Received: 14 September 2022 | Revised: 11 March 2023 | Accepted: 20 March 2023
DOI: 10.1002/mma.9248

RESEARCH ARTICLE

WILEY

Powers of Catalan generating functions for bounded operators

Pedro J. Miana[1] | **Natalia Romero**[2]

[1]Departamento de Matemáticas, Instituto Universitario de Matemáticas y Aplicaciones, Universidad de Zaragoza, Zaragoza, 50009, Spain

[2]Departamento de Matemáticas y Computación, Universidad de la Rioja, Logroño, 26006, Spain

Correspondence
Pedro J. Miana, Departamento de Matemáticas, Instituto Universitario de Matemáticas y Aplicaciones, Universidad de Zaragoza, 50009 Zaragoza, Spain.
Email: pjmiana@unizar.es

Communicated by: J. Vigo-Aguiar

Funding information
Pedro J. Miana has been partially supported by Project ID2019-105979GBI00, DGI-FEDER, of the MCEI and Project E48-20R, Gobierno de Aragón, Spain. Natalia Romero has been partially supported by the Project MTM2018-095896-B-C21 of the Spanish Ministry of Science.

Let $c = (C_n)_{n\geq 0}$ be the Catalan sequence and T a linear and bounded operator on a Banach space X such $4T$ is a power-bounded operator. The Catalan generating function is defined by the following Taylor series:

$$C(T) := \sum_{n=0}^{\infty} C_n T^n.$$

Note that the operator $C(T)$ is a solution of the quadratic equation $TY^2 - Y + I = 0$. In this paper, we define powers of the Catalan generating function $C(T)$ in terms of the Catalan triangle numbers. We obtain new formulae that involve Catalan triangle numbers: the spectrum of c^{*j} and the expression of c^{-*j} for $j \geq 1$ in terms of Catalan polynomials ($*$ is the usual convolution product in sequences). In the last section, we give some particular examples to illustrate our results and some ideas to continue this research in the future.

KEYWORDS
Catalan triangle numbers, generating function, powers of bounded operators, quadratic equation

MSC CLASSIFICATION
Primary 11B75, 47A05, Secondary 11D09, 47A10

1 | INTRODUCTION

The well-known Catalan numbers $(C_n)_{n\geq 0}$ are given by the combinatorial formula

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \quad n \geq 0.$$

They may be defined recursively by $C_0 = 1$ and

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i}, \qquad n \geq 1, \tag{1.1}$$

and first terms in this sequence are 1, 1, 2, 5, 14, 42, 132, … . They appear in a wide range of combinatorial problems: They count the number of ways to triangulate a regular polygon with $n+2$ sides, or the number of ways that $2n$ people seat around a circular table are simultaneously shaking hands with another person at the table in such a way that none of the arms cross each other, see for example [1, 2].

The generating function of the Catalan sequence $c = (C_n)_{n\geq 0}$ is defined by

$$C(z) := \sum_{n=0}^{\infty} C_n z^n = \frac{1-\sqrt{1-4z}}{2z}, \; z \in D\left(0, \frac{1}{4}\right) := \left\{ z \in \mathbb{C} \mid |z| < \frac{1}{4} \right\}. \tag{1.2}$$

This function satisfies the quadratic equation $zy^2 - y + 1 = 0$.

The main aim in Miana and Romero [3] is to consider the quadratic equation

$$TY^2 - Y + I = 0, \tag{1.3}$$

in the set of linear and bounded operators, $\mathcal{B}(X)$ on a Banach space X, where I is the identity on the Banach space, and $T, Y \in \mathcal{B}(X)$. Formally, some solutions of this vector-valued quadratic equations are expressed by

$$Y = \frac{1 \pm \sqrt{1-4T}}{2T},$$

which involves the (non-trivial) problems of the square root of operator $1 - 4T$ and the inverse of operator T.

In this paper, we are concerned about the powers of $(C(T))^n$ for $n \in \mathbb{Z}$ and it is organized as follows. In the second section, we consider the Catalan triangle sequences $(B_{n,k})_{n\geq 1, 1\leq k\leq n}$ and $(A_{n,k})_{n\geq 1, 1\leq k\leq n+1}$. We prove new formulae for these numbers (Lemma 2.2) and their asymptotic estimation (Lemma 2.3). We treat polynomials and generating formulae for these Catalan triangle numbers; see Definition 2.4 and Theorem 2.7.

In third section, we consider the Banach algebra $\left(\ell^1\left(\mathbb{N}^0, \frac{1}{4^n}\right), \|\ \|_{1,\frac{1}{4^n}}, *\right)$, where

$$\|a\|_{1,\frac{1}{4^n}} := \sum_{n=0}^{\infty} \frac{|a(n)|}{4^n} < \infty, \quad (a * b)(n) = \sum_{j=0}^{n} a(n-j)b(j), \; n \geq 0,$$

where $a, b \in \ell^1\left(\mathbb{N}^0, \frac{1}{4^n}\right)$. We consider Catalan triangle sequences $(a_k)_{k\geq 1}, (b_k)_{k\geq 1} \subset \ell^1\left(\mathbb{N}^0, \frac{1}{4^n}\right)$ (Definition 3.1). These sequences are powers of the Catalan sequence c in $\ell^1\left(\mathbb{N}^0, \frac{1}{4^n}\right)$ (Proposition 3.2); we describe their spectrum set in Proposition 3.3. An original and motivating results connects c^{-*k} and Catalan polynomials in Theorem 3.7.

The powers of the Catalan generating operator $C(T)$ are studied in fourth section. We transfer our results from the algebra $\ell^1\left(\mathbb{N}^0, \frac{1}{4^n}\right)$ to $\mathcal{B}(X)$ via the algebra homomorphism Φ,

$$\Phi(a)x := \sum_{n\geq 0} a_n T^n(x), \quad a = (a_n)_{n\geq 0} \in \ell^1\left(\mathbb{N}^0, \frac{1}{4^n}\right), \; x \in X.$$

Note that $\Phi(c) = C(T)$, $\Phi(b_k) = (C(T))^{2k}$ and $\Phi(a_k) = (C(T))^{2k-1}$ for $k \geq 1$. We describe $(C(T))^{-j}$ in terms of Catalan polynomials; we estimate their norms and describe $\sigma((C(T))^j)$ for $j \in \mathbb{Z}$ in Theorem 4.1.

In the last section, we illustrate our results with some concrete operators T in Equation (1.3). We consider the Euclidean space $\mathbb{C}^2$ and matrices

$$T = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}, \begin{pmatrix} \lambda & \mu \\ 0 & \lambda \end{pmatrix}.$$

We solve Equation (1.3) and calculate $(C(T))^j$ for these matrices and $j \in \mathbb{Z}$. We also check $(C(a))^j$ for some particular values of $a \in \ell^1\left(\mathbb{N}^0, \frac{1}{4^n}\right)$ and $j \geq 1$. Finally, we present some ideas to continue this research.

2 | SOME NEW RESULTS ABOUT CATALAN TRIANGLE NUMBERS

Calatan triangle numbers $(B_{n,k})_{n\geq 1, 1\leq k\leq n}$ were introduced in Shapiro [4]. These combinatorial numbers $B_{n,k}$ are the entries of the following Catalan triangle:

$n \setminus k$	1	2	3	4	5	6	…
1	1						
2	2	1					
3	5	4	1				
4	14	14	6	1			
5	42	48	27	8	1		
6	132	165	110	44	10	1	
…	…	…	…	…	…	…	…

(2.1)

which are given by

$$B_{n,k} := \frac{k}{n}\binom{2n}{n-k},\ n,k \in \mathbb{N},\ k \le n. \tag{2.2}$$

Numbers $B_{n,k}$ has several applications: They count the number of leaves at level $k+1$ in all ordered trees with $n+1$ edges; $B_{n,k}$ is also the number of walks of n steps, each in direction N, S, W, or E, starting at the origin, remaining in the upper half-plane and ending at height k; or $B_{n,k}$ denotes the number of pairs of non-intersecting paths of length n and distance k; see, for example, Shapiro [4] and sequence A039598 in Sloane [5]. Notice that $B_{n,1} = C_n$ and $B_{n,n} = 1$ for $n \ge 1$.

In the last years, Catalan triangle (2.1) has been studied in detail. These numbers $(B_{n,k})_{n\ge k\ge 1}$ have been analyzed in many ways. For instance, symmetric functions have been used in Chen and Chu [6], recurrence relations in Slavík [7], or in Guo and Zeng [8] the Newton interpolation formula, which is applied to conclude divisibility properties of sums of products of binomial coefficients.

Other combinatorial numbers $A_{n,k}$ defined as follows:

$$A_{n,k} := \frac{2k-1}{2n+1}\binom{2n+1}{n+1-k},\ n,k \in \mathbb{N},\ k \le n+1, \tag{2.3}$$

appear as the entries of this other Catalan triangle,

$n \setminus k$	1	2	3	4	5	6	…
0	1						
1	1	1					
2	2	3	1				
3	5	9	5	1			
4	14	28	20	7	1		
5	42	90	75	35	9	1	
6	132	297	275	154	54	11	1
…	…	…	…	…	…	…	…

(2.4)

which is considered in Miana and Romero [9]. These numbers also admit combinatorial interpretations: They count the number of lattice paths ending at a given height, in particular certain Grand–Dyck paths; see more details in Guy [10] and sequence A039599 in Sloane [5]. Notice that $A_{n,1} = C_n$ and $A_{n,n+1} = 1$ for $n \ge 1$.

The entries $B_{n,k}$ and $A_{n,k}$ of the above two particular Catalan triangles satisfy the recurrence relations

$$B_{n,k} = B_{n-1,k-1} + 2B_{n-1,k} + B_{n-1,k+1}, \qquad k \ge 2, \tag{2.5}$$

and

$$A_{n,k} = A_{n-1,k-1} + 2A_{n-1,k} + A_{n-1,k+1}, \qquad k \ge 2. \tag{2.6}$$

The generating function of the Catalan sequence $(C_n)_{n\ge 0}$ is defined by

$$C(z) := \sum_{n=0}^{\infty} C_n z^n = \frac{1-\sqrt{1-4z}}{2z},\ z \in D\left(0, \frac{1}{4}\right) := \left\{ z \in \mathbb{C} \mid |z| < \frac{1}{4} \right\}. \tag{2.7}$$

Note that $C\left(\frac{1}{4}\right) = 2$.

Theorem 2.1. *Take* $z \in D\left(0, \frac{1}{4}\right)$.

(i) For $\lambda \neq C(z)$,

$$\frac{1}{\lambda - C(z)} = \frac{\lambda z - 1 + zC(z)}{\lambda^2 z - \lambda + 1}.$$

(ii) For $w \in D\left(0, \frac{1}{4}\right)$ *and* $w \neq \frac{z}{(1+z)^2}$,

$$\frac{C^2(w)}{1 - zwC^2(w)} = \frac{C(w) - (z+1)}{w(1+z)^2 - z}.$$

Proof.

(i) Note that

$$(\lambda - C(z))(\lambda z - 1 + zC(z)) = z\lambda^2 - \lambda + C(z) - zC^2(z) = z\lambda^2 - \lambda + 1,$$

for $\lambda \in \mathbb{C}$.

(ii) By item (i), we get that

$$\frac{C^2(w)}{1 - zwC^2(w)} = \frac{C^2(w)}{z} \frac{1}{\frac{1+z}{z} - C(w)} = C^2(w) \frac{w(1+z) - z + wzC(w)}{w(1+z)^2 - z}$$
$$= \frac{C(w) - 1}{w} \frac{w(1+z) - z + wzC(w)}{w(1+z)^2 - z} = \frac{C(w) - (z+1)}{w(1+z)^2 - z},$$

where we have applied again the equality $wC^2(w) - C(w) + 1 = 0$.

□

As the following identity holds,

$$C(z)^q = \sum_{n \geq 0} \frac{q}{n+q} \binom{2n-1+q}{n} z^n, \ q \geq 1, \ z \in D\left(0, \frac{1}{4}\right),$$

(Stanley [2, Exercise A.32(a)]), we take $q = 2k$ and $q = 2k+1$ for $k \geq 1$ to obtain the generating functions for the columns of the Catalan triangles:

$$\sum_{n=k}^{\infty} B_{n,k} z^n = z^k C^{2k}(z) = (C(z) - 1)^k, \qquad k \geq 1, \tag{2.8}$$

$$\sum_{n=k}^{\infty} A_{n,k+1} z^n = z^k C^{2k+1}(z) = C(z)(C(z) - 1)^k, \qquad k \geq 0, \tag{2.9}$$

for $z \in D\left(0, \frac{1}{4}\right)$. Note that to get the second equality in both lines, we use the equality $zC^2(z) = C(z) - 1$.

We apply the formula (2.7) to get

$$\lim_{z \to \frac{1}{4}} C(z) = 2, \qquad \lim_{z \to -\frac{1}{4}} C(z) = 2(\sqrt{2} - 1),$$

(Stanley [2, Exercise A.66]). Also other direct applications of Abel's theorem allows us to prove the following result.

Lemma 2.2. *Given $k \geq 1$,*

$$\sum_{n=k}^{\infty} B_{n,k}\frac{1}{4^n} = 1, \qquad \sum_{n=k}^{\infty} B_{n,k}\frac{(-1)^n}{4^n} = (2\sqrt{2}-3)^k,$$
$$\sum_{n,k\geq 1}^{\infty} B_{n,k}\frac{1}{4^{n+k}} = \frac{1}{3}, \qquad \sum_{n,k\geq 1}^{\infty} B_{n,k}\frac{(-1)^n}{4^{n+k}} = \frac{8\sqrt{2}-13}{41},$$
$$\sum_{n=k}^{\infty} A_{n,k+1}\frac{1}{4^n} = 2, \qquad \sum_{n=k}^{\infty} A_{n,k+1}\frac{(-1)^n}{4^n} = 2(\sqrt{2}-1)(2\sqrt{2}-3)^k,$$
$$\sum_{n,k\geq 0}^{\infty} A_{n,k+1}\frac{1}{4^{n+k}} = \frac{8}{3}, \qquad \sum_{n,k\geq 0}^{\infty} A_{n,k+1}\frac{(-1)^n}{4^{n+k}} = \frac{8}{41}(5\sqrt{2}-3).$$

Proof. We apply formulae (2.8) and (2.9) in the points $z = \frac{1}{4}$ and $\frac{-1}{4}$. □

In the next lemma, we extend the asymptotic estimation for Catalan numbers:

$$C_n \sim \frac{4^n}{\sqrt{\pi} n^{\frac{3}{2}}}, \qquad n \to \infty,$$

(Stanley [2, Exercise A.64]) to Catalan triangle numbers.

Lemma 2.3. *Given $k \geq 1$,*

$$B_{n,k} \sim \frac{4^n}{\sqrt{\pi}}\frac{k}{n^{\frac{3}{2}}}, \quad n \to \infty, A_{n,k} \sim \frac{4^n}{\sqrt{\pi}}\frac{2k-1}{n^{\frac{3}{2}}}, \quad n \to \infty.$$

Proof. We use the well-known Stirling formula $n! \sim e^{-n}n^n\sqrt{2\pi n}$ to show both equivalences. □

We now introduce the generating functions for the rows of the Catalan triangle numbers.

Definition 2.4. Given $n \geq 0$, we define the polynomials

$$P_n(z) := \sum_{j=0}^{n} B_{n+1,j+1}z^j, \quad Q_n(z) := \sum_{j=0}^{n+1} A_{n+1,j+1}z^j.$$

The first values of these families of polynomials are given by

$$\begin{aligned} P_0(z) &= 1, & Q_0(z) &= 1+z, \\ P_1(z) &= 2+z, & Q_1(z) &= 2+3z+1, \\ P_2(z) &= 5+4z+z^2, & Q_2(z) &= 5+9z+5z^2+z^3, \\ P_3(z) &= 14+14z+6z^2+z^3, & Q_3(z) &= 14+28z+20z^2+7z^3+z^4. \end{aligned}$$

Theorem 2.5.

(i) The only solution of the recurrence system

$$\begin{cases} R_0(z) = 1, \\ zR_n(z) + C_n = (z+1)^2R_{n-1}(z), \ n \geq 1, \end{cases}$$

is the polynomial sequence $(P_n)_{n\geq 0}$ given in Definition 2.4.

(ii) The only solution of the recurrence system

$$\begin{cases} R_0(z) = 1 + z, \\ zR_n(z) + C_n = (z+1)^2 R_{n-1}(z), \; n \geq 1, \end{cases}$$

is the polynomial sequence $(Q_n)_{n\geq 0}$ given in Definition 2.4.

Proof. It is enough to check that the sequence $(P_n)_{n\geq 0}$ satisfies the recurrence relation. Similarly the polynomial sequence $(Q_n)_{n\geq 0}$ does. By the recurrence relation (2.5), we get

$$\begin{aligned} P_{n+1}(z) = \sum_{j=0}^{n+1} B_{n+2,j+1} z^j &= \sum_{j=0}^{n+1} (B_{n+1,j} + 2B_{n+1,j+1} + B_{n+1,j+2}) z^j \\ &= z \sum_{j=1}^{n+1} B_{n+1,j} z^{j-1} + 2 \sum_{j=0}^{n+1} B_{n+1,j+1} z^j + \frac{1}{z} \sum_{j=0}^{n+1} B_{n+1,j+2} z^{j+1} \\ &= (z+2) P_n(z) + \frac{1}{z} \left(\sum_{j=0}^{n} B_{n+1,j+1} z^j - B_{n+1,1} \right) \\ &= \frac{(z+1)^2}{z} P_n(z) - \frac{C_{n+1}}{z}, \end{aligned}$$

and we conclude the equality. □

Remark 2.6. The sequences of polynomials $(P_n)_{n\geq 0}$ and $(Q_n)_{n\geq 0}$ are useful to prove equalities for Catalan triangles numbers and other sequences of integer numbers. For example, taking $z = 1$ in Theorem 2.5, we prove easily by induction method that

$$\sum_{k=1}^{n} B_{n,k} = \frac{n+1}{2} C_n, \quad \sum_{k=1}^{n+1} A_{n,k} = (n+1) C_n, \quad n \geq 1.$$

Indeed, we claim that $P_{n-1}(1) = \frac{n+1}{2} C_n$, for $n \geq 1$. As $P_n(1) + C_n = 2^2 P_{n-1}(1) = 2(n+1) C_n$, we have that

$$P_n(1) = (2n+1) C_n = \frac{n+2}{2} C_{n+1},$$

and we conclude the proof. An alternative proof appears in Shapiro [4, Proposition 3.1]. Similarly for $z = -1$, we get that

$$\sum_{k=1}^{n} (-1)^k B_{n,k} = -C_{n-1}, \quad \sum_{k=1}^{n+1} (-1)^k A_{n,k} = 0, \quad n \geq 1,$$

see, for example, Miana and Romero [11, Theorems 2.1 and 2.2] and references therein.

For $z = \frac{1}{4}$, we follow similar ideas by induction method to get that

$$\sum_{k=1}^{n} B_{n,k} \left(\frac{1}{4} \right)^k = \frac{a(n)}{4^n}, \qquad \sum_{k=1}^{n+1} A_{n,k} \left(\frac{1}{4} \right)^k = \frac{b(n)}{4^{n+1}},$$

where $(a(n))_{n\geq 1}$ is the integer sequence A194725 and $(b(n))_{n\geq 0}$ is A130970 given in The On-Line Encyclopedia of Integer Sequences by Sloane [5].

Finally, for $z = \frac{-1}{4}$, we obtain that

$$\sum_{k=1}^{n} B_{n,k} \left(\frac{-1}{4} \right)^k = -\frac{d(n)}{(-4)^n}, \qquad \sum_{k=1}^{n+1} A_{n,k} \left(\frac{-1}{4} \right)^k = -\frac{e(n)}{4^{n+1}},$$

where $(d(n))_{n\geq 1}$ is the integer sequence A051550 and $(e(n))_{n\geq 0}$ is A132863 given in Sloane [5].

In the next theorem, we obtain the generating function for polynomial $(P_n)_{n\geq 0}$ and $(Q_n)_{n\geq 0}$ given in Definition 2.4.

Theorem 2.7. *For $n \geq 0$,*

$$P(z,w) := \sum_{n\geq 0} P_n(z)w^n = \frac{C(w)-(z+1)}{w(1+z)^2-z},$$
$$Q(z,w) := \sum_{n\geq 0} Q_n(z)w^n = \frac{(C(w)-(z+1))(z+1)}{w(1+z)^2-z} = P(z,w)(z+1).$$

Proof. We take $z, w \in \mathbb{C}$ such that the bivariate generating function for polynomial $(P_n)_{n\geq 0}$ converges. Then,

$$P(z,w) = \sum_{n\geq 0} P_n(z)w^n = \sum_{n\geq 0}\sum_{j=0}^{n} B_{n+1,j+1}z^j w^n = \sum_{j\geq 0} z^j \sum_{n=j}^{\infty} B_{n+1,j+1}w^n$$
$$= \sum_{j\geq 0} z^j w^j C^{2j+2}(w) = \frac{C^2(w)}{1-zwC^2(w)} = \frac{C(w)-(z+1)}{w(1+z)^2-z},$$

where we have applied Equation (2.8) and Theorem 2.1 (ii).

Similarly,

$$Q(z,w) = \sum_{n\geq 0} Q_n(z)w^n = \sum_{n\geq -1}\sum_{j=0}^{n+1} A_{n+1,j+1}z^j w^n - \frac{1}{w}$$
$$= \sum_{j\geq 0} z^j \sum_{n=j-1}^{\infty} A_{n+1,j+1}w^n - \frac{1}{w} = \sum_{j\geq 0} z^j w^{j-1} C^{2j+1}(w) - \frac{1}{w}$$
$$= \frac{1}{w}\frac{C(w)-1+zwC^2(w)}{1-zwC^2(w)} = \frac{(1+z)C^2(w)}{1-zwC^2(w)}$$
$$= \frac{(C(w)-(z+1))(z+1)}{w(1+z)^2-z} = P(z,w)(z+1),$$

where we have applied Equation (2.9) and Theorem 2.1 (ii). □

Remark 2.8. Note that for $|w| \leq \frac{1}{4}$ and $|z| < 1$, functions $P(z,w)$ and $Q(z,w)$ are well-defined due to

$$|P(z,w)| \leq \sum_{n\geq 0} |P_n(z)|\frac{1}{4^n} = 4\sum_{j\geq 0} |z|^j = 4\frac{1}{1-|z|}.$$

Formulae given in Theorem 2.7 extend several known generating formula, for example, for Catalan numbers

$$P(0,w) = \sum_{n\geq 0} P_n(0)w^n = \sum_{n\geq 0} B_{n+1,1}w^n = \sum_{n\geq 0} C_{n+1}w^n = \frac{C(w)-1}{w},$$
$$Q(0,w) = \sum_{n\geq 0} Q_n(0)w^n = \sum_{n\geq 0} A_{n+1,1}w^n = \sum_{n\geq 0} C_{n+1}w^n = \frac{C(w)-1}{w}.$$

Other generating functions for integer natural sequences, see Remark 2.6, are also obtained.

3 | SEQUENCES OF CATALAN TRIANGLE NUMBERS

In this section, we consider the weighted Banach algebra $\ell^1\left(\mathbb{N}^0, \frac{1}{4^n}\right)$. This algebra is formed by sequences $a = (a(n))_{n\geq 0}$ such that

$$\|a\|_{1,\frac{1}{4^n}} := \sum_{n=0}^{\infty} \frac{|a(n)|}{4^n} < \infty,$$

and the product is the usual convolution $*$ defined by

$$(a * b)(n) = \sum_{j=0}^{n} a(n-j)b(j), \qquad a, b \in \ell^1\left(\mathbb{N}^0, \frac{1}{4^n}\right).$$

For $a, b \in \ell^1\left(\mathbb{N}^0, \frac{1}{4^n}\right)$, note that

$$\begin{aligned} \|a * b\|_{1,\frac{1}{4^n}} &= \sum_{n=0}^{\infty} \frac{|(a*b)(n)|}{4^n} \le \sum_{n=0}^{\infty} \frac{1}{4^n} \sum_{j=0}^{n} |a(n-j)|\,|b(j)| \\ &= \sum_{j=0}^{\infty} |b(j)| \sum_{n=j}^{\infty} \frac{|a(n-j)|}{4^n} = \|a\|_{1,\frac{1}{4^n}} \|b\|_{1,\frac{1}{4^n}}. \end{aligned}$$

We write $a^{*0} = a$ and $a^{*n} = a^{*(n-1)} * a$ for $n \in \mathbb{N}$.

The canonical base $\{\delta_j\}_{j\ge 0}$ is formed by sequences such that $(\delta_j)(n) := \delta_{j,n}$ is the known delta Kronecker. Note that $\delta_1^{*n} = \delta_n$ for $n \in \mathbb{N}$. This Banach algebra has identity element, δ_0; its spectrum set is the closed disc $\overline{D\left(0, \frac{1}{4}\right)}$; and its Gelfand transform is given by the Z-transform

$$Z(a)(z) := \sum_{n=0}^{\infty} a(n) z^n, \quad z \in \overline{D\left(0, \frac{1}{4}\right)},$$

(Muscat [12, Example 14.35]). It is straightforward to check that $Z(\delta_n)(z) = z^n$ for $n \ge 0$ (see, e.g., Larsen [13, pp. 21–22]).

We recall that the resolvent set of $a \in \ell^1\left(\mathbb{N}^0, \frac{1}{4^n}\right)$, denoted as $\rho(a)$, is defined by

$$\rho(a) := \left\{ \lambda \in \mathbb{C} \; : \; (\lambda\delta_0 - a)^{-1} \in \ell^1\left(\mathbb{N}^0, \frac{1}{4^n}\right) \right\},$$

and the spectrum set of a is denoted by $\sigma(a)$ and given by $\sigma(a) := \mathbb{C}\backslash\rho(a)$.

The Catalan numbers may be defined recursively by $C_0 = 1$ and

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i}, \quad n \ge 1. \tag{3.1}$$

We write $c = (C_n)_{n\ge 0}$ and then $\|c\|_{1,\frac{1}{4^n}} = 2$ and $C(z) = Z(c)(z)$ for $z \in D\left(0, \frac{1}{4}\right)$. We may interpret the equality (2.1) in terms of convolution product in the following closed form:

$$\delta_1 * c^{*1} - c + \delta_0 = 0,$$

where we deduce that

$$c^{-1} = \delta_0 - \delta_1 * c. \tag{3.2}$$

Definition 3.1. Given the Catalan triangle numbers $(B_{n,k})_{n,k}$ and $(A_{n,k})_{n,k}$ considered in Section 3, we define the Catalan triangle sequences a_k and b_k by

$$a_k(n) := A_{n+k-1,k}, \quad b_k(n) := B_{n+k,k}, \quad n \ge 0,$$

for $k \ge 1$. Note that $a_1(n) = A_{n,1} = C_n$ and $b_1(n) = B_{n+1,1} = C_{n+1}$ for $n \ge 0$.

Proposition 3.2. *For $k \geq 1$, consider the sequences a_k and b_k given in Definition 3.1. Then,*

(i) $a_k, b_k \in \ell^1(\mathbb{N}^*, \frac{1}{4^n})$ *and*

$$\|a_k\|_{1,\frac{1}{4^n}} = 2^{2k-1}, \qquad \|b_k\|_{1,\frac{1}{4^n}} = 2^{2k}.$$

(ii) $Z(a_k)(z) = (C(z))^{2k-1}$ *and* $Z(b_k)(z) = (C(z))^{2k}$ *for* $z \in D\left(0, \frac{1}{4}\right)$.

(iii) $a_k = c^{*(2k-2)}$ *and* $b_k = c^{*(2k-1)}$.

Proof. Item (i) is a consequence of Lemma 2.2. To check (ii), note that

$$Z(a_k)(z) = \sum_{n=0} A_{n+k-1,k} z^n = z^{-k+1} \sum_{m=k-1} A_{m,k} z^m = C^{2k-1}(z),$$
$$Z(b_k)(z) = \sum_{n=0} B_{n+k,k} z^n = z^{-k} \sum_{m=k} B_{m,k} z^m = C^{2k}(z),$$

where we have applied formulae (2.8) and (2.9). Item (iii) is a straightforward consequence of (ii). □

We may get an alternative proof of Proposition 3.2 (i) from item (ii). Note that

$$\|a_k\|_{1,\frac{1}{4^n}} = Z(a_k)\left(\frac{1}{4}\right) = (C(z))^{2k-1}\left(\frac{1}{4}\right) = 2^{2k-1},$$
$$\|b_k\|_{1,\frac{1}{4^n}} = Z(b_k)\left(\frac{1}{4}\right) = (C(z))^{2k}\left(\frac{1}{4}\right) = 2^{2k},$$

for $k \geq 1$.

Proposition 3.3. *The spectra of the Catalan triangle sequences $(a_k)_{k\geq 1}$ and $(b_k)_{k\geq 1}$ in the algebra $\ell^1\left(\mathbb{N}^0, \frac{1}{4^n}\right)$ are given by*

$$\sigma(a_k) = \left(C\left(\overline{D\left(0, \frac{1}{4}\right)}\right)\right)^{2k-1}, \qquad \sigma(b_k) = \left(C\left(\overline{D\left(0, \frac{1}{4}\right)}\right)\right)^{2k},$$

for $k \geq 1$. Their boundary is given by

$$\partial(\sigma(a_k)) = \left\{ 2^{2k-1} e^{-i(2k-1)\theta} \left(1 - \sqrt{2\left|\sin\left(\frac{\theta}{2}\right)\right|} e^{\frac{i(\pi-\theta)}{4}}\right)^{2k-1} \ : \ \theta \in (-\pi, \pi) \right\},$$
$$\partial(\sigma(b_k)) = \left\{ 2^{2k} e^{-i2k\theta} \left(1 - \sqrt{2\left|\sin\left(\frac{\theta}{2}\right)\right|} e^{\frac{i(\pi-\theta)}{4}}\right)^{2k} \ : \ \theta \in (-\pi, \pi) \right\}.$$

Proof. As the algebra $\ell^1\left(\mathbb{N}^0, \frac{1}{4^n}\right)$ has identity, the spectrum of an element equals the range of its Gelfand transform (Larsen [13, Theorem 3.4.1]). Moreover, as $\sigma(c) = C\left(\overline{D\left(0, \frac{1}{4}\right)}\right)$ (Miana and Romero [3, Proposition 3.2]), we apply Proposition 3.2 (ii) to get both first equalities, that is,

$$\sigma(a_k) = Z(a_k)(\overline{D(0, \frac{1}{4})}) = \left(C\left(\overline{D\left(0, \frac{1}{4}\right)}\right)\right)^{2k-1},$$
$$\sigma(b_k) = Z(a_k)\left(\overline{D\left(0, \frac{1}{4}\right)}\right) = \left(C\left(\overline{D\left(0, \frac{1}{4}\right)}\right)\right)^{2k},$$

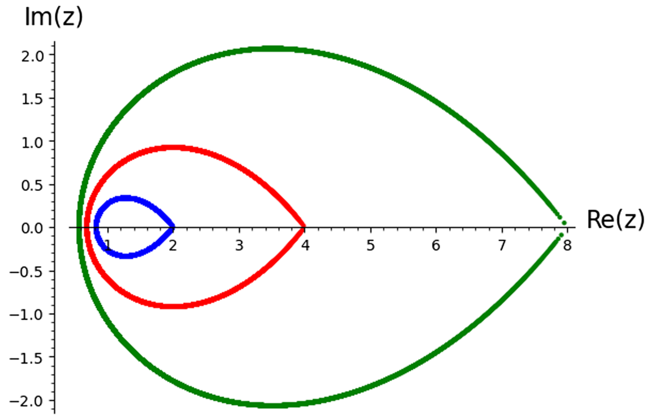

FIGURE 1 Sets $\partial(\sigma(c))$ in blue, $\partial(\sigma(b_1))$ in red, and $\partial(\sigma(a_2))$ in green. [Colour figure can be viewed at wileyonlinelibrary.com]

for $k \geq 1$. As

$$\partial(\sigma(c)) = \left\{ 2e^{-i\theta}\left(1-\sqrt{2|\sin(\frac{\theta}{2})|e^{\frac{i(\pi-\theta)}{4}}}\right) \;:\; \theta \in (-\pi,\pi) \right\},$$

see Miana and Romero [3, Proposition 3.2], we obtain second equalities from previous ones. □

Remark 3.4. In the Figure 1, we plot the sets $\partial(\sigma(c))$, $\partial(\sigma(b_1))$, and $\partial(\sigma(a_2))$.

Catalan polynomials are defined by the following linear recurrence relation:

$$\mathcal{P}_{k+2}(z) = \mathcal{P}_{k+1}(z) - z\mathcal{P}_k(z), \qquad k \geq 2, \tag{3.3}$$

and the starting values $\mathcal{P}_0(z) = \mathcal{P}_1(z) = 1$. The first values obtained are $\mathcal{P}_2(z) = 1-z$, $\mathcal{P}_3(z) = 1-2z$, and $\mathcal{P}_4(z) = 1-3z+z^2$. The closed form of $\mathcal{P}_k$ is given by the formula

$$\mathcal{P}_k(z) = \frac{(1+\sqrt{1-4z})^{k+1} - (1-\sqrt{1-4z})^{k+1}}{2^{k+1}\sqrt{1-4z}},$$

for $k \geq 0$. The bivariate generating function is

$$\frac{1}{1-t+zt^2} = \sum_{k\geq 0} \mathcal{P}_k(z)t^k,$$

see these and other properties in Jarvis et al. [14]. Other interesting property of Catalan polynomials is the following:

$$\frac{d\mathcal{P}_k(z)}{dz} = \frac{-1}{2^{k-1}}\sum_{l=0}^{k-2}(l+2)2^l\mathcal{P}_l(z), \qquad k \geq 2,$$

(Clapperton et al. [15, Identity II]). By induction method, we conclude that the coefficients of $\mathcal{P}_k(z)$ has alternative signs.

In the next results, we use the usual notation $P(\delta_1)$ where

$$P(\delta_1) := \sum_{k=0}^{n} a_k\delta_1^{*k} = \sum_{k=0}^{n} a_k\delta_k,$$

and P is the polynomial, $P(z) = \sum_{k=0}^{n} a_k z^k$.

Lemma 3.5. *Take the Catalan sequence polynomials $(\mathcal{P}_k)_{k\geq 0}$. Then, $\mathcal{P}_k(\delta_1) \in \ell^1\left(\mathbb{N}^0, \frac{1}{4^n}\right)$, $\|\mathcal{P}_0(\delta_1)\|_{1,\frac{1}{4^n}} = 1$, and*

$$\|\mathcal{P}_k(\delta_1)\|_{1,\frac{1}{4^n}} = \mathcal{P}_k\left(\frac{-1}{4}\right) = \frac{\alpha_k}{4^{k-1}}, \qquad k \geq 1,$$

where $\alpha_1 = 1$, $\alpha_2 = 5$, and $\alpha_k = 4(\alpha_{k-1} + \alpha_{k-2})$ for $k \geq 3$.

Proof. It is clear that $\mathcal{P}_k(\delta_1) \in \ell^1\left(\mathbb{N}^0, \frac{1}{4^n}\right)$ and $\|\mathcal{P}_0(\delta_1)\|_{1,\frac{1}{4^n}} = 1$. As the coefficients of polynomials $(\mathcal{P}_k)_{k\geq 0}$ have alternative signs, we conclude that

$$\begin{aligned}\|\mathcal{P}_k(\delta_1)\|_{1,\frac{1}{4^n}} &= \sum_{j=0}^{k} a_j \left(\frac{-1}{4}\right)^j = \mathcal{P}_k\left(\frac{-1}{4}\right)\\ &= \frac{\left(1+\sqrt{2}\right)^{k+1} - \left(1-\sqrt{2}\right)^{k+1}}{\sqrt{2}2^{k+1}} = \frac{\alpha_k}{4^{k-1}},\end{aligned}$$

where the integer sequence $(\alpha_k)_{k\geq 1}$ is numbered as A086347 in Sloane [5] and treated in detail there. □

Remark 3.6. The first values of the sequence $(\alpha_k)_{k\geq 1}$ are $1, 5, 24, 116, 560 \ldots$. This sequence is an example of generalized Fibonacci numbers $g(k) = cg(k-1) + dg(k-2)$ for $k \geq 2$ and seed values $g(0) = a$ and $g(1) = b$ $(a, b, c, d \in \mathbb{N}.)$

Theorem 3.7. *For $k \geq 1$,*

$$(c^{*k})^{-1} = \mathcal{P}_{k+1}(\delta_1) + (-c * \delta_1) * \mathcal{P}_k(\delta_1).$$

Moreover, $\|(c*c)^{-1}\|_{1,\frac{1}{4^n}} = \frac{3}{2}$ and $\|(c^{*k})^{-1}\|_{1,\frac{1}{4^n}} \leq \frac{1}{4^k}(\alpha_{k+1} + 2\alpha_k)$ for $k \geq 1$, where $(\alpha_k)_{k\geq 1}$ are defined in Lemma 3.5.

Proof. Note that $c^{-1} = \delta_0 - \delta_1 * c$, see formula (2.2), and then

$$\begin{aligned}(c*c)^{-1} &= c^{-1} * c^{-1} = \delta_0 - 2\delta_1 * c + \delta_1 * (\delta_1 * c * c)\\ &= \delta_0 - \delta_1 - \delta_1 * c = \mathcal{P}_2(\delta_1) + (-c * \delta_1) * \mathcal{P}_1(\delta_1),\end{aligned}$$

where we have applied that $\delta_1 * c^{*1} = c - \delta_0$. By induction, we have that

$$\begin{aligned}(c^{*(k+1)})^{-1} &= c^{-1} * (c^{*k})^{-1} = (\delta_0 - \delta_1 * c) * (\mathcal{P}_{k+1}(\delta_1) + (-c * \delta_1) * \mathcal{P}_k(\delta_1))\\ &= \mathcal{P}_{k+1}(\delta_1) - \delta_1 * c * \mathcal{P}_{k+1}(\delta_1) - \delta_1 * \mathcal{P}_k(\delta_1)\\ &= \mathcal{P}_{k+2}(\delta_1) + (-c * \delta_1) * \mathcal{P}_{k+1}(\delta_1),\end{aligned}$$

where we have applied the recurrence relation (2.3).

Finally, we apply Lemma 3.5 to get

$$\|(c^{*k})^{-1}\|_{1,\frac{1}{4^n}} \leq \|\mathcal{P}_{k+1}(\delta_1)\|_{1,\frac{1}{4^n}} + \frac{1}{2}\|\mathcal{P}_k(\delta_1)\|_{1,\frac{1}{4^n}} = \frac{1}{4^k}(\alpha_{k+1} + 2\alpha_k)$$

for $k \geq 1$. □

4 | POWERS OF CATALAN GENERATING FUNCTIONS FOR BOUNDED OPERATORS

In this section, we consider the particular case that T is a linear and bounded operator on the Banach space X, $T \in \mathcal{B}(X)$, such that

$$\sup_{n\geq 0} \|4^n T^n\| := M < \infty, \tag{4.1}$$

that is, $4T$ is a power-bounded operator. In this case, $\sigma(T) \subset \overline{D\left(0, \frac{1}{4}\right)}$. Under the condition (2.1), we define the Catalan generating function, $C(T)$, by

$$C(T) := \sum_{n\geq 0} C_n T^n, \tag{4.2}$$

see Miana and Romero [3, Section 5]. The bounded operator $C(T)$ may be considered as the image of the Catalan sequence $c = (C_n)_{n\geq 0}$ in the algebra homomorphism $\Phi : \ell^1\left(\mathbb{N}^0, \frac{1}{4^n}\right) \to \mathcal{B}(X)$ where

$$\Phi(a)x := \sum_{n\geq 0} a_n T^n(x), \quad a = (a_n)_{n\geq 0} \in \ell^1\left(\mathbb{N}^0, \frac{1}{4^n}\right),\ x \in X,$$

that is, $\Phi(c) = C(T)$. The Φ algebra homomorphism (also called functional calculus) is presented in some functional analysis textbooks, for example, Muscat [12, Chapters 13 and 14].

Theorem 4.1. *Given $T \in \mathcal{B}(X)$ such that $4T$ is power bounded and $c = (C_n)_{n\geq 0}$ the Catalan sequence, then*

(i) The powers $(C(T))^{2k-1} = \Phi(a_k)$ and $(C(T))^{2k} = \Phi(b_k)$ for $k \geq 1$, and

$$\|(C(T)^j\| \leq (C(\|T\|))^j, \qquad j \geq 1.$$

(ii) The operator $C(T)$ is invertible, $(C(T))^{-1} = I - TC(T)$,

$$(C(T))^{-(j+1)} = \mathcal{P}_j(T) - TC(T)\mathcal{P}_{j-1}(T) \qquad j \geq 1,$$

$\|C(T)^{-1}\| \leq 1 + \frac{1}{2}\sup_{n\geq 0}\|4^n T^n\|$, $\|C(T)^{-2}\| \leq \frac{3}{2}\sup_{n\geq 0}\|4^n T^n\|$ *and*

$$\|(C(T))^{-(j+1)}\| \leq \frac{1}{4^j} \sup_{n\geq 0} \|4^n T^n\| \left(\alpha_{j+1} + 2\alpha_j\right), \qquad j \geq 1,$$

where $(\alpha_j)_{j\geq 1}$ are defined in Lemma 3.5.

(iii) Take $(P_n)_{n\geq 0}$ and $(Q_n)_{n\geq 0}$ polynomials given in Definition 2.4. Then,

$$\sum_{n\geq 0} P_n(z)T^n = \frac{C(T) - (z+1)I}{T(1+z)^2 - zI},$$
$$\sum_{n\geq 0} Q_n(z)T^n = \frac{(C(T) - (z+1)I)(z+1)}{T(1+z)^2 - zI},$$

for $|z| < 1$.

(iv) The spectral mapping theorem holds for $(C(T))^n$, that is, $\sigma((C(T))^n) = C^n(\sigma(T))$ for $n \in \mathbb{Z}$.

Proof.

(i) From (2.2), $\Phi(c) = C(T) \in \mathcal{B}(X)$ as we have commented above. By Proposition 3.2 (iii), we have

$$(C(T))^{2k-1} = (\Phi(c))^{2k-1} = \Phi(c^{*(2k-2)}) = \Phi(a_k),$$
$$(C(T))^{2k} = (\Phi(c))^{2k} = \Phi(c^{*(2k-1)}) = \Phi(b_k),$$

for $k \geq 1$. By Proposition 3.2 (ii), we get

$$\|(C(T))^{2k-1}\| = \|\Phi(a_k)\| \leq \sum_{j\geq 0} a_k(j)\|T\|^j = (C(\|T\|))^{2k-1},$$
$$\|(C(T))^{2k}\| = \|\Phi(b_k)\| \leq \sum_{j\geq 0} b_k(j)\|T\|^j = (C(\|T\|))^{2k},$$

for $k \geq 1$, and we conclude the proof of (i).

(ii) As the homomorphism Φ is continuous, we apply the formula (2.2) to get

$$C(T)(I - TC(T)) = \Phi(c)(\Phi(\delta_0 - \delta_1 * c)) = \Phi(c - \delta_1 * c^{*1}) = \Phi(\delta_0) = I.$$

In fact $(C(T))^{-1} = \Phi(c^{-1})$ and

$$(C(T))^{-(j+1)} = \Phi((c^{-1})^{*j}) = \Phi(c^{*j})^{-1} = \mathcal{P}_{j+1}(T) - TC(T)\mathcal{P}_j(T), \qquad j \geq 1,$$

where we have applied Theorem 3.7 and Φ is an algebra homomorphism. The estimation of $\|(C(T))^{-(j+1)}\|$ follows also from Theorem 3.7.

(iii) We follow similar ideas to those shown in Theorem 2.7, and we check

$$\sum_{n\geq 0} P_n(z)T^n = \frac{C(T) - (z+1)I}{T(1+z)^2 - zI}, \qquad \sum_{n\geq 0} Q_n(z)T^n = \frac{(C(T) - (z+1)I)(z+1)}{T(1+z)^2 - zI},$$

for $|z| < 1$.

(iv) Since $4T$ is power bounded, the spectral mapping theorem for $C^n(T)$ may found in Dungey [16, Theorem 2.1] and then $\sigma((C(T))^n) = C^n(\sigma(T))$ for $n \in \mathbb{Z}$.

□

Remark 4.2. As $\sigma(T) \subset \overline{D\left(0, \frac{1}{4}\right)}$, we apply Proposition 3.3 to conclude that

$$\sigma(C^n(T)) \subset C^n\left(\overline{D\left(0, \frac{1}{4}\right)}\right), \qquad n \in \mathbb{Z}.$$

5 | EXAMPLES, APPLICATIONS, AND FINAL COMMENTS

In this section, we present some particular examples of operators T for which we solve Equation (1.3) and calculate $C(T)$ and $(C(T))^k$ for $k \in \mathbb{Z}$. In Section 5.1, we consider the Euclidean space $\mathbb{C}^2$ and some matrices T. To resolve this matrix equation, we need to solve a system of four quadratic equations. We also calculate $(C(T))^n$ for these matrices. In Section 5.2, we check $C(a)$ for some $a \in \ell^1\left(\mathbb{N}^0, \frac{1}{4^n}\right)$. Finally, we present some ideas to continue this research in Section 5.3.

5.1 | Matrices on $\mathbb{C}^2$

We consider the Euclidean space $\mathbb{C}^2$ and the operator $T = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, with $0 \neq \lambda, \mu \in \mathbb{C}$. For $\lambda = \mu$, the solution is presented in Miana and Romero [3, Subsection 6.1]. For $\lambda \neq \mu$, the solution of (1.3) is given by

$$Y = \begin{pmatrix} \frac{1\pm\sqrt{1-4\lambda}}{2\lambda} & 0 \\ 0 & \frac{1\pm\sqrt{1-4\mu}}{2\mu} \end{pmatrix},$$

where the allowed signs are all four combinations. In the case that $|\lambda|, |\mu| \leq \frac{1}{4}$, note that

$$(C(T))^j = \begin{pmatrix} (C(\lambda))^j & 0 \\ 0 & (C(\mu))^j \end{pmatrix},$$

for $j \in \mathbb{Z}$.

Now, we study the case $T = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}$ with $\lambda \in \mathbb{C}\backslash\{0\}$. When $|\lambda| \leq \frac{1}{4}$, we get that

$$C(T) = \begin{pmatrix} C_e(\lambda) & C_o(\lambda) \\ C_o(\lambda) & C_e(\lambda) \end{pmatrix}, \tag{5.1}$$

where functions C_e and C_o are functions given by

$$C_e(\lambda) := \sum_{n=0}^{\infty} C_{2n}\lambda^{2n} = \frac{\sqrt{1+4\lambda} - \sqrt{1-4\lambda}}{4\lambda},$$
$$C_o(\lambda) := \sum_{n=0}^{\infty} C_{2n+1}\lambda^{2n+1} = \frac{2-\sqrt{1+4\lambda} - \sqrt{1-4\lambda}}{4\lambda}.$$

Note that $C(T)$ is one of the four solutions of (1.3); see Miana and Romero [3, Section 6.1]. As

$$\begin{aligned} \begin{pmatrix} a & b \\ b & a \end{pmatrix}^{2n} &= \left(a^{2n} + \binom{2n}{2} a^{2(n-2)}b^2 + \dots \dots + b^{2n}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &\quad + \left(\binom{2n}{1} a^{2n-1}b + \dots \dots + \binom{2n}{1} ab^{2n-1}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \begin{pmatrix} a & b \\ b & a \end{pmatrix}^{2n+1} &= \left(a^{2n+1} + \dots \dots + \binom{2n+1}{2n} ab^{2n}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &\quad + \left(\binom{2n+1}{1} a^{2n}b + \dots \dots + b^{2n+1}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \end{aligned}$$

we use (2.1) to get new generating formulae for Catalan triangle numbers.

Theorem 5.1. *Take $n \geq 0$ and $z \in \overline{D\left(0, \frac{1}{4}\right)}$. Then,*

$$\begin{aligned} \sum_{k=n}^{\infty} B_{2k-n,n}z^{2k} &= z^{2n}\left(C_e^{2n}(z) + \binom{2n}{2} C_e^{2(n-2)}(z)C_o^2(z) + \dots \dots + C_o^2(z)\right), \\ \sum_{k=n}^{\infty} B_{2k+1-n,n}z^{2k+1} &= z^{2n}\left(\binom{2n}{1} C_e^{2n-1}(z)C_o(z) + \dots \dots + \binom{2n}{1} C_e(z)C_o^{2n-1}(z)\right), \\ \sum_{k=n}^{\infty} A_{2k-1-n,n}z^{2k} &= z^{2n}\left(C_e^{2n-1}(z) + \dots \dots + \binom{2n-1}{2n-2} C_e(z)C_o^{2n-2}(z)\right), \\ \sum_{k=n}^{\infty} A_{2k-n,n}z^{2k+1} &= z^{2n}\left(\binom{2n-1}{1} C_e^{2n-1}(z)C_o(z) + \dots \dots + C_o^{2n-1}(z)\right). \end{aligned}$$

Proof. Take $|z| \le \frac{1}{4}$ and we consider $T = \begin{pmatrix} 0 & z \\ z & 0 \end{pmatrix}$. We apply Theorem 4.1 to get

$$\begin{aligned} C(T)^{2n} &= \sum_{j=0}^{\infty} b_n(j) T^j = \sum_{l=0}^{\infty} b_n(2l) z^{2n} I + \sum_{l=0}^{\infty} b_n(2l+1) z^{2n+1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \sum_{l=0}^{\infty} B_{2l+n,n} z^{2n} I + \sum_{l=0}^{\infty} B_{2l+1+n,n} z^{2n+1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \frac{1}{z^{2n}} \left(\sum_{k=n}^{\infty} B_{2k-n,n} z^{2k} I + \sum_{k=n}^{\infty} B_{2k+1-n,n} z^{2k+1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right), \end{aligned}$$

and we conclude the first two equalities. Similarly, we consider $C(T)^{2n+1}$ and show the second two equalities. □

Finally, we study the case $T = \begin{pmatrix} \lambda & \mu \\ 0 & \lambda \end{pmatrix}$ with $\lambda, \mu \in \mathbb{C} \backslash \{0\}$. The solutions of (1.3) are given by

$$Y = \begin{pmatrix} a & \frac{\mu(a-1)}{\lambda(1-2\lambda a)} \\ 0 & a \end{pmatrix},$$

where a is any solution of the quadratic Catalan equation $\lambda a^2 - a + 1 = 0$. In the case that $|\lambda| \le \frac{1}{4}$, we get that

$$C(T) = \begin{pmatrix} C(\lambda) & \frac{\mu((C(\lambda)-1)}{\lambda(1-2\lambda C(\lambda))} \\ 0 & C(\lambda) \end{pmatrix},$$

and

$$(C(T))^j = \begin{pmatrix} (C(\lambda))^j & n(C(\lambda))^{j-1} \frac{\mu((C(\lambda)-1)}{\lambda(1-2\lambda C(\lambda))} \\ 0 & (C(\lambda))^j \end{pmatrix},$$

for $j \ge 1$. As $(C(T))^{-1} = \frac{1}{(C(\lambda))^2} \begin{pmatrix} C(\lambda) & -\frac{\mu((C(\lambda)-1)}{\lambda(1-2\lambda C(\lambda))} \\ 0 & C(\lambda) \end{pmatrix}$, we get that

$$(C(T))^{-j} = \frac{1}{(C(\lambda))^{2j}} \begin{pmatrix} (C(\lambda))^j & -n(C(\lambda))^{j-1} \frac{\mu((C(\lambda)-1)}{\lambda(1-2\lambda C(\lambda))} \\ 0 & (C(\lambda))^j \end{pmatrix},$$

for $j \ge 1$.

5.2 | Catalan operators on ℓ^p

We consider the space of sequences $\ell^p\left(\mathbb{N}^0, \frac{1}{4^n}\right)$ where

$$\|a\|_{p,\frac{1}{4^n}} := \left(\sum_{n=0}^{\infty} \frac{|a_n|^p}{4^{np}} \right)^{\frac{1}{p}} < \infty,$$

for $1 \le p < \infty$ and $\ell^\infty\left(\mathbb{N}^0, \frac{1}{4^n}\right)$ the space of sequences embedded with the norm

$$\|a\|_{\infty,\frac{1}{4^n}} := \sup_{n \ge 0} \frac{|a_n|}{4^n} < \infty.$$

Note that $\ell^1\left(\mathbb{N}^0, \frac{1}{4^n}\right) \hookrightarrow \ell^p\left(\mathbb{N}^0, \frac{1}{4^n}\right) \hookrightarrow \ell^\infty\left(\mathbb{N}^0, \frac{1}{4^n}\right)$.

Now, we consider sequences $c, (a_k), (b_k) \in \ell^1\left(\mathbb{N}^0, \frac{1}{4^n}\right)$, the Catalan triangle sequences given in Definition 3.1; and convolution operators $C(f) := c * f$, $C^{2k}(f) = b_k * f$, and $C^{2k-1}(f) = a_k * f$ for $f \in \ell^p\left(\mathbb{N}^0, \frac{1}{4^n}\right)$ and $k \geq 1$ with $1 \leq p \leq \infty$. By Theorem 4.1 (iv), we get that

$$\sigma(C^n) = C^n(\sigma(\delta_1)) = C^n\left(\overline{D\left(0, \frac{1}{4}\right)}\right), \qquad n \geq 1.$$

Note that the set $\sigma(C^n)$ is independent on p and coincides with the spectrum of the power of Catalan sequence c in $\ell^1\left(\mathbb{N}^0, \frac{1}{4^n}\right)$ (Proposition 3.3).

5.3 | A future research

Given $a, b \neq 0 \in \mathbb{C}$, the quadratic equation

$$\frac{bz}{2}y^2 - y + \frac{a}{2b} = 0 \tag{5.2}$$

has two solutions given by

$$y = \frac{1 \pm \sqrt{1 - za}}{bu}.$$

We define $C^{a,b}(z) := \frac{1-\sqrt{1-za}}{bz}$; note that $C^{a,b}(z) = \frac{a}{2b}C(\frac{az}{4})$ and

$$C^{a,b}(z) = \sum_{n\geq 0} \frac{a^{n+1}}{2^{2n+1}b}C_n z^n.$$

It would be natural to consider a vector-valued version of Equation (2.2) for $a, b, z \in \mathcal{B}(X)$.

ACKNOWLEDGEMENTS

We thank the referee for his/her very careful review of this paper and for the comments, corrections, and suggestions that ensued. A major version of the paper has been carried out to take them into account, and the paper has been significantly improved.

CONFLICT OF INTEREST STATEMENT

This work does not have any conflicts of interest.

ORCID

Pedro J. Miana https://orcid.org/0000-0001-9430-343X
Natalia Romero https://orcid.org/0000-0002-0653-560X

REFERENCES

1. N. Sloane, *A handbook of integer sequences*, Academic Press, New Jersey, 1973.
2. R. P. Stanley, *Catalan numbers*, Cambridge University Press, Cambridge, 2015.
3. P. J. Miana and N. Romero, *Catalan generating functions for bounded operators*, 2022. Preprint.
4. L. W. Shapiro, *A Catalan triangle*, Discrete Math. **14** (1976), 83–90.
5. N. Sloane, *The on-line encyclopedia of integer sequences*. https://oeis.org/
6. X. Chen and W. Chu, *Moments on Catalan numbers*, J. Math. Anal. Appl. **349** (2009), no. 2, 311–316.
7. A. Slavík, *Identities with squares of binomial coefficients*, Ars Comb. **113** (2014), 377–383.
8. V. J. W. Guo and J. Zeng, *Factors of binomial sums from Catalan triangle*, J. Number Theory **130** (2010), no. 1, 172–186.
9. P. J. Miana and N. Romero, *Moments of combinatorial and Catalan numbers*, J. Number Theory **130** (2010), no. 8, 1876–1887.
10. R. K. Guy, *Catwalks, sandsteps and Pascal pyramids*, J. Integer Sequences **3** (2000), 00.1.6.
11. P. J. Miana and N. Romero, Moments of Catalan triangle numbers, *Number theory and its applications*, IntechOpen, London, UK, 2020.

12. J. Muscat, *Functional analysis. An introduction to metric spaces, Hilbert spaces and Banach algebras*, Springer, Berlin, 2014.
13. R. Larsen, *Banach algebras: An introduction*, Marcel Dekker, New York, 1973.
14. A. F. Jarvis, P. J. Larcombe, and E. J. Fennessey, *Some factorization and divisibility properties of Catalan polynomials*, Bull. ICA **71** (2014), 36–56.
15. J. A. Clapperton, P. J. Larcombe, and E. J. Fennessey, *Some new identities for Catalan polynomials*, Util. Math. **80** (2009), 1–8.
16. N. Dungey, *Subordinated discrete semigroups of operators*, Trans. Amer. Math. Soc. **363** (2011), no. 4, 1721–1741.

How to cite this article: P. J. Miana and N. Romero, *Powers of Catalan generating functions for bounded operators*, Math. Meth. Appl. Sci. **46** (2023), 13262–13278. DOI 10.1002/mma.9248