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Full Length Article

Mean convergence of Fourier–Akhiezer–Chebyshev series

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Abstract

We prove mean convergence of the Fourier series in Akhiezer–Chebyshev polynomials in L^p , $p > 1$, using a weighted inequality for the Hilbert transform in an arc of the unit circle.

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1. Introduction

One of the cornerstone results in harmonic analysis has been the continuity of the conjugate operator or Hilbert transform in $L^p(0, 2\pi)$ for $1 < p < \infty$ by Marcel Riesz [16]. This result implies the convergence of the Fourier series in the mentioned function space. Riesz's results opened several questions of research such as finding closed systems of functions in L^p .

The goal of this paper is to prove mean convergence of Fourier–Akhiezer–Chebyshev series. The Akhiezer–Chebyshev polynomials were introduced by Akhiezer [1] and a rigorous exposition was given by Golinskii [6]. These polynomials are orthogonal with respect to a measure supported on an arc of the unit circle. At the present, L^p convergence of Fourier series of orthogonal polynomials have been established only for specific classes of weights

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in the real line, such as Hermite, Laguerre, Jacobi (see, for example, [3,4,8–12,14,15]). Our results are limited to a type of Akhiezer–Chebyshev weights.

Set $\alpha \in (0, \pi)$, $\Delta_\alpha = \{e^{i\vartheta} : \alpha < \vartheta < 2\pi - \alpha\}$. Let $w_\alpha(\vartheta)$ be the Akhiezer–Chebyshev weight in Δ_α given by

$$w_\alpha(\vartheta) = \begin{cases} \frac{\sin(\alpha/2)}{2 \sin(\vartheta/2) \sqrt{\cos^2(\alpha/2) - \cos^2(\vartheta/2)}}, & \vartheta \in (\alpha, 2\pi - \alpha), \\ 0, & \text{otherwise.} \end{cases} \tag{1.1}$$

Let $\{\psi_n\}_{n=0}^\infty$ be the sequence of orthonormal polynomials with the inner product defined by

$$\langle f, g \rangle = \frac{1}{2\pi} \int_\alpha^{2\pi-\alpha} f(e^{i\vartheta}) \overline{g(e^{i\vartheta})} w_\alpha(\vartheta) d\vartheta,$$

Consider the Lebesgue space $L^p(w_\alpha) = L^p((\alpha, 2\pi - \alpha), w_\alpha)$, $1 \leq p < \infty$, i.e. the set of all Borel measurable functions $f : \Delta_\alpha \rightarrow \mathbb{C}$ such that

$$\int_\alpha^{2\pi-\alpha} |f(e^{i\vartheta})|^p w_\alpha(\vartheta) d\vartheta < \infty$$

For $f \in L^p(w_\alpha)$, consider its n -th partial Fourier sum in terms of $\{\psi_n\}_{n=0}^\infty$

$$\mathcal{S}_n(f, z) = \sum_{j=0}^n \langle f, \psi_j \rangle \psi_j(z).$$

Then we prove the following result.

Theorem 1.1. *Let $p \in (1, \infty)$, and $f \in L^p(w_\alpha)$. Then we have*

$$\lim_{n \rightarrow \infty} \int_\alpha^{2\pi-\alpha} |f(e^{i\vartheta}) - \mathcal{S}_n(f, e^{i\vartheta})|^p w_\alpha(\vartheta) d\vartheta = 0. \tag{1.2}$$

As main auxiliary results we give an estimate for the sequence of para-orthonormal polynomials associated to Akhiezer–Chebyshev weight and we also obtain the following weighted inequality for the Hilbert transform in an arc of the unit circle.

Theorem 1.2. *If $1 < p < \infty$, then there exists a constant C such that for all $f \in L^p(w_\alpha)$ we have*

$$\int_\alpha^{2\pi-\alpha} \left| \int_\alpha^{2\pi-\alpha} \frac{f(e^{i\tau})}{e^{i\tau} - e^{i\vartheta}} d\tau \right|^p w_\alpha(\tau) d\tau \leq C \int_\alpha^{2\pi-\alpha} |f(e^{i\tau})|^p w_\alpha(\tau) d\tau, \tag{1.3}$$

where the singular integral $\int_\alpha^{2\pi-\alpha} \frac{f(e^{i\tau})}{e^{i\tau} - e^{i\vartheta}} d\tau$ exists almost everywhere in principal value sense.

The paper is organized as follows. First, for an easy reading, we include some calculations for Akhiezer–Chebyshev polynomials in Section 2. Then, in Section 3 we prove Theorems 1.1 and 1.2. Finally, Section 4 contains some results on mean convergence of series in orthogonal polynomial with respect to Akhiezer–Chebyshev weight multiplied by a function with nice properties.

2. Akhiezer–Chebyshev polynomials

Several results in this section can be found in [6]. We include the proofs of the main results in order to make the paper self-contained. First, we consider a conformal mapping from the unit disk onto the complement of Δ_α . This function is written in terms of the product of two Möbius transforms.

Lemma 2.1 ([6]). Let $\beta = i \tan \frac{\pi-\alpha}{4}$ and $V(z) = (z - e^{i\alpha})(z - e^{-i\alpha})$.

(i) The function

$$z = h(v) = \frac{(v - \beta)(\beta v - 1)}{(v + \beta)(\beta v + 1)} \tag{2.1}$$

is a conformal mapping from the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ onto $\overline{\mathbb{C}} \setminus \Delta_\alpha$. Moreover, $e^{i\vartheta} = h(e^{i\omega})$ for $\omega \in [0, 2\pi]$ sweeps the arc $e^{i\vartheta} \in \Delta_\alpha$ twice.

(ii) The functions

$$\chi_\pm(z) = \frac{z + 1 \pm \sqrt{V(z)}}{2 \cos(\alpha/2)} \tag{2.2}$$

are conformal mappings from $\overline{\mathbb{C}} \setminus \Delta_\alpha$ to $\mathbb{E} = \{v \in \mathbb{C} : |v| > 1\}$ or \mathbb{D} according to either sign plus or minus is chosen, where we take the branch of the root such that $\sqrt{1} = 1$.

(iii) Let f be a Borel function. We have $f \in L^1(w_\alpha)$ if and only if $f \circ h \in L^1(0, \pi)$. In that case, we get

$$\int_\alpha^{2\pi-\alpha} f(e^{i\vartheta}) w_\alpha(\vartheta) d\vartheta = \int_0^\pi f(h(e^{i\omega})) d\omega.$$

Proof. The Möbius transform

$$w = w(v) = i \frac{1 - \beta v}{v + \beta} \tag{2.3}$$

maps the exterior of the unit circle \mathbb{E} onto the interior of the unit circle, whereas $w(\frac{1}{v}) = i \frac{v-\beta}{\beta v+1}$ maps the interior of the unit circle \mathbb{D} onto itself. The inverse transform of $w = w(v)$ is

$$v = v(w) = \frac{i - \beta w}{w + i\beta}, \tag{2.4}$$

which maps \mathbb{E} onto \mathbb{D} .

Observe that

$$z = h(v) = \frac{(v - \beta)(\beta v - 1)}{(v + \beta)(\beta v + 1)} = w(v) w\left(\frac{1}{v}\right).$$

So we have

$$h\left(\frac{1}{v}\right) = h(v), \quad h(-v) = \frac{1}{h(v)}, \quad h(\bar{v}) = \frac{1}{\overline{h(v)}}. \tag{2.5}$$

In particular,

$$h(e^{-i\omega}) = h(e^{i\omega}), \quad |h(e^{i\omega})| = 1, \quad \omega \in \mathbb{R}. \tag{2.6}$$

As both numerator and denominator of h are polynomials of degree 2, each point of $z \in \overline{\mathbb{C}}$ is the image by $z = h(v)$ of exactly two points. Since $h(1/v) = h(v)$, h is injective in the unit disk, and conformal since the derivative of h is

$$h'(v) = 2(\beta + \beta^{-1}) \frac{v^2 - 1}{(v + \beta)^2(v + \beta^{-1})^2} \neq 0, \quad v \in \mathbb{D}.$$

Moreover,

$$h(1) = -\left(\frac{1-\beta}{1+\beta}\right)^2 = e^{i\alpha}, \quad h(-1) = e^{-i\alpha}, \quad h(\pm i) = -1, \quad h(0) = h(\infty) = 1,$$

$$|h(v)| = 1, \quad v \in \mathbb{R}.$$

Since

$$h(i\varepsilon) = \frac{1 - 2 \tan(\alpha/2)\varepsilon - \varepsilon^2}{1 + 2 \tan(\alpha/2)\varepsilon - \varepsilon^2},$$

for $\varepsilon > 0$ small, we have $|h(i\varepsilon)| < 1$. Hence, $z = h(v)$ maps \mathbb{D} onto $\overline{\mathbb{C}} \setminus \Delta_\alpha$; and $e^{i\vartheta} = h(e^{i\omega})$ for $\omega \in [0, 2\pi]$ sweeps the arc Δ_α twice.

Next we get (ii). Taking into account (2.4), the composition $z = h(v(w))$ maps \mathbb{E} onto $\overline{\mathbb{C}} \setminus \Delta_\alpha$ given by

$$z = h(v(w)) = \frac{\cos(\alpha/2)w^2 - w}{w - \cos(\alpha/2)}, \tag{2.7}$$

which has the property

$$\lim_{w \rightarrow \infty} \frac{h(v(w))}{w} = \cos(\alpha/2).$$

Taking the branch of the square root $\sqrt{1} = 1$, then the functions $\chi_+(z)$ and $\chi_-(z)$ given in (2.2), which are the solutions of (2.7) for each z , are conformal mappings from $\overline{\mathbb{C}} \setminus \Delta_\alpha$ onto \mathbb{E} and \mathbb{D} , respectively. The number $\gamma = \cos(\alpha/2)$ is the transfinite diameter of Δ_α . Observe that

$$V(z) = (z + 1)^2 - 4z \cos^2(\alpha/2) = z^2 - 2z \cos \alpha + 1.$$

Now we check (iii). If $e^{i\vartheta} = h(e^{i\omega})$, then

$$e^{2i\omega} - i(\beta + \beta^{-1}) \cot \frac{\vartheta}{2} e^{i\omega} + 1 = 0, \tag{2.8}$$

and this equation for $\omega \in (0, \pi)$ (i.e. $\sin \omega > 0$) has solution

$$e^{i\omega} = \frac{\tan \frac{\alpha}{2}}{\tan \frac{\vartheta}{2}} + i \sqrt{1 - \left(\frac{\tan \frac{\alpha}{2}}{\tan \frac{\vartheta}{2}}\right)^2} \tag{2.9}$$

and for $\omega \in (\pi, 2\pi)$,

$$e^{i\omega} = \frac{\tan \frac{\alpha}{2}}{\tan \frac{\vartheta}{2}} - i \sqrt{1 - \left(\frac{\tan \frac{\alpha}{2}}{\tan \frac{\vartheta}{2}}\right)^2}.$$

Thus, we have

$$2ie^{2i\omega} \frac{d\omega}{d\vartheta} - i(\beta + \beta^{-1}) \left(-\frac{1}{2 \sin^2(\vartheta/2)} e^{i\omega} + i \cot(\vartheta/2) e^{i\omega} \frac{d\omega}{d\vartheta}\right) = 0,$$

and

$$\frac{d\omega}{d\vartheta} = \frac{\sin(\alpha/2)}{2 \sin(\vartheta/2) \sqrt{\cos^2(\alpha/2) - \cos^2(\vartheta/2)}}, \quad \vartheta \in (\alpha, 2\pi - \alpha). \tag{2.10}$$

Therefore, statement (iii) follows straightforward. \square

Let $K = \sqrt{\frac{2 \sin(\alpha/2)}{1 + \sin(\alpha/2)}}$. The next lemma gives the most important relations of such polynomials for our interest in this paper.

Lemma 2.2 ([6]). For $z \in \mathbb{C} \setminus \Delta_\alpha$ we have

$$K^{-1} \psi_n(z) = \frac{i\beta}{1 + \beta^2} (\chi_+(z)^{n-1} + \chi_-(z)^{n-1}) + \frac{1}{1 + \beta^2} (\chi_+(z)^n + \chi_-(z)^n), \quad n \geq 1. \tag{2.11}$$

Also, for $\vartheta \in (\alpha, 2\pi - \alpha)$ we have

$$K^{-1} \psi_n(e^{i\vartheta}) = 2e^{in\vartheta/2} \left\{ \frac{i\beta}{1 + \beta^2} e^{-i\vartheta/2} \cos((n-1)\lambda_\vartheta) + \frac{1}{1 + \beta^2} \cos(n\lambda_\vartheta) \right\}, \tag{2.12}$$

$$K^{-1} \psi_n^*(e^{i\vartheta}) = 2e^{in\vartheta/2} \left\{ \frac{i\beta}{1 + \beta^2} e^{i\vartheta/2} \cos((n-1)\lambda_\vartheta) + \frac{1}{1 + \beta^2} \cos(n\lambda_\vartheta) \right\}, \quad n \geq 1, \tag{2.13}$$

where $\psi_n^*(e^{i\vartheta}) = e^{in\vartheta} \overline{\psi_n(e^{i\vartheta})}$, and $\cos \lambda_\vartheta = \frac{\cos(\vartheta/2)}{\cos(\alpha/2)}$, $\lambda_\vartheta \in [0, \pi]$, $\vartheta \in [\alpha, 2\pi - \alpha]$. Moreover, there exists a constant $C > 0$ such that¹

$$|\psi_n(e^{i\vartheta})| \leq C, \quad \text{for all } \vartheta \in [\alpha, 2\pi - \alpha]. \tag{2.14}$$

Proof. We have

$$\chi_+(z) + \chi_-(z) = \frac{z + 1}{\cos(\alpha/2)}, \quad \chi_+(z)\chi_-(z) = z,$$

Thus, for each $z \in \overline{\mathbb{C}} \setminus \Delta_\alpha$ there exists v with $|v| < 1$ such that

$$\chi_+(z) = w(v), \quad \chi_-(z) = w(1/v).$$

Let $g_n(z)$ be the right hand side of (2.11). By (2.2), we get

$$\begin{aligned} &\chi_+^{n+1}(z) + \chi_-^{n+1}(z) \\ &= (\chi_+^n(z) + \chi_-^n(z)) \frac{z + 1}{\cos(\alpha/2)} - z (\chi_+^{n-1}(z) + \chi_-^{n-1}(z)), \end{aligned}$$

which by induction yields that $g_n(z)$ is a polynomial of degree n in z . Moreover,

$$\frac{1}{1 - \beta v} = i \frac{\beta}{1 + \beta^2} \frac{1}{w(v)} + \frac{1}{1 + \beta^2}, \quad \frac{v}{v - \beta} = i \frac{\beta}{1 + \beta^2} \frac{1}{w(1/v)} + \frac{1}{1 + \beta^2}. \tag{2.15}$$

Thus,

$$g_n(z) = \frac{1}{1 - \beta v} w(v)^n + \frac{v}{v - \beta} w(1/v)^n.$$

Since $z = h(v) = w(v)w(1/v)$, the leading coefficient α_n of $g_n(z)$ is

$$\begin{aligned} \alpha_n &= \lim_{z \rightarrow \infty} \frac{K g_n(z)}{z^n} = \lim_{v \rightarrow -\beta} \left(\frac{v}{v - \beta} \frac{1}{w(v)^n} + \frac{1}{1 - \beta v} \frac{1}{w(1/v)^n} \right) \\ &= K \frac{1 + \sin(\alpha/2)}{2 \sin(\alpha/2)} \frac{1}{\cos^n(\alpha/2)} = \frac{1}{K \cos^n(\alpha/2)}. \end{aligned}$$

¹ We use the same letter C for different constants independent of either the counter n , a point in an interval or the function in a specified class. Even the appearance of the same letter C in two consecutive inequalities can refer to different values.

From Lemma 2.1(iii) and Cauchy’s theorem, for $m < n$ we obtain

$$\begin{aligned} & \int_{\alpha}^{2\pi-\alpha} g_n(e^{i\vartheta})e^{-im\vartheta} w_{\alpha}(\vartheta) d\vartheta \\ &= \int_0^{\pi} \left\{ \frac{w(e^{i\omega})^n}{1-\beta e^{i\omega}} + \frac{e^{i\omega}w(e^{-i\omega})^n}{e^{i\omega}-\beta} \right\} \frac{1}{h(e^{i\omega})^m} d\omega \\ &= i^{n-2m} \int_0^{\pi} \left(\frac{1-\beta e^{i\omega}}{e^{i\omega}+\beta} \right)^{n-m} \left(\frac{1+\beta e^{i\omega}}{e^{i\omega}-\beta} \right)^m \frac{d\omega}{(1-\beta e^{i\omega})} \\ &\quad + i^{n-2m} \int_0^{\pi} \left(\frac{e^{i\omega}-\beta}{1+\beta e^{i\omega}} \right)^{n-m} \left(\frac{e^{i\omega}+\beta}{1-\beta e^{i\omega}} \right)^m \frac{e^{i\omega}d\omega}{(e^{i\omega}-\beta)} \\ &= i^{n-2m-1} \int_{\mathbb{T}} \left(\frac{\zeta-\beta}{1+\beta\zeta} \right)^{n-m} \left(\frac{\zeta+\beta}{1-\beta\zeta} \right)^m \frac{d\zeta}{(\zeta-\beta)} = 0. \end{aligned}$$

The same calculation for $m = n$ yields

$$\begin{aligned} & \int_{\alpha}^{2\pi-\alpha} g_n(e^{i\vartheta})e^{-in\vartheta} w_{\alpha}(\vartheta)d\vartheta = i^{-n-1} \int_{\mathbb{T}} \left(\frac{\zeta+\beta}{1-\beta\zeta} \right)^n \frac{d\zeta}{(\zeta-\beta)} \\ &= i^{-n}2\pi \left(\frac{2\beta}{1-\beta^2} \right)^n. \end{aligned}$$

Therefore, with the correct normalization we obtain (2.11).

Moreover, since $V(0) = 1$ and

$$\begin{aligned} \chi_{\pm}(e^{i\vartheta}) &= e^{i\vartheta/2} \frac{\cos(\vartheta/2)}{\cos(\alpha/2)} \pm \frac{1}{\cos(\alpha/2)} \sqrt{e^{i\vartheta}i^2 \sin\left(\frac{\vartheta-\alpha}{2}\right) \sin\left(\frac{\vartheta+\alpha}{2}\right)} \\ &= e^{i\vartheta/2} \frac{\cos(\vartheta/2)}{\cos(\alpha/2)} \pm e^{i\vartheta/2}i \sqrt{1 - \frac{\cos^2(\vartheta/2)}{\cos^2(\alpha/2)}} = e^{i\vartheta/2 \pm \lambda_{\vartheta}i}, \end{aligned}$$

where $\cos \lambda_{\vartheta} = \frac{\cos(\vartheta/2)}{\cos(\alpha/2)}$ for $\vartheta \in (\alpha, 2\pi - \alpha)$, we get

$$\chi_{+}(z)^j + \chi_{-}(z)^j = 2e^{ij\vartheta/2} \cos(j\lambda_{\vartheta}), \quad j \geq 1.$$

Therefore, (2.12) follows intermediately from (2.11).

Statement (2.14) is clear from (2.12). The relation in (2.13) follows from the definition of the reverse polynomial. \square

Next we consider para-orthogonal polynomials associated to Akhiezer–Chebyshev weight. These polynomials allow us to work on the diagonal of integration region to estimate the $L^p(w_{\alpha})$ -norm of \mathcal{S}_n . The para-orthogonal polynomial associated to Akhiezer–Chebyshev polynomials with a zero at $e^{\pm i\alpha}$ is defined by

$$A_n^{(\pm\alpha)}(z) = \psi_n^*(z) - \frac{\psi_n^*(e^{\pm i\alpha})}{\psi_n(e^{\pm i\alpha})} \psi_n(z).$$

Observe that $A_n^{(\pm\alpha)}(e^{\pm i\alpha}) = 0$ and $|\frac{\psi_n^*(e^{\pm i\alpha})}{\psi_n(e^{\pm i\alpha})}| = 1$, so by (2.14) the sequence of para-orthogonal polynomials $\{A_n^{(\pm\alpha)}\}_{n=0}^{\infty}$ is uniformly bounded in the Δ_{α} . Moreover, we can say much more with the following estimate.

Lemma 2.3. *There exists a constant C independent of n and ϑ such that*

$$|A_n^{(\pm\alpha)}(e^{i\vartheta})| \leq C|e^{i\vartheta} - e^{\pm i\alpha}|^{1/2}, \quad \text{for all } n \in \mathbb{N}, \quad \vartheta \in (\alpha, 2\pi - \alpha). \tag{2.16}$$

Proof. We give only the computation to show that there exists a constant $C > 0$ such that

$$|A_n^{(\alpha)}(e^{i\vartheta})| \leq C|e^{i\vartheta} - e^{i\alpha}|^{1/2}, \quad \text{for all } n \in \mathbb{N}, \quad \vartheta \in (\alpha, 2\pi - \alpha), \tag{2.17}$$

since the other inequality is proved in the same manner. Let λ_ϑ be the value in $[0, \pi]$ such that

$$\cos \lambda_\vartheta = \frac{\cos(\vartheta/2)}{\cos(\alpha/2)}, \quad \vartheta \in [\alpha, 2\pi - \alpha].$$

Of course, $\lambda_\alpha = 0$, so

$$\cos(n\lambda_\alpha) = 1, \quad \text{for all } n \in \mathbb{N}. \tag{2.18}$$

Moreover, from (2.12), (2.13), and (2.18) it follows that

$$\frac{\psi_n^*(e^{i\alpha})}{\psi_n(e^{i\alpha})} = \frac{i\beta e^{i\alpha/2} + 1}{i\beta e^{-i\alpha/2} + 1}. \tag{2.19}$$

Denote by $\Upsilon = \frac{\psi_n^*(e^{i\alpha})}{\psi_n(e^{i\alpha})}$ the value above. Observe that

$$1 - \Upsilon = i\beta \frac{e^{-i\alpha/2} - e^{i\alpha/2}}{i\beta e^{-i\alpha/2} + 1}. \tag{2.20}$$

Thus from (2.12) and (2.13) we obtain

$$\begin{aligned} \frac{K^{-1}}{2} |A_n^{(\alpha)}(e^{i\vartheta})| &= \left| \frac{i\beta}{1 + \beta^2} \cos((n - 1)\lambda_\vartheta)(e^{-i\vartheta/2} - \Upsilon e^{i\vartheta/2}) \right. \\ &\quad \left. + \frac{1}{1 + \beta^2} \cos(n\lambda_\vartheta)(1 - \Upsilon) \right| \\ &= \left| \cos((n - 1)\lambda_\vartheta) \left(\frac{i\beta}{1 + \beta^2} (e^{-i\vartheta/2} - \Upsilon e^{i\vartheta/2}) + \frac{\cos \lambda_\vartheta}{1 + \beta^2} (1 - \Upsilon) \right) \right. \\ &\quad \left. - \frac{\sin \lambda_\vartheta \sin((n - 1)\lambda_\vartheta)}{1 + \beta^2} (1 - \Upsilon) \right|. \end{aligned} \tag{2.21}$$

Since $\sin \lambda_\vartheta = \frac{\sqrt{\cos^2(\alpha/2) - \cos^2(\vartheta/2)}}{\cos(\alpha/2)}$ and $\frac{i\beta}{1 + \beta^2} (e^{-i\vartheta/2} - \Upsilon e^{i\vartheta/2}) + \frac{\cos \lambda_\vartheta}{1 + \beta^2} (1 - \Upsilon) = e^{-i\vartheta/2} P(e^{i\vartheta})$, where P is a polynomial of degree 1 which equals zero at $\vartheta = \alpha$, there exists a constant C independent of ϑ such that

$$\sin \lambda_\vartheta \leq C|e^{i\vartheta} - e^{i\alpha}|^{1/2},$$

and

$$\left| \frac{i\beta}{1 + \beta^2} (e^{-i\vartheta/2} - \Upsilon e^{i\vartheta/2}) + \frac{\cos \lambda_\vartheta}{1 + \beta^2} (1 - \Upsilon) \right| \leq C|e^{i\vartheta} - e^{i\alpha}|^{1/2}.$$

Therefore, (2.17) follows immediately from (2.21) and the inequalities above. \square

3. Proof of Theorems 1.1 and 1.2

We need several results on the Hilbert transform on an arc of the unit circle. Let us recall Riesz’s theorem [16] on this issue. This result states that the harmonic conjugate

$$\tilde{f}(x) = \frac{1}{2\pi} \text{PV} \int_0^{2\pi} f(t) \cot \frac{x - t}{2} dt, \quad x \in (0, 2\pi),$$

is a bounded operator from $L^p(0, 2\pi)$ to itself, where the integral is defined as the Cauchy principal value at $t = x$; it means that \tilde{f} exists almost everywhere and there exists a constant $C > 0$ such that for all $f \in L^p(0, 2\pi)$, $p > 1$, it holds

$$\int_0^{2\pi} |\tilde{f}(x)|^p dx \leq C \int_0^{2\pi} |f(x)|^p dx. \tag{3.1}$$

Observe that if f is a 2π -periodic even function, then

$$\tilde{f}(x) = \frac{1}{2\pi} \text{PV} \int_0^\pi f(t) \left(\cot \frac{x+t}{2} - \cot \frac{t-x}{2} \right) dt.$$

In this case, Riesz’s result (3.1) can be written as

$$\int_0^\pi |\tilde{f}(x)|^p dx \leq C \int_0^\pi |f(x)|^p dx, \tag{3.2}$$

for all $f \in L^p(0, \pi)$.

Let $f \in L^p(w_\alpha)$. The (unweighted) Hilbert transform in the arc Δ_α can be written as

$$\begin{aligned} \mathcal{H}_1(f)(e^{i\tau}) &= \frac{1}{2\pi} \text{PV} \int_\alpha^{2\pi-\alpha} \frac{f(e^{i\vartheta})}{e^{i\tau} - e^{i\vartheta}} d\vartheta \\ &= \frac{1}{2\pi} \lim_{\varepsilon \rightarrow 0^+} \int_{(\alpha, 2\pi-\alpha) \setminus (\tau-\varepsilon, \tau+\varepsilon)} \frac{f(e^{i\vartheta})}{e^{i\tau} - e^{i\vartheta}} d\vartheta, \end{aligned}$$

where $\tau \in (\alpha, 2\pi - \alpha)$. The topic of Hilbert transform along curves has a long history in harmonic analysis that it can be seen in [17]. To get an weighted inequality for this transform we shall use the following Muckenhoupt’s inequality (see [11, Lemma 8, p. 440 of the second paper]).

Lemma 3.1. *If $1 < p < \infty$, $r > -1/p$, $s < 1 - 1/p$, $R < 1 - 1/p$, $S > -1/p$, $r \geq R$ and $s \leq S$, then there exists a constant C , independent of f , such that*

$$\int_{-\infty}^\infty \left| \int_{-\infty}^\infty \frac{f(y)}{x-y} |x|^r (1+|x|)^{s-r} dy \right|^p dx \leq C \int_{-\infty}^\infty |f(y)| |y|^R (1+|y|)^{S-R} |y|^p dy. \tag{3.3}$$

Next we prove the weighted inequality for the Hilbert transform in an arc given by Theorem 1.2.

Proof of Theorem 1.2. Easy computations give us

$$h(e^{it}) - h(e^{is}) = \frac{2(\beta + \beta^{-1})(e^{it} - e^{is})(e^{i(s+t)} - 1)}{(e^{it} + \beta)(e^{is} + \beta)(e^{it} + \beta^{-1})(e^{is} + \beta^{-1})},$$

$$e^{it} - e^{is} = 2ie^{i(s+t)/2} \sin \frac{t-s}{2},$$

$$e^{i(s+t)} - 1 = 2ie^{i(s+t)/2} \sin \frac{t+s}{2},$$

and

$$\begin{aligned} h(e^{it}) - h(e^{is}) &= \frac{8(\beta + \beta^{-1})e^{i(s+t)}}{(e^{it} + \beta)(e^{is} + \beta)(e^{it} + \beta^{-1})(e^{is} + \beta^{-1})} \frac{\sin s}{\cot \frac{t+s}{2} - \cot \frac{t-s}{2}} \\ &= \frac{4(\beta + \beta^{-1})e^{i(s+t)}(\cos s - \cos t)}{(e^{it} + \beta)(e^{is} + \beta)(e^{it} + \beta^{-1})(e^{is} + \beta^{-1})}. \end{aligned} \tag{3.4}$$

So, we can write

$$\frac{f(h(e^{it}))}{h(e^{is}) - h(e^{it})} = \frac{f(h(e^{it}))(e^{it} + \beta)(e^{is} + \beta)(e^{it} + \beta^{-1})(e^{is} + \beta^{-1})}{4(\beta + \beta^{-1})e^{i(s+t)}(\cos s - \cos t)}. \tag{3.5}$$

Using (2.9), we also have

$$\sin t = \frac{\sqrt{\cos^2(\alpha/2) - \cos^2(\vartheta/2)}}{\cos(\alpha/2) \sin(\vartheta/2)} = \frac{\tan(\alpha/2)}{2 \sin^2(\vartheta/2) w_\alpha(\vartheta)}. \tag{3.6}$$

Hence, doing the change of variables $e^{i\tau} = h(e^{is})$, $e^{i\vartheta} = h(e^{it})$ and $s, t \in (0, \pi)$, from Lemma 2.1(iii), we get

$$\begin{aligned} & \int_\alpha^{2\pi-\alpha} |\mathcal{H}_1(f)(e^{i\tau})|^p w_\alpha(\tau) d\tau \\ &= \int_\alpha^{2\pi-\alpha} \left| \int_\alpha^{2\pi-\alpha} \frac{f(e^{i\vartheta})}{e^{i\tau} - e^{i\vartheta}} d\vartheta \right|^p w_\alpha(\tau) d\tau \\ &= \int_0^\pi \left| \int_0^\pi \frac{2f(h(e^{it})) \sin t \cot(\alpha/2) \sin^2(\vartheta/2)}{h(e^{is}) - h(e^{it})} dt \right|^p ds. \end{aligned} \tag{3.7}$$

Since there exist positive constants C_1, C_2 such that

$$C_1 \leq \sin(\vartheta/2) \leq C_2, \quad \vartheta \in (\alpha, 2\pi - \alpha),$$

$$C_1 \leq \left| \frac{(e^{it} + \beta)(e^{is} + \beta)(e^{it} + \beta^{-1})(e^{is} + \beta^{-1})}{e^{i(s+t)}} \right| \leq C_2, \quad s, t \in (0, \pi),$$

and by (3.5) and (3.7) we obtain that there exists a constant $C > 0$ for all $f \in L^p(w_\alpha)$ such that

$$\int_\alpha^{2\pi-\alpha} |\mathcal{H}_1(f)(e^{i\tau})|^p w_\alpha(\tau) d\tau \leq C \int_0^\pi \left| \int_0^\pi \frac{f(h(e^{it})) \sin t}{\cos t - \cos s} dt \right|^p ds. \tag{3.8}$$

Next, we change the variables of integration to $\sqrt{u} = \tan t/2$, $\sqrt{v} = \tan s/2$ in the right hand side of (3.8),

$$\begin{aligned} & C_1 \int_0^\pi \left| \int_0^\pi \frac{f(h(e^{it})) \sin t}{\cos t - \cos s} dt \right|^p ds \\ &= \int_0^\infty \left| \int_0^\infty \frac{f(h(e^{it}))}{v-u} \frac{1+v}{1+u} \frac{du}{1+u} \right|^p \frac{dv}{\sqrt{v}(1+v)} \\ &\leq C_2 \left(\int_0^\infty \left| \int_0^\infty \frac{f(h(e^{it}))}{(1+u)^2} du \right|^p \frac{dv}{\sqrt{v}(1+v)} \right. \\ &\quad \left. + \int_0^\infty \left| \int_0^\infty \frac{f(h(e^{it}))}{v-u} du \right|^p \frac{dv}{\sqrt{v}(1+v)} \right), \end{aligned} \tag{3.9}$$

where C_1, C_2 are constants independent of f and in the inequality above we have used that $\frac{1+v}{1+u} = \frac{v-u}{1+u} + 1$. Moreover, $f \in L^p((\alpha, 2\pi - \alpha), w_\alpha)$ if and only if $f(h(e^{2i \arctan \sqrt{v}})) \in L^p((0, \infty), \frac{1}{\sqrt{v}(1+v)})$, and

$$\int_\alpha^{2\pi-\alpha} |f(e^{i\tau})|^p w_\alpha(\tau) d\tau = \int_0^\infty \left| f(h(e^{2i \arctan \sqrt{v}})) \right|^p \frac{dv}{\sqrt{v}(1+v)}.$$

Obviously, in that case we also have $f(h(e^{i2i \arctan \sqrt{v}}))/(1+v) \in L^p((0, \infty), \frac{1}{\sqrt{v(1+v)}})$ and by Hölder's inequality there exists C independent of f such that

$$\left| \int_0^\infty \frac{f(h(e^{2i \arctan \sqrt{u}}))}{(1+u)^2} du \right| \leq C \left(\int_0^\infty \left| \frac{f(h(e^{2i \arctan \sqrt{u}}))}{\sqrt{u(1+u)}} \right|^p du \right)^{1/p}. \tag{3.10}$$

Furthermore, applying Muckenhoupt's inequality (3.3) to the function which equals $f(h(e^{2i \arctan \sqrt{u}}))/(1+u)$ for $u > 0$ and zero for $u \leq 0$ with $s = -3/(2p)$, $S = 1 - 3/(2p)$, $r = R = -1/(2p)$, we obtain

$$\begin{aligned} & \int_0^\infty \left| \int_0^\infty \frac{f(h(e^{2i \arctan \sqrt{u}}))}{v-u} du \right|^p \frac{dv}{\sqrt{v(1+v)}} \\ & \leq C \int_0^\infty \left| \frac{f(h(e^{2i \arctan \sqrt{v}}))}{\sqrt{v(1+v)}} \right|^p dv. \end{aligned} \tag{3.11}$$

Combining relations (3.8), (3.9), (3.10), and (3.11), we get (1.3). \square

Now we can prove Theorem 1.1.

Poof of Theorem 1.1. If we prove that for each $f \in L^p(w_\alpha)$ there exists a constant $C = C(f)$ such that

$$\int_\alpha^{2\pi-\alpha} |\mathcal{S}_n(f, e^{i\tau})|^p w_\alpha(\tau) d\tau \leq C \int_\alpha^{2\pi-\alpha} |f(e^{i\tau})|^p w_\alpha(\tau) d\tau, \tag{3.12}$$

for all $n \in \mathbb{N}$, then by the Banach–Steinhaus theorem actually there exists a constant C independent of $f \in L^p(w_\alpha)$ such that the inequality above holds. Once we know that the constant C in (3.12) is independent of f , by Szegő–Kolmogorov–Krein's theorem [2, Addenda B], [5, chapter 1], [7, chapter 3] the algebraic polynomials are dense in $L^p(w_\alpha)$ and therefore from (3.12) the statement in Theorem 1.1 follows.

Moreover, it is sufficient to prove (3.12) with the integral on the left taken over (α, π) and on $(\pi, 2\pi - \alpha)$; the second one is obtained with analogous arguments to the first one. So, for $1 < p < \infty$ and $f \in L^p(w_\alpha)$, we check only that there exists a constant $C = C(f, p)$ such that we have

$$\int_\alpha^\pi |\mathcal{S}_n(f, e^{i\tau})|^p w_\alpha(\tau) d\tau \leq C \int_\alpha^{2\pi-\alpha} |f(e^{i\tau})|^p w_\alpha(\tau) d\tau, \tag{3.13}$$

for all $n \in \mathbb{N}$.

According to Christoffel–Darboux formula it holds

$$K_n(\vartheta, \tau) = \sum_{j=0}^n \overline{\psi_j(e^{i\vartheta})} \psi_j(e^{i\tau}) = \frac{\overline{\psi_{n+1}^*(e^{i\vartheta})} \psi_{n+1}^*(e^{i\tau}) - \overline{\psi_{n+1}(e^{i\vartheta})} \psi_{n+1}(e^{i\tau})}{1 - \overline{e^{i\vartheta}} e^{i\tau}},$$

and

$$\mathcal{S}_n(f, e^{i\tau}) = \frac{1}{2\pi} \int_\alpha^{2\pi-\alpha} f(e^{i\vartheta}) \overline{K_n(\tau, \vartheta)} w_\alpha(\vartheta) d\vartheta. \tag{3.14}$$

Since

$$\frac{\overline{\psi_{n+1}^*(e^{i\alpha})}}{\psi_{n+1}(e^{i\alpha})} = \frac{e^{in\alpha} \overline{\psi_{n+1}(e^{i\alpha})}}{\psi_{n+1}(e^{i\alpha})} = \frac{\overline{\psi_{n+1}(e^{i\alpha})}}{\psi_{n+1}^*(e^{i\alpha})},$$

we have

$$(1 - e^{i\vartheta} e^{i\tau})K_n(\vartheta, \tau) = \overline{\psi_{n+1}^*(e^{i\vartheta})} \Lambda_{n+1}^{(\alpha)}(e^{i\tau}) + \psi_{n+1}(e^{i\tau}) \frac{\psi_{n+1}^*(e^{i\alpha})}{\psi_{n+1}(e^{i\alpha})} \overline{\Lambda_{n+1}^{(\alpha)}(e^{i\vartheta})}. \tag{3.15}$$

Let $\delta > 0$ be small enough (smaller than $\pi - \alpha$). We have

$$\begin{aligned} \mathcal{S}_n(f, e^{i\tau}) &= \frac{1}{2\pi} \int_{\alpha}^{\pi+\delta} f(e^{i\vartheta})K_n(\vartheta, \tau)w_{\alpha}(\vartheta) d\vartheta \\ &+ \frac{1}{2\pi} \int_{\pi+\delta}^{2\pi-\alpha} f(e^{i\vartheta})K_n(\vartheta, \tau)w_{\alpha}(\vartheta) d\vartheta. \end{aligned} \tag{3.16}$$

The integral

$$\int_{\alpha}^{\pi} \left| \frac{1}{2\pi} \int_{\pi+\delta}^{2\pi-\alpha} f(e^{i\vartheta})K_n(\vartheta, \tau)w_{\alpha}(\vartheta) d\vartheta \right|^p w_{\alpha}(\tau) d\tau$$

is not a singular integral and this is a bounded sequence because of the sequence of polynomials $\{\psi_n\}_{n=0}^{\infty}$ also is. Thus, there exists a constant $C > 0$ such that

$$\begin{aligned} \int_{\alpha}^{\pi} \left| \frac{1}{2\pi} \int_{\pi+\delta}^{2\pi-\alpha} f(e^{i\vartheta})K_n(\vartheta, \tau)w_{\alpha}(\vartheta) d\vartheta \right|^p w_{\alpha}(\tau) d\tau \\ \leq C \int_{\alpha}^{2\pi-\alpha} |f(e^{i\tau})|^p w_{\alpha}(\tau) d\tau. \end{aligned} \tag{3.17}$$

Therefore, it is enough to work with

$$\int_{\alpha}^{\pi} \left| \int_{\alpha}^{\pi+\delta} f(e^{i\vartheta})K_n(\vartheta, \tau)w_{\alpha}(\vartheta) d\vartheta \right|^p w_{\alpha}(\tau) d\tau.$$

By (3.15) we obtain

$$\begin{aligned} \int_{\alpha}^{\pi} \left| \int_{\alpha}^{\pi+\delta} f(e^{i\vartheta})K_n(\vartheta, \tau)w_{\alpha}(\vartheta) d\vartheta \right|^p w_{\alpha}(\tau) \\ \leq C \left(\int_{\alpha}^{\pi} \left| \Lambda_{n+1}^{(\alpha)}(e^{i\tau}) \int_{\alpha}^{\pi+\delta} f(e^{i\vartheta}) \frac{\overline{\psi_{n+1}^*(e^{i\vartheta})}}{1 - e^{i\vartheta} e^{i\tau}} w_{\alpha}(\vartheta) d\vartheta \right|^p w_{\alpha}(\tau) d\tau \right. \\ \left. + \int_{\alpha}^{\pi} \left| \psi_{n+1}(e^{i\tau}) \int_{\alpha}^{\pi+\delta} f(e^{i\vartheta}) \frac{\psi_{n+1}^*(e^{i\alpha})}{\psi_{n+1}(e^{i\alpha})} \frac{\overline{\Lambda_{n+1}^{(\alpha)}(e^{i\vartheta})}}{1 - e^{i\vartheta} e^{i\tau}} w_{\alpha}(\vartheta) d\vartheta \right|^p w_{\alpha}(\tau) d\tau \right). \end{aligned} \tag{3.18}$$

According to Lemma 2.3, we know that there exists $C > 0$ such that

$$\left| \frac{\overline{\psi_{n+1}^*(e^{i\alpha})}}{\psi_{n+1}(e^{i\alpha})} \overline{\Lambda_{n+1}^{(\alpha)}(e^{i\vartheta})} w_{\alpha}(\vartheta) \right| \leq C, \quad \text{for all } \vartheta \in (\alpha, \pi + \delta),$$

and from (2.14) we know that

$$|\psi_{n+1}(e^{i\tau})| \leq C, \quad \text{for all } \tau \in (0, \pi).$$

Therefore, by Theorem 1.2 there exists $C > 0$ such that for all $f \in L^p(w_\alpha)$ we have

$$\int_\alpha^\pi \left| \psi_{n+1}(e^{i\tau}) \int_\alpha^{\pi+\delta} f(e^{i\vartheta}) \frac{\overline{\psi_{n+1}^*(e^{i\alpha})} \Lambda_{n+1}^{(\alpha)}(e^{i\vartheta})}{1 - e^{i\vartheta} e^{i\tau}} w_\alpha(\vartheta) d\vartheta \right|^p w_\alpha(\tau) \leq C \int_\alpha^{2\pi-\alpha} |f(e^{i\vartheta})|^p w_\alpha(\vartheta) d\vartheta. \tag{3.19}$$

On the other hand, doing the change of variables $e^{i\tau} = h(e^{is})$, $e^{i\vartheta} = h(e^{it})$ and $s, t \in (0, \pi/2)$ and by the first equality in (3.4),

$$\int_\alpha^\pi \left| \Lambda_{n+1}^{(\alpha)}(e^{i\tau}) \int_\alpha^{\pi+\delta} f(e^{i\vartheta}) \frac{\overline{\psi_{n+1}^*(e^{i\vartheta})}}{1 - e^{i\vartheta} e^{i\tau}} w_\alpha(\vartheta) d\vartheta \right|^p w_\alpha(\tau) \leq C \int_0^{\pi/2} \left| \frac{\Lambda_{n+1}^{(\alpha)}(h(e^{is}))}{\sin s} \int_0^{\pi/2+\widehat{\delta}} f(h(e^{it})) \overline{\psi_{n+1}^*(h(e^{it}))} (\cot \frac{t+s}{2} - \cot \frac{t-s}{s}) dt \right|^p ds,$$

where the arc in the unit circle from 1 to $e^{i(\pi/2+\widehat{\delta})}$ has image by h the arc from $e^{i\alpha}$ to $e^{i(\pi+\delta)}$. From (3.6) and Lemma 2.3 we know that there exists a constant C such that

$$\left| \frac{\Lambda_{n+1}^{(\alpha)}(h(e^{is}))}{\sin s} \right| \leq C, \quad \text{for all } s \in (0, \frac{\pi}{2}).$$

Then using Riesz’s result (3.2), it follows the inequality

$$\int_\alpha^\pi \left| \Lambda_{n+1}^{(\alpha)}(e^{i\tau}) \int_\alpha^{\pi+\delta} f(e^{i\vartheta}) \frac{\overline{\psi_{n+1}^*(e^{i\vartheta})}}{1 - e^{i\vartheta} e^{i\tau}} w_\alpha(\vartheta) d\vartheta \right|^p w_\alpha(\tau) \leq C \int_\alpha^{2\pi-\alpha} |f(e^{i\tau})|^p w_\alpha(\tau) d\tau. \tag{3.20}$$

Plugging (3.19) and (3.20) into (3.18), we finish the proof of (3.13). \square

4. Additional results

This section contains some results about mean convergence of series in orthogonal polynomials with respect to Akhiezer–Chebyshev weight multiplied by a function with nice properties. Assume that $k(\vartheta) \geq k > 0$ for all $\vartheta \in (\alpha, 2\pi - \alpha)$ and let $k(\vartheta)$ satisfy the Lipschitz condition in Δ_α , i.e.

$$|k(\vartheta_1) - k(\vartheta_2)| \leq \lambda |\vartheta_1 - \vartheta_2|, \quad \vartheta_1, \vartheta_2 \in (\alpha, 2\pi - \alpha), \tag{4.1}$$

where λ is a positive constant. We consider the measure

$$d\mu_\alpha(\vartheta) = k(\vartheta) w_\alpha(\vartheta) d\vartheta, \quad \vartheta \in (\alpha, 2\pi - \alpha). \tag{4.2}$$

Observe that since the function $k(\vartheta)$ satisfies the Lipschitz condition, it is also bounded above in the interval $(\alpha, 2\pi - \alpha)$.

To obtain our main results in this section we shall need some auxiliary results and definitions. The weighted Hilbert transform is defined by

$$\mathcal{H}_2(f)(e^{i\tau}) = \mathcal{H}_1(f w_\alpha)(e^{i\tau}) = \frac{1}{2\pi} \lim_{\varepsilon \rightarrow 0^+} \int_{(0,\pi) \setminus (s-\varepsilon, s+\varepsilon)} \frac{f(h(e^{it}))}{h(e^{it}) - h(e^{is})} dt,$$

where $\tau \in (\alpha, 2\pi - \alpha)$, $e^{i\tau} = h(e^{is})$, $e^{i\vartheta} = h(e^{it})$ and $s, t \in (0, \pi)$.

For the weighted Hilbert transform we have the following inequality.

Lemma 4.1. *There exists a constant C such that for all $f \in L^p(w_\alpha)$ we have*

$$\begin{aligned} & \int_\alpha^{2\pi-\alpha} \left| \mathcal{H}_2(f)(e^{i\tau}) \sqrt{\cos^2(\alpha/2) - \cos^2(\tau/2)} \right|^p w_\alpha(\tau) d\vartheta \\ & \leq C \int_\alpha^{2\pi-\alpha} |f(e^{i\tau})|^p w_\alpha(\tau) d\tau. \end{aligned}$$

Proof. Following the same steps of the proof of [Theorem 1.2](#) until [\(3.8\)](#) we get

$$\begin{aligned} & \int_\alpha^{2\pi-\alpha} \left| \mathcal{H}_2(f)(e^{i\tau}) \sqrt{\cos^2(\alpha/2) - \cos^2(\tau/2)} \right|^p w_\alpha(\tau) d\tau \\ & = \int_\alpha^{2\pi-\alpha} \left| \int_\alpha^{2\pi-\alpha} \frac{f(e^{i\vartheta}) \sqrt{\cos^2(\alpha/2) - \cos^2(\tau/2)}}{e^{i\vartheta} - e^{i\tau}} w_\alpha(\vartheta) d\vartheta \right|^p w_\alpha(\tau) d\tau \\ & = \int_0^\pi \left| \int_0^\pi \frac{f(h(e^{it})) \sin s \cos(\alpha/2) (\sin \tau/2)}{h(e^{is}) - h(e^{it})} dt \right|^p ds. \end{aligned}$$

Then we obtain

$$\begin{aligned} & \left(\int_\alpha^{2\pi-\alpha} \left| \mathcal{H}_2(f)(e^{i\tau}) \sqrt{\cos^2(\alpha/2) - \cos^2(\tau/2)} \right|^p w_\alpha(\tau) d\tau \right)^{1/p} \\ & \leq C \left(\int_0^\pi |f^*(s)|^p ds \right)^{1/p} \end{aligned} \tag{4.3}$$

where $f^*(s) = f(h(e^{is}))$ and the constant $C > 0$ is independent of f . Therefore, the lemma follows from [\(3.2\)](#), [\(4.3\)](#) and [Lemma 2.1\(iii\)](#). \square

Finally, we need to state a type of Korus’ lemma [[18](#), p. 162].

Lemma 4.2. *Let μ_α be the measure given by [\(4.2\)](#) with $k(\vartheta)$ such that $k(\vartheta) \geq k > 0$ for all $\vartheta \in (\alpha, 2\pi - \alpha)$ and let $k(\vartheta)$ satisfy the Lipschitz condition in Δ_α given by [\(4.1\)](#). Let $\{\varphi_n\}_{n=0}^\infty$ denote sequence of orthonormal polynomials with respect to μ_α . Then there exists a constant C such that*

$$|\varphi_n(e^{i\vartheta})| \leq C, \quad \text{for all } \vartheta \in (\alpha, 2\pi - \alpha), \text{ and all } n \in \mathbb{N}.$$

Proof. Let κ_n denote the leading coefficient of φ_n . By the reproducing property of Christoffel kernel and Christoffel–Darboux formula we have

$$\begin{aligned} \varphi_n(e^{i\tau}) &= \frac{1}{2\pi} \int_{\alpha}^{2\pi-\alpha} \varphi_n(e^{i\vartheta}) \sum_{j=0}^n \psi_j(e^{i\tau}) \overline{\psi_j(e^{i\vartheta})} w_{\alpha}(\vartheta) d\vartheta \\ &= \frac{\kappa_n}{\alpha_n} \psi_n(e^{i\tau}) + \frac{1}{2\pi} \int_{\alpha}^{2\pi-\alpha} \varphi_n(e^{i\vartheta}) \sum_{j=0}^{n-1} \psi_j(e^{i\tau}) \overline{\psi_j(e^{i\vartheta})} w_{\alpha}(\vartheta) \left(1 - \frac{k(\vartheta)}{k(\tau)}\right) d\vartheta \\ &= \frac{\kappa_n}{\alpha_n} \psi_n(e^{i\tau}) \\ &\quad + \frac{1}{2\pi k(\tau)} \int_{\alpha}^{2\pi-\alpha} \varphi_n(e^{i\vartheta}) \left(\overline{\psi_n^*(e^{i\vartheta})} \psi_n^*(e^{i\tau}) - \overline{\psi_n(e^{i\vartheta})} \psi_n(e^{i\tau})\right) \frac{k(\tau) - k(\vartheta)}{1 - e^{i(\tau-\vartheta)}} w_{\alpha}(\vartheta) d\vartheta. \end{aligned}$$

Since²

$$|1 - e^{i(\tau-\vartheta)}| = 2 \left| \sin \frac{\tau - \vartheta}{2} \right| \sim |\tau - \vartheta|, \quad \alpha < \tau, \vartheta < 2\pi - \alpha,$$

by (4.1), there exists a constant $C > 0$ such that

$$\left| \frac{k(\tau) - k(\vartheta)}{1 - e^{i(\tau-\vartheta)}} \right| \leq C.$$

By Cauchy–Schwarz’s inequality

$$\begin{aligned} \frac{\kappa_n}{\alpha_n} &= \frac{1}{2\pi} \int_{\alpha}^{2\pi-\alpha} \varphi_n(e^{i\vartheta}) \overline{\psi_n(e^{i\vartheta})} w_{\alpha}(\vartheta) d\vartheta \\ &\leq \left(\frac{1}{2\pi} \int_{\alpha}^{2\pi-\alpha} |\varphi_n(e^{i\vartheta})|^2 \frac{d\mu_{\alpha}(\vartheta)}{k(\vartheta)} \right)^{1/2} \left(\frac{1}{2\pi} \int_{\alpha}^{2\pi-\alpha} |\psi_n(e^{i\vartheta})|^2 w_{\alpha}(\vartheta) d\vartheta \right)^{1/2} \leq k^{-1/2}. \end{aligned}$$

Taking into account Lemma 2.2 we known that $\{\psi_n(e^{i\vartheta})\}_{n=0}^{\infty}$ is uniformly bounded for $\vartheta \in (\alpha, 2\pi - \alpha)$, therefore the above inequalities prove the lemma. \square

Let $S_n(f, \cdot)$ denote the n -th partial Fourier sums in terms of $\{\varphi_j\}_{j=0}^{\infty}$ for the function $f \in L^p(\mu_{\alpha})$.

Theorem 4.3. Let μ_{α} be the measure given by (4.2) with $k(\vartheta)$ such that $k(\vartheta) \geq k > 0$ for all $\vartheta \in (\alpha, 2\pi - \alpha)$ and let $k(\vartheta)$ satisfy the Lipschitz condition in Δ_{α} given by (4.1). Then for all $f \in L^p(\mu_{\alpha})$, $1 < p < \infty$, we have

$$\lim_{n \rightarrow \infty} \int_{\alpha}^{2\pi-\alpha} |(f(e^{i\tau}) - S_n(f, e^{i\tau})) \sqrt{\cos^2(\alpha/2) - \cos^2(\tau/2)}|^p d\mu_{\alpha}(\tau) = 0.$$

Proof. According to Christoffel–Darboux formula

$$\begin{aligned} S_n(f, z) &= \frac{\varphi_{n+1}^*(z)}{2\pi} \int_{\alpha}^{2\pi-\alpha} \frac{f(e^{i\vartheta}) \overline{\varphi_{n+1}^*(e^{i\vartheta})}}{1 - e^{i\vartheta} z} d\mu_{\alpha}(\vartheta) \\ &\quad - \frac{\varphi_{n+1}(z)}{2\pi} \int_{\alpha}^{2\pi-\alpha} \frac{f(e^{i\vartheta}) \overline{\varphi_{n+1}(e^{i\vartheta})}}{1 - e^{i\vartheta} z} d\mu_{\alpha}(\vartheta). \end{aligned} \tag{4.4}$$

² As usual, the notation $a(x) \sim b(x)$ for x in an interval I means that there exist positive constants C_1, C_2 independent of x such that $C_1 \leq \frac{a(x)}{b(x)} \leq C_2$ for all $x \in I$.

Because of [Lemma 4.2](#) we know that the Akhiezer–Chebyshev type polynomials $\{\varphi_n\}_{n=0}^\infty$ is a uniformly bounded sequence on the arc Δ_α . Combining [Lemma 4.1](#), the hypothesis on the function k , and [\(4.4\)](#), for all $p > 1$ there exists a constant $C = C(p) > 0$ such that for all $f \in L^p(w_\alpha)$ we have

$$\int_\alpha^{2\pi-\alpha} |S_n(f, e^{i\tau})\sqrt{\cos^2(\alpha/2) - \cos^2(\tau/2)}|^p d\mu_\alpha(t) \leq C \int_0^\pi |f(e^{i\tau})|^p d\mu_\alpha(\tau), \tag{4.5}$$

where C is a constant independent of $f \in L^p(\mu_\alpha)$. From the Szegő–Kolmogorov–Krein theorem, the algebraic polynomials are dense in $L^p(\mu_\alpha)$. Thus, there exists a sequence of algebraic polynomials $\{p_n\}$ with $\deg(p_n) \leq n$ such that

$$\lim_{n \rightarrow \infty} \int_\alpha^{2\pi-\alpha} |f(e^{i\tau}) - p_n(e^{i\tau})|^p d\mu_\alpha(\tau) = 0.$$

By [\(4.5\)](#), we have

$$\begin{aligned} & \left(\int_\alpha^{2\pi-\alpha} |(S_n(f, e^{i\tau}) - f(e^{i\tau}))\sqrt{\cos^2(\alpha/2) - \cos^2(\tau/2)}|^p d\mu_\alpha(\vartheta) \right)^{1/p} \\ & \leq \left(\int_\alpha^{2\pi-\alpha} |(S_n(f - p_n, e^{i\tau}))\sqrt{\cos^2(\alpha/2) - \cos^2(\tau/2)}|^p d\mu_\alpha(\tau) \right)^{1/p} \\ & \quad + \left(\int_\alpha^{2\pi-\alpha} |(p_n(e^{i\tau}) - f(e^{i\tau}))\sqrt{\cos^2(\alpha/2) - \cos^2(\tau/2)}|^p d\mu_\alpha(\tau) \right)^{1/p} \\ & \leq C \left(\int_\alpha^{2\pi-\alpha} |f(e^{i\tau}) - p_n(e^{i\tau})|^p d\mu_\alpha(\tau) \right)^{1/p}, \end{aligned}$$

and the conclusion of the theorem follows. \square

Remark 4.4. Since the operator which maps $f \in L^\infty(\mu_\alpha)$ into its n -th partial Fourier series is a projection operator, according to Losinski–Kharshilarze–Nikolaev’s theorem [[13](#), Appendix 3], this operator is not bounded from $L^\infty(\mu_\alpha)$ to $L^\infty(\mu_\alpha)$. By duality, it is also not bounded from $L^1(\mu_\alpha)$ to itself.

Of course, [Theorems 4.3](#) is not sharp for $p = 2$. An improvement of [Theorem 4.3](#) would be obtained if an inequality like in [Lemma 2.3](#) is proved for para-orthogonal polynomials associated to the measure μ_α .

Data availability

No data was used for the research described in the article.

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