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About the existence and uniqueness of solutions for some second-order nonlinear BVPs[☆]

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ABSTRACT

The significance of our work is to solve some second-order nonlinear boundary value problems. To do this, we take into account the equivalence of the problems considered with certain integral equations, we will obtain a fixed-point-type result for these integral equations. This result provides us the existence and uniqueness of solutions for the second-order nonlinear boundary value problems considered. As a novelty, we will use for this fixed-point-type result a family of third order iterative processes to approximate the solution, instead of the usually considered method of Successive Approximations of linear convergence.

1. Introduction

In Science and Engineering, many real life challenging problems require the solution of the following nonlinear equation

$$\mathcal{H}(v) = 0 \tag{1}$$

where $\mathcal{H} : \Delta \subset \mathcal{Y} \to \mathcal{Z}$ is a continuously differentiable Fréchet operator in the nonempty convex domain Δ of the Banach space $\mathcal{Y}$ to the Banach space $\mathcal{Z}$. In general, these kind of equations can be found in various fields such as Optimization,

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Elasticity, Equilibrium theory, etc., in the form of differential equations, matrix equations, integral equations or system of nonlinear equations (see [1,2]). A great variety of the above mentioned applied and real problems are modeled by boundary value problems. The exact solution of these equations are rarely found in their analytical form, thus we need to apply iterative processes to approximate it. In addition, nowadays the use of computers to program these algorithms is now giving to this research area an important treatment, providing very efficient and accurate results (see [7,8]). An important class of problems for ordinary differential equations is to find a solution $v^* \in \mathcal{C}^2([\alpha,\beta])$ of the second-order differential equation

$$\frac{d^2v(s)}{ds^2} - H(v(s)) = 0 \tag{2}$$

that satisfies the boundary conditions

$$v(\alpha) = P, \qquad v(\beta) = Q, \tag{3}$$

where $H(v)$ is a differentiable operator (see [11]).

For this purpose, we define a new operator $\widetilde{\mathcal{H}} : \mathcal{C}^2([\alpha,\beta]) \longrightarrow \mathcal{C}([\alpha,\beta])$ as follows:

$$[\widetilde{\mathcal{H}}(v)](s) = \frac{d^2v(s)}{ds^2} - H(v(s)), \tag{4}$$

thus, as in (1), we set out to locate a solution v^* of the equation $\widetilde{\mathcal{H}}(v) = 0$, such that the boundary conditions $v(\alpha) = P$ and $v(\beta) = Q$ are satisfied.

A common way to approximate a solution for the previous equations is the fixed point method (see [4]), for this we write the equation $\widetilde{\mathcal{H}}(v) = 0$ as $v = \mathcal{G}(v)$ with $\mathcal{G}(v) = v - \widetilde{\mathcal{H}}(v)$. Next, if $\mathcal{G}$ is contractive in a domain convex and compact set Ω, then the sequence of Successive Approximations, generated by $v_{n+1} = \mathcal{G}(v_n)$, $n \geq 0$, converges to the unique fixed point v^* for the operator $\mathcal{G}$, which is a solution of the equation $\widetilde{\mathcal{H}}(v) = 0$. Therefore, when $\widetilde{\mathcal{H}}$ is given by (4), v^* is a solution of second-order boundary value problem given by (2)-(3). This result has two outstanding properties: the Successive Approximations method is globally convergent in Ω and guarantees the existence and uniqueness of a fixed point. However, this iterative process has linear rate of convergence.

The main objective of this work is to prove the existence and uniqueness of a solution of the boundary value problem given by (2)-(3) in a closed ball, which provides us with a favorable location of the said solution. In addition, we will prove that a family of iterative processes, with cubic convergence, globally converges to the solution in the said closed ball. Obtaining in this way a fixed-point-type result.

Turning to the generic equation given by (1), Ezquerro and Hernández-Verón [5] have presented the family of third order iterative processes which iteration functions is as follows:

$$\begin{cases} v_0 \text{ given in } \Delta, \\ s_n = v_n - \Upsilon_n \mathcal{H}(v_n) \\ t_n = v_n + q(s_n - v_n), \qquad q \in (0,1], \\ v_{n+1} = s_n - \dfrac{1}{q^2}\Upsilon_n((q-1)\mathcal{H}(v_n) + \mathcal{H}(t_n)), \quad n \geq 0, \end{cases} \tag{5}$$

where $\Upsilon_n = [\mathcal{H}'(v_n)]^{-1}$. This family depends upon the parameter $q \in (0,1]$, when $q = 1$ it becomes the two step Newton's method with frozen derivative which is more efficient than Chebyshev and Newton method. We will use this family of iterative processes with cubic convergence (see [5]) for obtaining a closed ball of global convergence for the solution of the problem.

The paper is structured as follows. In Section 2, we present some preliminaries where we include the relationship of the boundary value problems considered with certain integral equations. In Section 3, we analyze the global convergence for the family of iterative processes (5), using recurrence relations under a ω-condition. Moreover, the existence and uniqueness theorem and the domain of global convergence for the solution is also provided. In Section 4, a fixed-point-type result is obtained. In Section 5, to show the efficacy of our work, numerical examples are solved. In them, we study the existence and uniqueness of solutions for two nonlinear second-order boundary value problems. In Section 6, the conclusion of the work is included.

In addition, we indicate $\overline{B(v,\varrho)} = \{s \in \mathcal{Y};\ \|s - v\| \leq \varrho\}$ and $B(v,\varrho) = \{s \in \mathcal{Y};\ \|s - v\| < \varrho\}$ for the closed and open balls with center in v and radius $\varrho > 0$ respectively.

2. Preliminaries

It is known [10], that solving the previous boundary value problem, given by (2)-(3), is equivalent to solving a Fredholm integral equation of the form:

$$v(s) = \frac{Q-P}{\beta-\alpha}s + \frac{\beta P - \alpha Q}{\beta - \alpha} - \int_\alpha^\beta K(s,t)H(v(t))\,dt, \tag{6}$$

where the kernel K is the Green function in $[\alpha,\beta]\times[\alpha,\beta]$:

$$K(s,t)=\begin{cases}\dfrac{(\beta-s)(t-\alpha)}{\beta-\alpha}, & t\le s,\\ \dfrac{(s-\alpha)(\beta-t)}{\beta-\alpha}, & s\le t.\end{cases} \tag{7}$$

Therefore, we can consider the operator $\mathcal{H}:\mathcal{C}([\alpha,\beta])\longrightarrow\mathcal{C}([\alpha,\beta])$ such that

$$[\mathcal{H}(\upsilon)](s)=\upsilon(s)-\frac{Q-P}{\beta-\alpha}s-\frac{\beta P-\alpha Q}{\beta-\alpha}+\int_\alpha^\beta K(s,t)\mathcal{N}(\upsilon)(t))\,dt, \tag{8}$$

where the operator $\mathcal{N}:\mathcal{C}^2([\alpha,\beta])\to\mathcal{C}^2([\alpha,\beta])$, such that $\mathcal{N}(\upsilon)(t)=H(\upsilon(t)$, is called the Nemystkii operator. Thus, as in (1), we set out to locate a solution υ^* of the equation $\mathcal{H}(\upsilon)=0$, with $\mathcal{H}$ given by (8). As a result, we obtain a solution $\upsilon^*\in\mathcal{C}^2([\alpha,\beta])$ of the boundary value problem given by (2) and also satisfies the boundary condition (3).

Suppose that υ^* is solution of the equation $\mathcal{H}(\upsilon)=0$, with $\mathcal{H}$ given by (8). On one hand, as $K(\alpha,t)=0$ and $K(\beta,t)=0$, from (6) we have that $\upsilon^*(\alpha)=P$ and $\upsilon^*(\beta)=Q$. Thus, the boundary conditions (3) are satisfied for the solution υ^*.

On the other hand, from definition of Green function and (6), we have

$$\begin{aligned}\upsilon^*(s)&=\frac{Q-P}{\beta-\alpha}s+\frac{\beta P-\alpha Q}{\beta-\alpha}-\int_\alpha^\beta K(s,t)H(\upsilon^*(t))\,dt\\&=\frac{Q-P}{\beta-\alpha}s+\frac{\beta P-\alpha Q}{\beta-\alpha}-\int_\alpha^s\frac{(\beta-s)(t-\alpha)}{\beta-\alpha}H(\upsilon^*(t))\,dt-\int_s^\beta\frac{(s-\alpha)(\beta-t)}{\beta-\alpha}H(\upsilon^*(t))\,dt\\&=\frac{Q-P}{\beta-\alpha}s+\frac{\beta P-\alpha Q}{\beta-\alpha}-\frac{(\beta-s)}{\beta-\alpha}\int_\alpha^s(t-\alpha)H(\upsilon^*(t))\,dt-\frac{(s-\alpha)}{\beta-\alpha}\int_s^\beta(\beta-t)H(\upsilon^*(t))\,dt.\end{aligned}$$

Next, by applying the Fundamental Theorem of Calculus, we derive the previous expression and we have

$$\begin{aligned}(\upsilon^*)'(s)&=\frac{Q-P}{\beta-\alpha}+\frac{1}{\beta-\alpha}\int_\alpha^s(t-\alpha)H(\upsilon^*(t))\,dt-\frac{(\beta-s)}{\beta-\alpha}(s-\alpha)H(\upsilon^*(s))\\&\quad-\frac{1}{\beta-\alpha}\int_s^\beta(\beta-t)H(\upsilon^*(t))\,dt+\frac{(s-\alpha)}{\beta-\alpha}(\beta-s)H(\upsilon^*(s))\\&=\frac{Q-P}{\beta-\alpha}+\frac{1}{\beta-\alpha}\int_\alpha^s(t-\alpha)H(\upsilon^*(t))\,dt-\frac{1}{\beta-\alpha}\int_s^\beta(\beta-t)H(\upsilon^*(t))\,dt.\end{aligned}$$

Then, deriving again, we obtain

$$(\upsilon^*)''(s)=\frac{(s-\alpha)}{\beta-\alpha}H(\upsilon^*(s))+\frac{(\beta-s)}{\beta-\alpha}H(\upsilon^*(s))=H(\upsilon^*(s)).$$

So, the solution υ^* verifies that $\upsilon^*\in\mathcal{C}^2([\alpha,\beta])$ and also satisfies the boundary conditions (3).

Therefore, our goal is to prove a global convergence result, for the family of iterative processes (5), applied to the equation $\mathcal{H}(\upsilon)=0$ with $\mathcal{H}$ given by (8). Moreover, from this result, we obtain a result of fixed-point-type for the boundary value problem given by (2)-(3). So, from now and continuing with the notation established in (1), we consider Δ a nonempty convex domain in the Banach space $\mathcal{Y}=\mathcal{C}([\alpha,\beta])$ and the the Banach space $\mathcal{Z}=\mathcal{C}([\alpha,\beta])$. We consider the space $\mathcal{C}([\alpha,\beta])$ with the max-norm.

Ezquerro and Hernández-Verón [6]

$$[\mathcal{H}(\upsilon)](s)=\upsilon(s)-\frac{Q-P}{\beta-\alpha}s-\frac{\beta P-\alpha Q}{\beta-\alpha}+\int_\alpha^\beta K(s,t)\left(\sum_{l=1}^n H_l(\upsilon(t))\right)dt, \tag{9}$$

where each H_l, for $l=1,2,\dots,n$, satisfies $\|H_l'(x)-H_l'(y)\|\le N_l\|x-y\|^{u_l}$ with $u_l\in(0,1]$ and $x,y\in\Delta\subset\mathcal{C}([\alpha,\beta])$.

In this case, for each $y\in\Delta$, we have

$$[\mathcal{H}'(\upsilon)y](s)=y(s)+\int_\alpha^\beta K(s,\theta)\left(\sum_{l=1}^n H_l'(\upsilon(\theta))y(\theta)\right)d\theta,$$

then, we obtain

$$\|\mathcal{H}'(\upsilon)-\mathcal{H}'(t)\|\le M\sum_{l=1}^n N_l\|\upsilon-t\|^{u_l},\quad N_l\ge 0,u_l\in[0,1],\ \text{for all } \upsilon,t\in\Delta,$$

where $M=\max_{s\in[\alpha,\beta]}|\int_\alpha^\beta K(s,\theta)d\theta|$. Clearly, for the operator $\mathcal{H}$, Lipschitz and Hölder continuous conditions fail but it satisfies the following ω-condition [3,13]:

$$\|\mathcal{H}'(\upsilon)-\mathcal{H}'(s)\|\le\omega(\|\upsilon-s\|),\ \text{for}\,\upsilon,s\in\Delta, \tag{10}$$

where $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing continuous function such that $\omega(p\theta) \le p^u \omega(\theta)$ for $u \in (0,1]$, $\theta \in (0,\infty)$ and $p \in [0,1]$.

Our work is devoted to deal with the global convergence analysis of the family given in (5) under ω-condition on $\mathcal{H}'$.

3. Global convergence analysis for the family (5)

Our aim is to analyze the domain of restricted global convergence for family (5). For this, we consider $\mathcal{H} : \Delta \subset \mathcal{Y} \to \mathcal{Z}$ a continuously differentiable Fréchet operator in the nonempty convex domain Δ of the Banach space $\mathcal{Y}$ to the Banach space $\mathcal{Z}$. Next, we introduce the following assumptions:

(I) There exists $\tilde{\Upsilon} = [\mathcal{H}'(\tilde{v})]^{-1}$ with $\|\tilde{\Upsilon}\| \le \xi$ and $\|\tilde{\Upsilon}\mathcal{H}(\tilde{v})\| \le \eta$, for some $\tilde{v} \in \Delta$.
(II) $\mathcal{H}'$ satisfies the ω-condition given by (10).

If we consider condition (II), it is easy to observe that

$$\|\mathcal{H}'(v) - \mathcal{H}'(\tilde{v})\| \le \tilde{\omega}(\|v - \tilde{v}\|), \text{ for all } v \in \Delta$$

with $\tilde{\omega} \le \omega$, once $\tilde{v}$ is fixed in Δ. Now, to prove the global convergence for the family of iterative processes given by (5), first we describe some identities in the next Lemma.

Lemma 1. *Under notation and conditions (I) and (II) set in previous section the following items are satisfied:*

(i) $\mathcal{H}(v_0) = \mathcal{H}(\tilde{v}) + \mathcal{H}'(\tilde{v})(v_0 - \tilde{v}) + \int_{\tilde{v}}^{v_0} (\mathcal{H}'(v) - \mathcal{H}'(\tilde{v}))dv$ *with* $v_0 \in \Delta$.

(ii) $\mathcal{H}(t_n) = (1-q)\mathcal{H}(v_n) + \int_{v_n}^{t_n} (\mathcal{H}'(v) - \mathcal{H}'(v_n))dv$ *with* $v_n, s_n \in \Delta$.

(iii) $\mathcal{H}(v_{n+1}) = \mathcal{H}(s_n) + \mathcal{H}'(s_n)(v_{n+1} - s_n) + \int_{s_n}^{v_{n+1}} (\mathcal{H}'(v) - \mathcal{H}'(s_n))dv$ *with* $s_n, v_{n+1} \in \Delta$.

(iv) $\mathcal{H}(v_n) + \mathcal{H}'(v_n)(\tilde{v} - v_n) = \mathcal{H}(\tilde{v}) - \int_{v_n}^{\tilde{v}} (\mathcal{H}'(v) - \mathcal{H}'(v_n))dv$, *with* $v_n \in \Delta$.

Proof. The proof is immediate, hence the details are skipped. □

Furthermore, let us assume that there exist $\varrho > 0$ such that

$$\xi\tilde{\omega}(\varrho) < 1 \tag{11}$$

and $v \in \overline{B(\tilde{v},\varrho)} \subset \Delta$. Now, from $\|I - \tilde{\Upsilon}\mathcal{H}'(v)\| \le \|\tilde{\Upsilon}(\mathcal{H}'(\tilde{v}) - \mathcal{H}'(v))\| \le \xi\tilde{\omega}(\varrho) < 1$, by applying Banach's Lemma [9], we get

$$\|[\mathcal{H}'(v)]^{-1}\| \le \frac{\xi}{1-\xi\tilde{\omega}(\varrho)} = \tau \qquad \text{and} \qquad \|[\mathcal{H}'(v)]^{-1}\tilde{\Upsilon}\| \le \frac{1}{1-\xi\tilde{\omega}(\varrho)}. \tag{12}$$

3.1. Recurrence relations

Next, through conditions (I) and (II), we build the recurrence relations that will allow us to ensure the global convergence of the family of iterative processes given in (5).

If $v_0 \in \overline{B(\tilde{v},\varrho)}$, using (11) and (12), as a consequence of (i) and (iv) of Lemma 1, we have

$$\begin{aligned}\|s_0 - v_0\| &\le \|\Upsilon_0\mathcal{H}'(\tilde{v})\|\|\tilde{\Upsilon}\mathcal{H}(v_0)\| \le \frac{1}{1-\xi\tilde{\omega}(\varrho)}\left(\eta + \varrho + \frac{\xi\varrho\tilde{\omega}(\varrho)}{1+u}\right)\\ &\le \frac{(1+u)(\eta+\varrho) + \xi\varrho\tilde{\omega}(\varrho)}{(1+u)(1-\xi\tilde{\omega}(\varrho))} = \delta.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|s_0 - \tilde{v}\| &= \|-\Upsilon_0(\mathcal{H}(v_0) + \mathcal{H}'(v_0)(\tilde{v} - v_0))\|\\ &\le \|\Upsilon_0\mathcal{H}'(\tilde{v})\|\|\tilde{\Upsilon}\mathcal{H}(\tilde{v})\| + \frac{\|\Upsilon_0\|\omega(\|v_0 - \tilde{v}\|)}{1+u}\|v_0 - \tilde{v}\|\\ &\le \frac{\eta}{(1-\xi\tilde{\omega}(\varrho))} + \frac{\xi\varrho\omega(\varrho)}{(1+u)(1-\xi\tilde{\omega}(\varrho))} \le \frac{(1+u)\eta + \xi\varrho\omega(\varrho)}{(1+u)(1-\xi\tilde{\omega}(\varrho))}.\end{aligned}$$

Thus, $s_0 \in \overline{B(\tilde{v},\varrho)}$, if

$$\frac{(1+u)\eta + \xi\varrho\omega(\varrho)}{(1+u)(1-\xi\tilde{\omega}(\varrho))} \le \varrho. \tag{13}$$

In addition,

$$\|t_0 - v_0\| = q\|s_0 - v_0\| \le q\delta,$$
$$\|t_0 - \tilde{v}\| \le (1-q)\|v_0 - \tilde{v}\| + q\|s_0 - \tilde{v}\| \le \varrho.$$

Hence, $t_0 \in \overline{B(\tilde{v}, \varrho)}$, if (13) holds. Moreover,

$$v_1 - v_0 = s_0 - v_0 - \frac{1}{q^2}\Upsilon_0((q-1)\mathcal{H}(v_0) + \mathcal{H}(t_0)).$$

Again using (ii) and (iv) of Lemma 1, we get

$$v_1 - v_0 = s_0 - v_0 - \frac{1}{q^2}\Upsilon_0\Big(\int_0^1 (\mathcal{H}'(v_0 + \theta(t_0 - v_0)) - \mathcal{H}'(v_0))d\theta\Big).$$

Therefore,

$$\|v_1 - v_0\| \le \|s_0 - v_0\| + \frac{\xi\omega(\delta)}{q^{1-u}(1-\xi\tilde{\omega}(\varrho))(1+u)}\|s_0 - v_0\|$$
$$\le \Big(1 + \frac{\tau\omega(\delta)}{(1+u)q^{1-u}}\Big)\|s_0 - v_0\| \le \Big(1 + \frac{b_0}{1+u}\Big)\|s_0 - v_0\|,$$

where $\tau = \dfrac{\xi}{1-\xi\tilde{\omega}(\varrho)}$ and

$$\|v_1 - \tilde{v}\| \le \|s_0 - \tilde{v}\| + \frac{b_0}{1+u}\|s_0 - v_0\|$$
$$\le \frac{(1+u)\eta + \xi\varrho\omega(\varrho)}{(1+u)(1-\xi\tilde{\omega}(\varrho))} + \frac{b_0\delta}{1+u},$$

where $b_0 = \dfrac{\tau\omega(\delta)}{q^{1-u}}$. Hence, $v_1 \in \overline{B(\tilde{v}, \varrho)}$, if

$$\frac{(1+u)\eta + \xi\varrho\omega(\varrho)}{(1+u)(1-\xi\tilde{\omega}(\varrho))} + \frac{b_0\delta}{1+u} \le \varrho. \tag{14}$$

Notice that (13) holds if (14) is true. Now,

$$\|s_1 - v_1\| \le \|\Upsilon_1\|\|\mathcal{H}(v_1)\|$$
$$\le \tau\Big(\frac{\omega(\delta)}{1+u} + (q^{1-u}b_0 + 1)\frac{\omega(\delta)}{(1+u)q^{1-u}} + \frac{b_0^{1+u}\omega(\delta)}{(1+u)^{2+u}}\Big)\|s_0 - v_0\|$$
$$\le \Big(\frac{q^{1-u}b_0}{(1+u)} + \frac{(q^{1-u}b_0+1)b_0}{(1+u)} + \frac{q^{1-u}b_0^{2+u}}{(1+u)^{2+u}}\Big)\|s_0 - v_0\|$$
$$\le (q^{1-u}b_0 + (q^{1-u}b_0 + 1)b_0 + q^{1-u}b_0^{2+u})\|s_0 - v_0\|$$
$$\le (b_0 + (b_0+1)b_0 + b_0^2)\|s_0 - v_0\|.$$

Thus, we have $\|s_1 - v_1\| \le f(b_0)\|s_0 - v_0\|$, where $f(t) = 2t^2 + 2t$. Let $b_1 = f(b_0)^u b_0$ and define the sequence as

$$b_n = b_{n-1}f(b_{n-1})^u.$$

As, f(t) is an increasing function for $t > 0$, it follows that

$$\|s_1 - \tilde{v}\| \le \frac{(1+u)\eta + \xi\varrho\omega(\varrho)}{(1+u)(1-\xi\tilde{\omega}(\varrho))}$$
$$\|t_1 - v_1\| \le qf(b_0)\|s_0 - v_0\|$$
$$\|t_1 - \tilde{v}\| \le \varrho$$
$$\|v_2 - v_1\| \le \Big(1 + \frac{b_1}{1+u}\Big)\|s_1 - v_1\|$$
$$\|v_2 - \tilde{v}\| \le \frac{(1+u)\eta + \xi\varrho\omega(\varrho)}{(1+u)(1-\xi\tilde{\omega}(\varrho))} + \frac{b_1\delta}{1+u}.$$

Thus, $s_1, t_1, v_2 \in \overline{B(\tilde{v}, \varrho)}$ if

$$\frac{(1+u)\eta + \xi\varrho\omega(\varrho)}{(1+u)(1-\xi\tilde{\omega}(\varrho))} + \frac{b_1\delta}{1+u} \le \varrho.$$

We observe that this condition holds if (14) holds and $f(b_0) < 1$, since $\{b_n\}$ is a decreasing sequence. In addition, $f(b_0) < 1$ if $b_0 < 0.3660$.

Our next aim is to ensure that the iterative family of iterative processes defined in (5) is well defined. For this, we consider the following result.

Lemma 2. *Consider $f(t) = 2t^2 + 2t$ a scalar function. If there exists $\varrho > 0$ such that $b_0 < 0.3660$ and condition (14) holds, then the following relations hold for all $n \geq 1$.*

$$\begin{aligned}
&\|s_n - v_n\| \leq f(b_{n-1})\|s_{n-1} - v_{n-1}\| \\
&\|s_n - \tilde{v}\| \leq \frac{(1+u)\eta + \xi\varrho\omega(\varrho)}{(1+u)(1-\xi\tilde{\omega}(\varrho))} \\
&\|t_n - v_n\| \leq q f(b_{n-1})\|s_{n-1} - v_{n-1}\| \\
&\|v_{n+1} - v_n\| \leq \left(1 + \frac{b_n}{1+u}\right)\|s_n - v_n\| \\
&\|v_{n+1} - \tilde{v}\| \leq \frac{(1+u)\eta + \xi\varrho\omega(\varrho)}{(1+u)(1-\xi\tilde{\omega}(\varrho))} + \frac{b_0}{1+u}\left(\prod_{i=0}^{n-1} f(b_i)^{1+u}\right)\delta.
\end{aligned} \tag{15}$$

Proof. The recurrence relations can be established easily by using mathematical induction. □

Note that condition (14) is generally satisfied for several values of ϱ but we accept the most favourable value. Besides that, larger value of ϱ provides the domain of global convergence and smaller value provides the best location for the solution.

3.2. Global convergence study

Under assumptions (I) and (II), we prove the existence and uniqueness theorem by using Lemma 1, Lemma 2 and previously derived recurrence relations.

Theorem 1. *Let us assume that (I) and (II) hold, for some $\tilde{v} \in \Delta$. Suppose that there exists $\varrho > 0$ which satisfies conditions (11), (14), such that $\overline{B(\tilde{v}, \varrho)} \subset \Delta$. If $b_0 < 0.3660$, then there is only one solution v^* of $\mathcal{H}(v) = 0$ in the domain $\overline{B(\tilde{v}, \varrho)} \subset \Delta$. Moreover, the sequence $\{v_n\}$ generated by (5) is well-defined and converges to the solution v^* from every starting point $v_0 \in B(\tilde{v}, \varrho)$.*

Proof. To prove the global convergence, we prove that the sequence $\{v_n\}$ generated by (5) is a Cauchy sequence. By using (15), we get

$$\|v_{n+m} - v_n\| \leq \sum_{i=n}^{n+m-1} \|v_{i+1} - v_i\| < (1 + \frac{b_0}{1+u}) f(b_0)^n \delta \left(\frac{1 - f(b_0)^m}{1 - f(b_0)}\right).$$

Thus, $\{v_n\}$ is a Cauchy sequence if $f(b_0) < 1$. Further,

$$\|\mathcal{H}(v_n)\| \leq \|\mathcal{H}'(v_n)\| f(b_0)^n \|s_0 - v_0\|$$

and $f(b_0) < 1$, hence $\|\mathcal{H}(v_n)\| \to 0$ as $n \to \infty$. Therefore, there exists v^* such that $lim_{n\to\infty} v_n = v^*$. On the other hand, as $\|\mathcal{H}'(v_n)\| \leq \|\mathcal{H}'(\tilde{v})\| + \tilde{\omega}(\varrho)$ is bounded, we obtain $\mathcal{H}(v^*) = 0$ by using the continuity of $\mathcal{H}$ in the domain $\overline{B(\tilde{v}, \varrho)}$.

To prove the uniqueness, we suppose that t^* is another solution of $\mathcal{H}(v) = 0$ in $\overline{B(\tilde{v}, \varrho)}$. Then, from

$$0 = \tilde{\Upsilon}(\mathcal{H}(t^*) - \mathcal{H}(v^*)) = \left(\int_0^1 \tilde{\Upsilon}\mathcal{H}'(v^* + \theta(t^* - v^*))d\theta\right)(t^* - v^*) = \mathcal{T}(t^* - v^*).$$

If operator $\mathcal{T} = \int_0^1 \tilde{\Upsilon}\mathcal{H}'(v^* + \theta(t^* - v^*))d\theta$ is invertible, we get $v^* = t^*$. So, we consider,

$$\begin{aligned}
\|I - \mathcal{T}\| &\leq \|\tilde{\Upsilon}\| \| \int_0^1 \mathcal{H}'(\tilde{v}) - \mathcal{H}'(v^* + \theta(t^* - v^*))d\theta\| \\
&\leq \xi \int_0^1 \tilde{\omega}(\|\tilde{v} - v^* - \theta(t^* - v^*)\|)d\theta \\
&\leq \xi \int_0^1 \tilde{\omega}((1-\theta)\|\tilde{v} - v^*\| + \theta\|\tilde{v} - t^*\|)d\theta \\
&\leq \xi\tilde{\omega}(\varrho) < 1.
\end{aligned}$$

Thus, by applying Banach lemma, $\mathcal{T}$ is invertible and hence uniqueness follows. □

The following result provides us with a generalization of the uniqueness result seen in the previous theorem.

Theorem 2. *Under conditions of Theorem 1, if the scalar equation*

$$2\xi\tilde{\omega}(\varrho+\epsilon)\left(1-\frac{1}{2^{1+u}}\right)=1+u$$

has at least one positive root and we denote $\tilde{\epsilon}$ the smallest one, then the solution v^* is unique in $B(\tilde{v},\tilde{\epsilon})\cap\Delta$.

Proof. Proceeding as in the proof of uniqueness seen above, we have that

$$\begin{aligned}
\|I-\mathcal{T}\| &\le \|\tilde{\Upsilon}\|\|\int_0^1 \mathcal{H}'(\tilde{v})-\mathcal{H}'(v^*+\theta(t^*-v^*))d\theta\|\\
&\le \xi\int_0^1 \tilde{\omega}(\|\tilde{v}-v^*-\theta(t^*-v^*)\|)d\theta\\
&\le \xi\int_0^1 \tilde{\omega}((1-\theta)\|\tilde{v}-v^*\|+\theta\|t^*-v^*\|)d\theta\\
&< \xi\int_0^{1/2}(1-\theta)^u\tilde{\omega}(\varrho+\tilde{\epsilon})d\theta+\int_{1/2}^1\theta^u\tilde{\omega}(\varrho+\tilde{\epsilon})d\theta\\
&= 2\xi\tilde{\omega}(\varrho+\tilde{\epsilon})\int_{1/2}^1\theta^u d\theta\\
&= \frac{2\xi}{1+u}\tilde{\omega}(\varrho+\tilde{\epsilon})\left(1-\frac{1}{2^{1+u}}\right)=1.
\end{aligned}$$

Thus, by appliyng Banach lemma, $\mathcal{T}$ is invertible and hence the uniqueness follows. □

4. A fixed-point-type result for BVP given by (2) and (3)

Now, by bearing in mind that solving the previous boundary value problem, given by (2) and (3), is equivalent to solving a Fredholm integral equation (6) and, therefore, solving the equation $\mathcal{H}(v)=0$, with $\mathcal{H}$ given by (9), we can apply the global convergence theorem seen in Theorem 1 and we get the following result.

Theorem 3. *Let us assume that (I) and (II) hold, for some $\tilde{v}\in\Delta$. Suppose that there exist $\varrho>0$ which satisfies conditions (11), (14), such that $\overline{B(\tilde{v},\varrho)}\subset\Delta$. If $b_0<0.3660$, then there is only one solution v^* of the boundary value problem, given by (2)-(3), in the domain $\overline{B(\tilde{v},\varrho)}\subset\Delta$. Moreover, the family of iterative processes defined in (5) is well-defined and converges to the solution v^* from every starting point $v_0\in B(\tilde{v},\varrho)$.*

5. Numerical Examples

To show the applicability of our theoretical results we consider the following examples.

Example 1. We consider the boundary value problem given by

$$\begin{cases}\dfrac{d^2v(s)}{ds^2}+v(s)^{\frac{3}{2}}=0\\ v(0)=1 \text{ and } v(1)=1.\end{cases} \tag{16}$$

As we have already indicated, its resolution is equivalent to solve the integral equation given by

$$v(s)=1+\int_0^1 K(s,\theta)v(\theta)^{\frac{3}{2}}d\theta \tag{17}$$

where, $K(s,\theta)$ is Green's function defined as

$$K(s,\theta)=\begin{cases}s(1-\theta), & s\le\theta\\ \theta(1-s), & \theta\le s,\end{cases}$$

(see [3,13]).

It is evident that approximate a solution of the equation (17) can be done by approximating a solution of the equation $\mathcal{H}(v)=0$, where $\mathcal{H}:\mathcal{C}[0,1]\to\mathcal{C}[0,1]$, such that

$$[\mathcal{H}(v)](s)=v(s)-1-\int_0^1 K(s,\theta)v(\theta)^{\frac{3}{2}}d\theta. \tag{18}$$

Table 1
Radii obtained for different values of q.

q	Radius of existence ball	Radius of the global convergence ball
1	0.25575	0.90764
0.8	0.26177	0.83866
0.6	0.27955	0.68881
0.4	0.31635	0.51815

Table 2
Distance between two consecutive iterations generated by (5) for different values of q.

n	$q = 0.3$	$q = 0.6$	$q = 0.9$
1	$1.47807900402806 \times 10^{-1}$	$1.47807900402806 \times 10^{-1}$	$1.47807900402806 \times 10^{-1}$
2	$2.948606526873643 \times 10^{-6}$	$3.071277267672528 \times 10^{-6}$	$8.915844769379317 \times 10^{-6}$
3	$1.856486214335986 \times 10^{-20}$	$2.708398700798353 \times 10^{-20}$	$1.746418271005370 \times 10^{-18}$
4	$4.935422468608716 \times 10^{-63}$	$1.857505120219090 \times 10^{-62}$	$1.305810745902912 \times 10^{-56}$
5	$9.398733808695364 \times 10^{-191}$	$5.991516646112165 \times 10^{-189}$	$5.456358006477081 \times 10^{-171}$

Therefore,

$$[\mathcal{H}'(v)w](s) = w(s) - \frac{3}{2}\int_0^1 K(s,\theta)v(\theta)^{\frac{1}{2}}w(\theta)d\theta$$

satisfying that

$$\|\mathcal{H}'(v) - \mathcal{H}'(s)\| \le \frac{3}{16}\left(\|v - s\|^{\frac{1}{2}}\right)$$

Clearly, Lipschitz condition fails but it satisfies the Hölder's condition. For $\tilde{v}(s) = 1$, we get $\|I - \mathcal{H}'(\tilde{v})\| \le \frac{3}{16}$. Using Banach Lemma, we get $\xi = 16/13$ and $\eta = 2/13$ so conditions (11) and (14) holds for $\varrho \in [0.25575, 1.41179]$.

For $q = 1$, $b_0 < 0.3660$ holds for $\varrho \in [0.25575, 0.90764]$. Thus, conditions of Theorem 3 are satisfied and as a result there exists only one solution v^* of BVP given by (16) in $B(1, \varrho)$ for $\varrho \in [0.25575, 0.90764]$. Moreover, sequence generated by the family (5) is well defined and converges to v^* from any starting point in $B(1, \varrho)$ for $\varrho \in [0.25575, 0.90764]$.

Hence, the optimal ball of existence for the location of v^* is $B(1, 0.25575)$ and the optimal ball for the global convergence is $B(1, 0.90764)$. In addition, from Theorem 2, the optimal ball of uniqueness for v^* is $B(1, 28.2153)$.

As we can see in Table 1, by increasing the value of q, the convergence domains decreases (best location) and global convergence domains increases.

Next, to obtain the numerical solution of (18), we discretize the problem by approximating the integral by Gauss Legendre quadrature formula using eight nodes, that is:

$$\int_0^1 \mathcal{F}(\theta)d\theta \approx \sum_{l=1}^{8} W_l \mathcal{F}(\theta_l),$$

where, θ_l and W_l are the nodes and weights respectively. If we denote $v_l = v(\theta_l), l = 1, 2, 3, ..., 8$ then, (18) can be written as the following nonlinear system of $8x8$

$$v_i = 1 + \sum_{l=1}^{8} b_{il} v_l^{\frac{3}{2}}, \quad \text{for } i = 1, 2, ..., 8,$$

where, $b_{il} = \begin{cases} W_l\theta_l(-\theta_i + 1), & l \le i \\ W_l\theta_i(-\theta_l + 1), & i \le l. \end{cases}$

We denote the nth iteration defined by iterative method (1) with $v^{(n)} = (v_1^{(n)}, v_2^{(n)}, ..., v_8^{(n)})$ so, by taking the initial approximation $v^{(0)} = (1, 1, 1, 1, 1, 1, 1, 1)'$, in Table 2, we can see the distances between iterations, $\|v^{(n+1)} - v^{(n)}\|$ until the tolerance required is obtained for different values of q, we show the distances for the first six iterations. We can observe that as the value of q increases, the speed of convergence of iterative scheme given by (5) decreases slightly.

Example 2. We consider the boundary value problem given by

$$\begin{cases} \dfrac{d^2v(s)}{ds^2} + v(s)^{\frac{7}{5}} + \dfrac{v(s)^2}{10} = 0 \\ v(0) = 1 \text{ and } v(1) = 1. \end{cases} \tag{19}$$

Table 3
Radii obtained for different values of q.

q	Radius of existence ball	Radius of the global convergence ball
1	0.25575	0.90764
0.9	0.43375	0.67405
0.8	0.53290	0.59704

As we have already indicated, its resolution is equivalent to solve the integral equation given by

$$v(s) = 1 + \int_0^1 K(s,\theta)\Big(v(\theta)^{\frac{7}{5}} + \frac{v(\theta)^2}{10}\Big)d\theta \tag{20}$$

where, $K(s,\theta)$ is the Green's function defined as

$$K(s,\theta) = \begin{cases} s(1-\theta), & s \le \theta \\ \theta(1-s), & \theta \le s, \end{cases}$$

(see [12,13]).

Now, solving the equation (17) is done by considering $\mathcal{H}(v) = 0$, where $\mathcal{H} : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$, is given by

$$[\mathcal{H}(v)](s) = v(s) - 1 - \int_0^1 K(s,\theta)\Big(v(\theta)^{\frac{7}{5}} + \frac{v(\theta)^2}{10}\Big)d\theta$$

Therefore,

$$[\mathcal{H}'(v)w](s) = w(s) - \int_0^1 K(s,\theta)\Big(\frac{7}{5}v(\theta)^{\frac{2}{5}} + \frac{v(\theta)}{5}\Big)w(\theta)d\theta,$$

and

$$\|\mathcal{H}'(v) - \mathcal{H}'(s)\| \le \frac{1}{40}\Big(7\|v-s\|^{\frac{2}{5}} + \|v-s\|\Big),$$

where $\omega(t) = \frac{1}{40}(7t^{\frac{2}{5}} + t)$. Clearly, Lipschitz and Hölder's conditions fails but it satisfies the ω-condition. We consider $\omega(t) = \tilde{\omega}(t)$.

For $\tilde{v}(s) = 1$, we get $\|I - \mathcal{H}'(\tilde{v})\| \le \frac{1}{5}$. Using Banach Lemma, we get $\xi = 5/4$ and $\eta = 11/64$.

The condition (14) holds for $\varrho \in [0.39669, 0.89998]$. Moreover, for all these values of ϱ, $\xi\tilde{\omega}(\varrho) < 1$.

Taking $q = 1$, $b_0 < 0.3660$ true for $\varrho \in [0.39669, 0.74758]$. Thus, all conditions of Theorem 3 are satisfied and as a result there exists only one solution v^* of BVP, given by (19), in $B(1,\varrho)$ for $\varrho \in [0.39669, 0.74758]$. Moreover, the sequence generated by (5) is well defined and converges to v^* from any starting point v_0 in $B(1,\varrho)$, with $\varrho \in [0.39669, 0.74758]$.

Therefore, the optimal ball of existence for the location of v^* is $B(1, 0.39669)$ and the optimal ball for global convergence is $B(1, 0.74758)$. Moreover, from Theorem 2, the optimal ball of uniqueness for v^* is $B(1, 14.8535)$. Again, we can see in Table 3, by increasing the value of q, the convergence domains decreases (best location) and global convergence domains increases.

Next, as in the previous example, to obtain the numerical solution, we approximate the integral by Gauss Legendre quadrature formula using eight nodes. In this case, the nonlinear system obtained is given by:

$$v_i = 1 + \sum_{l=1}^{8} b_{il}\Big(\frac{7}{5}v_l^{\frac{2}{5}} + \frac{v_l}{5}\Big), \quad \text{for } i = 1,2,\dots,8,$$

where, $b_{il} = \begin{cases} W_l\theta_l(-\theta_i + 1), & l \le i \\ W_l\theta_i(-\theta_l + 1), & i \le l. \end{cases}$

Consider the initial approximation $v^{(0)} = (v_1^{(0)}, v_2^{(0)}, \dots, v_8^{(0)}) = (1,1,1,1,1,1,1,1)'$ and $q = 1$, in this case the family (5) becomes two-step Newton's method. After four iterations with tolerance 10^{-150}, the iterative process converges to the solution v^*, given in Table 4.

1

In Table 5, we can see the distances between iterations, $\|v^{(n+1)} - v^{(n)}\|$ until the tolerance is obtained and the norm of the value of the nonlinear operator $\|\mathcal{H}(v^{(n)})\|$ at the approximated solution, as can see in the results after 5 iterations we find a solution that reach the required tolerance.

Table 4
Approximated solution υ^* of equation (17).

j	υ_j^*	j	υ_j^*
1	1.012770492741183	5	1.164801816936698
2	1.060750791269023	6	1.122189825669643
3	1.122189825669643	7	1.060750791269023
4	1.164801816936698	8	1.012770492741183

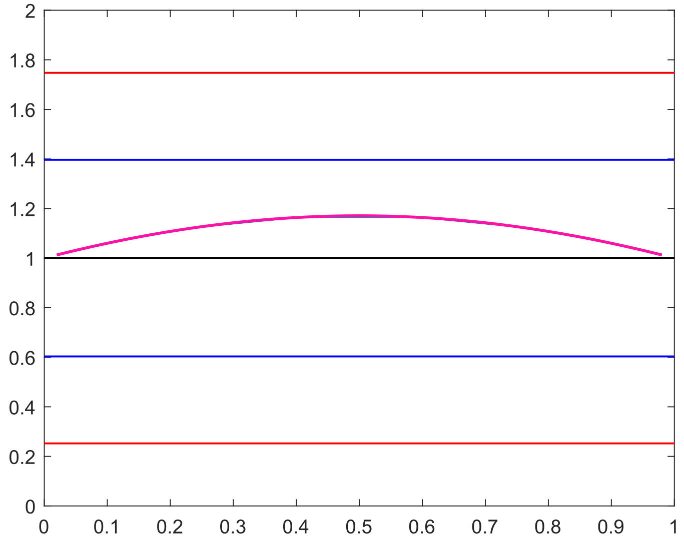

Fig. 1. Global convergence ball (red), Existence ball (blue), Interpolating solution (magenta).

Table 5
Numerical results obtained by the iterative process (5) for $q = 1$..

n	$\|\upsilon^{(n+1)} - \upsilon^{(n)}\|$	$\|\mathcal{H}(\upsilon^{(n)})\|$
1	$1.64786017364341 \times 10^{-1}$	$1.35914891280503 \times 10^{-1}$
2	$1.57995723568961 \times 10^{-5}$	$1.30256597428427 \times 10^{-5}$
3	$1.24725125073426 \times 10^{-17}$	$1.02886141216578 \times 10^{-17}$
4	$6.09824378478425 \times 10^{-54}$	$5.03074106559512 \times 10^{-54}$
5	$7.12352357855799 \times 10^{-163}$	$5.87657696678690 \times 10^{-163}$

6. Conclusions

In this work we have addressed the problem of obtaining a result of existence and uniqueness of solution for a special type of second-order nonlinear boundary value problems. Its characteristic is to be a fixed-point-type result, obtaining a global convergence domain through a family of iterative processes of cubic convergence, instead of the usually used method of Successive Approximations of linear convergence. To carry out this study, we have presented the global convergence of a family of third order iterative processes for solving nonlinear equations in Banach spaces. The existence and uniqueness results are established under ω-condition on the first Fréchet derivative of the operator, which is generalization of Lipschitz and Hölder's conditions. Thus, it covers several problems that the Lipschitz and Hölder continuity conditions fail.

Data availability

No data was used for the research described in the article.

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Further reading

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