume (T.b) to hold.

Although we have already covered the basic notions required to prove Theorem A, we need to refine some of them before we give the proof.

Definition 3.1 A basis $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^\infty$ of a Banach space $\mathbb{X}$ is *right dominant* if it satisfies (T.a). If this condition holds when replacing the assumption that $\max(\operatorname{supp}(f_j)) \le \min(\operatorname{supp}(g_j))$ with $\max(\operatorname{supp}(g_j)) \le \min(\operatorname{supp}(f_j))$, $\mathcal{X}$ is said to be *left dominant*.

Definition 3.2 A basis $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^\infty$ of a Banach space $\mathbb{X}$ is said to be *Schreier equivalent to the canonical basis of* ℓ_p $(1 \le p \le \infty)$ if it satisfies property (T.c).

It will be convenient to introduce a variation of Schreier equivalence.

Given a function $\alpha\colon \mathbb{N} \to \mathbb{N}$, we will use at several places the convention that $\alpha(0) = 0$.

Definition 3.3 Let $\gamma\colon \mathbb{N} \to \mathbb{N}$ be an increasing function. We will say that a basis $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^\infty$ of a Banach space $\mathbb{X}$ is *γ-asymptotically equivalent to the canonical basis of* ℓ_p $(1 \le p \le \infty)$ if there is a constant $C \ge 1$ such that $(\boldsymbol{x}_n)_{n\in A}$ is C-equivalent to the canonical basis of $\ell_p(A)$ for all $n \in \mathbb{N}$ and all $A \subseteq \mathbb{N}$ with $\gamma(n-1) < \min(A)$ and $|A| \le \gamma(n) - \gamma(n-1)$. If this condition holds for some increasing function $\gamma\colon \mathbb{N} \to \mathbb{N}$ with $\sup_n(\gamma(n) - \gamma(n-1)) = \infty$ we say that $\mathcal{X}$ is asymptotically equivalent to the canonical basis of ℓ_p.

The terminology in Definition 3.1 was introduced in [10], where it is proved that the unit vector system of $\mathcal{T}^{(p)}$, $1 \le p < \infty$, is right dominant. This result was previously stated without a proof in [9]. The authors of [10] also showed that any left or right dominant unconditional basis is asymptotically equivalent to the canonical basis of ℓ_p for some $1 \le p \le \infty$. The fact that the unit vector system of $\mathcal{T}^{(p)}$ is Schreier unconditional can be easily inferred from [9, Proposition 7.4 and Proposition 7.5]. We refer the reader to [11, Proposition IV.b.1] for the proof of an even stronger result.

Given $d \in \mathbb{N}$, $d \ge 2$, we put

$$\Gamma_d\colon \mathbb{N} \to \mathbb{N}, \quad \Gamma_d(n) = d^n.$$

Lemma 3.4 *Suppose that $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^\infty$ is an unconditional basis of a Banach space $\mathbb{X}$. Then $\mathcal{X}$ is Schreier equivalent to the canonical basis of ℓ_p $(1 \le p \le \infty)$ if and only if it is Γ_d-asymptotically equivalent to the canonical basis of ℓ_p for some $d \in \mathbb{N}$ with $d \ge 2$, in which case it is Γ_d-asymptotically equivalent to canonical ℓ_p basis for all $d \in \mathbb{N}$ with $d \ge 2$.*

Proof Let $\gamma\colon \mathbb{N} \to \mathbb{N}$ be an increasing function. Clearly, for each $D \in \mathbb{N}$ there is a constant C such that if $\mathcal{X}$ is Schreier equivalent to the canonical basis of ℓ_p (resp., γ-asymptotically equivalent to the canonical basis of ℓ_p) then $(\boldsymbol{x}_n)_{n\in A}$ is C-equivalent to the canonical basis of $\ell_p(A)$ whenever $A \subseteq \mathbb{N}$ satisfies $|A| \le D \min(A)$ (resp., $\gamma(n-1) < \min(A)$ and $|A| \le D(\gamma(n) - \gamma(n-1))$ for some $n \in \mathbb{N}$).

Pick $d \in \mathbb{N}$, $d \ge 2$, and $A \subseteq \mathbb{N}$. If $\Gamma_d(n) < \min A$ and $|A| \le \Gamma_d(n) - \Gamma_d(n-1)$ for some $n \in \mathbb{N}$, then $|A| \le D \min(A)$, where $D = d$. This gives Schreier equivalence implies Γ_d-asymptotically equivalence. Conversely, if $|A| \le \min(A)$ and we choose $n \in \mathbb{N}$ such that $\Gamma_d(n-1) < \min(A) \le \Gamma_d(n)$, then $|A| \le D(\Gamma_d(n) - \Gamma_d(n-1))$, where $D = \lceil d/(d-1) \rceil$. This gives that Γ_d-asymptotic equivalence implies Schreier equivalence. □

Definition 3.5 We will say that a basis $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^\infty$ of a Banach space $\mathbb{X}$ is *block-stable* if it satisfies property (T.d).

While there are known examples of right dominant unconditional bases other than the canonical basis of Tsirelson's space and its convexifications [10], block-stability seems to be a special property of the p-convexified Tsirelson's space and ℓ_p, $1 \le p < \infty$. Both left (or right) dominance and block-stability imply unconditionality. Any block-stable basis, as well as any basis which is Schreier equivalent to standard ℓ_p basis for some p is semi-normalized.

By a subbasis of an unconditional basis $\mathcal{X} = (\boldsymbol{x}_n)_{n \in \mathcal{N}}$ we mean a family $\mathcal{Y} = (\boldsymbol{x}_n)_{n \in \mathcal{J}}$ for some $\mathcal{J} \subseteq \mathcal{N}$. If $\mathcal{N} = \mathbb{N}$, $\mathcal{Y}$ can be rearranged according to the total ordering that $\mathcal{J}$ inherits from $\mathbb{N}$, so it becomes naturally indexed with the set of natural numbers. With this convention, block-stability, as well as left and right dominance and Schreier equivalence, plainly passes to subbases. To be precise, we have the following.

Lemma 3.6 *Let $(\boldsymbol{x}_n)_{n=1}^\infty$ be a block stable (resp., left or right dominant, or Schreier equivalent to the canonical basis of ℓ_p, $1 \le p \le \infty$) unconditional basis. Let $\varphi \colon \mathbb{N} \to \mathbb{N}$ be increasing. Then $(\boldsymbol{x}_{\varphi(n)})_{n=1}^\infty$ is block stable (resp., left or right dominant, or Schreier equivalent to the canonical basis of ℓ_p).*

Despite its simplicity, Lemma 3.6 will play an important role in our proof of Theorem A.

Lemma 3.7 *If $(\boldsymbol{x}_n)_{n=1}^\infty$ is a block-stable (unconditional) basis of a Banach space and $\gamma \colon \mathbb{N} \to \mathbb{N}$ is an increasing sequence, then $(\boldsymbol{x}_{\gamma(n)})_{n=1}^\infty$ is equivalent to $(\boldsymbol{x}_{\gamma(n+1)})_{n=1}^\infty$.*

Proof By Lemma 3.6, it suffices to consider the case when γ is the identity map. Applying block-stability with $\alpha(j) = 2j$ and $f_j = \boldsymbol{x}_{2j-1}$, $j \in \mathbb{N}$, gives that $(\boldsymbol{x}_{2j-1})_{j=1}^\infty$ and $(\boldsymbol{x}_{2j})_{j=1}^\infty$ are equivalent. In turn, applying block-stability with the same map α and $f_j = \boldsymbol{x}_{2j+1}$, $j \in \mathbb{N}$, gives that $(\boldsymbol{x}_{2j})_{j=1}^\infty$ and $(\boldsymbol{x}_{2j+1})_{j=1}^\infty$ are equivalent. By unconditionality, $(\boldsymbol{x}_n)_{n=1}^\infty$ and $(\boldsymbol{x}_{n+1})_{n=1}^\infty$ are equivalent. □

Lemma 3.8 *Let $\gamma \colon \mathbb{N} \to \mathbb{N}$ be an increasing function with $\gamma(1) \ge 2$. Define*

$$\delta \colon \mathbb{N} \to \mathbb{N}, \quad \delta(n) = \gamma^{(n-1)}(1), \quad n \in \mathbb{N}.$$

Suppose that an unconditional basis $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^\infty$ of a Banach space $\mathbb{X}$ is block-stable, Schreier equivalent to the canonical basis of ℓ_p for some $1 \le p \le \infty$, and γ-asymptotically equivalent to the canonical basis of ℓ_p. Then $\mathcal{X}$ is δ-asymptotically equivalent to the canonical basis of ℓ_p.

Proof Fix $n \in \mathbb{N}$ and $A \subseteq \mathbb{N}$ with $\delta(n) < \min(A)$ and $|A| = \delta(n+1) - \delta(n)$. Let

$$\rho\colon (\delta(n), \delta(n+1)] \to A$$

be an increasing bijection. Considering the partition

$$(\gamma(j-1), \gamma(j)] \cap \mathbb{Z}, \quad j \in [\delta(n-1), \delta(n)) \cap \mathbb{Z},$$

of $(\delta(n), \delta(n+1)] \cap \mathbb{Z}$, and successively using block-stability, Schreier equivalence and asymptotic equivalence we obtain

$$\begin{aligned}
\left\| \sum_{k=1+\delta(n)}^{\delta(n+1)} a_k\, \boldsymbol{x}_{\rho(k)} \right\| &\approx \left\| \sum_{j=\delta(n-1)}^{-1+\delta(n)} \left\| \sum_{k=1+\gamma(j-1)}^{\gamma(j)} a_k\, \boldsymbol{x}_{\rho(k)} \right\| \boldsymbol{x}_{\rho(\gamma(j))} \right\| \\
&\approx \left(\sum_{j=\delta(n-1)}^{-1+\delta(n)} \left\| \sum_{k=1+\gamma(j-1)}^{\gamma(j)} a_k\, \boldsymbol{x}_{\rho(k)} \right\|^p \right)^{1/p} \\
&\approx \left(\sum_{j=\delta(n-1)}^{-1+\delta(n)} \left(\sum_{k=1+\gamma(j-1)}^{\gamma(j)} |a_k|^p \right)^{p/p} \right)^{1/p} \\
&= \left(\sum_{k=1+\delta(n)}^{\delta(n+1)} |a_k|^p \right)^{1/p}.
\end{aligned}$$

□

Lemma 3.9 *Let $(\boldsymbol{x}_n)_{n=1}^\infty$ and $(\boldsymbol{y}_n)_{n=1}^\infty$ be block-stable (unconditional) bases of Banach spaces $\mathbb{X}$ and $\mathbb{Y}$, respectively. Let $(A_j)_{j=1}^\infty$ be an increasing partition of $\mathbb{N}$ and $(B_j)_{j=1}^\infty$ be an increasing sequence of subsets of $\mathbb{N}$. Let $(f_n)_{n=1}^\infty$ be a sequence in $\mathbb{Y}$ such that*

$$f_n \in \left[\boldsymbol{y}_k \colon k \in B_j\right], \quad n \in A_j,\ j \in \mathbb{N}.$$

Suppose that there is a constant C such $(f_n)_{n \in A_j}$ is C-equivalent to $(\boldsymbol{x}_n)_{n \in A_j}$ for all $j \in \mathbb{N}$. Suppose also that there are $\alpha, \beta\colon \mathbb{N} \to \mathbb{N}$ such that $\alpha(j) \in A_j$ and $\beta(j) \in B_j$ for all $j \in \mathbb{N}$, and $(\boldsymbol{x}_{\alpha(j)})_{j=1}^\infty$ is equivalent to $(\boldsymbol{y}_{\beta(j)})_{j=1}^\infty$. Then $(f_n)_{n=1}^\infty$ is equivalent to $(\boldsymbol{x}_n)_{n=1}^\infty$.

Proof Let C_1 and C_2 be the block-stability constants of $\mathcal{X}$ and $\mathcal{Y}$, respectively. Let C_3 be the equivalence constant for the subsequences. Let K be the unconditionality constant of one them. Then, for any $(a_n)_{n=1}^\infty \in c_{00}$,

$$\left\| \sum_{n=1}^{\infty} a_n\, f_n \right\| \approx \left\| \sum_{j=1}^{\infty} \left\| \sum_{n \in A_j} a_n\, f_n \right\| \boldsymbol{y}_{\beta(j)} \right\|$$

$$\approx \left\| \sum_{j=1}^{\infty} \left\| \sum_{n \in A_j} a_n \, \boldsymbol{x}_n \right\| \boldsymbol{x}_{\alpha(j)} \right\|$$
$$\approx \left\| \sum_{n=1}^{\infty} a_n \, \boldsymbol{x}_n \right\|,$$

where the constant involved in the first equivalence is C_2, the constant we use in the second is $C_3 K C$, and the one in the last equivalence is C_1. □

Proposition 3.10 *Suppose that an unconditional basis* $(\boldsymbol{x}_n)_{n=1}^{\infty}$ *of a Banach space* $\mathbb{X}$ *is block-stable and Schreier equivalent to the canonical basis of* ℓ_p *for some* $p \in [1, \infty]$. *Then for any* $d \in \mathbb{N}$, $d \geq 2$, *the subbasis* $(\boldsymbol{x}_{d^n})_{n=1}^{\infty}$ *is equivalent to* $(\boldsymbol{x}_n)_{n=1}^{\infty}$.

Proof Let δ be as in Lemma 3.8 relative to $\gamma := \Gamma_d$. Aiming to apply Lemma 3.9 we put

$$A_j = (\delta(j-1), \delta(j)], \quad j \in \mathbb{N},$$

and $B_j = A_{j+1}$ for all $j \in \mathbb{N}$. Set $\alpha(j) = \delta(j)$ and $\beta(j) = \delta(j+1)$ for all $j \in \mathbb{N}$. By Lemma 3.7, $(\boldsymbol{x}_{\alpha(j)})_{j=1}^{\infty}$ and $(\boldsymbol{x}_{\beta(j)})_{j=1}^{\infty}$ are equivalent. Set $f_n = \boldsymbol{x}_{\gamma(n)}$ for all $n \in \mathbb{N}$. Notice that if $n \in A_j$, then $\alpha(n) \in \alpha(A_j) \subseteq B_j$, so that $f_n \in \left[\boldsymbol{x}_n : n \in B_j\right]$. By Lemmas 3.4 and 3.8, there is a constant C such that both $(\boldsymbol{x}_n)_{n \in A_j}$ and $(f_n)_{n \in A_j}$ are C-equivalent to the canonical basis of $\ell_p(A)$. Hence, $(\boldsymbol{x}_n)_{n \in A_j}$ and $(f_n)_{n \in A_j}$ are C^2-equivalent. By Lemma 3.9, $(\boldsymbol{x}_n)_{n=1}^{\infty}$ and $(f_n)_{n=1}^{\infty}$ are equivalent. □

Given an atomic lattice $\mathcal{L}$ over $\mathcal{N}$ and a family $(\mathbb{X}_n)_{n \in \mathcal{N}}$ of Banach spaces, we will put

$$\mathbb{Y} := \left(\bigoplus_{n \in \mathcal{N}} \mathbb{X}_n \right)_{\mathcal{L}}$$

for the Banach space consisting of all families $f = (f(n))_{n \in \mathcal{N}}$ in $\prod_{n \in \mathcal{N}} \mathbb{X}_n$ such that

$$L(f) := (\|f(n)\|)_{n \in \mathcal{N}} \in \mathcal{L},$$

endowed with the norm $f \mapsto \|L(f)\|$. If $\mathbb{X}_n = \mathbb{X}$ for all $n \in \mathcal{N}$ and some Banach space $\mathbb{X}$, we put $\mathbb{Y} = \mathcal{L}(\mathbb{X})$. If each $\mathbb{X}_n$ is an atomic lattice over a set $\mathcal{J}_n$, then $\mathbb{Y}$ is an atomic lattice over the set

$$\mathcal{M} := \bigcup_{n \in \mathcal{N}} \{n\} \times \mathcal{J}_n.$$

If $\mathcal{N} = \mathbb{N}$ and each $\mathcal{J}_n$ is a finite subset of $\mathbb{N}$, then $\mathcal{M}$ can be naturally arranged using the unique bijection $\pi : \mathbb{N} \to \mathcal{M}$ that is increasing when we consider on $\mathcal{M}$ the lexicographical order.

Corollary 3.11 *Let $\mathcal{L}$ be an atomic lattice over $\mathbb{N}$. Suppose the unit vector system $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^{\infty}$ of $\mathcal{L}$ is block-stable and Schreier equivalent to the canonical basis of ℓ_p for some $1 \le p \le \infty$. Then given $d \in \mathbb{N}$, $d \ge 2$, the natural arrangement of the unit vector system of*

$$\left(\bigoplus_{n=1}^{\infty} \ell_p^{\Gamma_d(n)-\Gamma_d(n-1)} \right)_{\mathcal{L}}$$

is equivalent to $\mathcal{X}$.

Proof Put $A_j = (\Gamma_d(j-1), \Gamma_d(j)] \cap \mathbb{Z}$ for all $j \in \mathbb{N}$. By Proposition 3.10, $(\boldsymbol{x}_{\Gamma_d(n)})_{n=1}^{\infty}$ is equivalent to $\mathcal{X}$. In turn, by Lemma 3.4, $(\boldsymbol{x}_n)_{n\in A_j}$ is uniformly equivalent to the canonical basis of $\ell_p(A_j)$. Hence, using block-stability and unconditionality yields

$$\begin{aligned} \left\| \sum_{n=1}^{\infty} a_n \, \boldsymbol{x}_n \right\| &\approx \left\| \sum_{j=1}^{\infty} \left\| \sum_{n\in A_j} a_n \, \boldsymbol{x}_n \right\| \boldsymbol{x}_{\Gamma_d(j)} \right\| \\ &\approx \left\| \sum_{j=1}^{\infty} \left\| \sum_{n\in A_j} a_n \, \boldsymbol{x}_n \right\| \boldsymbol{x}_j \right\| \\ &\approx \left\| \sum_{j=1}^{\infty} \left(\sum_{n\in A_j} |a_n|^p \right)^{1/p} \boldsymbol{x}_j \right\| . \end{aligned}$$

The class of unconditional bases satisfies the Schröder–Bernstein principle for unconditional bases. As Wojtaszczyk pointed out in [29], the authors of [9] seemed to be unaware of this neat result, so they needed to circumvent it in several places, which added an extra level of complexity to some proofs.

Theorem 3.12 ([30, Corollary 1]) *Let $\mathcal{X}$ and $\mathcal{Y}$ be unconditional bases of Banach spaces $\mathbb{X}$ and $\mathbb{Y}$, respectively. Suppose that $\mathcal{X}$ is permutatively equivalent to a subbasis of $\mathcal{Y}$ and, conversely, $\mathcal{Y}$ is permutatively equivalent to a subbasis of $\mathcal{X}$. Then $\mathcal{X}$ and $\mathcal{Y}$ are permutatively equivalent.*

A family $(f_j)_{j\in J}$ of vectors in a Banach space $\mathbb{X}$ is said to be *complemented* in $\mathbb{X}$ if its closed linear span is a complemented subspace of $\mathbb{X}$. If $\mathcal{Y} = (\boldsymbol{y}_j)_{j\in\mathcal{J}}$ is a complemented unconditional basic sequence of $\mathbb{X}$, then there exist a constant $C \in [1, \infty)$ and linear functionals $\mathcal{Y}^* := (\boldsymbol{y}_j^*)_{j\in\mathcal{J}}$ in $\mathbb{X}^*$ such that

(i) $(\boldsymbol{y}_j, \boldsymbol{y}_j^*)_{j\in\mathcal{J}}$ is a biorthogonal system, and
(ii) for each sequence $(\lambda_j)_{j\in\mathcal{J}}$ with $\sup_j |\lambda_j| \le 1$, the linear map

$$T : \mathbb{X} \to \mathbb{X}, \quad f \mapsto \sum_{j\in\mathcal{J}} \lambda_j \, \boldsymbol{y}_j^*(f) \, \boldsymbol{y}_j$$

is bounded by C.

If (i) and (ii) hold, we say that $\mathcal{Y}$ is C-complemented in $\mathbb{X}$ and that $\mathcal{Y}^*$ is a family of C*-projecting functionals* for $\mathcal{Y}$.

Theorem 3.13 ([9, Lemma 7.6]) *Let $\mathbb{Y}$ be a finite-dimensional subspace of a Banach space $\mathbb{X}$, and let P be a bounded linear projection from $\mathbb{X}$ onto $\mathbb{Y}$. Let $(\boldsymbol{y}_j)_{j\in\mathcal{J}}$ be a complemented unconditional basic sequence of $\mathbb{X}$ with C-projecting functionals $(\boldsymbol{y}_j^*)_{j\in\mathcal{J}}$. Then given $c>0$ there is $d=d(c,C)\in\mathbb{N}$, $d\geq 2$, such that*

$$\left|\left\{j\in\mathcal{J}\colon \left|\boldsymbol{y}_j^*(P(\boldsymbol{y}_j))\right|\geq c\right\}\right|\leq d^{\dim(\mathbb{Y})}.$$

The *lattification* of a sequence $\mathcal{G}=(g_j)_{j\in\mathcal{J}}$ in a Banach lattice is the atomic lattice over $\mathcal{J}$ defined by

$$\boldsymbol{L}[\mathcal{G}]=\left\{(a_j)_{j\in\mathcal{J}}\in\mathbb{F}^{\mathcal{J}}\colon \sup_{\substack{F\subseteq\mathcal{J}\\ |F|<\infty}}\left\|\left(\sum_{j\in F}\left|a_j\right|^2\left|g_j\right|^2\right)^{1/2}\right\|<\infty\right\}.$$

If $\mathcal{G}$ is a complemented unconditional basic sequence in $\mathcal{L}$ or $\mathcal{G}$ is an a unconditional basic sequence and $\mathcal{L}$ satisfies some nontrivial concavity, then the unit vector system of $\boldsymbol{L}[\mathcal{G}]$ is equivalent to $\mathcal{G}$ (see [21, Theorem 1.d.6 and subsequent Remark 1]). So carrying on the lattification procedure is useful in the case when we are not sure whether $\mathcal{G}$ is an unconditional basic sequence.

Theorem 3.14 ([3, Theorem 4.2]) *Let $\mathcal{Y}=(\boldsymbol{y}_n)_{j\in\mathcal{J}}$ be a complemented unconditional basis of a Banach lattice $\mathcal{L}$ with $(\boldsymbol{y}_j^*)_{j\in\mathcal{J}}$ as projecting functionals for $\mathcal{Y}$. Let $\mathcal{U}=(\boldsymbol{u}_j)_{j\in\mathcal{J}}$ be a family in $\mathcal{L}$ such that*

- $\left|\boldsymbol{u}_j\right|\leq\left|\boldsymbol{y}_j\right|$ *for all $j\in\mathcal{J}$, and*
- $\inf_{j\in\mathcal{J}}\left|\boldsymbol{y}_j^*(\boldsymbol{u}_j)\right|>0.$

Then the unit vector system of $\boldsymbol{L}[\mathcal{U}]$ is equivalent to $\mathcal{Y}$.

The lattification of a family in a Köthe space can be regarded as a subspace of a Hilbert-valued lattice. Let us record this obvious fact for reference.

Lemma 3.15 *Let $\mathcal{G}=(g_j)_{j\in\mathcal{J}}$ be a family in an atomic lattice $\mathcal{L}$ over a set $\mathcal{N}$. For each $n\in\mathcal{N}$, let σ_n be a one-to-one map from the set*

$$J_n:=\left\{j\in\mathcal{J}\colon g_j(n)\neq 0\right\}$$

into $\mathbb{N}$. If we regard $\mathcal{L}(\ell_2)$ as an atomic lattice over $\mathcal{N}\times\mathbb{N}$, then the unit vector system of $\boldsymbol{L}[\mathcal{G}]$ is isometrically equivalent to the disjointly supported sequence $(h_j)_{j\in\mathcal{J}}$ of $\mathcal{L}(\ell_2)$ given by

$$h_j(n,k)=\begin{cases} g_j(n) & \text{if } k=\sigma_n(j) \text{ for some } j\in J_n,\\ 0 & \text{otherwise.}\end{cases}$$

Theorem A will be derived from the universality for complemented unconditional basic sequences of the canonical basis of the space, which is reflected in the following theorem.

Theorem 3.16 (cf. [9, Proposition 7.8]) *Let $\mathbb{X}$ be a Banach space with an unconditional basis $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^{\infty}$. Suppose that $\mathcal{X}$ is left or right dominant and Schreier equivalent to the canonical basis of ℓ_2. Then any semi-normalized complemented unconditional basic sequence of $\mathbb{X}$ is permutatively equivalent to a subbasis of $\mathcal{X}$.*

Proof Without loss of generality we may assume that $\mathcal{X}$ is the unit vector system of an atomic lattice $\mathcal{L}$ over $\mathbb{N}$. Given an interval I of the real line we denote by $S_I : \mathcal{L} \to \mathcal{L}$ the canonical projection on $I \cap \mathbb{N}$.

Let $\mathcal{Y} = (\boldsymbol{y}_j)_{j\in\mathcal{J}}$ be a semi-normalized complemented unconditional basic sequence with C-projecting functionals $(\boldsymbol{y}_j^*)_{j\in\mathcal{J}}$, $C \in [1, \infty)$. Put

$$\mu(j) = \max\left\{n \in \mathbb{N} \colon \left|\boldsymbol{y}_j^*\left(S_{[n,\infty)}(\boldsymbol{y}_j)\right)\right| \ge \frac{1}{2}\right\},$$
$$\nu(j) = \max\left\{n \in \mathbb{N} \colon \left|\boldsymbol{y}_j^*\left(S_{[n,\infty)}(\boldsymbol{y}_j)\right)\right| \ge \frac{1}{4}\right\}$$

for each $j \in \mathcal{J}$. Define $\mathcal{U} = (\boldsymbol{u}_j)_{j\in\mathcal{J}}$ and $\mathcal{V} = (\boldsymbol{v}_j)_{j\in\mathcal{J}}$ by

$$\boldsymbol{u}_j = S_{[\mu(j),\nu(j)]}(\boldsymbol{y}_j), \quad \boldsymbol{v}_j = S_{[\nu(j),\infty)}(\boldsymbol{y}_j).$$

By construction, $\left|\boldsymbol{y}_j^*(\boldsymbol{v}_j)\right| \ge 1/4$. Since $\left|\boldsymbol{y}_j^*\left(S_{[\nu(j)+1,\infty)}(\boldsymbol{y}_j)\right)\right| < 1/4$ and $\left|\boldsymbol{y}_j^*\left(S_{[\mu(j),\infty)}(\boldsymbol{y}_j)\right)\right| \ge 1/2$, we have $\left|\boldsymbol{y}_j^*(\boldsymbol{u}_j)\right| \ge 1/4$. Set

$$U = \left\{(n, j) \in \mathbb{N} \times \mathcal{J} \colon \boldsymbol{u}_j(n) \neq 0\right\}, \; V = \left\{(n, j) \in \mathbb{N} \times \mathcal{J} \colon \boldsymbol{v}_j(n) \neq 0\right\}.$$

For each $n \in \mathbb{N}$, let U_n and V_n be the nth section of U and V, respectively. If $j \in U_n \cup V_n$, then $n \ge \mu(j)$, and so

$$\left|\boldsymbol{y}_j^*\left(S_{[n+1,\infty)}(\boldsymbol{y}_j)\right)\right| < 1/2.$$

Hence, $\left|\boldsymbol{y}_j^*\left(S_{[1,n]}(\boldsymbol{y}_j)\right)\right| \ge 1/2$. By Theorem 3.13 there is $d = d(C) \in \mathbb{N}$, $d \ge 2$, such that $|U_n \cup V_n| \le \Gamma_d(n) - \Gamma_d(n-1)$ for all $n \in \mathbb{N}$. Therefore for each $n \in \mathbb{N}$ there is a one-to-one map

$$\sigma_n \colon U_n \cup V_n \to A_n := (\Gamma_d(n-1), \Gamma_d(n)].$$

Set $\mathbb{Y} = [\mathcal{Y}]$. Let us regard

$$\mathcal{R} := \left(\bigoplus_{n=1}^{\infty} \ell_2(A_n)\right)_{\mathcal{L}}$$

as an atomic lattice over $\mathcal{M} := \left\{(n, k) \in \mathbb{N}^2 \colon k \in A_n\right\}$. Let $(\boldsymbol{e}_{n,k})_{(n,k)\in\mathcal{M}}$ be the unit vector system of $\mathcal{R}$. For each $j \in \mathcal{J}$, let U^j and V^j denote the jth section of U and

V, respectively. By Theorem 3.14 and Lemma 3.15, there are isomorphic embeddings $T_u : \mathbb{Y} \to \mathcal{R}$ and $T_v : \boldsymbol{L}(\mathcal{V}) \to \mathcal{R}$ such that

$$T_u(\boldsymbol{y}_j) = \sum_{n \in U^j} \boldsymbol{u}_j(n)\, \boldsymbol{e}_{n,\sigma_n(j)}, \quad T_v(\boldsymbol{y}_j) = \sum_{n \in V^j} \boldsymbol{v}_j(n)\, \boldsymbol{e}_{n,\sigma_n(j)}, \quad j \in \mathcal{J}.$$

In turn, by Corollary 3.11, there is an isomorphic embedding $T : \mathcal{R} \to \mathcal{L}$ such that $T(\boldsymbol{e}_{n,k}) = \boldsymbol{x}_k$ for all $(n, k) \in \mathcal{M}$. The isomorphic embeddings $T \circ T_u$ and $T \circ T_v$ witness that $\mathcal{Y}$ is equivalent to both $\mathcal{U}' = (\boldsymbol{u}'_j)_{j \in \mathcal{J}}$ and $\mathcal{V}' = (\boldsymbol{v}'_j)_{j \in \mathcal{J}}$, where $\mathcal{U}'$ and $\mathcal{V}'$ are the disjointly supported families in $\mathcal{L}$ defined by

$$\boldsymbol{u}'_j = \sum_{n \in U^j} \boldsymbol{u}_j(n)\, \boldsymbol{x}_{\sigma_n(j)}, \quad \boldsymbol{v}'_j = \sum_{n \in V^j} \boldsymbol{v}_j(n)\, \boldsymbol{x}_{\sigma_n(j)}, \quad j \in \mathcal{J}.$$

In particular, $\mathcal{U}'$ and $\mathcal{V}'$ are semi-normalized.

Fix $j \in \mathcal{J}$. Let $n \in U^j$ and $m \in V^j$. We have $\sigma_n(j) \in A_n$, $\sigma_m(j) \in A_m$, and $n \le \nu(j) \le m$. Hence $\sigma_n(j) < \sigma_m(j)$ unless $n = m = \nu(j)$. Consequently, $\operatorname{supp}(\boldsymbol{u}'_j) \le \operatorname{supp}(\boldsymbol{v}'_j)$. Set $\alpha(j) = \max(\operatorname{supp}(\boldsymbol{u}'_j))$ for all $j \in \mathcal{J}$. If $\mathcal{X}$ is right (resp., left) dominant then $(\boldsymbol{x}_{\alpha(j)})_{j \in \mathcal{J}}$ dominates $\mathcal{U}'$ (resp., $\mathcal{V}'$) and is dominated by $\mathcal{V}'$ (resp., $\mathcal{U}'$). Therefore, $(\boldsymbol{x}_{\alpha(j)})_{j \in \mathcal{J}}$ is equivalent to $\mathcal{Y}$. □

Theorem 3.17 *Let $\mathbb{X}$ be a Banach space with an unconditional basis $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^\infty$. Suppose that $\mathcal{X}$ is left or right dominant, block-stable, and Schreier equivalent to the canonical basis of ℓ_2. Let $\mathbb{Y}$ be a complemented subspace of $\mathbb{X}$ with an unconditional basis $\mathcal{Y}$. Then $\mathbb{Y}$ has a (UTAP) unique unconditional basis.*

Proof Without loss of generality we assume that $\mathcal{Y}$ is semi-normalized. By Theorem 3.16, $\mathcal{Y}$ is permutatively equivalent to a subbasis of $\mathcal{X}$. Thus, by Lemma 3.6, a suitable rearrangement of $\mathcal{Y}$ is left or right dominant, block-stable, and Schreier equivalent to the canonical basis of ℓ_2. Let $\mathcal{U}$ be another semi-normalized unconditional basis of $\mathbb{Y}$. Applying again Theorem 3.16 and Lemma 3.6 gives that $\mathcal{U}$ is permutatively equivalent to a subbasis of $\mathcal{Y}$, whence a suitable rearrangement of $\mathcal{U}$ is left or right dominant, block-stable, and Schreier equivalent to the canonical basis of ℓ_2. Applying Theorem 3.16 a third time gives that $\mathcal{Y}$ is permutatively equivalent to a subbasis of $\mathcal{U}$. By Theorem 3.12, $\mathcal{Y}$ and $\mathcal{U}$ are permutatively equivalent. □

Proof of Theorem A Bearing in mind that the canonical basis of $\mathcal{T}^{(2)}$ is block stable, right dominant, and Schreier equivalent to the canonical basis of ℓ_2, the result follows from Theorem 3.17.

4 Closing remarks and related open problems

Unlike for Banach spaces, the uniqueness of unconditional basis in nonlocally convex quasi-Banach spaces seems to be the norm rather than the exception. Kalton showed in [15] that a wide class of nonlocally convex Orlicz sequence spaces including the spaces ℓ_p for $0 < p < 1$ have a unique unconditional basis. This positive results motivated further study with a number of authors contributing to the development of a coherent theory. An important advance was the paper [16] by Kalton et al. followed by the work of Leránoz [18], who, in the spirit of the problems left open in the *Memoir* proved that $c_0(\ell_p)$ has a (UTAP) unique unconditional basis for all $0 < p < 1$, and Wojtaszczyk [29], who proved that the Hardy space $H_p(\mathbb{T})$ for $0 < p < 1$ also does. Subsequently, it was proved that $\ell_p(\ell_2)$, $\ell_p(\ell_1)$, and $\ell_1(\ell_p)$ enjoy the property as well for all $0 < p < 1$ ([5, 6]), and the question arose of what can be said in this respect about the direct sums of Tsirelson-like spaces with other quasi-Banach spaces with (UTAP) unique unconditional basis. The authors proved in [7] that $H_p(\mathbb{T}) \oplus \mathcal{T}^{(2)}$ has a (UTAP) unique unconditional basis and the same hods true with the spaces $\ell_2 \oplus \mathcal{T}^{(2)}$, $\ell_1 \oplus \mathcal{T}^{(2)}$, and $c_0 \oplus \mathcal{T}^{(2)}$ (see [1, Theorem 4.4]).

Historically, Tsirelson's space provided a single Banach space with a complex array of properties against which functional analysts could prove or disprove many conjectures. In this context it was conveniently used in [2] to show that the p-Banach space $\ell_p(\mathcal{T}^*)$ has a (UTAP) unique unconditional basis for all $0 < p < 1$ but that, oddly enough, its Banach envelope $\ell_1(\mathcal{T}^*)$ does not! This was the first known example of a quasi-Banach space where this happens.

In this direction of work the following problems still remain open.

Problem 4.1 Does $\ell_p(\mathcal{T}^{(2)})$ have a (UTAP) unique unconditional basis for $0 < p < 1$?

Problem 4.2 Does the Banach envelope of the spaces in Problem 4.1, i.e., $\ell_1(\mathcal{T}^{(2)})$, have a (UTAP) unique unconditional basis?

Problem 4.3 Does $c_0(\mathcal{T}^{(2)})$ have a (UTAP) unique unconditional basis?

Problem 4.4 Does $\ell_2(\mathcal{T}^{(2)})$ have a (UTAP) unique unconditional basis?

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