Discrete Mathematics 346 (2023) 113636

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

# A new method to h-regularize finite topological spaces

Julián Cuevas-Rozo [b,*], Laureano Lambán [a], Ana Romero [a], Humberto Sarria [b]

[a] Department of Mathematics and Computer Science, University of La Rioja, Spain
[b] Department of Mathematics, National University of Colombia, Colombia

ARTICLE INFO

*Article history:*
Received 4 November 2021
Received in revised form 22 June 2023
Accepted 24 July 2023
Available online 7 August 2023

*Keywords:*
Finite topological spaces
h-regular finite spaces
Simple homotopy types

ABSTRACT

The most useful methods for computing homology of finite topological spaces require that certain regularity conditions are satisfied. In particular, the most suitable algorithmic methods, from the point of view of complexity, are only applicable to h-regular spaces. In this paper, we show a procedure to *h-regularize* a finite topological space $X$ of height at most 2, that is, to construct an h-regular space that is simple homotopy equivalent to $X$. This enables us to carry out some topological computations (homology groups) on the h-regular space obtained from this h-regularization process.



## 1. Introduction

Recently, several authors have studied finite topological spaces, not only for their intrinsic importance as objects with interesting properties to be discovered, but also for their relationship with other mathematical structures. One of the most significant results of this theory is the correspondence between finite topological spaces and finite partially ordered sets (finite posets), which was first considered by Alexandroff in [1]. Another example of the interactions between finite spaces and other structures is the claim that every finite simplicial complex is weak homotopy equivalent to a finite topological space (which is referred to as a *finite model* of the simplicial complex) [13], allowing to study topological properties (homotopical or homological) of simplicial complexes from a different perspective.

More generally, it is known that regular CW-complexes can be modeled by their face posets and the wider class of *h-regular CW-complexes* also admits finite models. A CW-complex is h-regular if the closed cells are contractible subcomplexes [2] and the face poset of an h-regular CW-complex satisfies Minian's definition of *h-regular poset* [14]. Moreover, Minian introduced a version of discrete Morse theory for h-regular spaces, allowing to study the topology of the order complexes of h-regular spaces from the critical points of admissible Morse matchings defined on the associated posets, which makes it possible to determine their homology groups by working on chain complexes with a smaller number of generators. In this sense, h-regular spaces permit the application of known techniques directly on them in order to study their topological properties. In [9], some algorithms and programs to compute topological invariants of finite spaces were shown, which are based on newly developed constructive versions of theoretical results found in [6], [14]. In particular, effective algorithms to compute homology groups and their generators of h-regular spaces were implemented, as well as the computation of homologically admissible Morse matchings on finite spaces, in order to use Minian's version of discrete Morse theory.

In the literature, there exist some modifications that can be applied to a finite topological space $X$ in order to obtain a smaller finite space with the same homotopy type as $X$. Eliminating beat points [18], weak points, and $\gamma$-points [2], as well

* Corresponding author.
*E-mail address:* jlcuevasr@unal.edu.co (J. Cuevas-Rozo).

https://doi.org/10.1016/j.disc.2023.113636

as qc-reductions and middle-reductions [11], are some of the alternatives to be applied on a finite space without modifying its weak homotopy type. However, the resulting spaces after such procedures are not h-regular in general.

Given a finite topological space $X$, its *barycentric subdivision* $X'$, which is the face poset of the order complex associated to $X$, is an h-regular finite space that is simple homotopy equivalent to $X$ (in particular, it is weak homotopy equivalent to $X$). Until now, examples of h-regular spaces in the literature have been restricted to face posets of h-regular complexes and regular CW-complexes (in particular, simplicial complexes). Indeed, algorithms designed to be applied on simplicial complexes, such as those in [11] and [15], are used to compute the homology groups of a finite $T_0$-space $X$, so that the order complex $\mathcal{K}(X)$ is used to find such invariants of $X$. Therefore, the barycentric subdivision $X' = \mathcal{X}(\mathcal{K}(X))$ is an h-regular space that, in principle, can be taken as input of algorithms in [9]. However, the size of barycentric subdivisions can become quite large, since each $n$-maximal chain of the original finite $T_0$-space produces $(n+1)!$ chains in the subdivision. This raises the question of whether there exist h-regular spaces, other than $X'$, that are weak homotopy equivalent to $X$ but have a smaller number of elements than $X'$ that can be used as input of our implemented algorithms.

In this paper, we aim to illustrate a procedure for constructing, from an arbitrary finite topological space $X$ of height at most 2, an h-regular space that is distinct from the barycentric subdivision of $X$, and is simple homotopy equivalent to $X$. This construction can be used to compute topological invariants of $X$ by applying homologically admissible vector fields, as in [9], or other techniques applicable to h-regular spaces that cannot be used on arbitrary finite spaces.

The paper is organized as follows. In Section 2, finite topological spaces, their relation with simplicial complexes, and basic point reduction techniques are discussed. In Section 3, the concept of *glueable pair* is introduced, and its relevance to the study of weak homotopy types of finite spaces is shown. Section 4 considers h-regular spaces and states some new results related to them. The main results of the paper are presented in Section 5, where the h-regularization process of finite spaces of height at most 2 is developed. Section 6 describes an implementation of the h-regularization process in the Kenzo system, providing some examples of computations. Finally, Section 7 presents the conclusions and outlines further work.

## 2. Preliminaries

In this section, we introduce some known concepts and results about finite topological spaces, which will be used throughout this paper. Details can be found in [2].

A topological space $X$ whose underlying point set is finite is called a *finite topological space* or *finite space* for short. The *minimal open set* containing an element $x \in X$, denoted by $U_x^X$, can be defined as the intersection of all the open sets that contain $x$ (observe that this concept is well-defined since arbitrary intersections of open subsets in a finite space are always open). The collection $\mathcal{U}_X = \{U_x^X\}_{x\in X}$ is the *minimal basis* of the finite space $X$.

A one-to-one correspondence between finite topological spaces and finite pre-ordered sets was provided by Alexandroff [1], wherein a pre-order relation on a finite topological space $X$ can be defined as follows:

$$x \leqslant y \Longleftrightarrow x \in U_y^X \Longleftrightarrow U_x^X \subseteq U_y^X. \tag{1}$$

The antisymmetry property of the relation $\leqslant$ corresponds to the separation axiom $T_0$ of $X$, thus finite $T_0$-spaces are equivalent to finite partially ordered sets (posets). Furthermore, the study of homotopical invariants can be restricted to the class of finite $T_0$-spaces, since any finite topological space is homotopy equivalent to a finite $T_0$-space [18].

**Notation 2.1.** Given two finite topological spaces $X$ and $Y$, we will use the symbols $X \overset{\text{hom}}{\approx} Y$, $X \overset{\text{he}}{\approx} Y$ and $X \overset{\text{we}}{\approx} Y$ when $X$ and $Y$ are homeomorphic, homotopy equivalent, and weak homotopy equivalent, respectively.

By using the pre-order in (1), minimal open subsets can be written as $U_x^X = \{z \in X : z \leqslant x\}$ and the (topological) *closure* of $\{x\}$ is denoted by $F_x^X = \{z \in X : x \leqslant z\}$; the set $C_x^X = U_x^X \cup F_x^X$ is called the *star* of $x$. The *reduced* versions $\widehat{U}_x^X = U_x^X - \{x\}$, $\widehat{F}_x^X = F_x^X - \{x\}$ and $\widehat{C}_x^X = C_x^X - \{x\}$ are often used; when there is no ambiguity about the finite space where we are working, we can omit the superscript of the above symbols.

Given a finite $T_0$-space $X$, the *Hasse diagram* $\mathcal{H}(X)$ of the poset $(X, \leqslant)$ is a digraph whose vertices are the points of $X$ and whose edges are the ordered pairs $(x, y)$ such that $x < y$ and there exists no $z$ such that $x < z < y$ (here $x < y$ means $x \leqslant y$ and $x \neq y$); the set of edges is denoted by $\mathrm{E}(\mathcal{H}(X))$. The *height* of $x$ in $X$, $h(x)$, is one less than the maximum cardinality of chains in $X$ containing $x$ as maximum; the cardinality of a set $A$, the set of its minimal elements, and the set of its maximal elements will be denoted by $\#A$, $\mathrm{mnl}(A)$, and $\mathrm{mxl}(A)$, respectively.

There is an important connection between finite topological spaces and simplicial complexes involving the following concepts:

**Definition 2.2.** The *order complex* $\mathcal{K}(X)$ of a finite $T_0$-space $X$ is the (finite) simplicial complex whose simplices are the non-empty chains of $X$. The *face poset* $\mathcal{X}(K)$ of a finite simplicial complex $K$ is the poset of simplices of $K$ ordered by inclusion.

Recall that a continuous map $f : X \longrightarrow Y$ between topological spaces is said to be a *weak homotopy equivalence* if it induces isomorphisms in all homotopy groups, that is, if $f_* : \pi_0(X) \longrightarrow \pi_0(Y)$ is a bijection and the maps $f_* : \pi_n(X, x_0) \longrightarrow \pi_n(Y, f(x_0))$ are isomorphisms for every $n \geqslant 1$ and every base point $x_0 \in X$. McCord [13] provides a weak homotopy equivalence between $X$ and $|\mathcal{K}(X)|$, where $|\mathcal{K}(X)|$ is the geometric realization of $\mathcal{K}(X)$. Conversely, if $K$ is a finite simplicial complex, McCord also proved that $|K|$ is weak homotopy equivalent to $\mathcal{X}(K)$. The simplicial complex $\mathcal{K}(\mathcal{X}(K))$ is the barycentric subdivision of $K$, and the poset $X' = \mathcal{X}(\mathcal{K}(X))$ is called the *barycentric subdivision* of $X$. It follows that $X' \overset{\text{we}}{\approx} X$.

The order complex $\mathcal{K}(X)$ can be used to compute topological invariants of $X$ by employing techniques on simplicial complexes. However, the size of $\mathcal{K}(X)$ often restricts the possible computations on it. Nevertheless, there exist methods that can be directly applied to finite topological spaces, which provide reductions to smaller spaces with the same topological invariants. For instance, Stong [18] proved that, given an element $x \in X$, if the subposet $\widehat{U}_x$ has a maximum or the subposet $\widehat{F}_x$ has a minimum, then $X \smallsetminus \{x\}$ is a strong deformation retract of $X$.

**Definition 2.3.** Let $X$ be a finite $T_0$-space. A point $x \in X$ is a *down beat point* if $\widehat{U}_x$ has a maximum; $x$ is an *up beat point* if $\widehat{F}_x$ has a minimum. In either of these cases, $x$ is a *beat point* of $X$ and we will say that there is an *elementary strong collapse* from $X$ to $X \smallsetminus \{x\}$. There is a *strong collapse* $X \searrow\!\!\searrow Y$ (or a *strong expansion* $Y \nearrow\!\!\nearrow X$) if there is a sequence of elementary strong collapses starting in $X$ and ending in $Y$. A *core* of $X$ is a strong deformation retract of $X$ which has no beat points.

The terms *elementary strong collapse* and *strong collapse*, along with their corresponding notations, were introduced in [4]. From a computational perspective, the key points of these concepts are that the core of a finite topological space $X$ is unique (up to homeomorphism) and that, when $x$ is a beat point, $X$ and $X \smallsetminus \{x\}$ are homotopy equivalent spaces [18]. Thus, it provides a method for computing a minimal space with the same homotopy type as $X$.

The notion of *weak point*, introduced in [3], generalizes that of beat point and has proven to be useful in the study of weak homotopy types of finite topological spaces.

**Definition 2.4.** Let $X$ be a finite $T_0$-space. A point $x \in X$ is a *down weak point* if $\widehat{U}_x$ is contractible; $x$ is an *up weak point* if $\widehat{F}_x$ is contractible. In either of these two cases, we say that $x$ is a *weak point* of $X$ and in this case we will say that there is an *elementary collapse* from $X$ to $X \smallsetminus \{x\}$. There is a collapse $X \searrow Y$ (or an *expansion* $Y \nearrow X$) if there is a sequence of elementary collapses starting in $X$ and ending in $Y$. Two finite $T_0$-spaces $X$ and $Y$ are *simple homotopy equivalent* if there is a sequence $X = X_1, X_2, \ldots, X_n = Y$ of finite $T_0$-spaces such that for each $1 \leqslant i < n$, $X_i \searrow X_{i+1}$ or $X_i \nearrow X_{i+1}$. We denote this case as $X \diagup\!\!\searrow Y$.

Simple homotopy equivalent finite spaces are, in particular, weak homotopy equivalent. Indeed, the deletion of a weak point $x \in X$ does not modify the weak homotopy type of $X$ since the inclusion map $\iota : X \smallsetminus \{x\} \longrightarrow X$ is a weak homotopy equivalence [2].

As a different process to modify a finite $T_0$-space in order to maintain its weak homotopy type, we can consider quotients by some subspaces. The next lemma provides a necessary and sufficient condition for the quotient by a subspace $A \subseteq X$ to be a $T_0$-space. The *open hull* of $A$ is $\underline{A} = \bigcup_{a \in A} U_a^X$, and its closure is $\overline{A} = \bigcup_{a \in A} F_a^X$.

**Lemma 2.5.** [2] *Let $A$ be a subspace of a finite $T_0$-space $X$, then $X/A$ is $T_0$ if and only if $\underline{A} \cap \overline{A} = A$.*

The minimal open subsets in a quotient space are given by the following result:

**Lemma 2.6.** [2] *Let $x \in X$ and let $q : X \longrightarrow X/A$ be the quotient map. If $x \in \overline{A}$, then $U_{q(x)} = q(U_x \cup \underline{A})$. If $x \notin \overline{A}$, then $U_{q(x)} = q(U_x)$.*

The standard way to check if a continuous map $f : X \longrightarrow Y$ between finite $T_0$-spaces is a weak homotopy equivalence is to verify that it is a local weak homotopy equivalence over a *basis-like open cover* of the codomain, that is, a basis for a topology in the underlying set of $Y$, which may be different from the original topology. In particular, the minimal basis $\mathcal{U} = \{U_y\}_{y \in Y}$ is a basis-like open cover of $Y$ and is the most commonly used.

With respect to the weak homotopy type, the next result is highly useful in practice, as the weak homotopy equivalence between finite topological spaces corresponds to a weak homotopy equivalence between the spaces of which they are models [2].

**Theorem 2.7.** [13] *Let $X$ and $Y$ be topological spaces and let $f : X \longrightarrow Y$ be a continuous map. Suppose that there exists a basis-like open cover $\mathcal{U}$ of $Y$ such that each restriction*

$$f|_{f^{-1}(U)} : f^{-1}(U) \longrightarrow U$$

*is a weak homotopy equivalence for every $U \in \mathcal{U}$. Then $f : X \longrightarrow Y$ is a weak homotopy equivalence.*

## 3. Glueable pairs

As mentioned above, there exist some techniques for reducing finite spaces that preserve certain topological invariants. In particular, some of the reductions found in the literature, in order to obtain weak homotopy equivalent spaces to a given finite $T_0$-space, are contained in the following three definitions.

**Definition 3.1.** [2, Section 11.2] Let $X$ be a finite $T_0$-space of height at most 2 and let $a, b \in X$ be two maximal elements of $X$ such that $U_a \cap U_b$ is contractible. We say that there is a *qc-reduction* from $X$ to $Y \setminus \{a, b\}$ where $Y = X \cup \{c\}$ with $a < c > b$. We say that $X$ is *qc-reducible* if we can obtain a space with a maximum by performing qc-reductions starting from $X$.

**Definition 3.2.** [11, Definition 3.2.5] Let $X$ be a finite $T_0$-space of height at most 2 and let $a, b \in X$ be neither maximal nor minimal points such that $U_a \cap U_b = \{*\}$. If, for every $x \in F_a \setminus F_b$, $U_b \cap U_x = \{*\}$, and for every $x \in F_b \setminus F_a$, $U_a \cap U_x = \{*\}$, then we say that there is a *middle-reduction* from $X$ to the quotient $X/\{a, b\}$. We say that $X$ is *middle-reducible* if it can be transformed into a connected space with a unique point (of height 1) by performing middle-reductions.

**Definition 3.3.** [11, Definition 3.2.10] Let $X$ be a finite $T_0$-space and let $e = (a, b)$ be an edge in the Hasse diagram $\mathcal{H}(X)$, with $b$ being a maximal element. If $U_b \setminus e$ is contractible, we say that there is an *edge-reduction* from $X$ to $X \setminus e$.

The above definitions are inspired by Theorem 2.7, a standard tool used to guarantee the existence of a weak homotopy equivalence between two finite $T_0$-spaces. We have adopted the term *glueable pair* to refer to a more general type of reduction.

**Definition 3.4.** Let $X$ be a finite $T_0$-space. A subset $A = \{a, b\} \subset X$ is a *glueable pair* if it satisfies the following conditions:

1. $\widehat{U}_a \cap \widehat{F}_b = \emptyset$ and $\widehat{U}_b \cap \widehat{F}_a = \emptyset$.
2. For every $x \in F_a \setminus F_b$, $U_x \cup U_b$ is homotopically trivial, and for every $x \in F_b \setminus F_a$, $U_a \cup U_x$ is homotopically trivial.

Note that if $\{a, b\}$ is a glueable pair, $U_a \cup U_b$ is homotopically trivial. In the particular case when $U_a \cup U_b$ is contractible, $U_a \cap U_b$ is also contractible, as stated in the next proposition.

**Proposition 3.5.** [7] *Let $X$ be a finite $T_0$-space and let $a, b \in X$. Then, $U_a \cap U_b$ is contractible if and only if $U_a \cup U_b$ is contractible.*

The following new result shows that the quotient of a space by a glueable pair does not alter its weak homotopy type.

**Proposition 3.6.** *Let $X$ be a finite $T_0$-space and $A = \{a, b\} \subset X$ be a glueable pair. Then $q : X \longrightarrow X/A$ is a weak homotopy equivalence.*

**Proof.** Observe that $\underline{A} = U_a \cup U_b$ and $\overline{A} = F_a \cup F_b$, then by condition 1 in Definition 3.4, it is clear that $\underline{A} \cap \overline{A} = A$. Hence, $X/A$ is a finite $T_0$-space by Lemma 2.5. In order to prove that $q : X \longrightarrow X/A$ is a weak homotopy equivalence, it is sufficient to show that for each $x \in X$, the restricted map $q|_{q^{-1}(U_{q(x)})} : q^{-1}(U_{q(x)}) \longrightarrow U_{q(x)}$ is a weak homotopy equivalence (Theorem 2.7). For this purpose, we have two cases. On the one hand, if $x \notin \overline{A}$, by Lemma 2.6, $U_{q(x)} = q(U_x)$. Since $x \notin F_a \cup F_b$, then $a \notin U_x$ and $b \notin U_x$, hence $q^{-1}(q(U_x)) = U_x$. On the other hand, if $x \in \overline{A}$, by Lemma 2.6, $U_{q(x)} = q(U_x \cup \underline{A})$. Note that if $x \in F_a \cap F_b$, $U_a \cup U_b \subseteq U_x$, then $q^{-1}(U_{q(x)}) = U_x$. Observe that $q^{-1}(q(U_x \cup U_b)) = U_x \cup U_b$ if $x \in F_a \setminus F_b$, and $q^{-1}(q(U_a \cup U_x)) = U_a \cup U_x$ if $x \in F_b \setminus F_a$. Therefore, for every $x \in \overline{A}$, $q^{-1}(U_{q(x)})$ is homotopically trivial, according to condition 2 in Definition 3.4.

In any case, the map $q|_{q^{-1}(U_{q(x)})} : q^{-1}(U_{q(x)}) \longrightarrow U_{q(x)}$ is a weak homotopy equivalence for each $x \in X$, as desired. □

**Example 3.7.** Fig. 1 shows how to obtain a weak homotopy equivalent space to $X$ (the face poset of the boundary of a triangle seen as a simplicial complex) by taking the quotient by the glueable pair $A = \{a, b\}$; after that, we delete $A$ as a beat point of $X/A$.

**Corollary 3.8.** *If $X$ is a finite $T_0$-space and $a, b \in X$ satisfy that $\widehat{F}_a = \widehat{F}_b$ and $U_a \cup U_b$ is homotopically trivial, then $X \overset{we}{\approx} X/\{a, b\}$.*

The above corollary is a generalization of the next proposition.

**Proposition 3.9.** [7] *Let $X$ be a finite $T_0$-space, and let $a$ and $b$ be maximal elements of $X$. If $U_a \cup U_b$ is homotopically trivial, then the quotient map $q : X \longrightarrow X/\{a, b\}$ is a weak homotopy equivalence.*

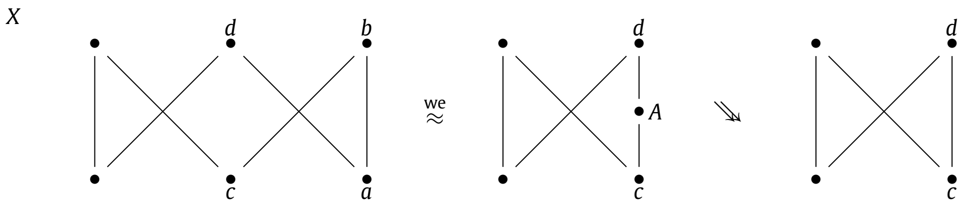



**Fig. 1.** The quotient of $X$ by the glueable pair $A = \{a, b\}$ followed by an elementary strong collapse.

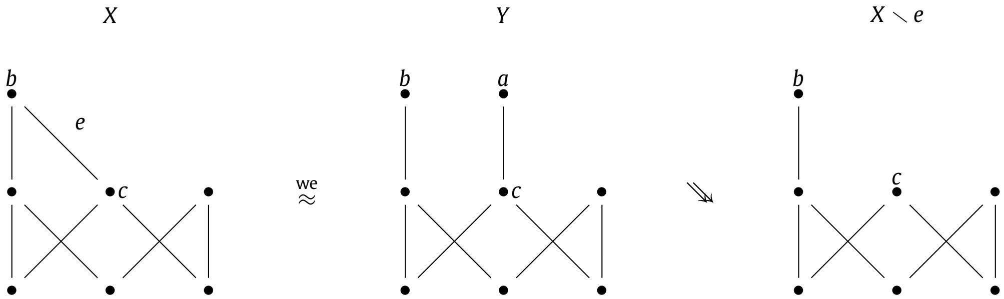



**Fig. 2.** An edge-reduction seen as a quotient by a glueable pair followed by an elementary strong collapse.

Some remarks can be drawn from Proposition 3.6, indicating that the reduction operations discussed at the start of this section can be expressed in terms of quotients by glueable pairs:

- Suppose that $b$ is a down beat point of $X$ and $a = \max(\widehat{U}_b)$. In particular, $(a, b)$ is an edge of the Hasse diagram of $X$, and thus $\widehat{U}_a \cap \widehat{F}_b = \emptyset$ and $\widehat{U}_b \cap \widehat{F}_a = \emptyset$. Additionally, $U_a \cup U_b = U_b$ is contractible, and thus $\{a, b\}$ is a glueable pair. A similar argument works when $b$ is an up beat point of $X$ and $a = \min(\widehat{F}_b)$. This implies that elementary strong collapses in finite $T_0$-spaces can be seen as quotients by glueable pairs.
- If $a, b \in X$ are maximal elements, $\widehat{F}_a = \emptyset = \widehat{F}_b$, then Proposition 3.6 generalizes Proposition 3.9. Bearing this in mind, edge-reductions can be seen as a sequence of quotients by some glueable pairs: if $b$ is a maximal element of $X$ and $e = (c, b)$ is an edge of $\mathcal{H}(X)$ such that $U_b^X \setminus e$ is contractible (as in Definition 3.3), we consider the space $Y = (X \setminus e) \cup \{a\}$ where $(c, a)$ is an edge of $\mathcal{H}(Y)$ and $a$ is a down beat point of $Y$, then the maximal elements $\{a, b\}$ form a glueable pair of $Y$ since $U_a^Y \cup U_b^Y = \{a\} \cup (U_b^X \setminus e) \searrow\!\!\searrow U_b^X \setminus e \searrow\!\!\searrow *$ and therefore $X \setminus e \nearrow\!\!\nearrow Y \overset{\text{we}}{\approx} Y/\{a, b\} \overset{\text{hom}}{\approx} X$. The sequence of spaces in Fig. 2 is an example of this situation.
- Regarding finite spaces of height at most 2, qc-reductions can be seen as quotients by glueable pairs (by definition) and the same occurs for middle-reductions: if $a, b \in X$ are neither maximal nor minimal points and $U_a \cap U_b = \{*\}$ (as in Definition 3.2), then it is not possible for there to exist $x \in X$ such that $a < x < b$ or $b < x < a$ (otherwise, either $a$ or $b$ would be either maximal or minimal).

Now, we are going to describe the opposite process: instead of gluing a pair (which decreases the given space by one point), we add a point under certain conditions. The following lemma captures this idea and will be useful in the subsequent sections.

**Lemma 3.10.** *Let $X$ be a finite $T_0$-space and $a \in X$. Consider the set $Z = X \cup \{b\}$, where $b \notin X$, and suppose that $Z$ is endowed with a topology that satisfies the following properties:*

1. $\widehat{F}_a^X = \widehat{F}_a^Z = \widehat{F}_b^Z$,
2. $\widehat{U}_a^X = \widehat{U}_a^Z \cup \widehat{U}_b^Z$,
3. *For* $x \in X \setminus \{a\}$, $U_x^Z = \begin{cases} U_x^X \cup \{b\} & , x \in \widehat{F}_a^X \\ U_x^X & , x \notin \widehat{F}_a^X \end{cases}$
4. $U_a^Z \cup U_b^Z$ *is homotopically trivial.*

*Then $Z$ is weak homotopy equivalent to $X$. Moreover, if $U_a^Z \cup U_b^Z$ is contractible, then $Z \nearrow\!\!\searrow X$.*

**Proof.** Properties *1* and *4* state that the map $q : Z \longrightarrow Z/\{a, b\}$ is a weak homotopy equivalence (by Corollary 3.8). It is clear that the quotient $Z/\{a, b\}$ and $X$ are homeomorphic, which is sufficient to conclude $Z \overset{\text{we}}{\approx} X$.

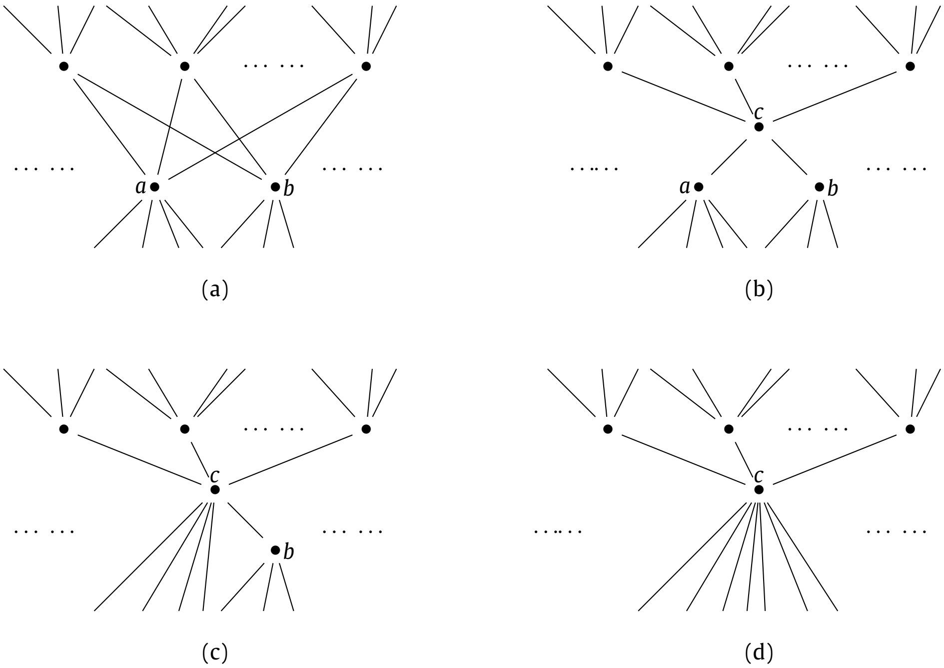



**Fig. 3.** Illustration of: (a) $C_a^Z \cup C_b^Z$, (b) $C_c^M$, (c) $C_c^{M \smallsetminus \{a\}}$, (d) $C_c^{M \smallsetminus \{a,b\}}$.

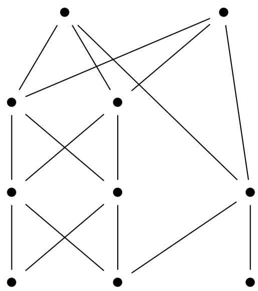


**Fig. 4.** An h-regular poset (shown in [14, Figure 1]).

Now, if $U_a^Z \cup U_b^Z$ is contractible, consider the space $M$ that it is obtained from $Z$ by adding a new point $c$ which covers $a$ and $b$ and is covered by all the elements that cover $a$ or $b$ in $Z$ (this construction is similar to the qc-reductions which appear in [2], see Fig. 3).

Note that $c$ is a weak point of $M$ and $a, b$ are beat points of $M$. Then $Z \nearrow M \searrow\!\!\searrow M \smallsetminus \{a\} \searrow\!\!\searrow M \smallsetminus \{a,b\}$. Finally, observe that $M \smallsetminus \{a,b\} \overset{\text{hom}}{\approx} X$ (the homeomorphism replaces the label $c$ in $M \smallsetminus \{a,b\}$ by the label $a$), then $Z \nearrow\!\!\searrow X$. $\square$

## 4. Some properties of h-regular spaces

The concept of *h-regular space* was introduced in [14].

**Definition 4.1.** A finite $T_0$-space $X$ is called *h-regular* if, for every $x \in X$, the order complex $\mathcal{K}(\widehat{U}_x)$ is homotopy equivalent to the $(h(x)-1)$-dimensional sphere, that is, $\widehat{U}_x \overset{\text{we}}{\approx} S^{h(x)-1}$.

The face poset $\mathcal{X}(K)$ of any h-regular CW-complex $K$ (in particular, of any finite simplicial complex) is h-regular. In [14], Fig. 4 is presented as an example of an h-regular space which is not the face poset of a regular CW-complex.

In this section, we present some new results regarding h-regular spaces. In particular, we characterize h-regular spaces of height 1, which is a useful tool for developing the *h-regularization* process described in Section 5. We begin by introducing the following new definition.

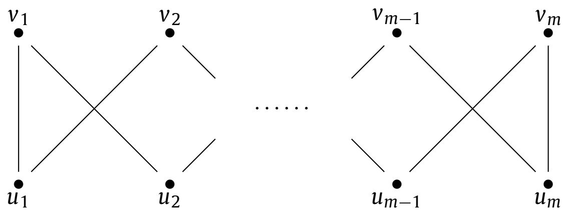



**Fig. 5.** Hasse diagram of a $2m$-crown.

**Definition 4.2.** Let $X$ be a finite $T_0$-space and let $n \in \mathbb{N}$. We say that $X$ is *n-h-regular* if the subposet $X^{(n)} = \{x \in X : h(x) \leqslant n\}$ is h-regular.

The next lemma is a straightforward consequence of the definitions previously mentioned.

**Lemma 4.3.** *Let $X$ be an $n$-h-regular space for some $n \in \mathbb{N}_0$, and let $A \subseteq X$ be an open subset. Then $A$ is $n$-h-regular. In particular, for all $x \in X$, $U_x^X$ and $\widehat{U}_x^X$ are $n$-h-regular spaces.*

The core of an h-regular space is not generally h-regular. However, in spaces of height 1, the h-regularity property is inherited by their cores.

**Lemma 4.4.** *Let $X$ be an h-regular space of height 1, and let $X_c$ be a core of $X$. Then $X_c$ is h-regular.*

**Proof.** Without loss of generality, assume that $X$ is connected (otherwise consider each connected component independently). If $X$ is contractible, there is nothing to proof. Suppose that $X$ is not contractible and let $x \in \mathrm{mxl}(X_c) \subseteq \mathrm{mxl}(X)$. Since $X_c$ is a subspace of $X$, $\widehat{U}_x^{X_c} = \widehat{U}_x^X \cap X_c$ so that $\#\widehat{U}_x^{X_c} \leqslant \#\widehat{U}_x^X = 2$. Since $h(x) = 1$, then $\#\widehat{U}_x^{X_c} \geqslant 1$, but $X_c$ has no beat points, so necessarily $\#\widehat{U}_x^{X_c} = 2$. $\square$

If $X$ is a connected finite $T_0$-space of height 1, $|\mathcal{K}(X)|$ is a connected graph, so that $X \overset{\text{we}}{\approx} \bigvee_{i=1}^{q} S^1$ if and only if $\mathcal{E}(X) = 1 - q$, where $\mathcal{E}(X) = \#X - \#\mathrm{E}(\mathcal{H}(X))$ is the Euler characteristic of $X$.

Observing the Euler characteristic characterizes the weak homotopy type of a connected graph, it is possible to characterize the h-regular spaces of height 1 by using this well-known fact. Note that an h-regular space of height 1 is the face poset of a regular CW-complex of dimension 1 (a graph). Since every connected graph is homotopy equivalent to a wedge of 1-spheres, and bearing in mind that a connected graph such that every vertex is in two edges is a circle, these remarks provide the following result.

**Lemma 4.5.** *Let $X$ be an h-regular connected finite $T_0$-space of height 1. Then,*

$$X \overset{we}{\approx} \bigvee_{i=1}^{q} S^1 \text{ if and only if } \#\mathrm{mnl}(X) - \#\mathrm{mxl}(X) = 1 - q.$$

*In particular, if $X^{op}$ is h-regular then $X \overset{we}{\approx} S^1$; the converse is true if $X$ has no beat points, and in this case $X^{op} \overset{hom}{\approx} X$.*

Lemma 4.5 allows us to establish the following definition.

**Definition 4.6.** A *2m-crown* is an h-regular finite model of $S^1$ of height 1 without beat points whose cardinal is $2m$. The Hasse diagram of a $2m$-crown looks like in Fig. 5.

Crown spaces are going to play a crucial role in the h-regularization process. It should be noted that if a point is deleted from a crown (which is a connected space), the resulting space remains connected. More generally, when a point of a crown that is contained in a connected finite $T_0$-space $Z$ is deleted from $Z$, the connectedness is preserved.

**Lemma 4.7.** *Let $Z$ be a connected finite $T_0$-space, and let $Y \subseteq Z$ be a crown. If $x \in Y$, then $Z \setminus \{x\}$ is connected.*

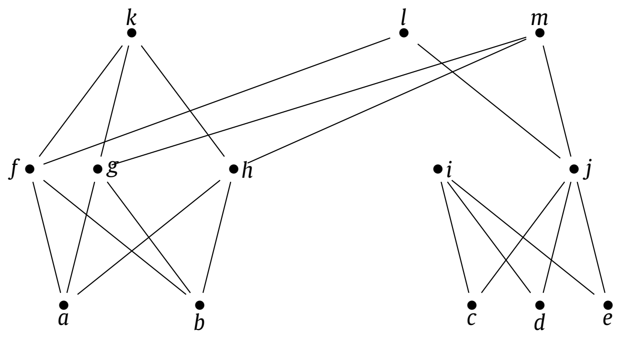



**Fig. 6.** Hasse diagram of the finite $T_0$-space $M$.

## 5. h-regularization of finite spaces

In this section, we show an effective method for constructing an h-regular space that is simple homotopy equivalent to a given finite $T_0$-space of height at most 2. We will illustrate the application of our results on the finite $T_0$-space $M$ in Fig. 6 (we will *h-regularize* the space $M$). Note that $M$ is not h-regular, as evidenced by the fact that $h(i) = 1$ but $\#\widehat{U}_i = 3$.

### 5.1. h-regularization of points of height 1

The process begins by modifying the minimal open sets of the elements of height 1. This procedure does not affect the elements of higher heights, so we adopt the following notation to emphasize this:

**Notation 5.1.** If $X$ and $Y$ are finite $T_0$-spaces such that $X \diagup\!\!\searrow Y$ and $X \searrow X^{(n)} \overset{\text{hom}}{\approx} Y \searrow Y^{(n)}$ for some $n \in \mathbb{N}$, we will write $X \approx_n Y$.

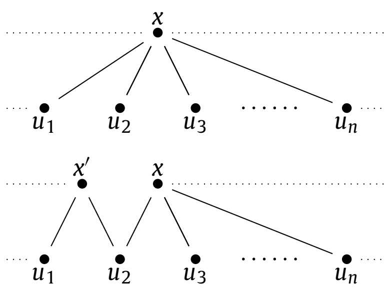
As previously established, each element $x$ of height 1 in an h-regular space must satisfy the condition $\#\widehat{U}_x = 2$ (the 0-dimensional sphere is a discrete poset with two elements). The next proposition allows us to separate the elements of $\widehat{U}_x$ when $\#\widehat{U}_x > 2$. Note that in the case $\#\widehat{U}_x = 1$, $x$ is a beat point and can be removed.

**Proposition 5.2.** *Let $X$ be a finite $T_0$-space and let $x \in X$ such that $h(x) = 1$ and $\#\widehat{U}_x^X = n \geqslant 3$. Then, there exists a finite $T_0$-space $X_x$ such that $\#\widehat{U}_x^{X_x} = n - 1$ and $X \approx_1 X_x$.*

**Proof.** Let $\widehat{U}_x^X = \{u_1, \dots, u_n\} \subseteq \mathrm{mnl}(X)$. Consider the space $X_x = X \cup \{x'\}$, with $x' \notin X$, whose minimal basis $\{U_z^{X_x}\}_{z \in X_x}$ is given by:

$X^{(1)} =$

$X_x^{(1)} =$




$$\widehat{U}_z^{X_x} = \begin{cases} \{u_1, u_2\} & , z = x' \\ \{u_2, \dots, u_n\} & , z = x \\ \widehat{U}_z^X \cup \{x'\} & , z \in \widehat{F}_x^X \\ \widehat{U}_z^X & , z \notin \widehat{F}_x^X. \end{cases}$$

By construction, $\widehat{F}_x^X = \widehat{F}_x^{X_x} = \widehat{F}_{x'}^{X_x}$. Furthermore, by Proposition 3.5, $U_x^{X_x} \cup U_{x'}^{X_x}$ is contractible since $U_x^{X_x} \cap U_{x'}^{X_x} = \{u_2\}$ is contractible, and thus, by Lemma 3.10 we have that $X_x \diagup\!\!\searrow X$. Finally, observe that $Z := X - X^{(1)}$ equals $X_x - X_x^{(1)}$ as sets; moreover, for all $y \in Z$, since $x' \notin Z$, then $U_y^Z = U_y^X \cap Z = U_y^{X_x} \cap Z$. Consequently, the topologies of $X$ and $X_x$ coincide on $Z$. $\square$

The above result allows us to modify the reduced minimal open set of a point of height 1 in $X$ that does not satisfy the h-regular property. Applying the process of Proposition 5.2 to each point of height 1 of $X$, we obtain a space $X_1$ that is 1-h-regular and $X_1 \approx_1 X$.

Regarding the space $M$ in Fig. 6, observe that $\#\widehat{U}_i = 3$ and $\#\widehat{U}_j = 3$, therefore we apply Proposition 5.2 to points $i$ and $j$, resulting in the 1-h-regular space in Fig. 7 (we have *1-h-regularized* the space $M$).

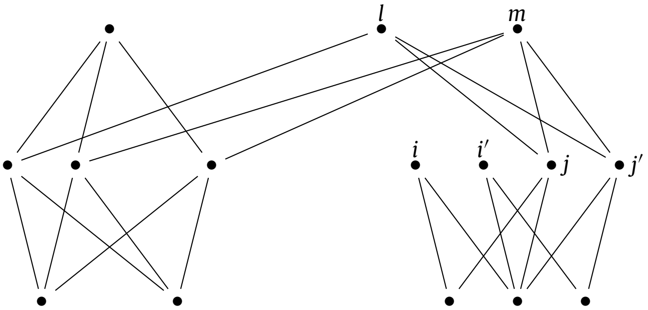



**Fig. 7.** 1-h-regularization of $M$.

### 5.2. h-regularization of points of height 2

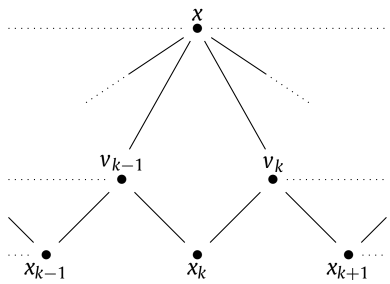
Once we have modified a given space so that its elements of height 1 satisfy the h-regularity property, we must ensure that for each element $x$ of height 2, the subspace $\widehat{U}_x$ to be weak homotopy equivalent to $S^1$. In particular, $\widehat{U}_x$ must be connected; if this is not the case, the following result allows us to modify the space in order to address this local situation.

**Proposition 5.3.** *Let $X$ be a finite $T_0$-space and let $x \in X$ with $h(x) \geqslant 2$. Then, there exists a finite $T_0$-space $X_x$ such that $\widehat{U}_x^{X_x}$ is connected and $X \approx_1 X_x$. Moreover for $n \in \mathbb{N}_0$, if $X$ is $n$-h-regular, then so is $X_x$.*

**Proof.** If $\widehat{U}_x^X$ is connected, take $X_x = X$; otherwise, consider $\widehat{U}_x^X = \bigsqcup_{k=1}^m C_k$, where $C_k$ are the connected components of $\widehat{U}_x^X$. Since $C_k \neq \emptyset$, fix any $x_k \in \operatorname{mnl}(C_k)$ for each $k = 1, \dots, m$. Consider the $T_0$-space $X_x = X \cup \{v_1, \dots, v_{m-1}\}$, whose minimal basis $\{U_z^{X_x}\}_{z \in X_x}$ is defined by:

$$\widehat{U}_z^{X_x} = \begin{cases} \{x_k, x_{k+1}\} & , z = v_k \text{ for } k = 1, \dots, m-1 \\ \widehat{U}_z^X \cup \{v_1, \dots, v_{m-1}\} & , z \in F_x^X \\ \widehat{U}_z^X & , z \notin F_x^X. \end{cases}$$

$X_x^{(h(x))} =$




Observe that $\widehat{F}_{v_k}^{X_x} = F_x^X$ for each $k = 1, \dots, m-1$. This implies that $v_k$ is an up beat point of $X_1$ and thus $X_x \searrow\!\!\searrow X$. To prove the connectedness of $\widehat{U}_x^{X_x}$ it is sufficient to observe that $x_1 < v_1 > x_2 < \cdots > x_{m-1} < v_{m-1} > x_m$ is a fence in $X_1$ connecting all the $C_k$. Finally, by construction, $X_x^{(0)} = X^{(0)}$, $X - X^{(1)} \overset{\text{hom}}{\approx} X_x - X_x^{(1)}$ and for $n \geqslant 1$, $X_x^{(n)} = X^{(n)} \cup \{v_1, \dots, v_{m-1}\}$, $h(v_k) = 1$ and $\#\widehat{U}_{v_k}^{X_x} = 2$. Therefore, if $X$ is $n$-h-regular, then $X_x$ is also $n$-h-regular. □

Considering the 1-h-regular space in Fig. 7, we can use Proposition 5.3 to make $\widehat{U}_l$ and $\widehat{U}_m$ connected subspaces by adding the points $l'$ and $m'$, as depicted in Fig. 8. This results in a 1-h-regular space with connected reduced minimal open sets for all elements of height 2.

Proposition 5.3 allows us to assume, from this point forward in this section, that the reduced minimal open sets of elements of height greater than 1 are connected. Now, once we have 1-h-regularized a given space, the subspace $\widehat{U}_x$ is an h-regular space of height 1, for every element $x$ of height 2, therefore $\widehat{U}_x$ is weak homotopy equivalent to a finite wedge of $q$ 1-dimensional spheres by Lemma 4.5. The case $q = 0$ indicates that $x$ is a weak point and we can remove it and if $q = 1$ the desired condition is satisfied. However, if $q > 1$ we can *separate* such 1-spheres in order to satisfy the (h-regular) condition $\widehat{U}_x \overset{\text{we}}{\approx} S^1$ as is shown in the next result.

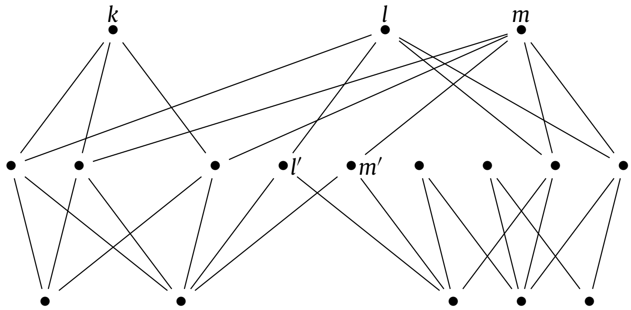


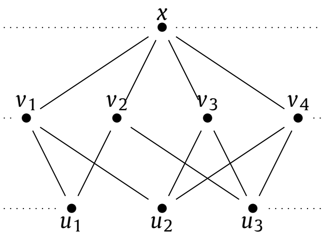

**Fig. 8.** Making $\widehat{U}_l$ and $\widehat{U}_m$ connected by adding the points $l'$ and $m'$.

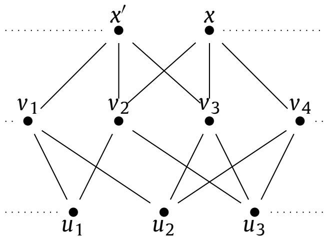
**Proposition 5.4.** *Let $X$ be a 1-h-regular finite $T_0$-space and let $x \in X$ such that $h(x) = 2$. Suppose $\widehat{U}^X_x \overset{we}{\approx} \bigvee_{i=1}^{q} S^1$ for $q > 1$. Then there exists a 1-h-regular finite $T_0$-space $X_x$ such that $\widehat{U}^{X_x}_x \overset{we}{\approx} \bigvee_{i=1}^{q-1} S^1$ and $X \approx_2 X_x$.*

**Proof.** Let $\#\mathrm{mnl}(\widehat{U}^X_x) = r$ and $\#\mathrm{mxl}(\widehat{U}^X_x) = R$. Let $X_c$ be a core of $\widehat{U}^X_x$, then $X_c \overset{we}{\approx} \bigvee_{i=1}^{q} S^1$. Also, by Lemmas 4.3 and 4.4, $X_c$ is h-regular. Observe that $X_c$ is a wedge of $q$ crowns, therefore we can take one of them: let $\{u_1, \dots, u_s, v_1, \dots, v_s\}$ be a $2s$-crown, $s > 1$, where $\{u_1, \dots, u_s\} \subseteq \mathrm{mnl}(M)$ and $\{v_1, \dots, v_s\} \subset \mathrm{mxl}(M)$; we label the rest of minimal and maximal elements of $\widehat{U}^X_x$ such that $\mathrm{mnl}(\widehat{U}^X_x) = \{u_1, \dots, u_s, u_{s+1}, \dots, u_r\}$ and $\mathrm{mxl}(\widehat{U}^X_x) = \{v_1, \dots, v_s, v_{s+1}, \dots, v_R\}$.

Consider the space $X_x = X \cup \{x'\}$, with $x' \notin X$, whose minimal basis $\{U^{X_x}_z\}_{z \in X_x}$ satisfies:

$$\widehat{U}^{X_x}_z = \begin{cases} \{u_1, \dots, u_s, v_1, \dots, v_s\} & , z = x' \\ \{u_1, \dots, u_r, v_2, \dots, v_R\} & , z = x \\ \widehat{U}^X_z \cup \{x'\} & , z \in \widehat{F}^X_x \\ \widehat{U}^X_z & , z \notin \widehat{F}^X_x. \end{cases}$$

$X^{(2)} =$ 


$X^{(2)}_x =$ 


By construction, $\widehat{F}^X_x = \widehat{F}^{X_x}_x = \widehat{F}^{X_x}_{x'}$. Furthermore, by Proposition 3.5, the subposet $U^{X_x}_{x'} \cup U^{X_x}_x$ is contractible, since $U^{X_x}_{x'} \cap U^{X_x}_x = \widehat{U}^{X_x}_{x'} \cap \widehat{U}^{X_x}_x = \{u_1, \dots, u_s, v_2, \dots, v_s\}$ is connected by Lemma 4.7 and

$$\#\mathrm{mxl}\left(\widehat{U}^{X_1}_{x'} \cap \widehat{U}^{X_1}_x\right) = \#\mathrm{mnl}\left(\widehat{U}^{X_1}_{x'} \cap \widehat{U}^{X_1}_x\right) - 1 \Longrightarrow \widehat{U}^{X_1}_{x'} \cap \widehat{U}^{X_1}_x \overset{we}{\approx} *.$$

Therefore, $X_x \nearrow\searrow X$ by Lemma 3.10. Note that, since $\widehat{U}^{X_x}_{x'} = \{u_1, \dots, u_s, v_1, \dots, v_s\}$ is a $2s$-crown, $\widehat{U}^{X_x}_{x'} \overset{we}{\approx} S^1$ (by using Lemma 4.5). On the other hand, taking $Z = \widehat{U}^X_x$, $Y = \widehat{U}^{X_x}_{x'}$ and $x = v_1$ in Lemma 4.7, we obtain the connectedness of $\widehat{U}^{X_x}_x = \{u_1, \dots, u_r, v_2, \dots, v_R\}$ and therefore, by Lemma 4.5

$$\#\mathrm{mxl}\left(\widehat{U}^{X_x}_x\right) = \#\mathrm{mnl}\left(\widehat{U}^{X_x}_x\right) + (q-1) - 1 \Longrightarrow \widehat{U}^{X_x}_x \overset{we}{\approx} \bigvee_{i=1}^{q-1} S^1.$$

Finally, observe that $W := X - X^{(2)}$ is equal to $X_x - X^{(2)}_x$ as sets; moreover, for all $y \in W$, since $x' \notin W$ then $U^W_y = U^X_y \cap W = U^{X_x}_y \cap W$, thus the topologies of $X$ and $X_x$ are the same on $W$. □

Repeating the procedure of Proposition 5.4 to each point of height 2 of $X$, we can obtain a space $X_h$ such that the reduced minimal open sets of the points of height 2 are weak homotopy equivalent to one 1-sphere, thus establishing the central result of this work:

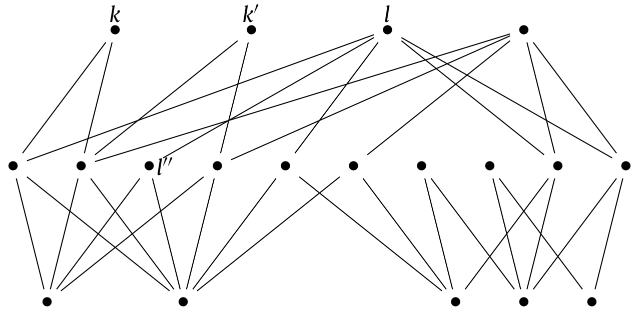



**Fig. 9.** 2-h-regularization of $M$.

**Theorem 5.5.** *Given a finite $T_0$-space $X$, the space $X_h$ constructed previously is 2-h-regular and $X_h \approx_2 X$. In particular, if $X$ has height 2, $X_h$ is h-regular.*

It is important to point out that the resulting space $X_h$ in Theorem 5.5, obtained after the h-regularization of a finite $T_0$-space, is not unique, as the involved constructions depend on expansions, collapses, and quotients by glueable pairs, which can be performed in different ways.

Continuing with our example, in order to 2-h-regularize the space in Fig. 8, we consider the elements of height 2. Note that $\widehat{U}_m$ has 5 minimal elements and 5 maximal elements so that by Lemma 4.5, such a subspace is weak homotopy equivalent to $S^1$, satisfying the h-regular property. On the other hand, $\widehat{U}_k$ is a finite model of $S^1 \vee S^1$, and $\widehat{U}_l$ is contractible. Therefore, by using the above construction, we obtain the h-regular space in Fig. 9.

Our construction of $X_h$ allows us to obtain from $X$ an h-regular space smaller than $X'$, thus enabling us to work with larger finite spaces in order to compute homotopical invariants, as we will show in Section 6. For example, observe that the space in Fig. 9 is an h-regular space with 19 elements that is simple homotopy equivalent to $M$; in contrast, the barycentric subdivision of $M$ has 63 elements.

The method of h-regularization has been completely developed for finite $T_0$-spaces of height at most 2. The complete process can be done for such spaces due to the fact that h-regular finite models of 1-spheres can be characterized by using Lemma 4.5. Unfortunately, there is not an analogous result on finite models of spheres in greater dimensions. However, most of the results that have been shown in this section can be applied to minimal finite $T_0$-spaces with no constraint on their heights. In particular, the construction of the space $X_h$ can be used as a first step in the search of h-regular spaces weak homotopy equivalent to finite $T_0$-spaces of height greater than two, as an alternative to the construction of barycentric subdivisions. Nevertheless, the developed procedure does not succeed in any circumstance for dimension greater than two.

## 6. Implementation and results

In Section 5, we have shown a procedure to obtain a simple homotopy equivalent h-regular space to a given one of height at most 2. In this section, we will describe an implementation of our algorithm in the *Kenzo system*, which allows for the h-regularization of finite spaces.

Kenzo [10] is a symbolic computation system devoted to algebraic topology. It was originally written in 1990 by Sergeraert and Rubio under the name of EAT (Effective Algebraic Topology) [17]. In 1998, it was rewritten by Sergeraert and Dousson, with the current name of Kenzo. The last official version dates from 2008, although there exists a more recent version maintained by G. Heber [12] with compatibility improvements and bug fixes.

The primary objective of Kenzo is to be able to handle spaces of infinite nature, encoded using the Common Lisp programming language and making extensive use of functional programming. It is the only program for algebraic topology that is capable of performing computations on infinite structures.

The program allows for the computation of homology and homotopy groups of complicated spaces, such as iterated loop spaces of a loop space modified by a cell attachment or components of complex Postnikov towers, which were previously unknown [16].

Kenzo has been enhanced with a module for computing invariants of finite topological spaces (instances of the class `FINITE-SPACE`) that was developed in [9] and is available at [8]. In this module, some methods have been implemented to identify beat points and weak points in finite spaces, allowing, in particular, the computation of cores and spaces with no weak points. Additionally, this module is able to compute homology groups of h-regular finite spaces working directly on the posets without having to go to the simplicial world. Moreover, the technique of discrete vector fields has been used to improve the computations of the homology by constructing a discrete vector field defined directly on the poset that can be applied to general h-regular finite spaces. Now, we have enhanced our module with a new function to compute the h-regular space by means of the algorithm presented in Section 5:

**Table 1**
Comparison of the cardinalities of the barycentric subdivision ($X'$) and our method of h-regularization ($X_h$). Twenty (20) random spaces of size $\#X$ and height 2, without beat points, were computed for each row.

| $\#X$ | Average $\#X_h$ | Average $\#X'$ | Average % reduction |
|---|---|---|---|
| 10 | 22.30 | 73.10 | 69.66 |
| 15 | 67.45 | 177.75 | 62.40 |
| 20 | 144.15 | 325.55 | 56.11 |
| 25 | 257.50 | 539.50 | 53.87 |
| 30 | 435.20 | 847.20 | 50.19 |
| 35 | 731.55 | 1266.75 | 43.14 |

`2-h-regularization` *minimal-finspace*
It returns the `FINITE-SPACE` $X_h$ in Theorem 5.5, i.e., a 2-h-regularization of *minimal-finspace*. The parameter *minimal-finspace* must be a `FINITE-SPACE` without beat points.

By using our implementation of the method in Kenzo, we tested the h-regularization of random finite $T_0$-spaces. We constructed 20 arbitrary finite $T_0$-spaces $X$ without beat points by using the function `random-2space` for some dimensions ($\#X$) (this function generates random finite $T_0$-spaces of height 2). We then used the method `2-h-regularization` to obtain instances of the class `FINITE-SPACE` representing the h-regularizations ($X_h$) of our testing examples. In the same way, the function `bar-subdivision` was used to compute the barycentric subdivisions ($X'$). After computing these spaces, we found the average of the cardinalities of the spaces $X$, $X_h$, and $X'$ in order to compare the results obtained. This information is summarized in Table 1.

It is clear that the execution of the h-regularization procedure entails a time cost which is usually greater than the time used to compute barycentric subdivisions. However, the smaller size of $X_h$ compensates for this disadvantage. Most importantly, the developed method allows for computations which, in practice, cannot be realized by applying barycentric subdivisions. As an example, the h-regularization method can be used to *repair* some perturbations applied to finite spaces coming from the simplicial complex world. For instance, we have imported into Kenzo the data of the facets of a random 2-dimensional simplicial complex with 25 vertices and 751 triangles, picked with probability 0.328, as described in the example *"rand2_n25_p0.328"* in [5], and we have constructed its face poset `K1`.

```
> (setf data_folder "...")
> (setf K1 (import-facets-to-finite-space "rand2_n25_p0.328"))
[K1 Finite-Space]
```

Then, we have taken a wedge of four copies of this finite $T_0$-space and a `random-2space` of 19 elements (a non-h-regular perturbation).

```
> (setf F2 (random-2space 19))
[K2 Finite-Space]
> (setf W (wedge-at-x1 K1 K1 K1 K1 F2))
[K3 Finite-Space]
> (cardinality W)
4373
```

As we can observe, the resulting space has 4373 points. Kenzo needed approximately two minutes to compute the h-regularization of `W` and its homology groups, while the barycentric subdivision of `W` could not be constructed (the function `h-regular-homology-sim` computes the homology groups of the input h-regular finite space of height at most 2).

```
> (time (2-h-regularization W))
Timing the evaluation of (2-H-REGULARIZATION W)

User time    =  0:01:04.046
System time  =         0.187
Elapsed time =  0:01:04.123
[K4 Finite-Space]

> (time (h-regular-homology-sim (k 4)))
Timing the evaluation of (H-REGULAR-HOMOLOGY-SIM (K 4))

Homology in dimension 0 : Z
Homology in dimension 1 :
```

```
Homology in dimension 2 :  Z ^ 1954
User time     =        57.015
System time   =         0.078
Elapsed time  =        57.257
```

## 7. Conclusions and further work

The algorithms developed in [9] for computing homology groups are applicable to h-regular finite spaces. In the literature, there are few examples of h-regular finite spaces that differ from face posets of simplicial complexes. The h-regularization process described in Section 5 produces a wide variety of h-regular finite spaces. As we have shown, any finite $T_0$-space of height at most 2 can be h-regularized, thus allowing for the consideration of new examples of this kind of spaces.

Some modifications to finite $T_0$-spaces had to be considered when searching for a correct implementation of h-regularization. In particular, when the reduced open minimal set of an element of height two is not connected, we *rectify* this by introducing a beat point, which makes the subspace connected in the new space. The most challenging part of our method was Proposition 5.4, where we showed an algorithmic way to *partition* the reduced open minimal set into 1-spheres, which was the key result to achieve the h-regularization of finite $T_0$-spaces that we had in mind.

Moreover, the modifications considered to h-regularize a finite $T_0$-space do not change the simple homotopy type, and all the spaces in the process are 3-deformations of the initial one. In the particular case when a homotopically trivial finite $T_0$-space of height two satisfies the Andrews-Curtis conjecture, all the spaces in the process of h-regularization satisfy it too, which could be used to attack the conjecture in a future work by using simple homotopy equivalent spaces to the given one, as in [11], where some potential counterexamples to the conjecture have been discarded by means of techniques on finite topological spaces.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Data availability

No data was used for the research described in the article.

## Acknowledgements

J. Cuevas-Rozo and A. Romero are supported by Grant PID2020-116641GB-I00 funded by MCIN/AEI/ 10.13039/501100011-033.

## References

[1] P. Alexandroff, Diskrete Räume, Mat. Sb. (N.S.) 2 (1937) 501–518.
[2] J.A. Barmak, Algebraic Topology of Finite Topological Spaces and Applications, Lecture Notes in Mathematics, vol. 2032, Springer, 2011.
[3] J.A. Barmak, G. Minian, Simple homotopy types and finite spaces, Adv. Math. 218 (2008) 87–104.
[4] J.A. Barmak, G. Minian, Strong homotopy types, nerves and collapses, Discrete Comput. Geom. 47 (2) (2012) 301–328.
[5] B. Benedetti, F.H. Lutz, Library of triangulations, http://page.math.tu-berlin.de/~lutz/stellar/library_of_triangulations/, 2017.
[6] N. Cianci, M. Ottina, A new spectral sequence for homology of posets, Topol. Appl. 217 (2017) 1–19.
[7] N. Cianci, M. Ottina, Poset splitting and minimality of finite models, J. Comb. Theory, Ser. A 157 (2018) 120–161.
[8] J. Cuevas-Rozo, Finite topological spaces in Kenzo, https://github.com/jcuevas-rozo/finite-topological-spaces, 2020.
[9] J. Cuevas-Rozo, L. Lambán, A. Romero, H. Sarria, Effective homological computations on finite topological spaces, Appl. Algebra Eng. Commun. Comput. (34) (2023) 34–56.
[10] X. Dousson, J. Rubio, F. Sergeraert, Y. Siret, The Kenzo Program, Institut Fourier, Grenoble, 1999, http://www-fourier.ujf-grenoble.fr/~sergerar/Kenzo/.
[11] X. Fernández, Métodos combinatorios y algoritmos en topología de dimensiones bajas y la conjetura de Andrews-Curtis, Ph.D. thesis, Universidad de Buenos Aires. Facultad de Ciencias Exactas y Naturales, 2017.
[12] G. Heber, A repackaged version of the Kenzo program by Francis Sergeraert and collaborators, https://github.com/gheber/kenzo, 2019.
[13] M.C. McCord, Singular homology groups and homotopy groups of finite topological spaces, Duke Math. J. 33 (3) (1966) 465–474.
[14] G. Minian, Some remarks on Morse theory for posets, homological Morse theory and finite manifolds, Topol. Appl. 159 (12) (2012) 2860–2869.
[15] J. Renders, Finite topological spaces in algebraic topology, Project of Master of Science in Mathematics, Ghent University, 2019.
[16] J. Rubio, F. Sergeraert, Constructive Homological Algebra and Applications, Lecture Notes Summer School on Mathematics, Algorithms, and Proofs, University of Genova, 2006, http://www-fourier.ujf-grenoble.fr/~sergerar/Papers/Genova-MAP-2006-v3.pdf.
[17] J. Rubio, F. Sergeraert, Y. Siret, EAT: Symbolic Software for Effective Homology Computation, Institut Fourier, Grenoble, 1997, https://www-fourier.ujf-grenoble.fr/~sergerar/Kenzo/EAT-program.zip.
[18] R.E. Stong, Finite topological spaces, Trans. Am. Math. Soc. 123 (2) (1966) 325–340.