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Derivative free processes of high order for nondifferentiable equations in Banach spaces

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In this article, we consider two parametric classes of derivative free iterative methods of high order of convergence in Banach spaces. We study local and semilocal convergence results in Banach spaces for both families of iterative processes. We carry out a numerical application to solve a nonlinear and non-differentiable Fredholm integral equation in the Banach space of continuous functions with the maximum norm.

KEYWORDS
Banach spaces, local convergence, nonlinear systems, semilocal convergence

MSC CLASSIFICATION
65Jxx

1 | INTRODUCTION

Mathematical modeling (*MM*) is often used to handle real applications in Computational Sciences or Engineering. As a result of the application of the *MM*,

$$F(x) = 0 \tag{1}$$

is usually solved, where F is an operator defined on some open set $D \subset E_1$, $F : D \to E_2$ and E_1, E_2 are abstract spaces. Here, in particular, E_1, E_2 are considered to be Banach spaces [1]. Throughout our study, we assume that there exists a first-order divided difference in the Banach space E_1 [2].

It is a great challenge to obtain a solution x^* of (1). Numerical functional analysis techniques are mainly developed to at least find an approximation to x^* iteratively [3], since the analytical solution is only attainable in rare occasions.

Researchers and practitioners mostly agree that Newton process (*NP*), expressed by

$$\begin{cases} x_0 \ \text{ given in } \ D, \\ x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \ n \geq 0, \end{cases} \tag{2}$$

is a preferred iterative process [4, 5].

NP is an iterative method with a quadratic order of convergence. But its implementation requires the linear operator inversion $F'(x_n)^{-1}$ in its iteration. But such an inversion may be computationally expensive or may not exist. If this is the case, NP cannot be used to solve (1) when the operator is not differentiable Fréchet. That is why the derivative in (2) has been replaced by the divided difference operator of order 1 [6]. For example, if $F'(x_n)$ is replaced by $[x_n + F(x_n), x_n; F]$ (for $E_1 = E_2$), the Steffensen's process (SP) is obtained, which is also of convergence order 2 [7, 8].

Another process which also has order of convergence two is Kurchatov's process (KP), obtained when $F'(x_n)$ in the process (2) is replaced by $[2x_n - x_{n-1}, x_{n-1}; F]$ [9–11].

Due to the exploding development of technology, the introduction of higher convergence order processes must be built that are also efficient and computationally affordable.

This has been done already by developing processes of convergence order 3 such as two-step NP or Traub's process (TP) [12–14] given as

$$\begin{cases} x_0 \text{ given in } D, \\ y_n = x_n - F'(x_n)^{-1}F(x_n), \\ x_{n+1} = y_n - F'(x_n)^{-1}F(y_n), \;\; n \geq 0. \end{cases}$$

or higher order extensions [15, 16] as, for example, the iterative process of order 4 given by

$$\begin{cases} x_0 \text{ given in } D, \\ y_n = x_n - F'(x_n)^{-1}F(x_n), \\ x_{n+1} = y_n - F'(y_n)^{-1}F(y_n), \;\; n \geq 0. \end{cases}$$

In order to avoid the inversion of the linear operators $F'(x_n), F'(y_n)$, as in the case of SP or KP, the following process is proposed in [17], for $E_1 = E_2 = \mathbb{R}^m$,

$$\begin{cases} x_0 \text{ given in } D, \\ w_n = x_n + \gamma F(x_n), \\ y_n = x_n - [w_n, x_n; F]^{-1}F(x_n), \\ \lambda_n = I - [w_n, x_n; F]^{-1}[y_n, w_n; F], \\ z_n = y_n - P(\lambda_n)[y_n, x_n; F]^{-1}F(y_n), \\ \delta_n = I - [w_n, x_n; F]^{-1}[z_n, y_n; F]P(\lambda_n), \\ x_{n+1} = z_n - Q(\lambda_n, \delta_n)[z_n, y_n; F]^{-1}F(z_n), \;\; n \geq 0, \end{cases} \tag{3}$$

where $\gamma \in \mathbb{R}$ and $P, Q : \mathcal{L}(E_1) \to \mathcal{L}(E_1)$ are linear weight operators and where $\mathcal{L}(E_1)$ is the domain of linear operators that are bounded; this is $\mathcal{L}(E_1) = \{\xi : E_1 \to E_1 \text{ linear operator } : \xi \text{ is bounded}\}$. This process is of convergence order 7 provided a number of constraints on the operators P and Q. Possible choices for P and Q are $P(.) = Q(.,.) = I$ or $P(\lambda_n) = [w_n, x_n; F]$ and $Q(\lambda_n, \delta_n) = [y_n, x_n; F]$ as long as the conditions (C_4) and (C_5) are satisfied, (local convergence), or **(**H_3**)** and **(**H_4**)** are satisfied, (semilocal convergence); other choices can be found in [8]; see also the Numerical Section 4.

Next, considering the real parameter γ as operators of the form $\gamma_n : D \times D \to \mathcal{L}(E_2, E_1)$, where $\mathcal{L}(E_2, E_1)$ is the domain of linear operators that are bounded, this is $\mathcal{L}(E_2, E_1) = \{\xi : E_2 \to E_1 \text{ linear operator } : \xi \text{ is bounded}\}$. So, we obtain the following processes with memory

$$\begin{cases} x_0 \text{ given in } D, \\ w_n = x_n + \gamma_n F(x_n), \\ y_n = x_n - [w_n, x_n; F]^{-1}F(x_n), \\ \lambda_n = I - [w_n, x_n; F]^{-1}[y_n, w_n; F], \\ z_n = y_n - P(\lambda_n)[y_n, x_n; F]^{-1}F(y_n), \\ \delta_n = I - [w_n, x_n; F]^{-1}[z_n, y_n; F]P(\lambda_n), \\ x_{n+1} = z_n - Q(\lambda_n, \delta_n)[z_n, y_n; F]^{-1}F(z_n), \;\; n \geq 0, \end{cases} \tag{4}$$

taking

$$\gamma_n = -[x_n, x_{n-1}; F]^{-1}, \tag{5}$$

$$\gamma_n = -[2x_n - x_{n-1}, x_{n-1}; F]^{-1}, \tag{6}$$

$$\gamma_n = -[x_n, y_{n-1}; F]^{-1}, \tag{7}$$

$$\gamma_n = -[2x_n - y_{n-1}, y_{n-1}; F]^{-1}. \tag{8}$$

Then, the corresponding convergence orders are $\frac{7+\sqrt{65}}{2}, 8, 4+\sqrt{17}$, and $\frac{9+\sqrt{89}}{2}$, respectively. That is, the convergence order increases from 7 to 9.21699.

A choice of the operators P and Q is

$$\begin{cases} P(\lambda) = \lambda^2 + \lambda + I \\ Q(\lambda, \delta) = I + \lambda\delta + \frac{13}{6}\lambda\delta^2, \end{cases}$$

where $\lambda, \delta \in \mathbb{R}$.

These choices satisfy the constraints imposed in [17]. Numerical applications for nonlinear systems are also implemented in [17], where processes (3) and (4) are applied.

1.1 | Motivation

The following concerns arise with the implementation of these important results limiting the utilization of these processes:

1. Although the processes (3) and (4) do not require the inversion of F', the proof of convergence is carried out by assuming that $F^{(4)}$ at least exists and is bounded. But consider the simple scalar function for $D = [-2, 3]$ given as $F(t) = -t^4 + t^5 + t^3 \ln t$ for $t \neq 0$ and $F(t) = 0$ for $t = 0$. Then, $F^{(3)}$ is unbounded on D. Thus, the results in [17] cannot assure the convergence to the solution $t^* = 1$ although the processes converge.
2. The results are shown on $\mathbb{R}^m$, with m denoting a natural number. But they can apply on equations defined on more general spaces such as Hilbert or Banach.
3. There are no results on the isolation of the solution x^* in a neighborhood containing it.
4. There is no a priori knowledge of how many iterations must be executed, so that a predetermined accuracy is obtained.
5. The results in [17] do not show the local convergence (LC) of these methods. LC results are useful, since they provide a realization of the challenge to find initial points x_0 assuring the convergence of these methods.

Problems (1)–(5) constitute our motivation for writing this article. Here is how we positively answer to these problems.

1.2 | Novelty

(a_1) The convergence conditions depend only on the operators which appear on the processes, that is, the divided difference $[.,.;F]$ and the operator F.
(a_2) The results are valid on Banach spaces.
(a_3) Isolation of the solution results is given.
(a_4) A priori estimates on $||x_n - x^*||$ determine the required number of iterations such that $||x_n - x^*||$ is less than a certain accuracy.
(a_5) New LC as well as semilocal convergence (SLC) results are developed using generalized conditions (ω-continuity).

The remaining sections include the LC of processes (3) and (4) in Section 2. The SLC for both processes appears in Section 3. The examples are in Section 4 followed by the conclusions in Section 5.

2 | LOCAL CONVERGENCE

From now, we consider F to be an operator defined on some open and nonempty set $D \subset E_1$, $F : D \to E_2$, where E_1, E_2 to be Banach spaces. As we have previously indicated, our convergence conditions are going to be as general as possible. To achieve this, we impose ω-continuity conditions on the operators that appear in the algorithm that defines the iterative method (3). Therefore, we consider the following hypotheses. Let us suppose the following.

(C_1) There exists an invertible linear operator L on E_1 such that

$$||L^{-1}([x,y;F]-L)|| \leq \phi_0(||x-x^*||, ||y-x^*||), \text{ for each } x,y \in D$$

and

$$||w_n - x^*|| \leq \phi_6(||x_n - x^*||),$$

where $\phi_0 : M \times M \to \mathbb{R}^+$, $\phi_6 : M \to \mathbb{R}^+$ are continuous nondecreasing functions (*CNF*) such that the scalar equation

$$\phi_0(\phi_6(t), t) - 1 = 0$$

has the smallest solution (*SS*) denoted by $\rho_0 \in M - \{0\}$.

(C_2) Let be $M_0 = [0, \rho_0)$ and $U = D \cap B(x^*, \rho_0)$, then

$$||L^{-1}([x,y;F]-[y,x^*;F])|| \leq \phi(||x-x^*||, ||y-x^*||), \text{ for each } x,y \in U,$$

where $\phi : M_0 \times M_0 \to \mathbb{R}^+$ is a *CNF*.

(C_3)

$$||L^{-1}([x,x^*;F]-L)|| \leq \phi_1(||x-x^*||) \text{ for each } x \in U,$$

where $\phi_1 : M_0 \to \mathbb{R}^+$ is a *CNF*.

(C_4)

$$||I - P(\lambda_n)|| \leq \phi_2(||x_n - x^*||, ||y_n - x^*||)$$

and

$$||P(\lambda_n)|| \leq \phi_3(||x_n - x^*||, ||y_n - x^*||),$$

where $\phi_2 : M_0 \times M_0 \to \mathbb{R}^+$, $\phi_3 : M_0 \times M_0 \to \mathbb{R}^+$, which are *CNF*.

(C_5)

$$||I - Q(\lambda_n, \delta_n)|| \leq \phi_4(||x_n - x^*||, ||y_n - x^*||, ||z_n - x^*||)$$

and

$$||Q(\lambda_n, \delta_n)|| \leq \phi_5(||x_n - x^*||, ||y_n - x^*||, ||z_n - x^*||),$$

where $\phi_4 : M_0 \times M_0 \times M_0 \to \mathbb{R}^+$, $\phi_5 : M_0 \times M_0 \times M_0 \to \mathbb{R}^+$, which are *CNF*.

As we shall see, some scalar functions are introduced that play a role in the convergence analysis of the method (3). These functions are later connected to the operators on the method (3). Given the influence of these scalar functions, we need the following technical result.

Lemma 1. *Under the above conditions* (C_1)–(C_5), *we suppose that*

(i) The scalar equation

$$g_1(t) - 1 = 0$$

has the smallest solution (SS) $s_1 \in M_0 - \{0\}$, *where* $g_1 : M_0 \to \mathbb{R}^+$ *is given by*

$$g_1(t) = \frac{\phi(\phi_6(t), t)}{1 - \phi_0(\phi_6(t), t)}.$$

(ii) The scalar equation

$$\phi_0(g_1(t)t, t) - 1 = 0$$

has a SS $\rho_1 \in M_0 - \{0\}$, *set* $M_1 = [0, \rho_1)$.

(iii) *The scalar equation*

$$g_2(t) - 1 = 0$$

has a SS $s_2 \in M_1 - \{0\}$, *where the function* $g_2 : M_1 \to \mathbb{R}^+$ *is given by*

$$g_2(t) = \left[\phi_2(t, g_1(t)t) + \frac{\phi_3(t, g_1(t)t)\phi(t, g_1(t)t)}{1 - \phi_0(g_1(t)t, t)}\right] g_1(t).$$

(iv) *The scalar equation*

$$\phi_0(g_2(t)t, g_1(t)t) - 1 = 0$$

has a SS $\rho_2 \in M_1 - \{0\}$, *set* $M_2 = [0, \rho_2)$.

(v) *The scalar equation*

$$g_3(t) - 1 = 0$$

has a SS $s_3 \in M_2 - \{0\}$, *where the function* $g_3 : M_2 \to \mathbb{R}^+$ *is given by*

$$g_3(t) = \left[\phi_4(t, g_1(t)t, g_2(t)t) + \frac{\phi_5(t, g_1(t)t, g_2(t)t)\phi(g_1(t)t, g_2(t)t)}{1 - \phi_0(g_2(t)t, g_1(t)t)}\right] g_2(t).$$

Let

$$s^* = \min_{i=1,2,3} \{s_i\} \tag{9}$$

and $M_3 = [0, s^*)$. *Under the previous conditions, the following hold for each* $t \in M_3$*:*

$$0 \le \phi_0(\phi_6(t), t) < 1, \tag{10}$$

$$0 \le \phi_0(g_1(t)t, t) < 1, \tag{11}$$

$$0 \le \phi_0(g_2(t)t, g_1(t)t) < 1, \tag{12}$$

$$0 \le g_i(t) < 1, \textit{ for } i = 1, 2, 3. \tag{13}$$

Proof.

(*a*) The number s_1 is by definition the smallest positive solution of the equation $g_1(t) - 1 = 0$. If $\phi_0(\phi_6(s_1), s_1) > 1$, this contradicts that the $\rho_0 > s_1$ is the smallest positive number for which $\phi_0(\phi_6(\rho_0), \rho_0) = 1$. Moreover, $\phi_0(\phi_6(s_1), s_1) = 1$ is not possible; since then, s_1 cannot be a solution of the equation $g_1(t) - 1 = 0$. Thus, $t \in M_3$ $\phi_0(\phi_6(t), t) < 1$, since ϕ_0 is *CNF*. Hence, (10) holds.

(*b*) If $g_1(t) = 1$ for $t < s_1$, then this contradicts that s_1 is the smallest solution of the equation $g_1(t) - 1 = 0$. Moreover, if $g_1(t) > 1$, then by (10) $\phi(\phi_6(t), t) > 1 - \phi_0(\phi_6(t), t)$ and since $s_1 > t$, we have $\phi(\phi_6(s_1), s_1) > 1 - \phi_0(\phi_6(s_1), s_1)$ contradicting that s_1 solves the equation $g_1(t) - 1 = 0$. Thus, (13) holds if $i = 1$.
Then, if we replace ϕ_6 by g_1 in (*a*), the proof of (11) is obtained. Moreover, (13) also holds for $i = 2$ by the proof of (*b*) and the definition of the function g_2.
Furthermore, we have by (11) that $0 \le \phi_0(g_1(t)t, g_2(t)t) \le \phi_0(g_1(t)t, t) < 1$. Hence, (12) holds. Finally, the proof of (13) for $i = 3$ follows as in (*b*).

□

Next, we study the iteration $n = 0$ for the algorithm given by method (3). Moreover, we suppose

(C_6)

$$B(x^*, \bar{s}^*) \subset D, \text{ for } \bar{s}^* = \max\{s^*, \phi_6(s^*)\}$$

and consider $x_0 \in B(x^*, s^*) \subset D$, with $x_0 \neq x^*$. Then, by the second condition in (C_1), we have that $w_0 \in B(x^*, \bar{s}^*) \subset D$. Moreover, using the first condition in (C_1) and Lemma 1, we obtain

$$||L^{-1}([w_0, x_0; F] - L)|| \leq \phi_0(||w_0 - x^*||, ||x_0 - x^*||) \leq \phi_0(\phi_6(s^*), s^*) < 1.$$

It follows, by (10) and Banach's lemma on linear and invertible operators [18], that $[w_0, x_0; F]^{-1}$ exists and

$$||[w_0, x_0; F]^{-1}L|| \leq \frac{1}{1 - \phi_0(||w_0 - x^*||, ||x_0 - x^*||)}. \tag{14}$$

So, the iterate y_0 is well-defined. In view of the second step of method (3), one can write

$$\begin{aligned} y_0 - x^* &= x_0 - x^* - [w_0, x_0; F]^{-1}F(x_0) \\ &= x_0 - x^* - [w_0, x_0; F]^{-1}[x_0, x^*; F](x_0 - x^*) \\ &= [w_0, x_0; F]^{-1}([w_0, x_0; F] - [x_0, x^*; F])(x_0 - x^*). \end{aligned} \tag{15}$$

By (9), (13) for $i = 1$, (14), (C_2), and (15), as $w_0, x_0 \in U$, we have that

$$\begin{aligned} ||y_0 - x^*|| &\leq ||[w_0, x_0; F]^{-1}L|| ||L^{-1}([w_0, x_0; F] - [x_0, x^*; F])|| ||x_0 - x^*|| \\ &\leq \frac{\phi(||w_0 - x^*||, ||x_0 - x^*||)}{1 - \phi_0(||w_0 - x^*||, ||x_0 - x^*||)} ||x_0 - x^*|| \\ &\leq \frac{\phi(\phi_6(||x_0 - x^*||), ||x_0 - x^*||)}{1 - \phi_0(\phi_6(||x_0 - x^*||), ||x_0 - x^*||)} ||x_0 - x^*|| \\ &\leq g_1(||x_0 - x^*||)||x_0 - x^*|| < ||x_0 - x^*|| < s^*. \end{aligned} \tag{16}$$

So, the iterate $y_0 \in B(x_0, s^*)$ and, proceeding as for the iterate w_0, from the first condition in (C_1), we have

$$||L^{-1}([y_0, x_0; F] - L)|| \leq \phi_0(||y_0 - x^*||, ||x_0 - x^*||) \leq \phi_0(g_1(s^*)s^*, s^*) < 1.$$

Then, by Banach's lemma, we obtain that there exists $[y_0, x_0; F]^{-1}$ with

$$||[y_0, x_0; F]^{-1}L|| \leq \frac{1}{1 - \phi_0(||y_0 - x^*||, ||x_0 - x^*||)}. \tag{17}$$

Thus, the iterate z_0 is well-defined. Then, as the operator λ_0 is well-defined, we can also write

$$\begin{aligned} z_0 - x^* &= y_0 - x^* - P(\lambda_0)[y_0, x_0; F]^{-1}F(y_0) \\ &= (I - P(\lambda_0)[y_0, x_0; F]^{-1}[y_0, x^*; F])(y_0 - x^*) \\ &= \left[I - P(\lambda_0)[y_0, x_0; F]^{-1}([y_0, x^*; F] + [y_0, x_0; F] - [y_0, x_0; F])\right](y_0 - x^*) \\ &= \left[I - P(\lambda_0) - P(\lambda_0)[y_0, x_0; F]^{-1}LL^{-1}([y_0, x^*; F] - [y_0, x_0; F])\right](y_0 - x^*) \\ &= (I - P(\lambda_0)) + P(\lambda_0))[y_0, x_0; F]^{-1}LL^{-1}([x_0, y_0; F] - [y_0, x^*; F])(y_0 - x^*) \end{aligned} \tag{18}$$

leading by (C_2), (C_4), (17) and (18) to

$$\begin{aligned} ||z_0 - x^*|| &\leq \left(\phi_2(||x_0 - x^*||, ||y_0 - x^*||) + \frac{\phi_3(||x_0 - x^*||, ||y_0 - x^*||)\phi(||x_0 - x^*||, ||y_0 - x^*||)}{1 - \phi_0(||y_0 - x^*||, ||x_0 - x^*||)}\right) ||y_0 - x^*|| \\ &\leq g_2(||x_0 - x^*||)||y_0 - x^*|| \leq g_2(||x_0 - x^*||)g_1(||x_0 - x^*||)||x_0 - x^*|| < s^*, \end{aligned}$$

since $x_0, y_0 \in U$.

Therefore, the iterate $z_0 \in B(x^*, s^*)$. So, using the first condition in (C_1) and Lemma 1, we have

$$||L^{-1}([z_0, y_0; F] - L)|| \leq \phi_0(||z_0 - x^*||, ||y_0 - x^*||) \leq \phi_0(g_2(s^*)s^*, g_1(s^*)s^*) < 1.$$

It follows, by (10) and Banach's lemma on linear and invertible operators, that $[w_0, x_0; F]^{-1}$ exists and

$$||[z_0, y_0; F]^{-1}L|| \leq \frac{1}{1 - \phi_0(||z_0 - x^*||, ||y_0 - x^*||)}. \tag{19}$$

Moreover, the iterate x_1 is well-defined.

Furthermore, as operator δ_0 is well-defined, we can write

$$\begin{aligned} x_1 - x^* &= z_0 - x^* - Q(\lambda_0, \delta_0)[z_0, y_0; F]^{-1}F(z_0) \\ &= [I - Q(\lambda_0, \delta_0)[z_0, y_0; F]^{-1}[z_0, x^*; F]](z_0 - x^*) \\ &= [I - Q(\lambda_0, \delta_0)[z_0, y_0; F]^{-1}([z_0, x^*; F] + [z_0, y_0; F] - [z_0, y_0; F])](z_0 - x^*) \\ &= [I - Q(\lambda_0, \delta_0) - Q(\lambda_0, \delta_0)[z_0, y_0; F]^{-1}LL^{-1}([z_0, x^*; F] - [z_0, y_0; F])](z_0 - x^*) \\ &= [I - Q(\lambda_0, \delta_0) - Q(\lambda_0, \delta_0)[z_0, y_0; F]^{-1}LL^{-1}([z_0, x^*; F] - [y_0, z_0; F])](z_0 - x^*), \end{aligned} \tag{20}$$

leading by (C_2), (C_5), (19) and (20) to

$$\begin{aligned} ||x_1 - x^*|| \leq [&\phi_4(||x_0 - x^*||, ||y_0 - x^*||, ||z_0 - x^*||) + \\ &+ \frac{\phi_5(||x_0 - x^*||, ||y_0 - x^*||, ||z_0 - x^*||)\phi(||y_0 - x^*||, ||z_0 - x^*||)}{1 - \phi_0(||z_0 - x^*||, ||y_0 - x^*||)}]||z_0 - x^*|| \\ &\leq g_3(||x_0 - x^*||)||x_0 - x^*|| < s^*, \end{aligned}$$

that is, $x_1 \in B(x^*, s^*)$.

From the study carried out, we can obtain the following result.

Lemma 2. *Under conditions (C_1)–(C_6) further if $x_0 \in B(x^*, s^*)$, then the following items hold $y_n, z_n, x_{n+1} \in D$, for $n \geqslant 0$. Moreover,*

$$||y_n - x^*|| \leq g_1(||x_n - x^*||)||x_n - x^*||, \tag{21}$$

$$||z_n - x^*|| \leq g_2(||x_n - x^*||)||x_n - x^*||, \tag{22}$$

and

$$||x_{n+1} - x^*|| \leq g_3(||x_n - x^*||)||x_n - x^*|| \tag{23}$$

for $n \geqslant 0$.

Proof. Taking into account the study carried out previously, items (21)–(23) hold if $n = 0$. The induction for items (21)–(23) is completed provided that iterates x_0, w_0, y_0, z_0, and x_1 are switched by x_n, w_n, y_n, z_n, and x_{n+1}, respectively, in the preceding calculations. □

Theorem 1. *Suppose the conditions (C_1)–(C_6) hold and those of Lemma 1; x^* is a solution of equation $F(x) = 0$. Then, for any $x_0 \in B(x^*, s^*) - \{x^*\}$, the sequence $\{x_n\}$ given in (3) verifies that $\{x_n\} \subset B(x^*, s^*)$ and converges to x^*. Moreover,*

$$\|x_{n+1} - x^*\| \leq K^{n+1}\|x_0 - x^*\|,$$

where $K = \max_{t \in M_3}\{g_3(t)\}$.

Proof. On the one hand, from Lemma 2, the sequence $\{x_n\}$ given in (3) is well-defined and obviously $\{x_n\} \subset B(x^*, s^*)$.

On the other hand, it is clear that there exists $K > 0$ such that $\max_{t \in M_3}\{g_3(t)\} = K < 1$. Next, from Lemma 2, we obtain

$$\begin{aligned}||x_{n+1} - x^*|| &\leq g_3(||x_n - x^*||)||x_n - x^*|| \leq K||x_n - x^*|| \\ &\leq K g_3(||x_{n-1} - x^*||)||x_{n-1} - x^*|| \leq K^2||x_{n-1} - x^*|| \leq \dots \\ &\leq K^{n+1}||x_0 - x^*||;\end{aligned}$$

therefore, the sequence $\{x_n\}$ given in (3) converges to x^*. □

Theorem 2. *Let us assume the following.*

(*i*) *There exists a solution y^* of Equation (1), with $y^* \in B(x^*, \rho_3)$ for some $\rho_3 > 0$.*
(*ii*) *The condition (C_1) holds on the ball $B(x^*, \rho_3)$.*
(*iii*) *There exists $\rho_4 \geqslant \rho_3$ such that*

$$\phi_0(\rho_4, \rho_3) < 1. \tag{24}$$

Let $\tilde{U} = D \cap B(x^*, \rho_4)$, then Equation (1) is uniquely solvable by y^* in the region $\tilde{U}$.

Proof. Let $z^* \in \tilde{U}$ such that $F(z^*) = 0$. Define the linear operator $T = [y^*, z^*; F]$. Then, we obtain in turn (34) that

$$||L^{-1}(T - L)|| \leq \phi_0(||y^* - x^*||, ||z^* - x^*||) \leq \phi_0(\rho_3, \rho_4) < 1;$$

thus, T^{-1} exists and

$$z^* - y^* = T^{-1}(F(z^*) - F(y^*)) = T^{-1}(0) = 0,$$

leading to $z^* = y^*$. □

Proposition 1. *We did not assume that x^* is a solution of Equation (1) in Theorem 2. But if we do, then $y^* = x^*$.*

Proposition 2. *Conditions (C_1)–(C_6) are left uncluttered and very general. Let us drop the following.*

1. *The second condition in (C_1). We get*

$$w_n - x^* = x_n - x^* + \gamma F(x_n) = x_n - x^* + \gamma[x_n, x^*; F](x_n - x^*) = (I + \gamma[x_n, x^*; F])(x_n - x^*).$$

But

$$I + \gamma[x_n, x^*; F] = I + \gamma L L^{-1}[x_n, x^*; F] = I + \gamma L L^{-1}([x_n, x^*; F] - L + L) = I + \gamma L + \gamma L L^{-1}([x_n, x^*; F] - L),$$

so

$$||I + \gamma[x_n, x^*; F]|| \leq ||I + \gamma L|| + |\gamma|||L||\phi_1(||x_n - x^*||).$$

Consequently,

$$||w_n - x^*|| \leq (||I + \gamma L|| + |\gamma|||L||\phi_1(||x_n - x^*))||x_n - x^*|| = \phi_6(||x_n - x^*||),$$

where $\phi_6(t) = (||I + \gamma L|| + |\gamma|||L||\phi_1(t))t$.

2. *Let us specialize the operators P, Q to find a possible choice for the real functions ϕ_2, ϕ_3, ϕ_4, and ϕ_5. As in [8], consider the weight operators*

$$P(\lambda) = I + \lambda + \lambda^2 \tag{25}$$

and

$$Q(\lambda, \delta) = I + \lambda\delta + \frac{13}{6}\lambda\delta^2. \tag{26}$$

Then, we have in turn that

$$P(\lambda) - I = (I - [w_n, x_n; F]^{-1}[y_n, w_n; F]) + (I - [w_n, x_n; F]^{-1}[y_n, w_n; F])(I - [w_n, x_n; F]^{-1}[y_n, w_n; F]).$$

But

$$I - [w_n, x_n; F]^{-1}[y_n, w_n; F] = [w_n, x_n; F]^{-1}([w_n, x_n; F] - [y_n, w_n; F]).$$

Suppose

(C₇)

$$||L^{-1}([x, y; F] - [z, x; F])|| \leq \phi_7(||x - x^*||, ||y - x^*||, ||z - x^*||), \text{ for each } x, y, z \in U_0.$$

where $\phi_7 : M_0 \times M_0 \times M_0 \to \mathbb{R}$ *is a CNF.*

Then,

$$\begin{aligned} ||\lambda_n|| &\leq ||[w_n, x_n; F]^{-1}([w_n, x_n; F] - [y_n, w_n; F])|| \\ &\leq \frac{\phi_7(||x_n - x^*||, ||y_n - x^*||, ||w_n - x^*||)}{1 - \phi_0(\phi_6(||x_n - x^*||), ||x_n - x^*||)} \\ &\leq \frac{\phi_7(||x_n - x^*||, ||y_n - x^*||, \phi_6(||x_n - x^*||)}{1 - \phi_0(\phi_6(||x_n - x^*||), ||x_n - x^*||)} \\ &= \phi_8(||x_n - x^*||, ||y_n - x^*||), \end{aligned}$$

where

$$\phi_8(t, s) = \frac{\phi_7(t, s, \phi_6(t))}{1 - \phi_0(\phi_6(t), t)};$$

thus,

$$||I - P(\lambda)|| \leq \phi_8(t, s) + \phi_8(t, s)^2.$$

so

$$\phi_2(t, s) = \phi_8(t, s) + \phi_8(t, s)^2.$$

Then,

$$\phi_3(t, s) = 1 + \phi_8(t, s) + \phi_8(t, s)^2 = 1 + \phi_2(t, s).$$

Under these specializations of the functions ϕ_2 *and* ϕ_3*, condition* (C_4) *is dropped.*
In order to drop condition (C_5)*, let us first see that*

$$Q(\lambda_n, \delta_n) - I = \lambda_n \delta_n + \frac{13}{6} \lambda_n \delta_n^2.$$

Next, we need to estimate on $||\delta_n||$*:*

$$||\delta_n|| \leq ||I - [w_n, x_n; F][z_n, y_n; F]P(\lambda_n)|| = ||[w_n, x_n; F]^{-1}]([w_n, x_n; F] - [z_n, y_n; F])P(\lambda_n)||.$$

Suppose

(C₈)

$$||L^{-1}([x, y; F] - [z, u; F])|| \leq \phi_9(||x - x^*||, ||y - x^*||, ||z - x^*||, ||u - x^*||), \text{ for each } x, y, z, u \in U,$$

where $\phi_9 : M_0 \times M_0 \times M_0 \times M_0 \to \mathbb{R}$ *is a CN symmetric real function.*

Then, we get in turn

$$\begin{aligned} ||\delta_n|| &\leq \frac{\phi_9(||x_n - x^*||, ||y_n - x^*||, ||z_n - x^*||, ||w_n - x^*||)||P(\lambda_n)||}{1 - \phi_0(\phi_6(||x_n - x^*||), ||x_n - x^*||)} \\ &\leq \frac{\phi_9(||x_n - x^*||, ||y_n - x^*||, ||z_n - x^*||, \phi_6(||x_n - x^*||)\phi_3(||x_n - x^*||, ||y_n - x^*||)}{1 - \phi_0(\phi_6(||x_n - x^*||), ||x_n - x^*||)} \\ &= \phi_{10}(||x_n - x^*||, ||y_n - x^*||, ||z_n - x^*||), \end{aligned}$$

where

$$\phi_{10}(t,s,r) = \frac{\phi_9(t,s,r,\phi_6(t))\phi_3(t,s)}{1-\phi_0(\phi_6(t),t)},$$

so

$$||Q(\lambda_n,\delta_n) - I|| \leq \frac{13}{6}||\lambda_n||||\delta_n||^2 + ||\lambda_n||||\delta_n||;$$

thus,

$$\phi_4(t,s,r) = \frac{13}{6}\phi_8(t,s)\phi_{10}(t,s,r)^2 + \phi_8(t,s)\phi_{10}(t,s,r)$$

and

$$\phi_5(t,s,r) = 1 + \phi_4(t,s,r).$$

Hence, according to Proposition 2, we arrived at the following specialization of Theorem 1.

Theorem 3. *Under conditions* (C_1) *(first point of this condition),* $(C_2), (C_3), (C_6), (C_7)$*, and* (C_8)*, the conclusions of Theorem 1 hold for method (3).*

Proposition 3. *The first condition in* (C_1) *can be dropped if we choose*

$$L = F'(x^*) \text{ or } L = [x_{-1}, x_0; F];$$

other choices are possible [8].

Let us look at the selection of the function ϕ_0 *in the two cases. Notice though that the choice* $L = [x_{-1}, x_0; F]$ *seems to be more interesting, since it allows the handling of equations when the operator* F *is not necessarily differentiable.*

If we consider $L = F'(x^*)$*, then the first condition in* (C_1) *reduces to the standard center Lipschitz condition for divided differences. It is specialized further as follows.*

(C_9*)*

$$||F'(x^*)^{-1}([x,y;F] - F'(x^*))|| \leq l_1||x-x^*|| + l_2||y-x^*||, \text{ for each } x,y \in D \text{ and some } l_1, l_2 \geqslant 0.$$

Then, choose $\phi_0(s,t) = l_1 s + l_2 t$.

If we consider $L = [x_{-1}, x_0; F]$*, with* $x_{-1}, x_0 \in D$

(C_{10}*)*

$$||[x_{-1},x_0;F]^{-1}([x,y;F] - [x_{-1},x_0;F])|| \leq l_3||x-x_1|| + l_4||y-x_0||,$$

for each $x, y \in B(x^*, \tilde{r}) \subset D$ *and some* $l_3, l_4, \tilde{r} \geqslant 0$.

In view of the calculation

$$l_3||x-x_1|| + l_4||y-x_0| \leq l_3||x-x^*|| + l_3||x_1-x^*|| + l_4||y-x^*|| + l_4||x_0-x^*||,$$

we can choose $\phi_0(s,t) = l_3 s + l_4 t + (l_3+l_4)\tilde{r}$*. A possible choice for* $\tilde{r}$ *is* $\tilde{r} = \sup\{t > 0 : B(x_0,t) \subset D\}$.

Proposition 4. *The preceding results are immediately extended to cover the case of the convergence of method (4). Suppose*

(C_{11}*)*

$$||\gamma_n|| \leq \tilde{\gamma}, \text{ for each } n = 0,1,2,\ldots \text{ and some } \tilde{\gamma} > 0.$$

Then, the preceding results hold for method (4) if $\tilde{\gamma}$ replaces γ. Next, we present a possible choice for the parameter $\tilde{\gamma}$. Let $||x_{-1} - x^*|| \leq \epsilon_{-1}$ and $||x_0 - x^*|| \leq \epsilon_0$ and suppose $\phi_0(\epsilon_0, \epsilon_{-1}) < 1$. In view of condition (C_1), we get in turn for the choice of γ_n given by (5)–(8) that

$$|| - \gamma_n|| = ||\gamma_n L L^{-1}|| \leq \frac{||L^{-1}||}{1 - \phi_0(||x_n - x^*||, ||x_{n-1} - x^*||)},$$

since

$$||L^{-1}(\gamma_n - L)|| \leq \phi_0(||x_n - x^*||, ||x_{n-1} - x^*||) \leq \phi_0(\epsilon_0, \epsilon_{-1}) < 1.$$

So

$$||\gamma_n L|| \leq \frac{1}{1 - \phi_0(||x_n - x^*||, ||x_{n-1} - x^*||)} \leq \frac{1}{1 - \phi_0(\epsilon_0, \epsilon_{-1})}.$$

Therefore, the parameter $\tilde{\gamma}$ can be chosen as $\tilde{\gamma} = \frac{||L^{-1}||}{1-\phi_0(\epsilon_0, \epsilon_{-1})}$. The other choices of γ_n are handled similarly.

3 | SEMILOCAL CONVERGENCE

The semilocal convergence analysis is recovered by switching the role of x^* with x_0 in the previous section and utilizing majorizing sequences [18].

As in the local convergence analysis, the operators, which are on the method (3), are associated to the following real functions:

(H_1) There exists an invertible linear operator L on E_1 such that

$$\|L^{-1}([x, y; F] - L)\| \leq \psi_0(\|x - x_0\|, \|y - x_0\|), \text{ for each } x, y \in D,$$

and

$$\|w_n - x_0\| \leq \psi_6(\|x_n - x_0\|),$$

where $\psi_0 : M \times M \to \mathbb{R}^+$, $\psi_6 : M \to \mathbb{R}^+$, which are *CNF* and such that the scalar equation

$$\psi_0(\psi_6(t), t) - 1 = 0$$

has a least a solution $t_0 \in M - \{0\}$. Set $M_0 = [0, t_0)$ and there exists the smallest solution $t_1 \in [0, t_0] - \{0\}$ of the equation

$$\psi_0(t, t) - 1 = 0. \tag{27}$$

(H_2) Let the set be $V = D \cap B(x_0, t_0)$ provided that t_0 exists.

$$\|L^{-1}([y, x; F] - [w, x; F])\| \leq \psi(\|x - x_0\|, \|w - x_0\|, \|y - x_0\|) \text{ for each } w, y \in V,$$

where $\psi : M_0 \times M_0 \times M_0 \to \mathbb{R}^+$ is a *CNF*.

(H_3)

$$\|P(\lambda_n)\| \leq \psi_3(\|x_n - x_0\|, \|y_n - x_0\|),$$

where $\psi_3 : [0, t_1) \times [0, t_1) \to \mathbb{R}^+$ is a *CNF*.

(H_4)

$$\|Q(\lambda_n, \delta_n)\| \leq \psi_5(\|x_n - x_0\|, \|y_n - x_0\|, \|z_n - x_0\|),$$

where $\psi_5 : [0, t_1) \times [0, t_1) \times [0, t_1) \to \mathbb{R}^+$ is a *CNF*.

(H_5) $B(x_0, \alpha^*) \subset D$.

It is important for the reader to realize the association of conditions (C_1)–(C_7) to the conditions listed above.
Next, we define the following real sequences provided that $a_0 = 0$, $b_0 \geq 0$.

$$\begin{aligned} c_n &= \psi(a_n, \psi_6(a_n), b_n)(b_n - a_n), \\ d_n &= b_n + \frac{\psi_3(a_n, b_n)c_n}{1 - \psi_0(a_n, b_n)}, \\ e_n &= (1 + \psi_0(b_n, d_n))(d_n - b_n) + c_n, \\ a_{n+1} &= d_n + \frac{\psi_5(a_n, b_n, d_n)c_n}{1 - \psi_0(b_n, d_n)}, \\ \xi_{n+1} &= (1 + \psi_0(a_n, a_{n+1}))(a_{n+1} - a_n) + (1 + \psi_0(a_n, \psi_6(a_n)))(b_n - a_n), \\ b_{n+1} &= a_{n+1} + \frac{\xi_{n+1}}{1 - \psi_0(\psi_6(a_{n+1}), a_{n+1})}. \end{aligned} \tag{28}$$

This sequence so defined is a majorizing sequence under certain conditions (see the Theorem 4). However, below certain conditions shall assure the convergence of it.

Lemma 3. *Under conditions (H_1)–(H_5), let us suppose that there exists a parameter $t_2 \in [0, t_1)$ such that*

$$\begin{aligned} &\psi_0(\psi_6(a_n), a_n) < 1, \\ &\psi_0(a_n, b_n) < 1, \\ &\psi_0(b_n, d_n) < 1, \\ &a_n \leq t_2. \end{aligned} \tag{29}$$

Then, the following items hold

$$0 \leq a_n \leq b_n \leq d_n \leq a_{n+1} \leq a^* \leq t_2, \tag{30}$$

where $a^* = \lim_{n\to\infty} a_n \in [0, t_2]$.

Proof. Item (30) is a consequence of formula (28) and condition (29). Then, by item (30), there exists $a^* = \lim_{n\to\infty} a_n$. □

Next, we relate the real auxiliary sequences constructed with the iterations given by method (3) in the following result.

Lemma 4. *Under conditions (H_1)–(H_5) and (29), if there exists $[w_0, x_0; F]^{-1}$, so that $\|[w_0, x_0; F]^{-1}F(x_0)\| \leq \beta_0$, the following recurrence relations hold:*

(I_n) $\|z_n - y_n\| \leq d_n - b_n$ *and* $z_n \in B(x_0, a^*)$.
(II_n) $\|x_{n+1} - z_n\| \leq a_{n+1} - d_n$ *and* $x_{n+1} \in B(x_0, a^*)$.
(IIL_n) $\|y_{n+1} - x_{n+1}\| \leq b_{n+1} - a_{n+1}$ *and* $y_{n+1} \in B(x_0, a^*)$.

Proof. Then, we have in turn as in the local convergence case but exchanging x^*, (C_1)–(C_7) with x_0, (H_1)–(H_5).

We consider $n = 0$. So, we take $x_0 \in D$ and by hypothesis y_0 is well-defined and $\|y_0 - x_0\| \leq b_0 < a^*$, then $y_0 \in B(x_0, a^*) \subset D$. Now, by (H_1), we get

$$||L^{-1}([y_0, x_0; F] - L)|| \leq \psi_0(||y_0 - x_0||, ||x_0 - x_0||) \leq \psi_0(b_0, a_0) < 1.$$

Then, by the Banach lemma for invertible operators, we obtain that there exists $[y_0, x_0; F]^{-1}$ with

$$||[y_0, x_0; F]^{-1}L|| \leq \frac{1}{1 - \psi_0(b_0, a_0)}; \tag{31}$$

thus, the iterate z_0 is well-defined. Next, from method (3), we have

$$F(y_0) = F(y_0) - F(x_0) - [w_0, x_0; F](y_0 - x_0) = ([y_0, x_0; F] - [w_0, x_0; F])(y_0 - x_0).$$

Then, as the operator λ_0 is well-defined, from (H_3), we can also write

$$\begin{aligned}\|z_0 - y_0\| &= \|P(\lambda_0)\|\|[y_0, x_0; F]^{-1}F(y_n)\| \\ &\leq \|P(\lambda_0)\|\|[y_0, x_0; F]^{-1}L\|\|L^{-1}F(y_0)\| \\ &\leq \frac{\psi_3(\|x_0 - x_0\|, \|y_0 - x_0\|)\psi(\|x_0 - x_0\|, \|w_0 - x_0\|, \|y_0 - x_0\|)\|y_0 - x_0\|}{1 - \psi_0(\|x_0 - x_0\|, \|y_n - x_0\|)} \\ &\leq \frac{\psi_3(a_0, b_0)\psi(a_0, \psi_6(a_0), b_0)(b_0 - a_0)}{1 - \psi_0(a_0, b_0)} = d_0 - b_0,\end{aligned}$$

where we also used

$$\|L^{-1}([y_0, x_0; F] - [w_0, x_0; F])\| \leq \psi(\|x_0 - x_0\|, \|w_0 - x_0\|, \|y_0 - x_0\|) \leq \psi(a_0, \psi_6(a_0), b_0),$$

and

$$\|L^{-1}F(y_0)\| \leq \psi(a_0, \psi_6(a_0), b_0)(b_0 - a_0) = c_0$$

since (H_2).

On the other hand, we obtain

$$\|z_0 - x_0\| \leq \|z_0 - y_0\| + \|y_0 - x_0\| \leq d_0 - b_0 + b_0 - a_0 = d_0 < a^*.$$

So, the iterate $z_0 \in B(x_0, a^*)$ and (I_0) is proved.

Next, to prove (II_0), by (H_1), we get

$$||L^{-1}([z_0, y_0; F] - L)|| \leq \psi_0(||z_0 - x_0||, ||y_0 - x_0||) \leq \psi_0(d_0, b_0) < 1.$$

Then, by the Banach lemma for invertible operators, we obtain that there exists $[z_0, y_0; F]^{-1}$ with

$$||[z_0, y_0; F]^{-1}L|| \leq \frac{1}{1 - \psi_0(d_0, b_0)}. \tag{32}$$

Then, as δ_0 is well-defined, x_1 is well-defined too. So, we consider

$$\|x_1 - z_0\| \leq \|Q(\lambda_0, \delta_0)\|\|[z_0, y_0; F]^{-1}L\|\|L^{-1}F(z_0)\|,$$

and as

$$\begin{aligned}\|L^{-1}F(z_0)\| &\leq \|L^{-1}\left(F(z_0) - F(y_0) + F(y_0)\right)\| \\ &\leq \|L^{-1}\left([z_0, y_0; F](z_0 - y_0) + F(y_0)\right)\| \\ &\leq \|I - L^{-1}\left([z_0, y_0; F] - L\right)\|\|z_0 - y_0\| + \|L^{-1}F(y_0)\| \\ &\leq (1 + \psi_0(\|z_0 - x_0\|, \|y_0 - x_0\|))\|z_0 - y_0\| + \|L^{-1}F(y_0)\| \\ &\leq (1 + \psi_0(d_0, b_0))(d_0 - b_0) + \psi(a_0, \psi_6(a_0), b_0)(b_0 - a_0) \\ &\leq (1 + \psi_0(d_0, b_0))(d_0 - b_0) + c_0 = e_0.\end{aligned}$$

Therefore, from (H_4) and (32), we get

$$\begin{aligned}\|x_1 - z_0\| &\leq \frac{\psi_5(\|x_n - x_0\|, \|y_n - x_0\|, \|z_n - x_0\|)e_0}{1 - \psi_0(d_0, b_0)} \\ &\leq \frac{\psi_5(a_0, b_0, d_0)e_0}{1 - \psi_0(d_0, b_0)} = a_1 - d_0.\end{aligned}$$

Moreover,

$$\|x_1 - x_0\| \leq \|x_1 - z_0\| + \|z_0 - x_0\| \leq a_1 - d_0 + d_0 - a_0 = a_1 - a_0 = a_1 < a^*.$$

Thus, the iterate $x_{n+1} \in B(x_0, a^*)$ and (II_0) is proved.

Next, to prove (III_0), by (H_1), we get

$$||L^{-1}([w_1, x_1; F] - L)|| \leq \psi_0(||w_1 - x_0||, ||x_1 - x_0||) \leq \psi_0(\psi_6(a_1), a_1) < 1.$$

Then, by the Banach lemma for invertible operators, we obtain that there exists $[w_1, x_1; F]^{-1}$ with

$$||[w_1, x_1; F]^{-1}L|| \leq \frac{1}{1 - \psi_0(\psi_6(a_1), a_1)}.$$

Next, as previously, we consider

$$\begin{aligned} \|L^{-1}F(x_1)\| =& \|L^{-1}\left(F(x_1) - F(x_0) - [w_0, x_0; F](y_0 - x_0)\right)\| \\ \leq& \|L^{-1}[x_1, x_0; F]\|\|x_1 - x_0\| + \|L^{-1}[w_0, x_0; F]\|\|y_0 - x_0\| \\ \leq& \|I + L^{-1}([x_1, x_0; F] - L)\|\|x_1 - x_0\| + \|I + L^{-1}([w_0, x_0; F] - L)\|\|y_0 - x_0\| \\ \leq& (1 + \psi_0(\|x_1 - x_0\|, \|x_0 - x_0\|))\|x_1 - x_0\| + (1 + \psi_0(\|w_0 - x_0\|, \|x_0 - x_0\|))\|y_0 - x_0\| \\ \leq& (1 + \psi_0(a_1, a_0))(a_1 - a_0) + (1 + \psi_0(\psi_6(a_0), a_0)(b_0 - a_0) = \xi_1; \end{aligned} \tag{33}$$

thus,

$$\begin{aligned} \|y_1 - x_1\| &\leq \|[w_1, x_1; F]^{-1}L\|\|L^{-1}F(x_1)\| \\ &\leq \frac{\xi_1}{1 - \psi_0(\psi_6(a_1), a_1)} = b_1 - a_1. \end{aligned}$$

Moreover,

$$\|y_1 - x_0\| \leq \|y_1 - x_1\| + \|x_1 - x_0\| \leq b_1 - a_1 + a_1 - a_0 = b_1 - a_0 = b_1 < a^*.$$

Then, items (I_0), (II_0), and (III_0) hold.

Next, by an inductive procedure, the result is proved. □

Therefore, we arrive at the following theorem.

Theorem 4. *Under conditions of Lemma 3 if the conditions (H_1)–(H_5) hold, then there exists a solution $x^* \in \overline{B(x_0, \alpha^*)}$ of the equation $F(x) = 0$. Moreover, the sequence $\{x_n\}$, given in (3), belongs to $B(x_0, \alpha^*)$ and converges to x^*. Moreover,*

$$\|x^* - x_n\| \leq a^* - a_n.$$

Proof. From the previous lemma, it is easy to check that

$$\|x_{n+1} - x_n\| \leq a_{n+1} - a_n,$$

as the real sequence $\{a_n\}$ converges to a^*, then the sequence $\{x_n\}$ is a Cauchy sequence in a Banach space, so there exists $x^* = \lim_{n \to +\infty} x_n \in \overline{B(x_0, \alpha^*)}$.

Next, taking into account (3), for $n + 1$, we have

$$\|L^{-1}F(x_{n+1})\| \leq (1 + \psi_0(a_{n+1}, a_n))(a_{n+1} - a_n) + (1 + \psi_0(\psi_6(a_n), a_n)(b_n - a_n),$$

and for $n \to +\infty$, we obtain that $F(x^*) = 0$. On the other hand, as $\|x_{n+m} - x_n\| \leq a_{n+m} - a_n$ for all $n, m \geqslant 0$, taking limits when $n \to +\infty$, we obtain that $\|x^* - x_n\| \leq a^* - a_n$, for all $n \geqslant 0$. □

Next, we have in turn as in the local convergence case the following result of uniqueness.

Theorem 5. *Under conditions of previous theorem, suppose we have the following:*

(*i*) *There exists a solution y^* of Equation (1), with $y^* \in B(x^*, t_3)$ for some $t_3 < 0$.*
(*ii*) *Condition (H_1) holds on the ball $B(x^*, t_3)$.*

(*iii*) *There exists* $t_4 \geqslant t_3$ *such that*

$$\psi_0(t_4, t_3) < 1. \tag{34}$$

Let $\tilde{V} = D \cap B(x^*, t_4)$, *then Equation (1) is uniquely solvable by* y^* *in the region* $\tilde{V}$.

Proof. Let $z^* \in \tilde{U}$ such that $F(z^*) = 0$. Define the linear operator $T = [y^*, z^*; F]$. Then, we obtain in turn (34) that

$$||L^{-1}(T - L)|| \leq \phi_0(||y^* - x^*||, ||z^* - x^*||) \leq \phi_0(\rho_3, \rho_4) < 1;$$

thus, T^{-1} exists and

$$z^* - y^* = T^{-1}(F(z^*) - F(y^*)) = T^{-1}(0) = 0,$$

leading to $z^* = y^*$. □

We did not assume that x^* is a solution of Equation (1) in Theorem 5. But if we do, then $y^* = x^*$.

Proposition 5. *Some choices for the functions* ψ *can be made as in the local case (see Propositions 2 and 3). As an example, the second condition in* (H_1) *can be dropped as follows:*

$$w_n - x_0 = x_n - x_0 + \gamma F(x_n) = x_n - x_0 + \gamma [x_n, x_0; F](x_n - x_0) + \gamma F(x_0) = (I + \gamma [x_n, x_0; F])(x_n - x_0) + \gamma F(x_0).$$

But

$$I + \gamma[x_n, x_0; F] = I + \gamma LL^{-1}[x_n, x_0; F] = I + \gamma LL^{-1}([x_n, x_0; F] - L + L) = I + \gamma L + \gamma LL^{-1}([x_n, x_0; F] - L),$$

so

$$||I + \gamma[x_n, x_0; F]|| \leq ||I + \gamma L|| + |\gamma|||L||\phi_0(||x_n - x_0\|, 0).$$

Consequently,

$$||w_n - x_0|| \leq (||I + \gamma L|| + |\gamma|||L||\phi_0(||x_n - x_0\|, 0))||x_n - x_0|| + |\gamma|\|F(x_0)\| = \phi_7(||x_n - x^*||),$$

where $\phi_7(t) = (||I + \gamma L|| + |\gamma|||L||\phi_0(t, 0))\, t + |\gamma|\|F(x_0)\|$.

4 | NUMERICAL EXPERIMENTS

Now, we want to apply the iterative method family analyzed in the previous sections in order to approximate the solution of a nonlinear problem defined in generic Banach spaces, as we already mention in the motivation section; see (P_2).

Specifically, we consider the nonlinear integral equation

$$x(s) = g(s) + \lambda p(s) \int_a^b q(t)[\Phi(x)](t)\, dt; \tag{35}$$

we apply the iterative scheme (3) to solve the equation $F(x)(s) = 0$, where

$$[F(x)](s) = x(s) - g(s) - \lambda p(s) \int_a^b q(t)[\Phi(x)](t) dt, \tag{36}$$

with $F : \Omega \subseteq \mathscr{C}([a, b]) \to \mathscr{C}([a, b])$, considering the set $\mathscr{C}([a, b])$ with the max-norm as a Banach space, functions $x(s), g(s), p(s)$, and $q(s)$ are continuous functions defined in $[a, b]$, and $[\Phi(x)](t) = \phi((t)$ is a continuous function defined in Ω, with $\phi : \mathbb{R} \to \mathbb{R}$ a continuous real function.

In order to obtain an approximation for a solution of (36), we deal with the approximation of the derivative operator $F'(x(s))$ by divided differences operator; see [15] and [9], where $[x, y; F] \in \mathcal{L}(\mathscr{C}([a, b]), \mathscr{C}([a, b]))$ that must be verified [2] as follows:

$$[x, y; F]\,(x - y) = F(x) - F(y), \tag{37}$$

with $\mathcal{L}(\mathscr{C}([a, b]), \mathscr{C}([a, b]))$ denotes the space of bounded linear operators in the defined Banach space.

We work with nondifferentiable problems, so we can define for each $x, y \in \mathscr{C}([a, b])$ the following function:

$$\psi[x, y](t) = \begin{cases} \frac{\phi(x(t))-\phi(y(t))}{x(t)-y(t)} & \text{if } t \in [a, b] \text{ with } x(t) \neq y(t), \\ 0 & \text{if } t \in [a, b] \text{ with } x(t) = y(t). \end{cases}$$

Then, we have the following divided difference operator:

$$[x, y; F]u(s) = u(s) - \lambda p(s) \int_a^b q(t)\psi[x, y](t)u(t)\,dt = \omega(s), \tag{38}$$

so we can characterize the inverse operator as follows:

$$u(s) = [x, y; F]^{-1}\omega(s) = \omega(s) + \lambda p(s)\mathbb{E}(x, y, w), \tag{39}$$

where $\mathbb{E}(x, y, w) = \int_a^b q(t)\psi[x_n, y_n](t)u(t)\,dt$.

That is, to obtain $[x, y; F]^{-1}$ explicitly an independent of $u(t)$, we multiply (39) by $q(s)\psi[x, y](s)$ and integrate between a and b in order to obtain the following:

$$\mathbb{E}(x, y, w) = \int_a^b q(s)\psi[x, y](s)\omega(s)\,ds + \lambda \int_a^b p(s)q(s)\psi[x, y](s)\,ds\; \mathbb{E}(x, y, w).$$

If

$$C(x, y) = \int_a^b p(s)q(s)\psi[x, y](s)\,ds \text{ and } B(x, y, w) = \int_a^b q(s)\psi[x, y](s)\omega(s)\,ds, \tag{40}$$

then

$$\mathbb{E}(x, y, w) = \frac{B(x, y, w)}{1 - \lambda C(x, y)}.$$

Thus, we can define the action of $[x, y; F]^{-1}$, given by

$$[x, y; F]^{-1}\omega(s) = \omega(s) + \lambda p(s)\mathbb{E}(x, y, w).$$

So the application of the family of iterative schemes (3) is given by the following algorithm.

Fixed $x_{-1}(s), x_0(s) \in \Omega \subseteq \mathscr{C}([a, b])$, for $n \geqslant 0$:

- **First step:** Calculate

$$F(x_n)(s) = x_n(s) - g(s) - \lambda\, p(s) \int_a^b q(t)\Phi(x_n)(t)\,dt.$$

- **Second step:** Calculate $\psi[x_{n-1}, x_n](s)$; by (40), we obtain $C(x_{n-1}, x_n)$ and $B(x_{n-1}, x_n, F(x_n))$ and then

$$w_n(s) = x_n(s) - F(x_n)(s) - \lambda p(s)\mathbb{E}(x_{n-1}, x_n, F(x_n)).$$

- **Third step:** Calculate $\psi[w_n, x_n](s)$, then obtain $C(w_n, x_n)$ and $B(w_n, x_n, F(x_n))$ and then

$$y_n(s) = x_n(s) - F(x_n)(s) - \lambda p(s)\mathbb{E}(w_n, x_n, F(x_n)).$$

- **Fourth step:** Notice how the operator μ_n works and let I the identity matrix, then

$$\begin{aligned}
\mu_n(x(s)) &= (I - [w_n, x_n, F]^{-1}[y_n, w_n, F])(x(s)) \\
&= x(s) - [w_n, x_n, F]^{-1}(x(s) - \lambda p(s) \int_a^b q(t)\psi[y_n, w_n](t)x(t)\,dt) \\
&= x(s) - (x(s) + \lambda p(s)\mathbb{E}(w_n, x_n, x(s)) + [w_n, x_n, F]^{-1}(\tilde{x}(s)) \\
&= -\lambda p(s)\mathbb{E}(w_n, x_n, x(s)) + \tilde{x}(s) + \lambda p(s)\mathbb{E}(w_n, x_n, \tilde{x}(s)),
\end{aligned}$$

where $\tilde{x}(s) = \lambda p(s) \int_a^b q(t)\psi[y_n, w_n](t)x(t)\,dt$.

- **Fifth step:** Calculate

$$F(y_n)(s) = y_n(s) - g(s) - \lambda p(s) \int_a^b q(t)\Phi(y_n)(t)\,dt$$

and $\psi[y_n, x_n](s)$, then obtain $C(y_n, x_n)$ and $B(y_n, x_n, F(y_n))$; next,

$$z_n(s) = y_n(S) - A(\mu_n)(F(y_n)(s) + \lambda p(s)\mathbb{E}(y_n, x_n, F(y_n)).$$

That is, for $A(\mu_n) = \mu^2 + \mu_n + I$, if we denote $h_n(s) = (F(y_n)(s) + \lambda p(s)\mathbb{E}(y_n, x_n, F(y_n))$, we have the following:

$$\mu_n(h_n(s)) = -\lambda p(s)\mathbb{E}(w_n, x_n, h_n(s)) + \widetilde{h_n}(s) + \lambda p(s)\mathbb{E}(w_n, x_n, \widetilde{h_n}(s)),$$

where $\widetilde{h_n}(s) = \lambda p(s) \int_a^b q(t)\psi[y_n, w_n](t)h_n(t)\,dt$.

So, we obtain the following:

$$z_n(s) = y_n(s) - \mu_n^2(h_n(s)) - \mu_n(h_n(s)) - h_n(s).$$

- **Sixth step:** Notice how the operator δ_n works:

$$\begin{aligned}
\delta_n(x(s)) &= (I - [w_n, x_n, F]^{-1}[z_n, y_n, F])A(\mu_n(x(s))) \\
&= A(\mu_n(x(s))) - [w_n, x_n, F]^{-1}(A(\mu_n(x(s))) - \lambda p(s) \int_a^b q(t)\psi[z_n, y_n](t)A(\mu_n(x(s)))\,dt) \\
&= -\lambda p(s)\mathbb{E}(w_n, x_n, A(\mu_n(x(s))) + [w_n, x_n, F]^{-1}(\tilde{A}(\mu_n(x(s))) \\
&= -\lambda p(s)\mathbb{E}(w_n, x_n, A(\mu_n(x(s))) + \tilde{A}(\mu_n(x(s))) + \lambda p(s)\mathbb{E}(w_n, x_n, \tilde{A}(\mu_n(x(s))),
\end{aligned}$$

where $\tilde{A}(\mu_n(x(s))) = \lambda p(s) \int_a^b b(t)\psi[z_n, y_n](t)A(\mu_n(x(s)))\,dt$.

- **Seventh step:** Calculate

$$F(z_n)(s) = z_n(s) - g(s) - \lambda p(s) \int_a^b q(t)\Phi(z_n)(t)\,dt$$

and $\psi[z_n, y_n](s)$, then obtain $C(z_n, y_n)$ and $B(z_n, y_n, F(z_n))$ so

$$x_{n+1}(s) = z_n(s) - D(\mu_n, \delta_n)(F(z_n)(s) + \lambda p(s)\mathbb{E}(z_n, y_n, F(z_n)),$$

TABLE 1 Different algorithms depending of the choice of γ_n.

I. Method	Equation	γ_n	Second step
ALG_1	(5)	$-[x_n, x_{n-1}; F]^{-1}$	$\psi(x_n, x_{n-1})$
ALG_2	(6)	$-[2x_n - x_{n-1}, x_{n-1}; F]^{-1}$	$\psi(2x_n - x_{n-1}, x_{n-1})$
ALG_3	(7)	$-[x_n, y_{n-1}; F]^{-1}$	$\psi(x_n, y_{n-1})$
ALG_4	(8)	$-[2x_n - y_{n-1}, y_{n-1}; F]^{-1}$	$\psi(2x_n - y_{n-1}, y_{n-1})$

TABLE 2 Numerical results for different values of γ_n.

I. Method	*iter*	$\|x_{n+1}(s) - x_n(s)\|$	$\|F(x_{n+1}(s))\|$	p
ALG_1	5	6.42499e-1950	1.29133e-5007	7.46643
ALG_2	4	7.56024e-341	1.02825e-2729	8.18241
ALG_3	5	6.78516e-896	4.11821e-5008	6.67125
ALG_4	4	2.40361e-316	3.06247e-2363	7.51117

with $D(\mu_n, \delta_n) = I + \mu_n \circ \delta_n + 13/6\mu_n \circ \delta_n^2$, so let be $f_n(s) = F(z_n)(s) + \lambda p(s)\mathbb{E}(z_n, y_n, F(z_n))$ then we have the following:

$$x_{n+1}(s) = z_n(s) - f_n(s) - \mu_n(\delta_n(f_n(s))) - 13/6\mu_n(\delta_n^2(f_n(s))).$$

Remark:

Notice that the algorithm has been described for case $\gamma_n = [x_{n-1}, x_n; F]^{-1}$, but for obtaining the iterates in different cases, given by (6)–(8), we have only to change in second step the parameters of function ψ, so we have Table 1.

We compare the behavior of these algorithms applying them to a particular example.

4.1 | Particular example

Now, in (36), we take $g(s) = (1 - 11/80\lambda)s - 1/2$, $\lambda = 1$, $p(s) = s$, $q(t) = t$ and $\Phi(x(t)) = x^3(t) + |x(t)|$, so we have the nonlinear integral equation:

$$[F(x)](s) = x(s) - ((1 - 11/80)s - 1/2) - s\int_a^b t(x^3(t) + |x(t)|)dt, \tag{41}$$

in which the exact solution is $x^*(s) = s - 1/2$.

Then, by taking starting functions $x_0(s) = s$ and $x_{-1}(s) = 1/3$ with $s \in [0, 1]$, we apply iterative schemes given in Table 1, by following the first to the seventh steps described above, where all the integrals have been approximated by Simpson quadrature with 200 nodes.

We work with MATLAB R 2019a with 5000 digits, by imposing the stopping criteria $\|x_{n+1}(s) - x_n(s)\| \leq 10^{-300}$. Notice that we have to work with variable precision arithmetic for running high-order methods in order to reach the approximated computational order of convergence; see [7] that is shown in the numerical results of Table 2 by p.

We observe in the second column of Table 2 that $ALG2$ and $ALG4$ methods need one less iteration for reaching the required tolerance; also, we show in the table the distance between the last two iterations and the value of the nonlinear operator F at the approximated solution.

5 | CONCLUSIONS

The main objective of this paper is to obtain theoretical results of local and semilocal convergence for a higher order method that can be applied to nondifferentiable problems, so all the auxiliary lemmas and corresponding theorems are proved working with the operator of divided differences that approximates the derivative of the nonlinear operator involved in the main problem. Finally, we also approximate the solution of a nondifferentiable problem by working in the infinite dimensional space of the continuous functions in a closed interval. Moreover, the methodology of this paper can be used to extend the applicability of other methods (single or multistep) by using inverse or linear operators such Steffensen's, Kurchatov's, and Stirling's. This is the direction of our future research.

AUTHOR CONTRIBUTIONS

All authors have contributed equally to this paper.

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CONFLICT OF INTEREST STATEMENT

This work does not have any conflicts of interest.

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