Proceedings of the 16th International Conference on Computational and Mathematical Methods in Science and Engineering, CMMSE 2016
4–8 July, 2016.

Convergence of Newton's method under Vertgeim conditions: new extensions using restricted convergence domains

I. K. Argyros[1], J. A. Ezquerro[2], M. A. Hernández-Verón[2] and Á. A. Magreñán[3]

[1] *Department of Mathematics Sciences, Cameron University, USA.*

[2] *Department of Mathematics and Computation, University of La Rioja, Spain.*

[3] *School of Engineering, International University of La Rioja, Spain.*

emails: iargyros@cameron.edu, jezquer@unirioja.es, mahernan@unirioja.es, alberto.magrenan@unir.net

Abstract

We present new sufficient convergence conditions for the semilocal convergence of Newton's method to a locally unique solution of a nonlinear equation in a Banach space. We use Hölder and center Hölder conditions, instead of just Hölder conditions, for the first derivative of the operator involved in the convergence study. The new convergence conditions are weaker than others required in earlier studies.

Key words: Newton's method, semilocal convergence, recurrent functions, Hölder Hölder condition, center Hölder condition.
MSC 2000: 47H99, 65H10, 65J15.

1 Introduction

In general, when we consider the calculation of a solution of a nonlinear equation $F(x) = 0$, it is difficult to find an exact solution because there are hardly procedures that allow us to find it. For this reason, we usually apply iterative methods to approximate the solution. The best known and most used iterative method in practice is Newton's method, whose algorithm is:

$$\begin{cases} x_0 \text{ given}, \\ x_n = x_{n-1} - [F'(x_{n-1})]^{-1}F(x_{n-1}), \quad n \in \mathbb{N}. \end{cases}$$

So, the idea is clear: starting from a good initial approximation x_0 to a solution x^* of the equation $F(x) = 0$, we construct a sequence $\{x_n\}$ such that $x^* = \lim_{n\to\infty} x_n$, so that it is needed to impose enough conditions for the sequence is convergent.

To give sufficient generality to our study, we consider that $F : \Omega \subseteq X \longrightarrow Y$ is a nonlinear operator defined on a nonempty open convex subset Ω of a Banach space X with values in a Banach space Y. Many problems from computational sciences, physics and other disciplines can be brought into a form similar to equation $F(x) = 0$ using mathematical modelling. So, the unknowns of this equation can be functions (difference, differential, and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real/complex numbers (single algebraic equations with single unknowns).

The fact of considering our study in Banach spaces has the advantage that all sequences of Cauchy in Banach spaces are convergent. Therefore, in our study, it deals with proving that $\{x_n\}$ is a sequence of Cauchy. For this, we study the semilocal convergence of Newton's method, so that conditions on the starting point x_0 and on the operator involved F are required, along with a condition that guarantee that the sequence $\{x_n\}$ is of Cauchy from the conditions required to x_0 and F.

2 Semilocal convergence of Newton's method

The first semilocal convergence result for Newton's method in Banach spaces was given by Kantorovich [4] under the following conditions:

(C1) There exists $\Gamma_0 = [F'(x_0)]^{-1} \in \mathcal{L}(Y, X)$, for some $x_0 \in \Omega$, with $\|\Gamma_0\| \leq \beta$ and $\|\Gamma_0 F(x_0)\| \leq \eta$, where $\mathcal{L}(Y, X)$ is the set of bounded linear operators from Y to X.

(C2) $\|F''(x)\| \leq K$ for $x \in \Omega$.

(C3) $h = K\beta\eta \leq \frac{1}{2}$ and $B(x_0, t^*) \subset \Omega$, where $t^* = \frac{1-\sqrt{1-2h}}{h}\,\eta$.

A few years later, Ortega observes that the second derivative of the operator involved, F'', is not in the algorithm of Newton's method and that condition (C2) implies that F' is Lipschitz continuous in Ω. In addition, he presents in [6] a variant of the result given by Kantorovich where (C2) is replaced by a Lipschitz condition for F' in Ω; that is:

(C2b) $\|F'(x) - F'(y)\| \leq K\|x - y\|$ for $x, y \in \Omega$.

Later, other authors ([1, 3, 5, 7]) consider the following generalization of the last condition:

(C2c) $\|F'(x) - F'(y)\| \leq \ell\|x - y\|^p$ for $x, y \in \Omega$ and $p \in [0, 1]$,

 ISBN: 978-84-608-6082-2

which is known as a Hölder condition for F' in Ω. Obviously, if $p = 1$, then condition (C2c) is reduced to condition (C2b).

This generalization of condition (C2b) leads to a modification of condition (C3) given for the parameters appeared previously. So, for example, Hernández proves in [3] the semilocal convergence of Newton's method under conditions (C1), (C2c) and

(C3b) $a = \ell\beta\eta^p \leq z^*$, where z^* is the unique zero of the function

$$f(t) = (1+p)^p(1-t)^{1+p} - t^p$$

in the interval $(0, 1/2]$ and $B(x_0, r) \subset \Omega$, where $r = \frac{(1+p)(1-a)}{(1+p)-(2+p)a}\,\eta$.

In this work, by means of some modifications of condition (C2c), we try to weaken condition (C3b). Observe that if condition (C2c) is verified in a domain $\Omega_0 \subset \Omega$, the value of the constant ℓ is obviously lower and, therefore, condition (C3b) is more easily verifiable. Even if we consider this situation in the extreme case, $\Omega_0 = \{x_0\}$, that is:

$$\|F'(x) - F'(x_0)\| \leq \ell_0\|x - x_0\|^p \quad \text{with} \quad x \in \Omega \quad \text{and} \quad p \in [0, 1],$$

it is clear that $\ell_0 \leq \ell$. Note that the last condition is called: center Hölder condition for F' in Ω. As a consequence, we do a brief reminder of the main modifications obtained for the previous condition and, using recurrent functions [2], obtain a result which improves the previously ones obtained by other authors.

Acknowledgements

This work has been partially supported by the project MTM2014-52016-C2-1-P of Spanish Ministry of Economy and Competitiveness.

References

[1] I. K. Argyros, *Remarks on the convergence of Newton's method under Hölder continuity conditions*, Tamkang J. Math. **23** (1992) 269–277.

[2] I. K. Argyros and S. Hilout, *On the weakening of the convergence of Newton's method using recurrent functions*, J. Complexity **25** (2009) 530–543.

[3] M. A. Hernández, *The Newton method for operators with Hölder continuous first derivative*, J. Optim. Theory Appl. **109** (2001) 631–648.

[4] L. V. Kantorovich and G. P. Akilov, *Functional analysis*, Pergamon Press, Oxford, 1982.

[5] H. B. Keller, *Numerical methods for two-point boundary value problems*, Dover Pub., New York, 1992.

[6] J. M. Ortega, *The Newton-Kantorovich theorem*, Amer. Math. Monthly **75** (1968) 658–660.

[7] J. Rokne, *Newton's method under mild differentiability conditions with error analysis*, Numer. Math. **18** (1972) 401–412.

  ISBN: 978-84-608-6082-2