Full Length Article

Mean convergence of Lagrange interpolation

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Abstract

In this note we prove mean convergence of Lagrange interpolation at the zeros of para-orthogonal polynomials for measures on the unit circle which do not belong to Szegő's class. When the measure is in Szegő's class, mean convergence of Lagrange interpolation is proved for functions in the disk algebra.
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1. Introduction

Mean convergence of Lagrange interpolation on zeros of orthogonal polynomials on the real line is a topic which has been extensively studied. Just to name a few works, let us recall the foundational paper of Erdös and Turán [6], the trilogy [15–17] of Nevai, and also [1,14,20]. Marcinkiewicz [13] showed that there is a strong connection between mean convergence of Lagrange interpolation and mean convergence of Fourier series.

Interpolation with nodes on the unit circle has been studied for functions in the disk algebra $A(\overline{\mathbb{D}})$ (i.e., functions analytic in the unit disk $\mathbb{D}$ which are continuous in $\overline{\mathbb{D}}$), mostly using unit roots or perturbations of these points (cf. [3,11,12,21]). Given a nontrivial finite positive Borel measure μ on the unit circle $\mathbb{T}$, the algebraic polynomials are dense in $L^2(\mu)$ if and only if $\log \mu' \notin L^1(\mathfrak{m})$, where $\mathfrak{m}$ is the Lebesgue measure in $\mathbb{T}$ and μ' is the Radon–Nikodym

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derivative of μ with respect to $\mathfrak{m}$. This is due to Szegő–Kolmogorov–Krein's theorem (see for example [7, Chapter 1] or [9, Chapter 3]). Therefore, since functions in $A(\overline{\mathbb{D}})$ are a uniform limit of algebraic polynomials in $\overline{\mathbb{D}}$, that could be one of the reasons to consider functions in $A(\overline{\mathbb{D}})$.

In this note we are concerned with the mean convergence of Lagrange interpolation at the zeros of para-orthogonal polynomials for measures on the unit circle. The zeros of para-orthogonal polynomials are the nodes of the Szegő's quadrature formula, which is the analogous on the unit circle to Gaussian quadrature formula (cf. [2,4,5,8,10]). These zeros share several properties with zeros of orthogonal polynomials with respect to a measure on the real line.

Consider a nontrivial finite positive Borel measure μ on the unit circle, and let $\{w_n\}_{n=0}^{\infty}$ be a sequence of points in $\mathbb{T}$. For each n, let $Z_n(w_n)$ denote the $n + 1$ zeros of para-orthogonal polynomial of degree $n + 1$ associated to w_n and μ. It is well known that $Z_n(w_n) \subset \mathbb{T}$. Given a function f defined in $\mathbb{T}$, let $L_n(f)$ denote the nth Lagrange interpolation polynomial of f with respect to $Z_n(w_n)$. We obtain mean convergence of $L_n(f)$ in two different settings.

Theorem 1.1. *Suppose that* $\log \mu' \notin L^1(\mathfrak{m})$. *If* $f \in C(\mathbb{T})$ *then for all* $p \in (0, 2]$ *we have*

$$\lim_{n \to \infty} \|f - L_n(f)\|_p = 0.$$

An analogous result holds for all nontrivial finite positive measures if we consider Lagrange interpolation of functions in the disk algebra $A(\overline{\mathbb{D}})$.

Theorem 1.2. *If* $f \in A(\overline{\mathbb{D}})$ *then for all* $p \in (0, 2]$ *we have*

$$\lim_{n \to \infty} \|f - L_n(f)\|_p = 0.$$

Recall that mean convergence implies the existence of a subsequence of $\{L_n(f)\}$ which converges almost everywhere to f. This contrasts with [19], where Vértesi proved that for any infinite triangular array on the unit circle, there exists a function in $A(\overline{\mathbb{D}})$ such that the sequence of its Lagrange interpolants diverge almost everywhere on the unit circle.

Our results delve into the close relationship between the zeros of orthogonal polynomials on the real line and the zeros of para-orthogonal polynomials.

The paper is organized as follows. In Section 2 we recall some definitions and well known facts about Lagrange interpolation. In Section 3 we prove both theorems above.

2. Lagrange interpolation

Let μ be a finite positive Borel measure on the unit circle $\mathbb{T}$ which is nontrivial i.e. its support has infinitely many points, and consider the Hilbert space $L^2(\mu)$ of μ-square-integrable functions with inner product

$$\langle f, g \rangle := \int_{\mathbb{T}} f \bar{g} \, d\mu,$$

and associated norm[1] when $1 \le p < \infty$,

$$\|f\|_p = \left(\int |f|^p \, d\mu \right)^{1/p}.$$

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[1] For $0 < p < 1$ we also use the same notation.

Let $\{\varphi_n\}_{n=0}^{\infty}$ be the sequence of orthonormal polynomials with respect to μ with positive leading coefficients; that is, these are polynomials of the form

$$\varphi_n(z) = \kappa_n z^n + \text{lower degree terms}, \quad \kappa_n > 0,$$

satisfying the orthonormality relations $\langle \varphi_n, \varphi_m \rangle = \delta_{nm}$ for all $n, m \geq 0$.

We denote by Π_n the set of polynomials of degree less than or equal to n. The Christoffel kernel is defined by

$$K_n(w, z) = \sum_{j=0}^{n} \overline{\varphi_j(w)} \varphi_j(z).$$

The reproducing kernel property of K_n says that for every $p \in \Pi_n$ we have

$$\langle p, K_n(w, \cdot) \rangle = \int p(z) \overline{K_n(w, z)} \, d\mu(z) = p(w). \tag{2.1}$$

By the Christoffel–Darboux formula (cf. [18, Theorem 11.4.2]),

$$K_n(w, z) = \frac{\overline{\varphi_{n+1}^*(w)} \varphi_{n+1}^*(z) - \overline{\varphi_{n+1}(w)} \varphi_{n+1}(z)}{1 - \overline{w} z}, \tag{2.2}$$

for $1 \neq \overline{w} z$, where $\varphi_{n+1}^*(z) := z^{n+1} \overline{\varphi_{n+1}(1/\overline{z})}$. Para-orthogonal polynomials are defined as

$$B_{n+1}(w, z) := (1 - \overline{w} z) K_n(w, z) = \overline{\varphi_{n+1}^*(w)} \varphi_{n+1}^*(z) - \overline{\varphi_{n+1}(w)} \varphi_{n+1}(z).$$

These polynomials were introduced in [10, Section 6].

For fixed $n \in \mathbb{N}$ and $w_n \in \mathbb{T}$, we denote by $Z_n(w_n)$ the zeros of $B_{n+1}(w_n, \cdot)$. It is well known that these zeros are simple and lie on the unit circle (cf. [2,8,10]). Hence

$$Z_n(w_n) = \{\zeta_{0n}, \zeta_{1n}, \ldots, \zeta_{nn}\} \subset \mathbb{T}.$$

Clearly, one of the zeros is w_n.

Next, we recall a symmetry property concerning the zeros of para-orthogonal polynomials. We include a short proof for an easy reading and to emphasize the connection with the formulas above.

Lemma 2.1 (*See [8]*)**.** *The identity*

$$K_n(\zeta_{jn}, \zeta_{mn}) = 0$$

holds for all $j, m \in \{0, 1, \ldots, n\}$ distinct. In particular, for all $j, m \in \{0, 1, \ldots, n\}$ we have

$$Z_n(\zeta_{jn}) = Z_n(\zeta_{mn}).$$

Proof. Take distinct indexes j, m. Since $\zeta_{jn}, \zeta_{mn} \in Z_n(w_n)$, we have

$$\frac{\varphi_{n+1}(\zeta_{jn})}{\varphi_{n+1}^*(\zeta_{jn})} = \frac{\overline{\varphi_{n+1}^*(w)}}{\overline{\varphi_{n+1}(w)}} = \frac{\varphi_{n+1}(\zeta_{mn})}{\varphi_{n+1}^*(\zeta_{mn})}$$

Since these quotients have modulus 1, we also know that

$$\frac{\varphi_{n+1}(\zeta_{jn})}{\varphi_{n+1}^*(\zeta_{jn})} = \frac{\overline{\varphi_{n+1}^*(\zeta_{jn})}}{\overline{\varphi_{n+1}(\zeta_{jn})}}.$$

Hence the statements follow using (2.2). $\quad\square$

Now let us consider the polynomials

$$L_{jn}(z) = \frac{K_n(\zeta_{jn}, z)}{\sqrt{K_n(\zeta_{jn}, \zeta_{jn})}}, \quad j = 0, 1, \ldots, n.$$

The following well known result will be crucial in the proofs of our two theorems in Section 3.

Lemma 2.2 (*See [8]*). *The polynomials* $\{L_{jn}\}_{j=0}^n$ *form an orthonormal basis of* Π_n.

Proof. Using (2.1) and Lemma 2.1 we have

$$\int L_{jn}\overline{L_{mn}}\, d\mu = \frac{K_n(\zeta_{jn}, \zeta_{mn})}{\sqrt{K_n(\zeta_{jn}, \zeta_{jn})}\sqrt{K_n(\zeta_{mn}, \zeta_{mn})}} = \delta_{jm}. \quad \square$$

Let μ_n be the measure associated to $Z_n(w_n)$ given by

$$d\mu_n = \sum_{j=0}^n \frac{d\delta_{\zeta_{jn}}}{K_n(\zeta_{jn}, \zeta_{jn})}. \tag{2.3}$$

The Szegő quadrature (cf. [10]) is defined by

$$Q_n(f) := \int f\, d\mu_n = \sum_{j=0}^n \frac{f(\zeta_{jn})}{K_n(\zeta_{jn}, \zeta_{jn})}.$$

Lemma 2.3 (*See [8]*). *If* $p, q \in \Pi_n$ *then*

$$Q_n(p\bar{q}) = \int p\bar{q}\, d\mu_n = \int p\bar{q}\, d\mu.$$

Proof. By definition of μ_n we have

$$\int p\bar{q}\, d\mu_n = \sum_{j=0}^n \frac{p(\zeta_{jn})}{\sqrt{K_n(\zeta_{jn}, \zeta_{jn})}} \frac{\overline{q(\zeta_{jn})}}{\sqrt{K_n(\zeta_{jn}, \zeta_{jn})}}.$$

Using (2.1) we get

$$\int p\overline{L_{jn}}\, d\mu = \frac{p(\zeta_{jn})}{\sqrt{K_n(\zeta_{jn}, \zeta_{jn})}}.$$

Now the statement follows using Lemma 2.2. $\quad \square$

Corollary 2.4 (*[10]*). *If* Λ *is a Laurent polynomial of degree* n *in* z *and* $1/z$, *then*

$$Q_n(\Lambda) = \int \Lambda\, d\mu_n = \int \Lambda\, d\mu.$$

Proof. By the fundamental theorem of algebra there exist polynomials p and q in Π_n such that $\Lambda(z, 1/z) = p(z)\overline{q(z)}$ for $|z| = 1$. Hence the statement follows directly from Lemma 2.3. $\quad \square$

Corollary 2.5 (*[10]*). *If* $f \in C(\mathbb{T})$ *then*

$$\lim_{n\to\infty} Q_n(f) = \int f\, d\mu.$$

Proof. This follows immediately from the density in the uniform norm of Laurent polynomials in $C(\mathbb{T})$ and Corollary 2.4. □

Remark 2.6. Notice that taking $p = q = 1$ in Lemma 2.3 we get

$$\mu_n(\mathbb{T}) = \mu(\mathbb{T}) =: M < \infty. \tag{2.4}$$

Let us denote the fundamental polynomials for $Z_n(w_n)$ by

$$\ell_{jn}(z) = \frac{K_n(\zeta_{jn}, z)}{K_n(\zeta_{jn}, \zeta_{jn})}, \quad j = 0, 1, \ldots, n.$$

These are polynomials of degree n that satisfy

$$\ell_{jn}(\zeta_{kn}) = \delta_{jk}. \tag{2.5}$$

The $n-$th Lagrange interpolation polynomial with respect to $Z_n(w_n)$ of a function f on $\mathbb{T}$ is defined as

$$L_n(f)(z) := \sum_{j=0}^{n} f(\zeta_{jn})\ell_{jn}(z).$$

Remark 2.7. Note that

$$\ell_{jn}\sqrt{K_n(\zeta_{jn}, \zeta_{jn})} = L_{jn}, \quad j = 0, 1, \ldots, n,$$

so due to Lemma 2.2 the fundamental polynomials form an orthogonal basis of Π_n. Since

$$\sum_{j=0}^{n} \ell_{jn}(z) = L_n(1) = 1$$

for all $z \in \mathbb{T}$, we also have

$$\sum_{j=0}^{n} \|\ell_{jn}\|_2^2 = \left\| \sum_{j=0}^{n} \ell_{jn} \right\|_2^2 = \mu(\mathbb{T}) = M.$$

Lemma 2.8. *For any function* $f : \mathbb{T} \to \mathbb{C}$ *we have*

$$\|L_n(f)\|_2^2 = Q_n(|f|^2).$$

Proof. Using Lemma 2.3 we obtain

$$\|L_n(f)\|_2^2 = \int |L_n(f)|^2 \, d\mu = \int |L_n(f)|^2 \, d\mu_n = Q_n(|f|^2). \quad \square$$

3. Proofs of Theorems 1.1 and 1.2

In this section we prove our two main results. So let us consider a nontrivial finite positive Borel measure μ on $\mathbb{T}$, and fix a sequence of points $\{w_n\}_{n=0}^{\infty}$ in $\mathbb{T}$. As in Section 2, for each n, let $Z_n(w_n)$ denote the zeros of para-orthogonal polynomials associated to w_n and μ, and $L_n(f)$ denote the nth Lagrange interpolation polynomial with respect to $Z_n(w_n)$ of a function f.

Proof of Theorem 1.1. It is enough to consider the case $p = 2$. Indeed, once this case is proved, the case $0 < p < 2$ follows using Hölder's inequality. So let us focus on $p = 2$.

Let $f \in C(\mathbb{T})$. Fix $\varepsilon > 0$. Since $\log \mu' \notin L^2(\mathfrak{m})$, by the Szegő–Kolmogorov–Krein theorem there exists a polynomial Π such that

$$\|f - \Pi\|_2^2 < \varepsilon. \tag{3.1}$$

Taking n larger than the degree of Π, we have

$$\begin{aligned}
\|f - L_n(f)\|_2^2 &\le \left(\|f - \Pi\|_2 + \|L_n(\Pi - f)\|_2\right)^2 \\
&\le 2\left(\|f - \Pi\|_2^2 + \|L_n(\Pi - f)\|_2^2\right).
\end{aligned}$$

Thus, according to (3.1), Lemma 2.8, and Corollary 2.5, we get

$$\limsup_{n \to \infty} \|f - L_n(f)\|_2^2 \le 4\varepsilon.$$

Therefore $\|f - L_n(f)\|_2$ converges to 0, as we wanted to prove. $\square$

As an immediate consequence of the proof of Theorem 1.1, we obtain the following result.

Corollary 3.1. *We have*

$$\lim_{n \to \infty} \|f - L_n(f)\|_2 = 0, \quad \forall f \in C(\mathbb{T}), \tag{3.2}$$

if and only if polynomials are dense in $L^2(\mu)$.

Proof. By the proof of Theorem 1.1 we have that if polynomials are dense in $L^2(\mu)$ then (3.2) holds. The other implication is obvious since the continuous functions are dense in $L^2(\mu)$. $\square$

Proof of Theorem 1.2. As in the proof of Theorem 1.1, we just need to focus on the case $p = 2$.

By Mergelyan's theorem, functions in $A(\overline{\mathbb{D}})$ are a uniform limit of algebraic polynomials in $\overline{\mathbb{D}}$. Thus, given $f \in A(\overline{\mathbb{D}})$, there exists a sequence of algebraic polynomials $\{P_n\}$ such that $\deg(P_n) \le n$ and

$$\|f - P_n\|_\infty = \max_{z \in \mathbb{T}} |f(z) - P_n(z)| \xrightarrow[n \to \infty]{} 0.$$

Clearly, this implies

$$\|f - P_n\|_2 \xrightarrow[n \to \infty]{} 0.$$

Now, using that L_n is a projection on Π_n and Remark 2.7 we have

$$\begin{aligned}
\|L_n(f) - P_n\|_2^2 &= \|L_n(f - P_n)\|_2^2 \\
&= \left\| \sum_{j=0}^{n} (f(\zeta_{jn}) - P_n(\zeta_{jn}))\ell_{jn} \right\|_2^2 \\
&= \sum_{j=0}^{n} \left\| (f(\zeta_{jn}) - P_n(\zeta_{jn}))\ell_{jn} \right\|_2^2 \\
&\le M \|f - P_n\|_\infty^2
\end{aligned}$$

Therefore

$$\|f - L_n(f)\|_2 \le \|f - P_n\|_2 + \|L_n(f) - P_n\|_2 \underset{n\to\infty}{\longrightarrow} 0,$$

as we wanted to prove. □

Data availability

No data was used for the research described in the article.

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