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Two Dynamic Remarks on the Chebyshev–Halley Family of Iterative Methods for Solving Nonlinear Equations

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Abstract: The aim of this paper is to delve into the dynamic study of the well-known Chebyshev–Halley family of iterative methods for solving nonlinear equations. Our objectives are twofold: On the one hand, we are interested in characterizing the existence of extraneous attracting fixed points when the methods in the family are applied to polynomial equations. On the other hand, we are also interested in studying the free critical points of the methods in the family, as a previous step to determine the existence of attracting cycles. In both cases, we want to identify situations where the methods in the family have bad behavior from the root-finding point of view. Finally, and joining these two studies, we look for polynomials for which there are methods in the family where these two situations happen simultaneously. The rational map obtained by applying a method in the Chebyshev–Halley family to a polynomial has both super-attracting extraneous fixed points and super-attracting cycles different from the roots of the polynomial.

Keywords: Chebyshev–Halley family of iterative methods; extraneous fixed points; critical points; parameter plane; dynamical systems

MSC: 37F10; 65H05

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1. Introduction

The well-known Chebyshev–Halley family of iterative methods ($\mathcal{CH}$ from now on) for solving nonlinear scalar equations, $f(z) = 0$, was first introduced by Werner [1] in 1980. This family, initially defined for real-valued functions, has been extended to complex variables, systems of equations, or even equations defined in Banach spaces, as can be seen, for instance, in [2–5]. In this work, we focus our interest on functions defined on the complex plane $f : \mathbb{C} \to \mathbb{C}$.

Each member in $\mathcal{CH}$ is given by an iteration map, which can be seen as a modification of the Newton iteration map,

$$N(z) = z - \frac{f(z)}{f'(z)}, \tag{1}$$

with the inclusion of a parameter $\alpha \in \mathbb{C}$ and new evaluations of $f(z)$ and its derivatives up to order two. Specifically, we build a sequence $z_{n+1} = G_\alpha(z_n)$, $n \geq 0$, where

$$G_\alpha(z) = z - \left(1 + \frac{1}{2}\frac{L_f(z)}{1 - \alpha L_f(z)}\right)\frac{f(z)}{f'(z)}, \tag{2}$$

and

$$L_f(z) = \frac{f(z)f''(z)}{f'(z)^2}. \tag{3}$$

All methods in $\mathcal{CH}$ have the cubic order of convergence for simple roots. In the case of multiple roots, there exist several variants of $\mathcal{CH}$ that make it possible to recover the cubic order of convergence, as given by Ivanov [6] or Osada [7].

One of the reasons for considering the $\mathcal{CH}$ family is because it allows us to study in a unified way the most famous third-order iterative methods, such as Chebyshev's method ($\alpha = 0$), Halley's method ($\alpha = 1/2$), or super-Halley method ($\alpha = 1$). In general, it is not possible to establish a classification of the members of $\mathcal{CH}$ in terms of efficiency, velocity of convergence, and other similar numerical criteria because the behavior of $G_\alpha(z)$ depends on the considered function $f(z)$ through the asymptotic error constant (see [8] for more details). However, for particular problems, it is possible to find the optimal method in $\mathcal{CH}$ to approximate the solution. This is the case for the calculus of nth roots, (see Dubeau and Gnang [9] or Gutiérrez, Hernández, and Salanova [10] for a more detailed study), the computation of the matrix sign (Cordero et al. [11]), or the simultaneous calculus of all the roots of a polynomial (Osada [7]).

The dynamical study of the iteration maps arising from the application of the methods in $\mathcal{CH}$ (2) to polynomial equations is a problem that has attracted many researchers. For instance, Cordero, Torregrosa, and Vindel [12] studied the dynamics of the methods in $\mathcal{CH}$ applied to quadratic polynomials. In the analysis of the corresponding parameter planes appears a singular set, baptized as "the cat" by the authors, with curious similarities with the Mandelbrot set. In this same paper, the existence of members in $\mathcal{CH}$ with a pathological behavior as root-finding methods is emphasized; they can be attracted by limits that are not roots of the considered equation $f(z) = 0$, such as periodic orbits (cycles) or extraneous fixed points. In [13,14], it is proved that there exist methods in $\mathcal{CH}$ with attracting two-cycles; Campos et al. [15,16] studied the behavior of $\mathcal{CH}$ for polynomials in the form $z^n + c$, where c is a complex parameter. In particular, they characterized methods with Fatou components that are simply connected and, hence, the Julia set is connected; Gutiérrez, Magreñán, and Varona [17] characterized the universal Julia sets for the methods in $\mathcal{CH}$ applied to quadratic polynomials; in this same line, Babajee, Cordero, and Torregrosa [18] introduced the Cayley Quadratic Test as a first step in the study of the stability of families of iterative processes for solving nonlinear equations. In brief, this test allows us to check if the universal Julia set of an iterative process is conjugated with the unit circle or not.

In the rest of this work, we continue with the dynamic study of the methods of the $\mathcal{CH}$-family. Specifically, in Section 2, we characterize the existence of super-attracting fixed points for the methods in $\mathcal{CH}$. We place a special emphasis on the case of the Chebyshev and super-Halley methods. For these methods, we even prove the existence of polynomials with both super-attracting fixed points and super-attracting cycles. In Section 3, we study the number of critical points of the methods in $\mathcal{CH}$. In particular, we show that the graphical tool known as the parameter plane is useful only for two methods in the $\mathcal{CH}$ family: $\alpha = 0$ (Chebyshev's method) and $\alpha = 1/2$ (Halley's method). These two cases have been profusely studied by Gutiérrez and Varona [19] and by Roberts and Horgan-Kobelski [20], respectively. For the rest of the methods in the family, the high number of free critical points and the difficulty of obtaining them discourage the use of the parameter plane.

2. Fixed Points in the Family $\mathcal{CH}$

In this section, we apply the methods introduced in (2) to a polynomial equation

$$p(z) = 0, \quad p(z) = z^d + a_{d-1}z^{d-1} + \cdots + a_1 z + a_0, \tag{4}$$

where the coefficients a_j, $j = 0, 1, \ldots, d-1$ are constant complex numbers. We can assume, without loss of generality, that $p(z)$ is a monic polynomial. We use the notation $R_{\alpha,p}(z)$ for the rational map obtained in this case

$$R_{\alpha,p}(z) = z - \left(1 + \frac{1}{2}\frac{L_p(z)}{1 - \alpha L_p(z)}\right)\frac{p(z)}{p'(z)}, \tag{5}$$

where $L_p(z)$ is defined in (3). In addition, we consider the rational map related to Newton's method (1) in the polynomial case

$$R_p(z) = z - \frac{p(z)}{p'(z)}. \tag{6}$$

The first thing to complete for investigating the dynamics of the rational maps defined in (5) is to study its degree. Following Nayak and Pal [21], we can generalize the result they give for Chebyshev's method to all the methods in the family (5). The exact degree is given in terms of the number of distinct roots of p and in terms of certain types of critical points introduced by these authors. Indeed, given a polynomial $p(z)$, a critical point $\omega \in \mathbb{C}$ is called special if $p(\omega) \neq 0$ and $p''(\omega) = 0$.

Theorem 1. *Let $p(z)$ be the polynomial defined in* (4)*. Let m, n, and r denote the number of its distinct simple roots, double roots, and roots of multiplicity bigger than two, respectively. Let s be the number of distinct special critical points of $p(z)$. Then, for $\alpha \neq 1/2$,*

$$\deg(R_{\alpha,p}(z)) = 3(m+n+r) - 2 - B + s,$$

where B is the sum of the multiplicities of all the special critical points. If $p(z)$ has no special critical points, then $\deg(R_{\alpha,p}(z)) = 3(m+n+r) - 2$. If $p(z)$ has no special critical points or multiple roots, then $\deg(R_{\alpha,p}(z)) = 3d - 2$.

Proof. This proof mimics the one given by Nayak and Pal [21] for the case of Chebyshev's method, $R_{0,p}(z)$ given by $\alpha = 0$ in (5). Indeed, this proof is based on the following factorization of the polynomial $p(z)$, in terms of its simple, double, and multiple (with multiplicity bigger than two) roots:

$$p(z) = \prod_{i=1}^{m}(z-\alpha_i)\prod_{j=1}^{n}(z-\beta_j)^2\prod_{k=1}^{r}(z-\gamma_k)^{a_k},$$

with $\deg(p) = m + 2n + M$, $M = \sum_{k=1}^{r} a_k$.

So, let c_j, $j = 1, \ldots, s$ be the special critical points of $p(z)$, with multiplicities b_j, then the quotient $L_p(z)$ defined in (3) can be written in the form

$$L_p(z) = \frac{\tilde{h}(z)}{\tilde{g}(z)^2}\prod_{i=1}^{m}(z-\alpha_i)\frac{1}{\prod_{j=1}^{s}(z-c_j)^{b_j+1}},$$

where $\tilde{h}(z)$ and $\tilde{g}(z)$ are functions without common roots. It can be seen (in [21]) that

$$\deg L_p(z) = 2m + 2n + 2r - 2 - B - s.$$

Note that $\deg(R_{\alpha,p}(z))$ equals the number of fixed points of $R_{\alpha,p}(z)$ minus one (see [22]), and the number of fixed points of $R_{\alpha,p}(z))$ are as follows:

1. The number of different roots of $p(z)$: $m + n + r$.
2. The roots of the equation $L_p(z) = 2/(2\alpha - 1)$, $\alpha \neq 1/2$, that is, $2m + 2n + 2r - 2 - B + s$.
3. The infinity point.

Consequently, $\deg(R_{\alpha,p}(z)) = 3(m+n+r) - 2 - B + s$. □

As a root-finder method, it would be desirable that each attracting fixed point of $R_{\alpha,p}(z)$ would be a root of $p(z)$. However, this "ideal behavior" is disturbed by the appearance of other attracting phenomena, such as periodic orbits (cycles) or extraneous fixed points, which are fixed points of $R_{\alpha,p}(z)$ that are not roots of $p(z)$.

It is well-known that the only fixed points of Newton's method in the complex plane are the roots of $p(z)$ (see the classical book of Traub [23], for instance). If we consider the extended complex plane, it is also known that the infinity point is a repelling fixed point for Newton's method, with the multiplier $d/(d-1) > 1$, where d is the degree of $p(z)$. In addition, we have that simple roots of $p(z)$ are super-attracting fixed points of $R_p(z)$, whereas roots with the multiplicity $m > 1$ are attracting fixed points with the multiplier $(m-1)/m < 1$.

For the case of Halley's method (obtained for $\alpha = 1/2$ in (5)), it is known (see Kneisl [24]) that the only attracting fixed points are the roots of $p(z)$. There exist extraneous fixed points in the complex plane, but all of them are repelling with the multiplier $1 + 2/j$ for an adequate $j \in \mathbb{N}$. The infinity point is a repelling fixed point for Halley's method, with the multiplier $(d+1)/(d-1) > 1$, where d is the degree of $p(z)$.

For the case of the Chebyshev method ($\alpha = 0$ in (5)), the existence of attracting extraneous fixed points has been proven (see [24] Theorem 2.6.4 or Vrscay–Gilbert [25] p. 12). In addition, in [19], the existence of super-attracting extraneous fixed points has been established in terms of the quotients $L_p(z)$, defined in (4), and $L_{p'}(z)$, defined by

$$L_{p'}(z) = \frac{p'(z)p'''(z)}{p''(z)^2}. \tag{7}$$

In particular, ω is an extraneous fixed point of Chebyshev's method if $L_p(z) = -2$. ω is attracting if $\left|6 - 2L_{p'}(z)\right| < 1$ and super-attracting if $L_{p'}(z) = 3$.

In this section, we are going to generalize this result by determining sufficient conditions for the existence of super-attracting extraneous fixed points for the methods in $\mathcal{CH}$. First, we establish a preliminary technical result that will help us in further theoretical development.

Lemma 1. *Let $p(z)$ be a d-degree polynomial and $L_p(z)$, $L_{p'}(z)$ be the rational functions defined in (4) and (7), respectively. Then,*

$$L_p'(z)\frac{p(z)}{p'(z)} = L_p(z) - 2L_p(z)^2 + L_{p'}(z)L_p(z)^2. \tag{8}$$

Proof. The proof simply requires a process of derivation and the grouping of terms in an appropriate way. First,

$$L_p'(z) = \left(\frac{p(z)p''(z)}{p'(z)^2}\right)' = \frac{(p'(z)p''(z) + p(z)p'''(z))p'(z)^2 - 2p(z)p'(z)p''(z)^2}{p'(z)^4},$$

and next,

$$L_p'(z)\frac{p(z)}{p'(z)} = L_p(z) + \frac{p(z)^2p'''(z)}{p'(z)^3} - 2L_p(z)^2.$$

Just by multiplying and dividing by $p''(z)^2/p'(z)$ in the second term, and taking (7) into account, we arrive at the result. □

Theorem 2. *For $\alpha \neq 1/2$, let $\omega \in \mathbb{C}$ be a point such that $p(\omega) \neq 0$, $p'(\omega) \neq 0$, $p''(\omega) \neq 0$. Then, if*

$$L_p(\omega) = \frac{2}{2\alpha - 1}, \quad L_{p'}(\omega) = 3 - \alpha, \tag{9}$$

ω is a super-attracting extraneous fixed point of the rational map $R_{\alpha,p}(z)$ defined in (5) and, therefore, of the iterative method corresponding to the parameter α in $\mathcal{CH}$.

Proof. First, as $p(\omega) \neq 0$, ω is not a root of $p(z)$. In addition, ω is a fixed point of $R_{\alpha,p}(z)$ because

$$1 + \frac{1}{2}\frac{L_p(\omega)}{1 - \alpha L_p(\omega)} = 0, \tag{10}$$

just by taking into account the first equation in (9). So, ω is an extraneous fixed point of $R_{\alpha,p}(z)$.

For ω to be a super-attractor, it must be true that $R'_{\alpha,p}(\omega) = 0$. Then, taking into account (10), we have

$$R'_{\alpha,p}(\omega) = 1 - \left(1 + \frac{1}{2}\frac{L_p(z)}{1 - \alpha L_p(z)}\right)'\Bigg|_{z=\omega} \frac{p(\omega)}{p'(\omega)}.$$

Note that

$$\left(1 + \frac{1}{2}\frac{L_p(z)}{1 - \alpha L_p(z)}\right)' = \frac{1}{2}\frac{L'_p(z)}{(1 - \alpha L_p(z))^2},$$

and then, by Lemma 1,

$$R'_{\alpha,p}(\omega) = 1 - \frac{1}{2}\frac{L_p(\omega) - 2L_p(\omega)^2 + L_{p'}(\omega)L_p(\omega)^2}{(1 - \alpha L_p(\omega))^2}.$$

As $L_p(\omega) = 2/(2\alpha - 1)$, we obtain

$$R'_{\alpha,p}(\omega) = 2(3 - \alpha - L_{p'}(\omega)).$$

Finally, as $L_{p'}(\omega) = 3 - \alpha$, the condition $R'_{\alpha,p}(\omega) = 0$ also holds and, as a consequence, ω is a super-attracting extraneous fixed point of the rational map $R_{\alpha,p}(z)$. □

The previous theorem allows us to find, as long as its conditions are met, extraneous super-attractor fixed points for all the methods in the family $\mathcal{CH}$, with the exception of Halley's method ($\alpha = 1/2$). Note that the conditions $p(\omega) \neq 0$, $p'(\omega) \neq 0$, $p''(\omega) = 0$ imply $L_p(\omega) = 0$, which makes the existence of extraneous super-attractor fixed points impossible. If $\alpha \neq 1/2$, $p(\omega) \neq 0$ and $p'(\omega) = 0$, the expression of the corresponding method in (5) is not well-defined.

For Halley's method, we can give the following result, which was also proven by Kneisl ([24] Theorem 2.6.3). Previously, we wrote the corresponding iteration function with an alternative expression:

$$R_{1/2,p}(z) = z - \left(1 + \frac{1}{2}\frac{L_p(z)}{1 - L_p(z)/2}\right)\frac{p(z)}{p'(z)} = z - \frac{2p(z)p'(z)}{2p'(z)^2 - p(z)p''(z)}. \tag{11}$$

Theorem 3. *Let us consider Halley's method, obtained for $\alpha = 1/2$ in $\mathcal{CH}$ and whose iteration function is shown in (11). The extraneous fixed points of the rational function $R_{1/2,p}(z)$ are the solutions of $p'(z) = 0$, with $p(z) \neq 0$. All of them are repulsors.*

Proof. The fixed points of $R_{1/2,p}(z)$ are the roots of $p(z)$ and the solutions of $p'(z) = 0$. Then, ω is an extraneous fixed point of $R_{1/2,p}(z)$ if, and only if, $p'(\omega) = 0$ and $p(\omega) \neq 0$. Let $m \in \mathbb{N}$ be the multiplicity of ω as a root of $p'(z)$. We can then write

$$p'(z) = (z - \omega)^m g(z),$$

with $g(\omega) \neq 0$. After a few calculi in (11), we obtain

$$R'_{1/2,p}(\omega) = 1 + \frac{2}{m} > 1.$$

Therefore, ω is a repelling fixed point. □

Remark 1. *In the extended complex plane, the infinity point is a fixed point for the methods in $\mathcal{CH}$, with the multiplier*

$$\frac{2d(d(\alpha-1)-\alpha)}{(d-1)(2d(\alpha-1)-2\alpha+1)}.$$

In the case of the most famous methods in the family, infinity is a repelling fixed point. Indeed, for $\alpha = 0, 1/2, 1$, that is, the Chebyshev, Halley, and super-Halley methods, the multiplier of the infinity point is $2d^2/((2d-1)(d-1))$, $(d+1)/(d-1)$, and $2d/(d-1)$, respectively. However, we can see that infinity is not always a repelling fixed point. Even more, for each degree $d \geq 2$, we can obtain a method corresponding with the value of

$$\alpha = \frac{d}{d-1},$$

for which infinity is a super-attracting fixed point. In Figure 1, we show the basins of attraction related to the method in $\mathcal{CH}$ with $\alpha = 2$ applied to $p(z) = z^2 - 1$ and related to the method in $\mathcal{CH}$ with $\alpha = 3/2$ applied to $p(z) = z^3 - 1$. Together with the basins of the roots of these polynomials, we have plotted in white the basin of attraction of the infinity point.

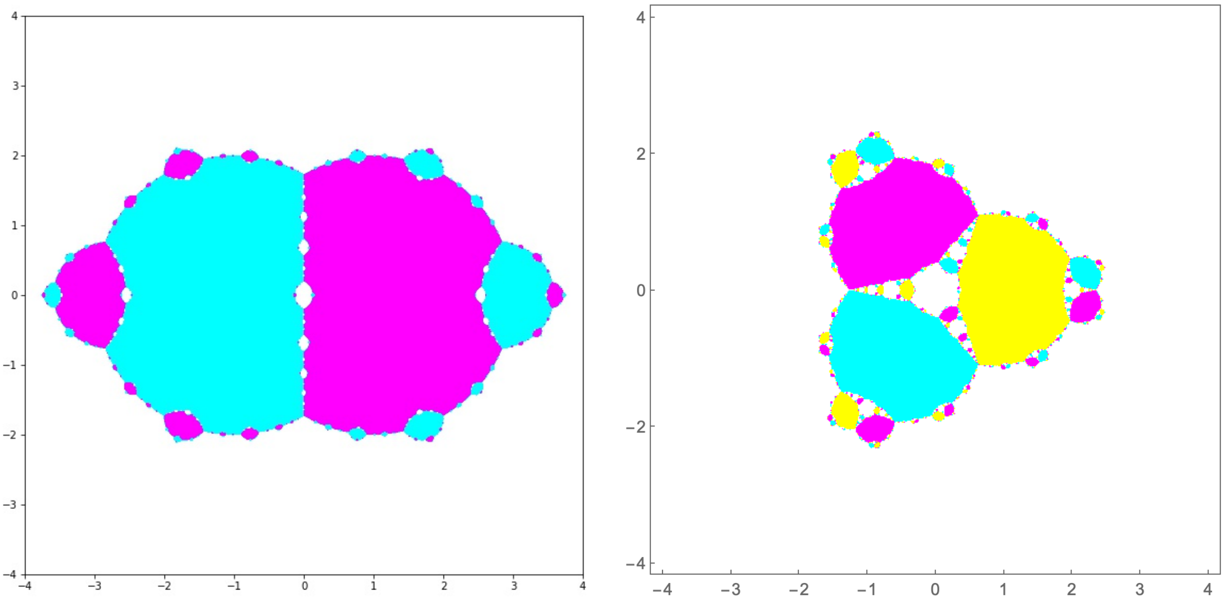

Figure 1. (On the **left**), basins of attraction of the method corresponding to $\alpha = 2$ applied to the polynomial $p(z) = z^2 - 1$. (On the **right**), basins of the method corresponding to $\alpha = 3/2$ are applied to the polynomial $p(z) = z^3 - 1$. In both cases, the basin of attraction of the infinity point can be seen in white, whereas the basins of the roots are colored cyan-magenta or cyan-magenta-yellow, respectively.

We are now going to characterize polynomials for which $z = 0$ is a strange super-attractor fixed point for the methods of the (5) family. To perform this, we consider the generic polynomial of degree d defined in (4).

Theorem 4. *Let $p(z)$ be the polynomial defined in (4) and $R_{\alpha,p}(z)$ be the rational map defined in (5), with $\alpha \neq 1/2$. Then, if*

$$a_1 = \beta, \quad a_2 = \frac{\beta^2}{2\alpha - 1}, \quad a_3 = \frac{2(3-\alpha)}{3}\frac{\beta^3}{(2\alpha-1)^2},$$

with $\beta \in \mathbb{C} \setminus \{0\}$, $z = 0$ is a super-attracting extraneous fixed point of $R_{\alpha,p}(z)$.

Proof. Note that since $z = 0$ must not be a root of $p(z)$, we can consider, without loss of generality, that $a_0 = 1$. Furthermore, for the conditions of Theorem 2 to be fulfilled, $p'(0) \neq 0$ and $p''(0) \neq 0$, it is also necessary that $a_1 \neq 0$ and $a_2 \neq 0$. The proof continues simply by solving the system given by Equation (9) which, in this case, is

$$L_p(0) = \frac{2a_2}{a_1^2} = \frac{2}{2\alpha - 1}, \quad L_{p'}(0) = \frac{3a_1a_3}{2a_2^2} = \frac{2}{2\alpha - 1}.$$

□

In particular cases, we can obtain polynomials with $z = 0$ as an extraneous super-attracting fixed point for Chebyshev's method (already known by García-Olivo et al. [26]). In this case,

$$a_1 = \beta \neq 0, \quad a_2 = -\beta^2, \quad a_3 = 2\beta^3. \tag{12}$$

The super-Halley method is another well-known method in $\mathcal{CH}$ ([27,28]). However, its dynamical properties have been less studied. We can obtain polynomials with $z = 0$ as an extraneous super-attracting fixed point for the super-Halley method:

$$a_1 = \beta \neq 0, \quad a_2 = \beta^2, \quad a_3 = \frac{4}{3}\beta^3. \tag{13}$$

In the left side of Figure 2, we show the basins of attraction of Chebyshev's method applied to the polynomial

$$p(z) = z^3 + z^2 + 2z - 4. \tag{14}$$

It is obtained by taking $\beta = 1/2$ in (12) and, next, by multiplying by four. The basin of the root $z = 1$ appears colored in cyan, that of the root $z = -1 + \sqrt{3}i$ in yellow, and that of the root $z = -1 - \sqrt{3}i$ in magenta. The basin of attraction of the extraneous fixed point appears in white.

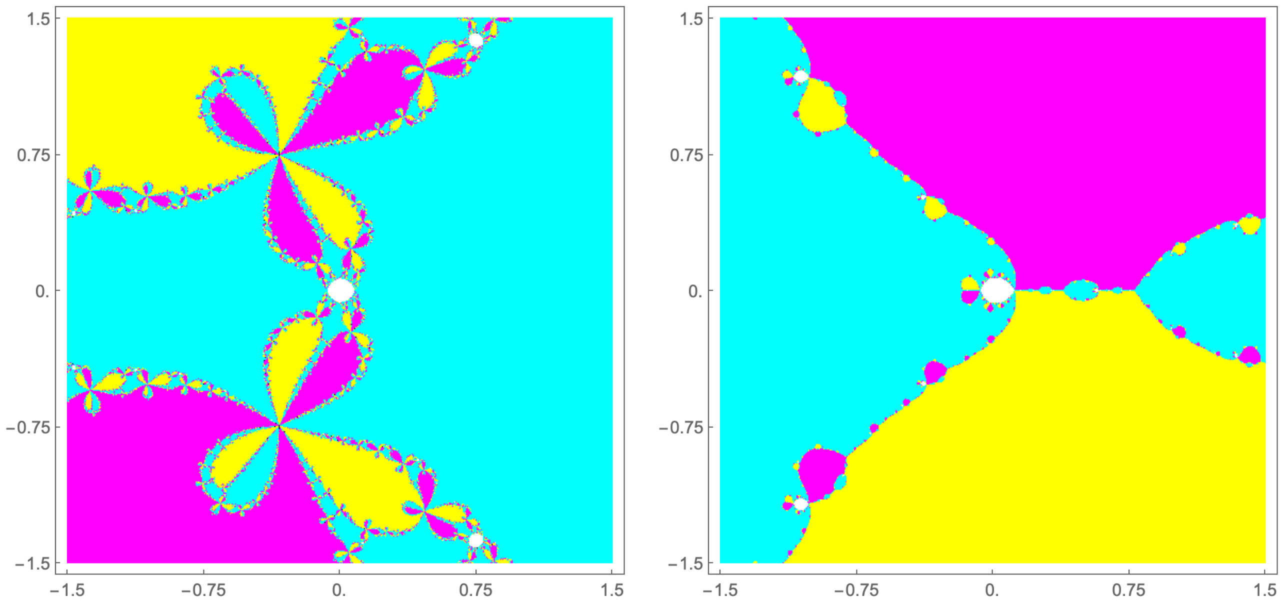

Figure 2. (On the **left**) the basins of attraction of the Chebyshev method applied to the polynomial (14). (On the **right**) the basins of attraction of the super-Halley method applied to the polynomial (15). In both cases, the basin of attraction of the extraneous fixed point at $z = 0$ can be seen in white.

In the right part of Figure 2, we show the basins of attraction of the super-Halley applied to the polynomial

$$p(z) = 4z^3 + 3z^2 + 3z + 3. \tag{15}$$

It is obtained by taking $\beta = 1$ in (13) and, next, by multiplying by three. The basin of the three roots $z = -0.873873$, $z = 0.0619363 - 0.924344i$, and $z = 0.0619363 + 0.924344i$ are colored in cyan, yellow, and magenta, respectively. The basin of attraction of the extraneous fixed point $z = 0$ appears in white.

In general, Chebyshev's method applied to polynomials in the form

$$p(z) = -1 + az + a^2z^2 + 2a^3z^3 + \sum_{j=4}^{d} a_j z^j \tag{16}$$

has a super-attracting extraneous fixed point at $z = 0$. Actually, we have

$$p(0) \neq 0, \quad L_p(0) = -2, \quad L_{p'}(0) = 3,$$

so the conditions in Theorem 2 are fulfilled.

In a similar way, the super-Halley method applied to polynomials in the form

$$p(z) = 1 + az + a^2z^2 + 4/3a^3z^3 + \sum_{j=4}^{d} a_j z^j \tag{17}$$

has a super-attracting extraneous fixed point at $z = 0$. Indeed, we have

$$p(0) \neq 0, \quad L_p(0) = L_{p'}(0) = 2,$$

so the conditions in Theorem 2 are satisfied.

3. Critical Points in the Family $\mathcal{CH}$

The parameter plane (space) is a very powerful graphical tool for better understanding the dynamic behavior of an iterative method for solving a family of nonlinear equations depending on a complex parameter. It is based on the Fatou–Julia Theorem [22], which says that the immediate basin of attraction of a (super) attractor cycle contains at least one critical point. Consequently, to determine the existence of attracting behaviors (fixed points and cycles), we must study the iterations of the critical points of the iteration function in question.

Let us restrict our interest to the case of iterative methods applied to polynomial Equation (4). In this case, a free critical point of an iterative method is a critical point of the corresponding iteration map that is not a root of the polynomial $p(z)$. Taking into account that the roots of $p(z)$ are (super) attracting fixed points of the iteration map, all of them have their own basin of attraction that is related to a critical point (the same root). Therefore, to detect attracting behaviors different from the root, we must follow the orbits of the free critical point.

For example, G. Roberts and J. Horgan-Kobelski [20] characterize cubic polynomials in the form

$$p_\lambda(z) = (z-1)(z+1)(z-\lambda), \quad \lambda \in \mathbb{C}, \tag{18}$$

for which Newton's method has super-attracting n-cycles. Specifically, they obtain (numerically) some values of the parameters $\lambda_n = \beta_n i$, with β_n given in Table 1, for which Newton's method applied to the polynomial $p_{\lambda_n}(z)$ defined in (18), with $\lambda_n = \beta_n i$ having a super-attracting n-cycle.

Table 1. Approximate values of the parameter β_n, for which Newton's method applied to the polynomial $p_{\lambda_n}(z)$ defined in (18), with $\lambda_n = \beta_n i$ having a super-attracting n-cycle (see [20]).

n	β_n	n	β_n	n	β_n
2	4.500116	6	1.982100	10	1.777227
3	2.938069	7	1.892459	20	1.732819
4	2.396385	8	1.836401	25	1.732152
5	2.131089	9	1.800522	50	1.732051

The strategy consists of coloring the parameter space $\lambda \in \mathbb{C}$ according to the convergence of the only free critical point $\lambda/3$, as performed in Figure 3. If the orbit of $\lambda/3$ converges to 1, -1, or λ, the value of the corresponding parameter λ is colored in cyan, magenta, or yellow, respectively.

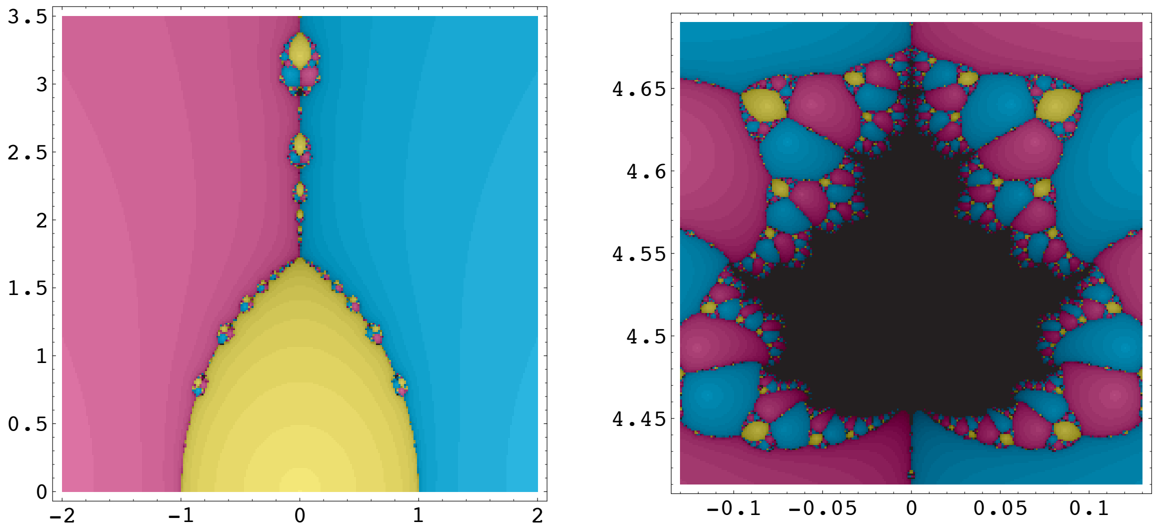

Figure 3. Graphic representation of the parameter plane of Newton's method applied to polynomials in the form (18). The figure on the right shows an enlargement of a black area where a Mandelbrot-type set can be seen.

The black-colored regions in the parameter space are formed by the values of λ, for which Newton's method applied to the corresponding polynomial $p_\lambda(z)$, have an attracting cycle that does not contain any root of the polynomial. The appearance of the Mandelbrot-type sets in the black areas of the parameter plane is a notable phenomenon.

Continuing in this line of work, Roberts and Horgan-Kobelski themselves [20] or, previously, E. R. Vrscay and W. J. Gilbert [25], prove the existence of polynomials with attracting cycles for Halley's method. Specifically, the table of values of β_n, for which Halley's method applied to the polynomial $p_{\lambda_n}(z)$ with $\lambda_n = \beta_n i$, has a super-attracting n-cycle and is shown in Table 2.

Table 2. Approximate values of the parameters β_n, for which Halley's method applied to the polynomial $p_{\lambda_n}(z)$ defined in (18) with $\lambda_n = \beta_n i$, have super-attracting n-cycle (see [20]).

n	β_n	n	β_n	n	β_n
2	1.342232060	6	1.018656160	10	1.0011575040
3	1.158338303	7	1.009291550	20	1.0000011298
4	1.076647075	8	1.004637455	25	1.0000000353
5	1.037611090	9	1.002316127	50	1.0000000011

The strategy to graphically represent the parameter space associated with Halley's method applied to polynomials of the form (18) changes slightly due to the appearance of two free critical points:

$$\rho_{\pm}(\lambda) = \frac{2\lambda \pm \sqrt{-2\lambda^2 - 6}}{6}.$$

As can be seen in [20], the range of colors in the parameter plane is expanded, according to the criteria shown in Table 3. For example, λ is colored blue if the orbit of $\rho_-(\lambda)$ converges to the root -1 and the orbit of $\rho_+(\lambda)$ converges to the root 1. The result of coloring the parameter plane of Halley's method in this way is shown in Figure 4.

Table 3. Color scheme for drawing the parameter plane associated with Halley's method applied to the polynomials defined in (18).

Color	$(\rho_-(\lambda), \rho_+(\lambda)) \to$	Color	$(\rho_-(\lambda), \rho_+(\lambda)) \to$
Yellow	$(-1, -1)$	Blue	$(-1, 1)$
Green	$(1, -1)$	Red	(λ, λ)
Brown	$(-1, \lambda)$	Pink	$(\lambda, -1)$
Orange	$(1, 1)$	Cyan	$(\lambda, 1)$
Purple	$(1, \lambda)$		

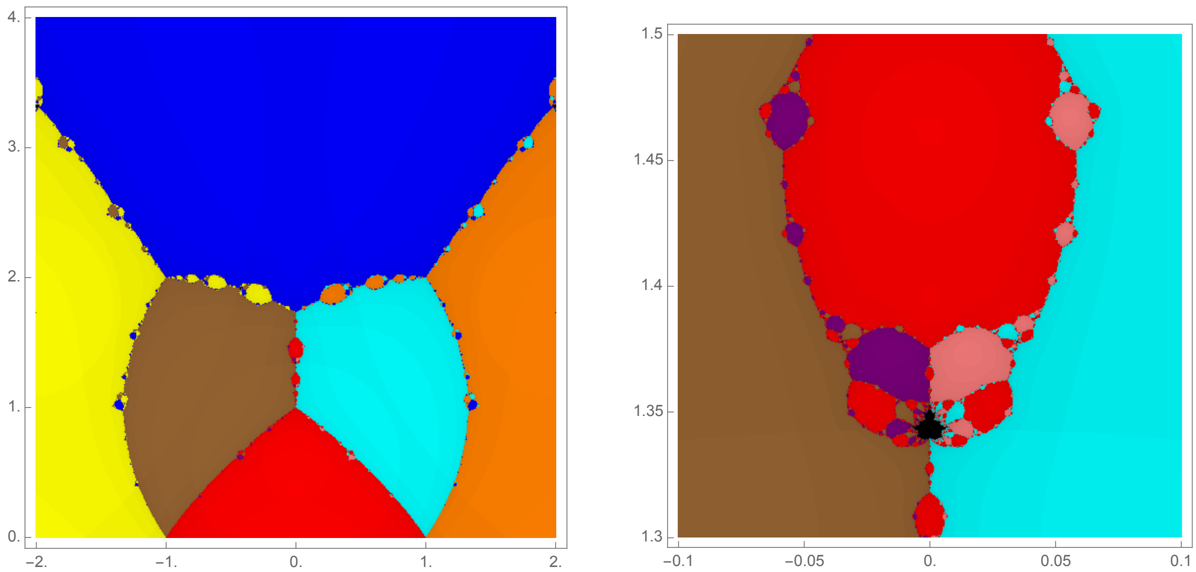

Figure 4. Parameter plane of Halley's method applied to the family of polynomials (18). In the figure on the right, we can see a Mandelbrot type set that appears in an enlargement of a black area.

The richness of the dynamic study of the methods of the family $\mathcal{CH}$ increases when we consider Chebyhev's method, as evidenced in the work of Gutiérrez and Varona [19]. With techniques similar to those used for the Newton or Halley methods, values of the parameter β_n can be given, for which Chebyshev's method applied to the polynomial $p_{\lambda_n}(z)$ defined in (18) with $\lambda_n = \beta_n i$, which have a super-attracting n-cycle (see Table 4).

Table 4. Approximate values of the parameter β_n, for which Chebyshev's method applied to the polynomial $p_{\lambda_n}(z)$ defined in (18) with $\lambda_n = \beta_n i$, have a super-attracting n-cycle (see [19]).

n	β_n	n	β_n	n	β_n
2	1.28657	6	1.48369	10	1.62056
3	1.34015	7	1.52557	20	1.72078
4	1.38943	8	1.56245	25	1.72856
5	1.43776	9	1.59405	50	1.72856

The strategy for coloring the parameter plane associated with Chebyshev's method applied to polynomials of the family (18) is the same as that followed for Halley's method (see [19] for more details), studying the orbits of the two fixed points which, in this case is

$$\rho_{\pm}(\lambda) = \frac{5\lambda \pm \sqrt{-5\lambda^2 - 15}}{15}.$$

The result is shown in Figure 5.

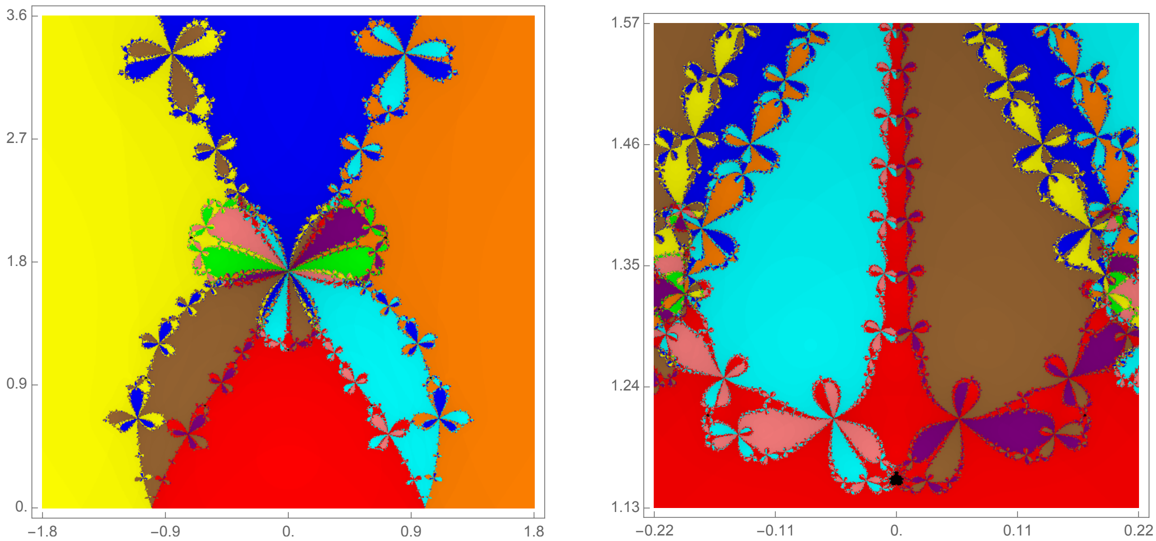

Figure 5. Parameter plane of Chebyshev's method applied to the family of polynomials (18). In the figure on the right, a channel of black holes around the imaginary axis can be seen.

Now, when analyzing the parameter plane associated with Chebyshev's method applied to the polynomials of the family (18), we have the "black holes" that appear in it, which are caused by two reasons: attracting cycles or attracting extraneous fixed points.

On the left side of Figure 5, the parameter plane of Chebyshev's method applied to the family of polynomials (18) is shown. The figure on the right shows an enlargement, around the imaginary axis, where a sort of channel of black holes can be seen. The black hole shown below is associated with a strange fixed point at $z = 2\sqrt{3}/3i$. However, the rest of the black holes in this channel are associated with attractor cycles.

Figure 6 shows two details of black holes: one associated with an extraneous fixed point (on the left) and another with an attracting two-cycle attractor (on the right).

What happens when we want to draw the parameter plane associated with other methods of the family $\mathcal{CH}$? The situation becomes considerably more complicated. The following results explain the reason.

Lemma 2. *Let $p(z)$ be a polynomial and let $L_p(z)$ and $L_{p'}(z)$ be the rational functions defined in (4) and (7), respectively. Suppose that $p'(z) \neq 0$ and that $p''(z) \neq 0$. Then, the free critical points of the rational function $R_{\alpha,p}(z)$ defined in (5) are solutions of the equation*

$$L_{p'}(z) = 3(1-\alpha) + \alpha(2\alpha - 1)L_p(z). \quad (19)$$

Proof. To calculate the free critical points associated with a method in $\mathcal{CH}$, we derive the rational function $R_{\alpha,p}(z)$ that appears in (5). Taking into account (7) and Lemma 1, we obtain

$$R'_{\alpha,p}(z) = \frac{L_p(z)^2\Big(3(1-\alpha) + \alpha(2\alpha-1)L_p(z) - L_{p'}(z)\Big)}{2(1-\alpha L_p(z))^2}. \quad (20)$$

Note that $L_p(z) \neq 0$ because we are assuming that $p''(z) \neq 0$ and $p(z) \neq 0$ because z is a free critical point (it is not the root of $p(z)$). The result follows by simply solving for $L_{p'}(z)$ in the equation $R'_{\alpha,p}(z) = 0$. $\square$

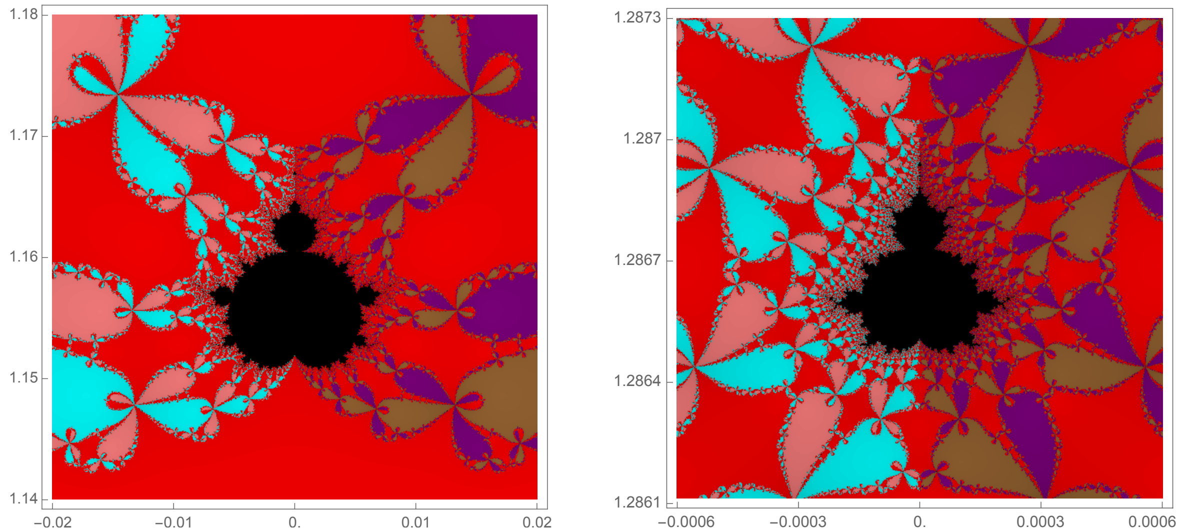

Figure 6. Two details of black holes of different nature: (on the **left**) one is associated with an extraneous fixed point and (on the **right**) one is associated with an attracting 2-cycle.

The analysis of Lemma 2 allows us to obtain some interesting conclusions about the number of free critical points in the family $\mathcal{CH}$. First, we note that there are two situations for which the Equation (19) has particularly simple solutions. These are the cases where $\alpha = 0$ (Chebyshev method) and $\alpha = 1/2$ (Halley method). In both cases, we arrive at an equation of the type $L_{p'}(z) = C$ with a constant C, which leads us to a polynomial equation of degree $2d - 4$, with d being the degree of the polynomial $p(z)$. In the case of cubic polynomials, such as those given in (18), the equation to be solved is quadratic. Consequently, to draw the parameter plane associated with Chebyshev's and Halley's methods, it is necessary to analyze the orbits of the two free critical points obtained, as has been performed, for example, in [19,20], respectively.

The number of free critical points in other methods in $\mathcal{CH}$ increases, which complicates their analysis from a dynamic point of view. For example, in another of the named methods of the family, such as the super-Halley method ($\alpha = 1$), the equation $L_{p'}(z) = L_p(z)$ is obtained. This equation leads to a polynomial equation of degree $4d - 6$. In the case of cubic polynomials (18), an equation of degree six must be solved, which greatly complicates the process. To be more specific, there is no formula that provides in an analytic way the roots in terms of the coefficients, as occurs in the cases of Chebyshev's and Halley's method, where a quadratic equation must be solved. Furthermore, with six possible free critical points and three roots, the range of colors to be handled is $3^6 = 729$, which makes it cumbersome to use, in the case of the super-Halley method, the strategy developed for Chebyshev's and Halley's methods. The situation is similar for the rest of the methods of the $\mathcal{CH}$ family, where the degree of the polynomial equation obtained to find the free critical points is also $4d - 6$.

4. Polynomials with Double Misbehavior

We have seen that some methods in $\mathcal{CH}$ can have a pathological behavior when they are applied to polynomial equations. By pathological, we mean the convergence to points or cycles that are not the roots of the polynomial. In particular, we have analyzed this bad behavior for the Chebyshev and super-Halley method. We have found polynomials for which each of these methods has extraneous fixed points or super-attracting cycles not including the roots. Now, we face the following question: is it possible to find polynomials and methods in $\mathcal{CH}$ so that the rational map obtained by applying the method to the polynomial has both extraneous fixed points and super-attracting cycles? Halley's method must be excluded from this search since it does not have attracting extraneous fixed points. Our first "candidate" is Chebyshev's method.

We follow a numerical strategy to find a polynomial in the form (16) (it has a super-attracting extraneous fixed point at zero), such that Chebyshev's method applied to it has a super-attracting two-cycle in $\{-1, 1\}$. To perform this, let $R_{0,p}(z)$ be the iteration map related to Chebyshev's method (see (5), with $\alpha = 0$). We look for a solution to the system of equations

$$\begin{cases} R_{0,p}(1) & = & -1, \\ R_{0,p}(-1) & = & 1, \\ R'_{0,p}(1) & = & 0, \\ R'_{0,p}(-1) & = & 0. \end{cases} \tag{21}$$

Taking into account (see (20) in Lemma 2)

$$R'_{0,p}(z) = \frac{1}{2}(3 - L_{p'}(z))L_p(z)^2,$$

the last two equations in (21) can be substituted by $L_{p'}(1) = 3$ and $L_{p'}(-1) = 3$. As we have four equations, we take four parameters a, a_4, a_5, and a_6 in (16). So, we solve numerically the nonlinear system (21), and we obtain two solutions with real coefficients:

$$a = 0.115238, \quad a_4 = 0.00106173, \quad a_5 = 0.000786477, \quad a_6 = 0.000284636,$$

$$a = -0.115238, \quad a_4 = 0.00106173, \quad a_5 = -0.000786477, \quad a_6 = 0.000284636.$$

In Figure 7, we show the basins of attraction of Chebyshev's method applied to the polynomial

$$p(z) = -1 + az + a^2z^2 + 2a^3z^3 + a_4z^4 + a_5z^5 + a_6z^6, \tag{22}$$

where a, a_4, a_5, and a_6 are the first of the above solutions. Together with the basins of the six roots of the polynomial, we can see (in white) the basin of the super-attracting extraneous fixed point $z = 0$. In yellow, we can see the basin of the super-attracting two-cycle $\{-1, 1\}$.

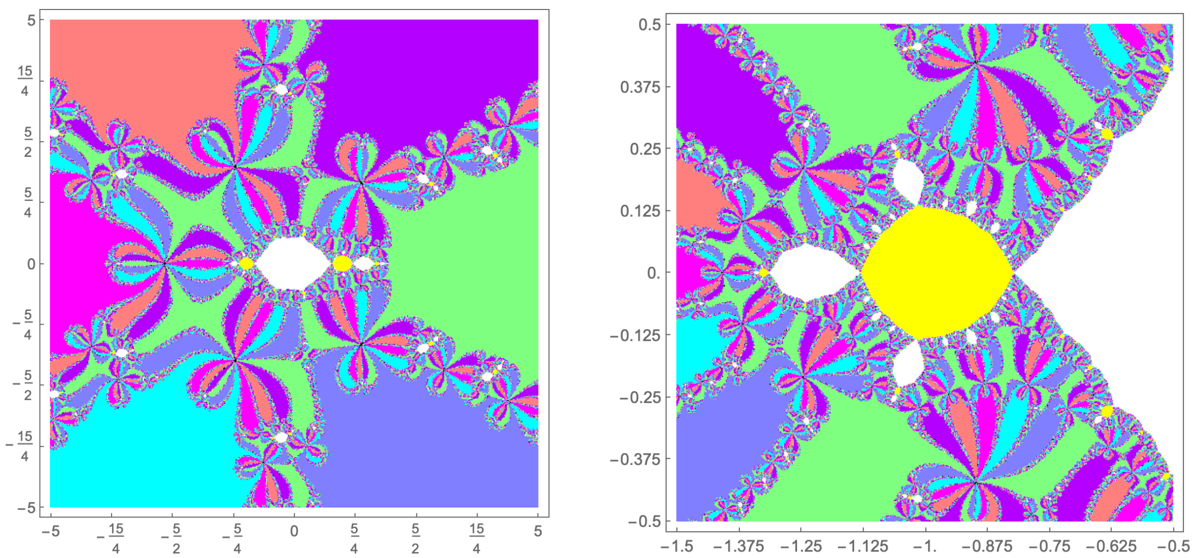

Figure 7. Two details of the basins of attraction of Chebyshev's method applied to the polynomial defined in (22), where we can see the basin of attraction of super-attracting extraneous fixed point (in white) and a super-attracting 2-cycle (in yellow). The basins of the other six roots are colored with the rest of colors (magenta, green, purple, pink, cyan, or blue).

In the case of the super-Halley method, we proceed in a similar way. In this case, we find a polynomial in the form (17) (it has a super-attracting extraneous fixed point at zero), such that the super-Halley method applied to it has a super-attracting two-cycle in $\{-1, 1\}$. To perform this, let $R_{1,p}(z)$ be the iteration map related to the super-Halley method (see (5), with $\alpha = 1$). We look for a solution to the system of equations

$$\begin{cases} R_{1,p}(1) & = & -1, \\ R_{1,p}(-1) & = & 1, \\ R'_{1,p}(1) & = & 0, \\ R'_{1,p}(-1) & = & 0. \end{cases} \tag{23}$$

Taking into account (see (20) in Lemma 2)

$$R'_{1,p}(z) = \frac{L_p(z) - L_{p'}(z)}{2(1 - L_p(z))^2} L_p(z)^2,$$

the last two equations in (23) can be substituted by $L_p(1) = L_{p'}(1)$ and $L_p(-1) = L_{p'}(-1)$. As we have four equations, we take four parameters a, a_4, a_5, and a_6 in (17). So, we solve numerically the nonlinear system (23), and we obtain the solution (not the only one)

$$a = 0.325178, \quad a_4 = -0.009199, \quad a_5 = -0.029499, \quad a_6 = -0.011043.$$

In Figure 8, we show the basins of attraction of the super-Halley method applied to the polynomial

$$p(z) = -1 + az + a^2 z^2 + 2a^3 z^3 + a_4 z^4 + a_5 z^5 + a_6 z^6, \tag{24}$$

where a, a_4, a_5, and a_6 are the above parameters. Together with the basins of the six roots of the polynomial, we can see (in white) the basin of the super-attracting extraneous fixed point $z = 0$. In yellow, we can see the basin of the super-attracting two-cycle $\{-1, 1\}$.

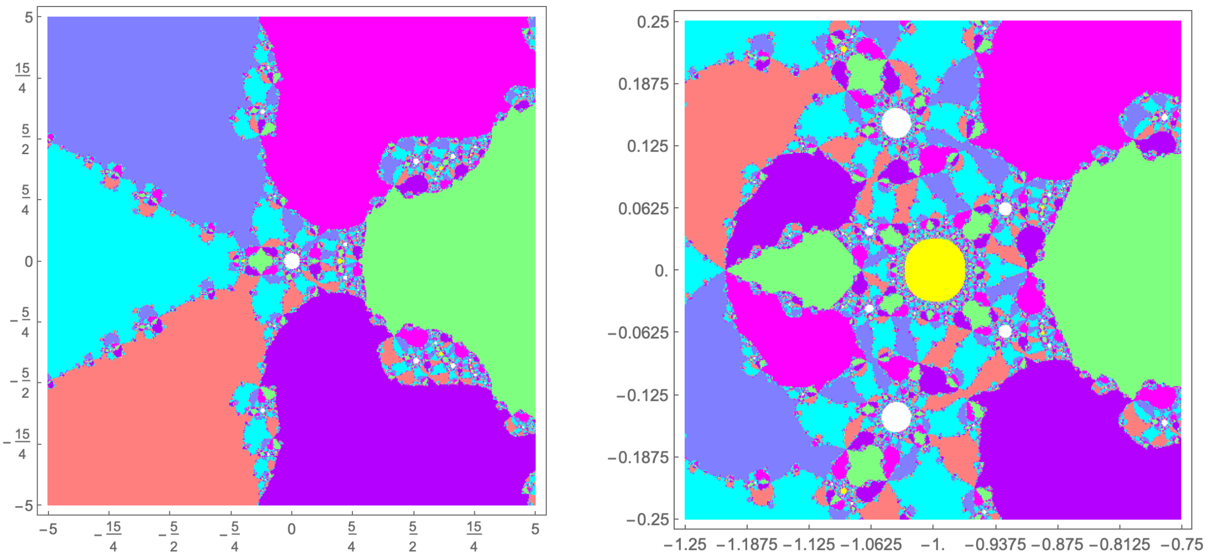

Figure 8. Two details of the basins of attraction of super-Halley method applied to the polynomial defined in (24), where we can see the basin of attraction of super-attracting extraneous fixed point (in white) and a super-attracting 2-cycle (in yellow). The basins of the other six roots are colored with the rest of colors (magenta, green, purple, pink, cyan, or blue).

5. Conclusions

In this article, we have made an incursion into the dynamic study of iterative processes for solving polynomial equations in the complex plane. We have focused our interest on the well-known Chebyshev–Halley family of root-finding methods. This is a one-parameter family, depending on the parameter $\alpha \in \mathbb{C}$, which includes the most famous third-order iterative methods, such as Chebyshev, Halley, or super-Halley methods. First, we have characterized the existence of extraneous fixed points (fixed points of the iteration map that are not roots of the corresponding polynomial) in terms of the quotients

$$L_p(z) = \frac{p(z)p''(z)}{p'(z)^2} \text{ and } L_{p'}(z) = \frac{p'(z)p'''(z)}{p''(z)^2}.$$

In addition, we have analyzed the behavior of the infinity point as an extraneous fixed point, reaching the conclusion that it is not always repulsive, as in the case of the aforementioned most famous iterative processes or even Newton's method.

The following part of this article is devoted to the study of the critical points of the methods in the family $\mathcal{CH}$. In particular, we have given a result that characterizes the existence and number of free critical points, that is, critical points that are not roots of the considered polynomial. This result plays a key role in the construction of the parameter plane related to the rational map obtained by applying the methods in $\mathcal{CH}$ to families of polynomials depending on a complex parameter (we have considered the case of cubic polynomials).As far as we know, only the parameter planes associated with the methods of Newton, Halley, or Chebyshev have been represented (see references [19,20], for instance). The main limitations for using the parameter plane with the rest of the methods in the family arise when solving the equations of critical points because there is not a "closed formula" for them. We have not managed to solve this difficulty and the problem is open for future research. Maybe, a good starting point to face this challenge is to consider the dynamic behavior of the super-Halley method, which has all the ingredients that we have not been able to solve.

Finally, we conclude with another new dynamic question: the problem of finding polynomials and methods in $\mathcal{CH}$ so that the rational map obtained by applying the method to the polynomial has both extraneous fixed points and super-attracting cycles. We have obtained some of these couples of polynomial methods by following a numerical strategy. In fact, we have found polynomials with double misbehavior for Chebyshev and super-Halley methods. In both cases, polynomials with a sixth-degree have been obtained. This problem opens a suggestive line of research, with issues such as whether it is possible to find couples of polynomial methods with double misbehavior with a lower degree. Is there a geometric, algebraic, or analytical strategy to find these polynomials?

This work can be also extended to other problems such as nonlinear systems of equations or nonlinear ordinary differential equations. For instance, in [29], we can find promising results in solving these problems using an iterative scheme.

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