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On Kantorovich's conditions for Newton's method

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Abstract

We study the semilocal and local convergence of Newton's method in Banach spaces under a generalization of the classic conditions used in mathematical literature until now. We illustrate this study with some application.

Key words: Newton's method, semilocal convergence, Kantorovich-type conditions, majorizing sequences, local convergence
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1 Introduction

A large number of problems in applied mathematics and engineering are solve by finding solutions of equations. The most commonly used solution methods are iterative: from one or several initial approximations a sequence is constructed that converges to a solution of an equation. Newton's method is the most well-known iterative method to solve equations because of its simplicity, easy implementation and efficiency.

To give sufficient generality to the problem of approximating a solution of a nonlinear equation by Newton's method, we consider equations of the form $F(x) = 0$, where F is a nonlinear operator, $F : \Omega \subseteq X \to Y$, defined on a non-empty open convex domain Ω of a Banach space X with values in a Banach space Y, which is usually known as the Newton-Kantorovich method and whose algorithm is

$$x_0 \text{ given in } \Omega; \qquad x_n = x_{n-1} - [F'(x_{n-1})]^{-1} F(x_{n-1}), \quad n \in \mathbb{N}. \tag{1}$$

We can do different analysis of the convergence of Newton's method: local, semilocal or global, depending on the required conditions. We are interested in the semilocal and local convergence. There is a large number of local and semilocal convergence results on

Newton's method in Banach spaces. We refer to reader to [1, 2] and the references there for the history and recent results on Newton's method. Basic results concerning the convergence of the method have been published under assumptions of Kantorovich-type. An abundant list of references can be found in [1] and [5], where several techniques for finding sufficient conditions for the convergence of Newton's method are given.

The first semilocal convergence result for Newton's method in Banach spaces is due to L. V. Kantorovich, which is usually known as the Newton-Kantorovich theorem and is proved under the following conditions for the operator F and the starting point x_0:

$$(i)\ \|\Gamma_0\| \leq \beta, \qquad (ii)\ \|\Gamma_0 F(x_0)\| \leq \eta, \qquad (iii)\ \|F''(x)\| \leq M,\ x \in \Omega, \qquad (iv)\ M\beta\eta \leq \frac{1}{2},$$

where it is supposed that the operator $\Gamma_0 = [F'(x_0)]^{-1} \in \mathcal{L}(Y, X)$ exists at some $x_0 \in \Omega$, where $\mathcal{L}(Y, X)$ is the set of bounded linear operators from Y into X.

There are several techniques to prove the semilocal convergence of Newton's method. In this paper, we use the majorant principle to prove it, which is based on the concept of majorizing sequence. This technique was first developed by Kantorovich [4] and used later by many authors to analyse the semilocal convergence of different iterative methods. The majority of results presented in the mathematical literature demand that $\|F''(x)\|$ is bounded in the domain Ω, where the solution x^* must exist. According to this, the number of equations that can be solved by Newton's method is limited, since it is not easy to see that $\|F''(x)\|$ is bounded in a general domain Ω. It is not easy either to locate a domain where $\|F''(x)\|$ is bounded and the solution x^* is contained.

The main aim of this paper is to generalize the semilocal convergence conditions given by Kantorovich for Newton's method, so that conditions (iii)–(iv) are relaxed in order to Newton's method can be applied for solving more equations. To do this, we follow a variation of Kantorovich's technique based on the majorant principle.

Next, we emphasize a particular case of our general convergence conditions, prove that the R-order of convergence of Newton's method is at least two and give some a priori error bounds.

Finally, we also provide some application where our results are applied in this study.

2 New convergence conditions

Our idea in this paper is to generalize the hypotheses of Kantorovich by modifying conditions (iii)–(iv) and, following Kantorovich's theory [4], construct a real function $f \in \mathcal{C}^{(2)}([t_0, t'])$ with $t_0, t' \in \mathbb{R}$ which satisfies:

(I) There exists the operator $\Gamma_0 = [F'(x_0)]^{-1} \in \mathcal{L}(Y, X)$, for some $x_0 \in \Omega$, with $\|\Gamma_0\| \leq -\frac{1}{f'(t_0)}$, $\|\Gamma_0 F(x_0)\| \leq -\frac{f(t_0)}{f'(t_0)}$, and $\|F^{(i)}(x_0)\| \leq f^{(i)}(t_0)$ for $i = 2, 3, \ldots, k-1$.

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(II) $\|F^{(k)}(x)\| \leq f^{(k)}(t)$, for $\|x - x_0\| \leq t - t_0$, $x \in \Omega$ and $t \in [t_0, t']$.

Obviously, the above generalization of Kantorovich's hypotheses leads to a variation in the technique of the majorant principle, which Kantorovich develops to prove the semilocal convergence of Newton's method under conditions (i)–(iv).

To establish the semilocal convergence of Newton's method, under conditions (i)–(iv) Kantorovich considers a quadratic polynomial and proves that the semilocal convergence of Newton's sequence in the Banach space X is guaranteed from the scalar majorizing sequence which is constructed from the quadratic polynomial (see [4]).

To obtain the sequence $\{t_n\}$ Kantorovich uses a real function $f(t)$ defined in $[t_0, t'] \subset \mathbb{R}$ as:

$$t_0 \text{ given}; \qquad t_n = t_{n-1} - \frac{f(t_{n-1})}{f'(t_{n-1})}, \quad n \in \mathbb{N},$$

considers that $f(t)$ is a second degree polynomial and fits its coefficients with conditions (i) and (ii), so that Kantorovich obtains the polynomial

$$f(t) = \frac{M}{2}t^2 - \frac{t}{\beta} + \frac{\eta}{\beta}\cdot \tag{2}$$

Observe that this problem is of interpolation fitting.

In our case, if we consider (I)–(II), we cannot obtain a real function by interpolation fitting, since (II) does not allow determining the class of functions where (I) can be applied. To solve this problem, we proceed differently. Observe that polynomial (2) can be obtained otherwise, without interpolation fitting, by solving the following initial value problem:

$$y''(s) = M; \qquad y(t_0) = \frac{\eta}{\beta}, \quad y'(t_0) = -\frac{1}{\beta}\cdot$$

The new way of getting polynomial (2) has the advantage of being able to be generalized to conditions (I)–(II), so that we can then construct real functions $f(t)$ under more general conditions.

3 A particular case

In the following, we see a situation that can be deduced as particular case of our general conditions (I)–(II). We suppose that conditions (I)–(II) are reduced to the following conditions:

(A_1) $\|\Gamma_0\| \leq \beta$, $\|\Gamma_0 F(x_0)\| \leq \eta$, $\|F^{(i)}(x_0)\| \leq M_i$, with $M_i \in \mathbb{R}_+$ and $i = 2, 3, \ldots, k-1$,

(A_2) $\|F^{(k)}(x)\| \leq \omega(\|x\|)$, $x \in \Omega$, where $\omega : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}$ is a non-decreasing continuous function such that $\omega(0) = 0$.

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In order to be able to use the new way above-mentioned of getting polynomial (2) with conditions (A_1)–(A_2), we first note that we have $\|F^{(k)}(x)\| \leq \omega(\|x\|) \leq \omega(t - t_0 + \|x_0\|) \equiv \omega(t; t_0)$, provided that $\|x\| - \|x_0\| \leq \|x - x_0\| \leq t - t_0$, since ω is non-decreasing. In consequence, instead of (A_2), we consider $\|F^{(k)}(x)\| \leq \omega(t; t_0)$ for $\|x - x_0\| \leq t - t_0$, where $\omega : [t_0, +\infty) \to \mathbb{R}$ is a continuous non-decreasing function such that $\omega(t_0; t_0) \geq 0$. The corresponding initial value problem to solve is then

$$y^{(k)}(t) - \omega(t; t_0) = 0; \quad y(t_0) = \frac{\eta}{\beta},\ y'(t_0) = -\frac{1}{\beta},\ y^{(i)}(t_0) = M_i, \text{ for } i = 2, 3, \ldots, k-1,$$

whose solution is:

$$f(t) = \int_{t_0}^{t} \int_{t_0}^{s_{k-1}} \cdots \int_{t_0}^{s_1} \omega(z; t_0)\, dz\, ds_1 \cdots ds_{k-1} + \frac{M_{k-1}}{(k-1)!}(t-t_0)^{k-1} + \cdots + \frac{M_2}{2!}(t-t_0)^2 - \frac{t-t_0}{\beta} + \frac{\eta}{\beta}.$$

In addition, from the ideas of Dennis and Schnabel in [3], we obtain a local convergence result that leads to R-quadratic convergence of Newton's method under condition (A_2). Moreover, some a priori error bounds are given.

Finally, the above developed theory is illustrated with some application.

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Construction of hybrid iterative methods with memory

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Abstract

We present simple modifications of one-point iterative methods with memory that are called hybrid iterative methods and consist of two methods with different speed of convergence. The principal feature of these methods is that the accessibility regions are extended until that of the iterative method with lower speed of convergence.

Key words: Nonlinear equation in Banach spaces, iterative method with memory, hybrid method, semilocal convergence, decreasing region, accessibility region.
MSC 2000: 47H99, 65H10.

1 Introduction

Many scientific and engineering problems can be brought in the form of a nonlinear equation $F(x) = 0$, where F is a nonlinear operator defined on a non-empty open convex subset Ω of a Banach space X with values in another Banach space Y. In general, if the operator F is nonlinear, iterative methods are used to solve $F(x) = 0$. A very important aspect in the study of iterative methods is the choice of good initial approximations. In general, iterative methods usually converge once the initial approximations satisfy certain conditions (that is, semilocal convergence). The most used iterative processes are the well-known one-point iterations, that are defined as follows:

$$\begin{cases} z_0 \text{ given in } \Omega, \\ z_n = G(z_{n-1}), \quad n \in \mathbb{N}, \end{cases}$$

where G is an operator defined on Ω with values in the Banach space X.