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On the set of initial guesses for the secant method

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The secant method is the most used iterative method to solve an operator equation where the operator involved is nondifferentiable. A known problem that arises when applying this method is its accessibility. Then, we try to improve it by using the technique of decomposition of the operator involved, so that the operator is in turn the sum of two operators, one differentiable and the other continuous but not differentiable. For this, we use a family of Newton-secant-type iterative methods that arises from Newton's method and from a well-known family of secant-type iterative methods. We study the accessibility of the new methods in two different ways: From the convergence balls of the methods, obtained from a local study of the convergence, and from a dynamic study of the methods. Some examples related to chemistry are also presented to prove the theoretical results.

KEYWORDS

accessibility, convergence ball, dynamics, Hammerstein integral equation, local convergence, Newton's method, the secant method

MSC CLASSIFICATION

45G10, 47H99, 65J15

1 | INTRODUCTION

Many problems of Mathematics, Mathematical Chemistry, or Engineering can be written as a nonlinear equation,[1,2] as for example, boundary value problems for differential equations, nonlinear integral equations arising in many contexts, such as the theory of elasticity, electrostatics, the potential theory, and radiative heat transfer problems, and systems of nonlinear equations resulting from the discretization of numerous problems. Since we can write all these problems as the operator equation $F(x) = 0$, we need to give the operator F some generality. So, we consider the operator $F : \Omega \subseteq X \to Y$, where X and Y are Banach spaces and Ω is a nonempty open convex subset of X.

In general, the roots of $F(x) = 0$ cannot be expressed in a closed form, so this problem is commonly solved by applying iterative methods. If the operator F is differentiable, Newton's method[3] is the most used iteration to solve $F(x) = 0$, due to its computational efficiency, which is given by

$$\begin{cases} x_0 \text{ given in } \Omega, \\ x_{n+1} = x_n - [F'(x_n)]^{-1}F(x_n), \quad n \geq 0, \end{cases} \tag{1}$$

and has quadratic convergence. As Newton's method needs the existence of F', this method cannot be applied when F is nondifferentiable; so the secant method[4] is widely used in this situation, since it is efficient. As we can see in its algorithm,

$$\begin{cases} x_{-1}, x_0 \text{ given in } \Omega, \\ x_{n+1} = x_n - [x_{n-1}, x_n; F]^{-1} F(x_n), \ n \geq 0, \end{cases} \tag{2}$$

the smoothness properties of the operator F and the use of the first-order divided difference of F play an important role in the application of the method. Moreover, it is superlinearly convergent with R-order of convergence at least $\frac{1}{2}\left(1+\sqrt{5}\right)$. Remember that a linear and bounded operator from X to Y, denoted by $[x, y; F]$, which satisfies the condition $[x, y; F](x-y) = F(x) - F(y)$, is called a first-order divided difference of F at a pair of distinct points x and y.[5] For the existence of divided differences in linear spaces, see Balazs and Goldner.[6]

On the one hand, taking into account the loss of the speed of convergence obtained when (2) is used instead of (1), the following uniparametric family of secant-type iterative methods is introduced in Hernández and Rubio:[7,8]

$$\begin{cases} x_{-1}, x_0 \text{ given in } \Omega, \\ y_n = \theta x_n + (1-\theta) x_{n-1}, \ \theta \in [0, 1), \\ x_{n+1} = x_n - [y_n, x_n; F]^{-1} F(x_n), \ n \geq 0. \end{cases} \tag{3}$$

This family can be considered as a combination of the secant method ($\theta = 0$) and Newton's method ($\theta = 1$) for differentiable operators. Notice that if F is a differentiable operator, then $[x, x; F] = F'(x)$. We see in Hernández and Rubio[9] that the R-order of convergence of (3) is for all θ at least the same as that of the secant method. But, in practice, the closer x_n and y_n, the higher the speed of convergence. Indeed, the speed of convergence of (3) increases with $\theta \in [0, 1]$, approaching the speed of convergence of Newton's method when θ is close to 1.

On the other hand, the accessibility of the secant method is usually a problem in the application of the method, since the set of initial guesses that guarantee the convergence of the method is very small. To try to improve it, we use the technique of decomposition of the operator F. So, we consider F as

$$F(x) = G(x) + H(x),$$

where $G, H : \Omega \subseteq X \to Y$ are nonlinear operators, G is differentiable, and H is continuous but nondifferentiable. Then, for approximating a root of $F(x) = 0$, we apply the following family of Newton-secant-type iterative methods:

$$\begin{cases} x_{-1}, x_0 \text{ given in } \Omega, \\ y_n = \theta x_n + (1-\theta) x_{n-1}, \ \theta \in [0, 1), \\ x_{n+1} = x_n - \left(G'(x_n) + [y_n, x_n; H]\right)^{-1} F(x_n), \ n \geq 0. \end{cases} \tag{4}$$

Notice that the iteration (4) is reduced to Newton's method if F is differentiable ($H(x) = 0$) and to the family of secant-type methods (3) if $G(x) = 0$. Moreover, if $\theta = 1$, then (3) and (4) are both reduced to Newton's method, since $y_n = x_n$ and $[x_n, x_n; F] = F'(x_n)$, so that only the case in which the operator F is differentiable can be considered.

The main aim of this work is to justify that the family (4) improves the accessibilities of the secant method and the iterations of family (3). Thus, from the study we do, we obtain that we can improve the speed of convergence and the accessibility of the secant method from the application of the family (4).

We can consider two situations to study the accessibility of an iterative method. First, from the study of its local convergence, we obtain the convergence ball that allows us to compare, according to its size, the accessibility of the iterative method. Second, from the dynamic study of the iterative method, we obtain the set of initial guesses that guarantee the convergence of the method, that is, its accessibility.

Throughout the paper, we suppose that there exists a first-order divided difference $[z, w; F]$ for each pair of distinct points $(z, w) \in \Omega \times \Omega$ and denote $\overline{B(x, \rho)} = \{y \in X; \|y - x\| \leq \rho\}$ and $B(x, \rho) = \{y \in X; \|y - x\| < \rho\}$, for $\rho > 0$. In the examples, we have considered the infinity norm.

2 | A STUDY OF THE LOCAL CONVERGENCE

Remember that the local convergence of an iterative method is obtained from conditions on the operator involved and the solution of the equation to solve and provides the so-called convergence ball of the sequence given by the iterative method, so that the accessibility to the solution is shown from the initial approximations belonging to the convergence ball.[10,11]

In order to prove the local convergence of the iteration (4), we suppose the following conditions:

- (C1) There exists two continuous and nondecreasing functions $\omega_1 : [0,+\infty) \to \mathbb{R}$ and $h : [0,1] \to \mathbb{R}$, such that $\omega_1(tz) \leq h(t)\omega_1(z)$, for all $t \in [0,1]$, $z \in [0,+\infty)$ and $\|G'(x) - G'(y)\| \leq \omega_1(\|x-y\|)$.
- (C2) There exists $[p,q;H]$ for every pair of distinct points $p,q \in \Omega$ such that $\|[x,y;H] - [u,v;H]\| \leq \omega_2(\|x-u\|, \|y-v\|)$, for all $x,y,u,v \in \Omega$, where $\omega_2 : [0,+\infty) \times [0,+\infty) \to \mathbb{R}$ is a continuous and nondecreasing function in both arguments.
- (C3) Let x^* be a solution of the equation $F(x) = 0$ and consider $\tilde{x} \in \Omega$ such that $\|\tilde{x} - x^*\| = \alpha > 0$, so that there exists the operator $D^{-1} = \left(G'(x^*) + [x^*,\tilde{x};H]\right)^{-1}$ and $\|D^{-1}\| \leq \delta$.
- (C4) There exists $\rho \geq 0$ such that $B(x^*,\rho) \subset \Omega$ and $\ell = \delta\left(\omega_1(\rho) + \omega_2(\rho,\rho+\alpha)\right) < 1$.

After that, we give two technical lemmas. The first is on the existence of the inverse operators involved.

Lemma 1. *Suppose the conditions (C1)–(C2)–(C3)–(C4). Then, the operator given by* $\left(G'(x) + [\theta x + (1-\theta)y, x;H]\right)$ *is invertible, for every pair of distinct points* $x, y \in B(x^*,\rho)$*, and*

$$\left\|\left(G'(x) + [\theta x + (1-\theta)y, x;H]\right)^{-1}\right\| \leq \frac{\delta}{1-\ell}. \tag{5}$$

Proof. First, the operator $[\theta x + (1-\theta)y, x;H]$ exists, since $x, y, \theta x + (1-\theta)y \in B(x^*,\rho)$ with $x \neq \theta x + (1-\theta)y$ and $\theta \in [0,1)$.

Second, from

$$\begin{aligned}
&\left\|I - D^{-1}\left(G'(x) + [\theta x + (1-\theta)y, x;H]\right)\right\| \\
&\quad \leq \|D^{-1}\|\left(\|G'(x^*) - G'(x)\| + \|[x^*,\tilde{x};H] - [\theta x + (1-\theta)y, x;H]\|\right) \\
&\quad \leq \delta\left(\omega_1(\rho) + \omega_2(\rho,\rho+\alpha)\right) \\
&\quad = \ell \\
&< 1
\end{aligned}$$

and the Banach lemma on invertible operators, it follows that there exists the operator $\left(G'(x) + [\theta x + (1-\theta)y, x;H]\right)^{-1}$ and satisfies (5). □

In the second technical lemma, we see that the sequence $\{x_n\}$ given by the iteration (4) is well-defined. Observe first that $\theta x_{n-1} + (1-\theta)x_{n-2} \in B(x^*,\rho)$, provided that $x_{n-1}, x_{n-2} \in B(x^*,\rho)$ and $x_{n-1} \neq x_{n-2}$, since $\theta x_{n-1} + (1-\theta)x_{n-2}$ is a point of the segment that joins x_{n-1} and x_{n-2}.

Lemma 2. *Suppose the conditions (C1)–(C2)–(C3)–(C4) and* $x_{n-1}, x_{n-2} \in B(x^*,\rho)$ *with* $x_{n-1} \neq x_{n-2}$*. Then, the sequence* $\{x_n\}$ *given by (4) is well defined and* $\|x_n - x^*\| \leq K\|x_{n-1} - x^*\|$*, where* $K = \frac{\sigma}{1-\ell}$, $\sigma = \delta\left(I_h\omega_1(\rho) + \omega_2(\rho,0)\right)$ *and* $I_h = \int_0^1 h(t)dt$.

Proof. As $x_{n-1} \neq x_{n-2}$ and $\theta \neq 1$, we have that the operator $[\theta x_{n-1} + (1-\theta)x_{n-2}, x_{n-1};H]$ exists. Next, we denote $D_{n-1} = G'(x_{n-1}) + [\theta x_{n-1} + (1-\theta)x_{n-2}, x_{n-1};H]$ and, from Lemma 1, see that D_{n-1} is invertible and $\|D_{n-1}^{-1}\| \leq \frac{\delta}{1-\ell}$, so that x_n is well defined.

After that, from (4), it follows that

$$\begin{aligned}
x_n - x^* &= x_{n-1} - D_{n-1}^{-1}F(x_{n-1}) - x^* \\
&= D_{n-1}^{-1}\left(\left(G'(x_{n-1}) + [\theta x_{n-1} + (1-\theta)x_{n-2}, x_{n-1};H]\right)(x_{n-1} - x^*) - G(x_{n-1}) - H(x_{n-1})\right) \\
&= D_{n-1}^{-1}\left(\int_0^1 \left(G'(x_{n-1} + \tau(x^* - x_{n-1})) - G'(x_{n-1})\right)(x_{n-1} - x^*)d\tau\right) \\
&\quad + D_{n-1}^{-1}\left([\theta x_{n-1} + (1-\theta)x_{n-2}, x_{n-1};H] - [x^*, x_{n-1};H]\right)(x_{n-1} - x^*)
\end{aligned}$$

and, taking norms, we obtain

$$\|x_n - x^*\| \leq \frac{\delta}{1-\ell}\left(I_h\omega_1(\rho) + \omega_2(\rho, 0)+\right)\|x_{n-1} - x^*\| = \frac{\sigma}{1-\ell}\|x_{n-1} - x^*\| = K\|x_{n-1} - x^*\|.$$

The proof is complete. □

Note that $K < 1$ provided that $\ell + \sigma < 1$, that is,

$$\delta\left((1 + I_h)\omega_1(\rho) + \omega_2(\rho, \alpha + \rho) + \omega_2(\rho, 0)\right) < 1$$

with $I_h = \int_0^1 h(t)dt$. Then, if the equation

$$\delta\left((1 + I_h)\omega_1(\rho) + \omega_2(\rho, \alpha + \rho) + \omega_2(\rho, 0)\right) - 1 = 0 \tag{6}$$

has at least one positive real root and we denote the smallest positive real root by r, then $\ell + \sigma < 1$ for all $\rho_\star \in \mathbb{R}_+$ such that $\rho_\star < r$.

Next, from the above mentioned, we can establish the following result of local convergence for the iteration (4).

Theorem 3. *Under the conditions (C1)–(C2)–(C3)–(C4), we suppose that Equation (6) has at least one positive real root and we denote the smallest positive real root by r and consider $\rho_\star \in \mathbb{R}_+$ such that $\rho_\star < r$ with $B(x^*, \rho_\star) \subset \Omega$. If $x_0 \in B(x^*, \rho_\star)$ and $x_{-1} \in B(x_0, \rho_\star - \eta)$, with $x_{-1} \neq x_0$ and $\eta = \|x_0 - x^*\|$, then the sequence given by (4) is well defined, $x_n \in B(x^*, \rho_\star)$, for all $n \geq 0$, and is convergent to a solution x^* of the equation $F(x) = 0$.*

Proof. As $x_{-1} \in B(x_0, \rho_\star - \eta)$, then $x_{-1} \in B(x^*, \rho_\star)$, since

$$\|x_{-1} - x^*\| \leq \|x_{-1} - x_0\| + \|x_0 - x^*\| \leq \rho_\star - \eta + \eta = \rho_\star.$$

Moreover, $\theta x_0 + (1 - \theta)x_{-1} \neq x_0$, since $\theta \neq 1$. Therefore, by Lemma 1, there exists the operator $D_0^{-1} = \left(G'(x_0) + [\theta x_0 + (1 - \theta)x_{-1}, x_0; H]\right)^{-1}$ and $\|D_0^{-1}\| \leq \frac{\delta}{1-\ell}$. As a consequence, x_1 is well defined and, by Lemma 2, we have

$$\|x_1 - x^*\| \leq K\|x_0 - x^*\| < \|x_0 - x^*\| < \rho_\star,$$

since $K < 1$. Then, $x_1 \in B(x^*, \rho_\star)$ with $x_0 \neq x_1$ and, obviously, $\theta x_1 + (1 - \theta)x_0 \in B(x^*, \rho_\star)$.

Then, by mathematical induction on n, it is easy to prove that $x_n \in B(x^*, \rho_\star)$. In addition, $\theta x_n + (1-\theta)x_{n-1} \in B(x^*, \rho_\star)$, provided that $x_n \neq x_{n-1}$, and there exists the operator D_{n+1}^{-1} and x_{n+1} is well defined. Thus, by Lemma 2, we have $\|x_{n+1} - x^*\| \leq K\|x_n - x^*\|$. Therefore, $x_{n+1} \in B(x^*, \rho_\star)$, for all $n \geq 0$, and $\|x_{n+1} - x^*\| \leq K^{n+1}\|x_0 - x^*\|$, so that $\{x_n\}$ is convergent to x^*. □

Remark 4. Notice that a simple choice of $\tilde{x}$ is x_0 and, in this case, $\alpha = \eta$.

3 | UNIQUENESS OF SOLUTION

In this section, we establish the uniqueness of solution from the following theorem.

Theorem 5. *Suppose the conditions (C1)–(C2)–(C3)–(C4) and the existence of $R \geq \rho$ such that $\delta\left(I_h\omega_1(R) + \omega_2(0, R + \alpha)\right) < 1$. Then, the solution x^* is unique in $\overline{B(x^*, R)} \cap \Omega$.*

Proof. Let $\zeta^* \in \overline{B(x^*, R)} \cap \Omega$ be a solution of $F(x) = 0$ and the operator $J = \int_0^1 G'(x^* + \tau(\zeta^* - x^*))d\tau + [x^*, \zeta^*; H]$. Then, from

$$\begin{aligned} \|D^{-1}J - I| &\le \|D^{-1}\|\|J - D\| \\ &\le \delta \left(\int_0^1 \omega_1(\|\tau(\zeta^* - x^*)\|)d\tau + \omega_2(0, R + \alpha) \right) \\ &\le \delta \left(I_h \omega_1(R) + \omega_2(0, R + \alpha) \right) \\ &< 1, \end{aligned}$$

we can guarantee the existence of J^{-1} by the Banach lemma on invertible operators. As a consequence, from $0 = F(x^*) - F(\zeta^*) = J(x^* - \zeta^*)$, we obtain that $x^* = \zeta^*$, so that x^* is unique in $\overline{B(x^*, R)} \cap \Omega$. □

4 | EXAMPLES

Now, we illustrate the previous study with three examples. In the first example, we analyze the dynamic behavior of the iteration (4) from Theorem 3 when it is applied to solve a complex equation. In the next two examples, nonlinear integral equations of mixed Hammerstein type are involved.

Example 6. We consider the complex function $F(z) = z^3 + z|z| - 2z$, which is clearly nondifferentiable. We can decompose the function F as $F(z) = G(z) + H(z)$, where $G(z) = z^3 - 2z$ is the differentiable part and $H(z) = z|z|$ is the nondifferentiable part. It is clear that the function F has three different zeros: $z^* = 0$, $z^{**} = -1$ and $z^{***} = 1$.

We consider the domain $\Omega = B(z^*, \varepsilon)$ and we then obtain

$$\omega_1(s) = 6\varepsilon s, \qquad h(t) = t, \qquad I_h = \frac{1}{2}, \qquad \omega_2(t, s) = 4\varepsilon.$$

Now, if we choose the auxiliary point $\tilde{z} = 1/10$, then $\alpha = 1/10$ and $\delta = \|D^{-1}\| = 10/19$. In this case, for $\varepsilon = 0.2$ and any $\theta \in [0, 1)$, the smallest positive real root of Equation (6) is $r = 0.1666\ \ldots$, and we can then consider $\rho_\star < 0.1666\ \ldots$. Next, if we choose $z_0 = 0 \in B(z^*, \rho_\star)$ and $z_{-1} = 0.1 \in B(0, \rho_\star - |z_0|)$, the sequence $\{z_n\}$ given by (4) is well defined, $z_n \in B(z^*, \rho_\star) \subseteq \Omega$, for all $n \ge 0$, and is convergent to the solution $z^* = 0$ of the equation $F(z) = 0$.

In Figure 1, we consider $z_{-1} = 0.1$ and z_0 free and show the basin of attraction (the set of points in the space such that initial conditions chosen in the set dynamically evolve to a particular attractor[12,13]) obtained with a tolerance of 10^{-20} and a maximum of six iterations. The strategy used is the following: The yellow color is assigned to basin of attraction of the root $z^* = 0$ and the color black is used if the iteration does not converge. Moreover, we see in Figure 1 the ball of convergence obtained, the red ball, which almost completes the domain Ω, whose border is colored white. Also, if we look at the black area, which represents the points from which there is no convergence to the solution, we see that the ball of convergence is quite accurate.

In the second example, we consider a Hammerstein integral equation of the second kind.[14] The Hammerstein equations have strong physical background and arise from the electromagnetic fluid dynamics. These equations appeared in the 30s of the 20th century as general models for the study of semilinear boundary value problems, where the kernel typically arises as the Green function of a differential operator. Also, these equations are applied in the theory of radiative transfer and the theory of neutron transport as well in the kinetic theory of gases. They also play a very significant role in several applications, as for example the dynamic models of chemical reactors, which are governed by control equations, justifying then their study and solution. Then, we obtain the convergence balls and uniqueness of solution from Theorems 3 and 5, respectively.

Example 7. We consider the following nonlinear integral equation of mixed Hammerstein type:

$$x(s) = f(s) + \int_a^b \mathcal{K}(s, t) \left(\lambda x(t)^2 + \mu |x(t)| \right) dt, \ s \in [a, b], \tag{7}$$

where $\lambda, \mu \in \mathbb{R}, -\infty < a < b < +\infty$, the function $f(s)$ is continuous on $[a, b]$ and given, the kernel $\mathcal{K}(s, t)$ is a known function in $[a, b] \times [a, b]$ and x is a solution to be determined.

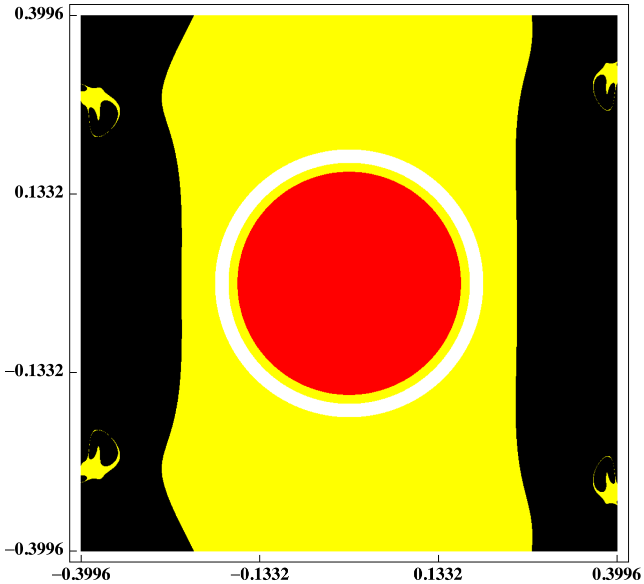


FIGURE 1 The convergence ball of the method (4) when it is applied to the equation $F(z) = z^3 + z|z| - 2z = 0$ [Colour figure can be viewed at wileyonlinelibrary.com]

Solving (7) is equivalent to solving the equation $\mathcal{F}(x) = 0$, where $\mathcal{F} : C[a,b] \to C[a,b]$,

$$[\mathcal{F}(x)](s) = f(s) + \int_a^b \mathcal{K}(s,t)\left(\lambda x(t)^2 + \mu|x(t)|\right) dt,\ s \in [a,b]. \tag{8}$$

We consider the kernel $\mathcal{K}(s,t)$ as the Green function in $[a,b] \times [a,b]$ and use a process of discretization to transform (7) into a finite dimensional problem by a Gauss–Legendre quadrature formula with m nodes

$$\int_a^b \chi(t)\,dt = \sum_{i=1}^m w_i \chi(t_i),$$

where the nodes t_i and the weights w_i are determined for $i = 1, 2, \ldots, m$. If we denote the approximations of $x(t_i)$ and $f(t_i)$ by x_i and f_i, respectively, with $i = 1, 2, \ldots, m$, then Equation (7) is equivalent to the following nonlinear system equations:

$$x_i = f_i + \sum_{j=1}^m a_{ij}\left(\lambda x_i^2 + \mu|x_i|\right),\ j = 1, 2, \ldots, m, \tag{9}$$

where

$$a_{ij} = w_j \mathcal{K}(t_i, t_j) = \begin{cases} w_j \frac{(b-t_i)(t_j-a)}{b-a}, & \text{if } j \le i, \\ w_j \frac{(b-t_j)(t_i-a)}{b-a}, & \text{if } j > i. \end{cases}$$

Now, the system (9) can be written as

$$F(\mathbf{x}) \equiv \mathbf{x} - \mathbf{f} - A\hat{\mathbf{x}} = 0, \qquad F : \mathbb{R}^m \to \mathbb{R}^m, \tag{10}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_m)^T$, $\mathbf{f} = (f_1, f_2, \ldots, f_m)^T$, $A = (a_{ij})_{i,j=1}^m$ and

$$\hat{\mathbf{x}} = \left(\lambda x_1^2 + \mu|x_1|, \lambda x_2^2 + \mu|x_2|, \ldots, \lambda x_m^2 + \mu|x_m|\right)^T.$$

In particular, if we choose $a = 0$, $b = 1$ and $\lambda = \mu = \frac{1}{2}$, then the system (10) is reduced to

$$F(\mathbf{x}) = G(\mathbf{x}) + H(\mathbf{x}) \text{ with } G(\mathbf{x}) = \mathbf{x} - \mathbf{f} - \frac{1}{2}A\dot{\mathbf{x}} \text{ and } H(\mathbf{x}) = -\frac{1}{2}A\ddot{\mathbf{x}}, \tag{11}$$

where $\dot{\mathbf{x}} = (x_1^2, x_2^2, \dots, x_m^2)^T$ and $\ddot{\mathbf{x}} = (|x_1|, |x_2|, \dots, |x_m|)^T$. So, the function F defined in (11) is nonlinear and nondifferentiable.

We consider divided differences of first-order in $\mathbb{R}^m$ that do not need that H is differentiable, so that we then use the divided difference of first-order given by $[\mathbf{p}, \mathbf{q}; H] = \left([\mathbf{p}, \mathbf{q}; H]_{ij}\right)_{i,j=1}^m$, where

$$[\mathbf{p}, \mathbf{q}; H]_{ij} = \frac{1}{p_j - q_j}\left(H_i(p_1, \dots, p_{j-1}, p_j, q_{j+1}, \dots, q_m) - H_i(p_1, \dots, p_{j-1}, q_j, q_{j+1}, \dots, q_m)\right),$$

$i, j = 1, 2, \dots, m$, $\mathbf{p} = (p_1, p_2, \dots, p_m)^T$ and $\mathbf{q} = (q_1, q_2, \dots, q_m)^T$.

If we consider $\mathbf{f} = 0$ in (11), then $\mathbf{x}^* = 0$ is obviously a solution of $F(\mathbf{x}) = 0$ and the operator F is reduced to

$$F(\mathbf{x}) = \mathbf{x} - \frac{1}{2}A\bar{\mathbf{x}} \text{ with } \bar{\mathbf{x}} = \left(x_1^2 + |x_1|, x_2^2 + |x_2|, \dots, x_m^2 + |x_m|\right)^T.$$

In this case, $G(\mathbf{x}) = \mathbf{x} - \frac{1}{2}A\dot{\mathbf{x}}$, $G'(\mathbf{x}) = I - A \ \mathrm{diag}\{x_1, x_2, \dots, x_m\}$, where I denotes the identity matrix, and

$$\|G'(\mathbf{x}) - G'(\mathbf{y})\| \le \|A\|\|\mathbf{x} - \mathbf{y}\|.$$

In addition, $H(\mathbf{x}) = -\frac{1}{2}A\ddot{\mathbf{x}}$,

$$[\mathbf{p}, \mathbf{q}; H] = -\frac{1}{2}A \ \mathrm{diag}\left\{\frac{|p_1| - |q_1|}{p_1 - q_1}, \frac{|p_2| - |q_2|}{p_2 - q_2}, \dots, \frac{|p_m| - |q_m|}{p_m - q_m}\right\}.$$

and $\|[\mathbf{x}, \mathbf{y}; H] - [\mathbf{u}, \mathbf{v}; H]\| \le \|A\|$. So, $\omega_1(z) = \|A\|z$, $I_h = 1/2$ and $\omega_2(s, t) = \|A\|$.

If we choose $m = 8$ and $\mathbf{x}_0 = \tilde{\mathbf{x}} = (1, 1, \dots, 1)^T$. Then, for Theorem 3, we have

$$\|A\| = 0.1235\ \dots, \qquad \alpha = 1, \qquad \delta = \|D^{-1}\| = 0.5326\ \dots,$$

and the smallest positive root of Equation (6) is $\rho = 8.7968\ \dots$, so that, from Theorem 5, we obtain that the convergence ball is $B(\mathbf{x}^*, \rho)$ with $\rho < 8.7968\ \dots$. Moreover, from Theorem 5, the solution is unique in the ball $\overline{B(\mathbf{x}^*, R)}$ with $R < 28.3904\ \dots$.

In the third example, that also appears in different chemistry problems, we see that the iteration (4) provides better approximations to the solution than the family of secant-type methods (3).

Example 8. We consider the integral equation (7) with $f(s) = \frac{1}{2}$, $a = 0$, $b = 1$, $\lambda = \mu = \frac{3}{4}$ and the kernel $\mathcal{K}(s, t)$ as the Green function in $[0, 1] \times [0, 1]$. Following the same process of discretization as for Example (7), we transform the integral equation in the system

$$F(\mathbf{x}) \equiv \mathbf{x} - \mathbf{u} - \frac{3}{4}A\bar{\mathbf{x}} = 0, \qquad F : \mathbb{R}^8 \to \mathbb{R}^8, \tag{12}$$

where $\mathbf{x} = (x_1, x_2, \dots, x_8)^T$, $\mathbf{u} = \left(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}\right)^T$, $A = (a_{ij})_{i,j=1}^m$ and

$$\bar{\mathbf{x}} = \left(x_1^2 + |x_1|, x_2^2 + |x_2|, \dots, x_8^2 + |x_8|\right)^T.$$

Next, we use the iterative methods given by (3) and (4) with $\theta = \frac{1}{2}$ and starting at $\mathbf{x}_{-1} = \left(\frac{2}{5}, \frac{2}{5}, \dots, \frac{2}{5}\right)^T$ and $\mathbf{x}_0 = \left(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}\right)^T$ to approximate a solution of (12) for different values of θ and see, through the errors $\|\mathbf{x}_n - \mathbf{x}^*\|$ obtained in Table 1 with the stopping criterion $\|\mathbf{x}_n - \mathbf{x}_{n-1}\| < 10^{-24}$, that method (4) provides better approximations to the solution than method (3). Similar results are obtained for different $\theta \in [0, 1)$. In addition, the computational order of

TABLE 1 Absolute errors obtained by methods (3) and (4) with $\theta = \frac{1}{2}$, respectively

n	$\|\|\mathbf{x}_n - \mathbf{x}^*\|\|$	$\|\|\mathbf{x}_n - \mathbf{x}^*\|\|$
1	$9.7710 \ldots \times 10^{-4}$	$5.8998 \ldots \times 10^{-4}$
2	$3.4775 \ldots \times 10^{-6}$	$2.8862 \ldots \times 10^{-8}$
3	$1.4162 \ldots \times 10^{-10}$	$6.8605 \ldots \times 10^{-17}$

convergence[15] of the family of secant-type methods (3) approximates the order of convergence $\frac{1+\sqrt{5}}{2} = 1.6180 \ldots$ of the secant method[4] and that of the iteration (4) to the order two of Newton's method.

5 | A DYNAMIC STUDY

In this section, we compare, experimentally, the accessibility of the families of iterative methods (3) and (4) for different values of θ. For this, we consider their dynamic behavior by means of the study of the attraction basins of each iterative method when they are applied to solve a complex equation $F(z) = 0$, where $F : \mathbb{C} \to \mathbb{C}$. The number of studies that introduce the dynamic analysis has increased in the last years[16–19] as it gives interesting information about the behavior of the method. In this case, we consider again the complex function $F(z) = z^3 - z|z| - 2z$ of Example 6. Remember that the function F has three different zeros $z^* = 0$, $z^{**} = -1$ and $z^{***} = 1$. We show the fractal pictures that are generated to approximate the three solutions by means of the iterative methods (3) and (4). The strategy used is the following: A color is assigned to each basin of attraction of a zero and the color black is used if the iteration does not converge. To draw the convergence planes, we choose as yellow the convergence to z^*, cyan to z^{**}, and magenta to z^{***}. In all the cases, the tolerance 10^{-6} and a maximum of N iterations are used. So, given a point of the convergence plane, if we have not obtained the desired tolerance with N iterations, we do not continue and decide that the iterative method does not converge to any zero and the point is colored black. If the tolerance required for a number of iterations less than or equal to N, with respect to one of the solutions, is reached for the starting point, then the point is colored in the color indicated for the basin of attraction of the solution. The graphics are generated with Mathematica 12 and using similar algorithms than those appearing in previous works.[20–22]

To place the pair of starting points (z_{-1}, z_0) as a point of the convergence plane, we consider two strategies. First, z_{-1} is fixed and z_0 is free, and, in the second place, z_{-1} and z_0 are free.

5.1 | First strategy: z_{-1} is fixed and z_0 is free

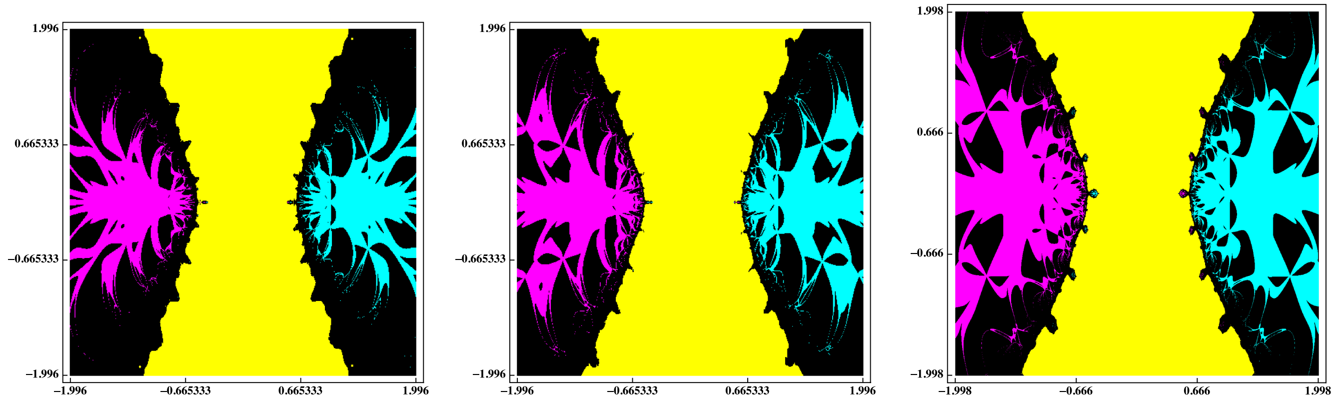
It is known[23,24] that when an iterative method with memory, as (3) or (4), are applied, the starting points z_{-1} and z_0 are considered sufficiently close. In our study, we consider $z_{-1} = z_0 - \frac{1}{10}$ and z_0 free. So, we use the convergence plane algorithm presented in Magreñán,[21] in which the horizontal axis is chosen for the values of the real part of z_0 and the vertical axis for the values of the imaginary part of z_0.

In Figures 2 and 3, we show the dynamic behaviors of both methods when $\theta = 0$ (the secant method), $\theta = 1/3$ and $\theta = 2/3$. We observe that method (4) has a better dynamic behavior than method (3) when $\theta = 0$ and $\theta = 1/3$. Moreover, both methods have a similar dynamic behavior as the parameter θ increases and approaches $\theta = 1$. This is logical, since both methods with $\theta = 1$ coincide with Newton's method. Therefore, we can affirm that the family of iterative methods (4) has better accessibility than the family of iterative methods (3).

Once the accessibility of the families (3) and (4) has been graphically analyzed, we see their behavior in a numerical way. For this, we compute the percentage of points that converge, after $N = 10$ iterations, to any of the zeros with a tolerance of 10^{-6}. We collect this information in Table 2, where we see that the accessibility of the methods of the family (4) is better than that of the methods of the family (3).

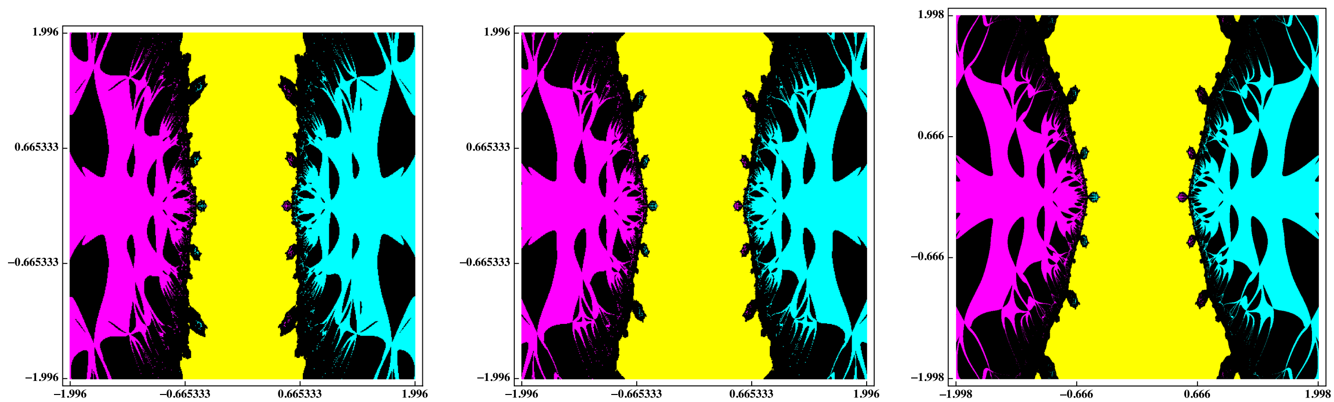
5.2 | Second strategy: z_{-1} and z_0 are free

In this case, we consider the behavior of the real dynamics[21] of the families (3) and (4) to study the accessibility of both families. For this, since the roots of the equation $F(z) = z^2 + 2z|z| - 2z = 0$ are real numbers, we consider z_{-1} and z_0 as real numbers. We represent the values of z_0 on the horizontal axis of the plane and the values of z_{-1} on the vertical axis. With the strategy indicated above of a tolerance of 10^{-6} and a maximum number N of iterations, we draw the convergence planes for both families of iterative methods.



Iteration (3) with $\theta = 0$ Iteration (3) with $\theta = 1/3$ Iteration (3) with $\theta = 2/3$

FIGURE 2 Basins of attraction of the three zeros of $F(z) = z^3 - z|z| - 2z$ when the iteration (3) is applied and $N = 10$ [Colour figure can be viewed at wileyonlinelibrary.com]



Iteration (4) with $\theta = 0$ Iteration (4) with $\theta = 1/3$ Iteration (4) with $\theta = 2/3$

FIGURE 3 Basins of attraction of the three zeros of $F(z) = z^3 - z|z| - 2z$ when the iteration (4) is applied and $N = 10$ [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 2 Percentage of the convergence points for $F(z) = z^3 - z|z| - 2z$

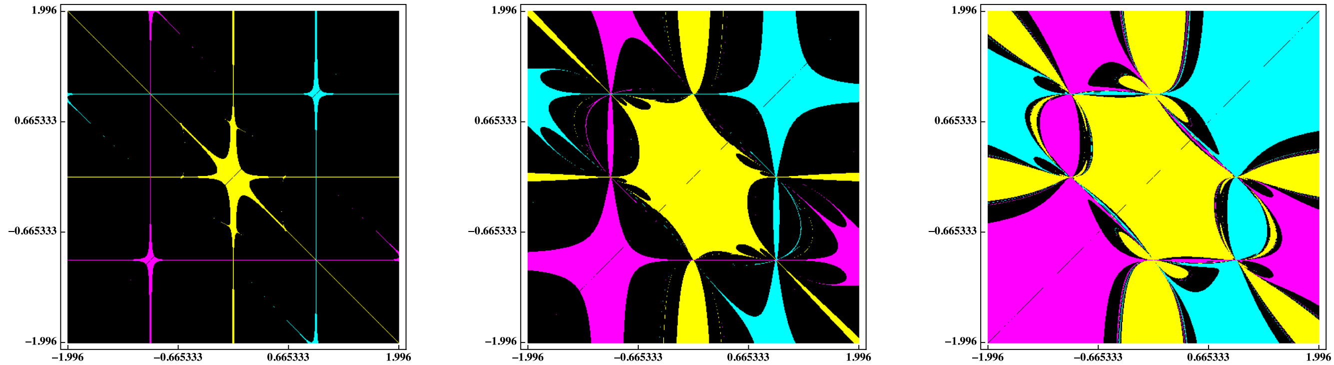
θ	Method (3)	Method (4)
0	57.05 %	62.30 %
1/3	61.75 %	63.86 %
2/3	66.91 %	63.99 %

In Figures 4–9, the real dynamic behaviors of both methods, (3) and (4), for $\theta = 0, 1/3, 2/3$ and $N = 10, 20, 30$ are shown. We observe graphically that the dynamic behavior of the family (4) is better than that of the family (3), so that the accessibility of the family (4) is better than that of the family (3). Observe that, by increasing the value of N, when the value of $\theta = 1$ is approximated, both real dynamics are similar.

Once the accessibility has been analyzed graphically, we see the real dynamic behavior in a numerical way. For this, we compute the percentage of points that converges to any of the roots after N iterations with a tolerance of 10^{-6}. We collect this information in Table 3, where we see that the accessibility of the family (4) is better than that of the family (3).

6 | PARTICULAR CASES

Remember that the iteration (4) is reduced to Newton's method for $\theta = 1$ if the operator F is differentiable, namely, $H \equiv 0$, and to the family of secant-type methods (3) if $G \equiv 0$. As a consequence, the previous study of the local convergence allows obtaining results of local convergence for Newton's method and the family of secant-type methods (3). The result

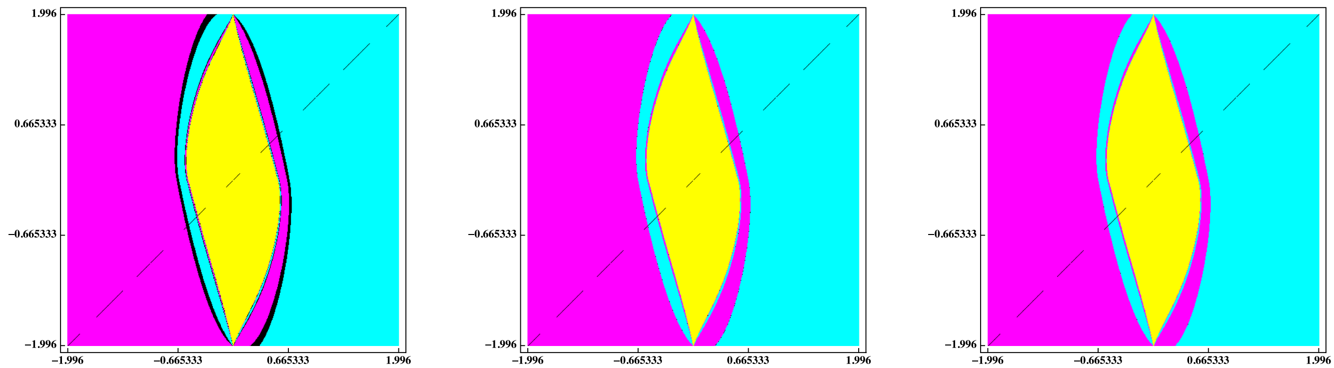


Iteration (3) with $\theta = 0$ and $N = 10$

Iteration (3) with $\theta = 0$ and $N = 20$

Iteration (3) with $\theta = 0$ and $N = 30$

FIGURE 4 Basins of attraction of the three zeros of $F(z) = z^3 - z|z| - 2z$ when the iteration (3) is applied [Colour figure can be viewed at wileyonlinelibrary.com]

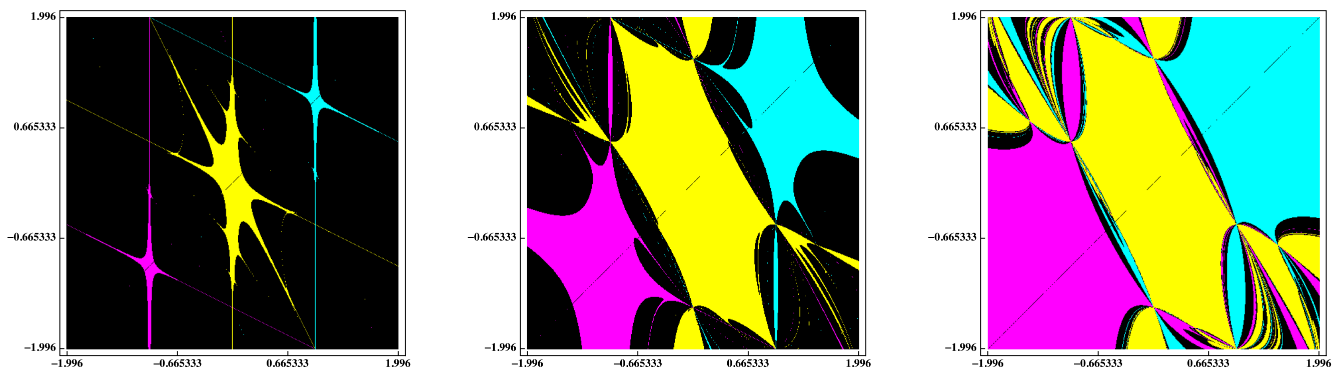


Iteration (4) with $\theta = 0$ and $N = 10$

Iteration (4) with $\theta = 0$ and $N = 20$

Iteration (4) with $\theta = 0$ and $N = 30$

FIGURE 5 Basins of attraction of the three zeros of $F(z) = z^3 - z|z| - 2z$ when the iteration (4) is applied [Colour figure can be viewed at wileyonlinelibrary.com]



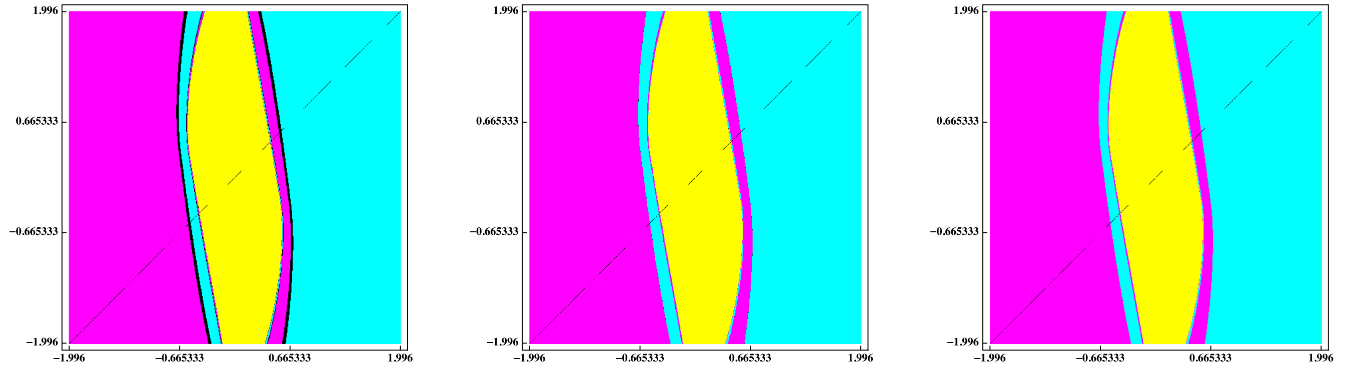
Iteration (3) with $\theta = \frac{1}{3}$ and $N = 10$

Iteration (3) with $\theta = \frac{1}{3}$ and $N = 20$

Iteration (3) with $\theta = \frac{1}{3}$ and $N = 30$

FIGURE 6 Basins of attraction of the three zeros of $F(z) = z^3 - z|z| - 2z$ when the iteration (3) is applied [Colour figure can be viewed at wileyonlinelibrary.com]

of local convergence obtained for Newton's method is in line with those obtained by other authors[10,25] and that obtained for the family (3) is a result for nondifferentiable operators.

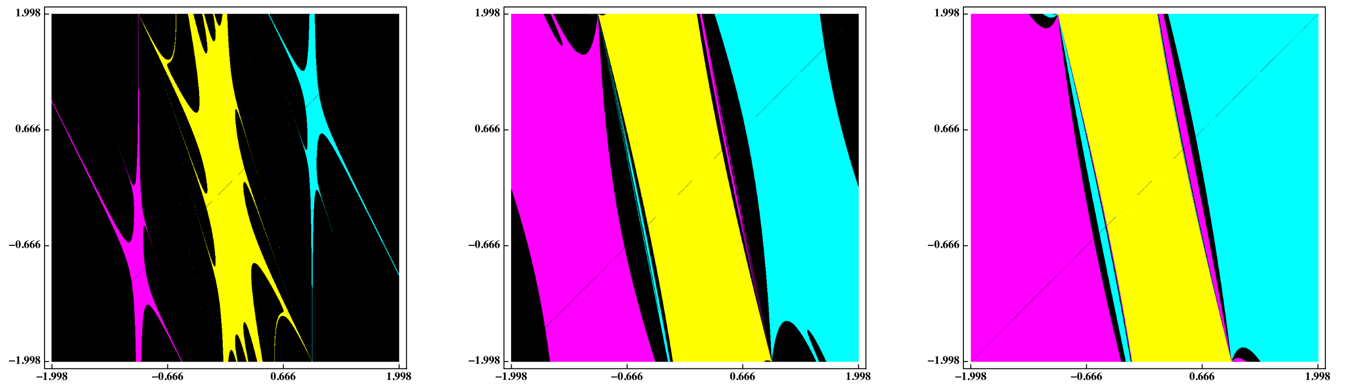


Iteration (4) with $\theta = \frac{1}{3}$ and $N = 10$

Iteration (4) with $\theta = \frac{1}{3}$ and $N = 20$

Iteration (4) with $\theta = \frac{1}{3}$ and $N = 30$

FIGURE 7 Basins of attraction of the three zeros of $F(z) = z^3 - z|z| - 2z$ when the iteration (4) is applied [Colour figure can be viewed at wileyonlinelibrary.com]

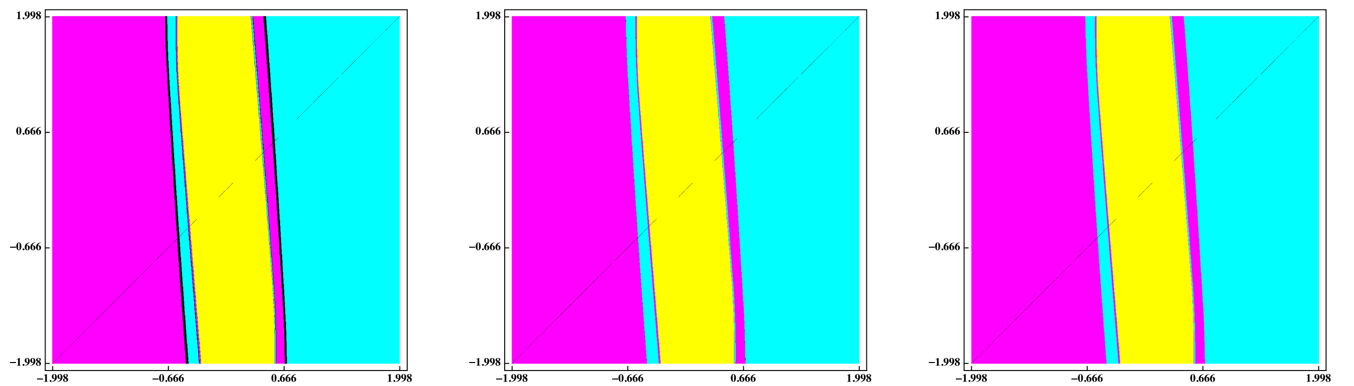


Iteration (3) with $\theta = \frac{2}{3}$ and $N = 10$

Iteration (3) with $\theta = \frac{2}{3}$ and $N = 20$

Iteration (3) with $\theta = \frac{2}{3}$ and $N = 30$

FIGURE 8 Basins of attraction of the three zeros of $F(z) = z^3 - z|z| - 2z$ when the iteration (3) is applied [Colour figure can be viewed at wileyonlinelibrary.com]



Iteration (4) with $\theta = \frac{2}{3}$ and $N = 10$

Iteration (4) with $\theta = \frac{2}{3}$ and $N = 20$

Iteration (4) with $\theta = \frac{2}{3}$ and $N = 30$

FIGURE 9 Basins of attraction of the three zeros of $F(z) = z^3 - z|z| - 2z$ when the iteration (4) is applied [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 3 Percentage of convergence points for $F(z) = z^3 + 2z|z| - 2z$

θ	N	Method (3)	Method (4)
	10	3.96 %	97.35 %
0	20	38.45 %	99.85 %
	30	85.58 %	99.88 %
	10	5.89 %	97.92 %
1/3	20	50.65 %	99.86 %
	30	88.27 %	99.89 %
	10	15.62 %	98.34 %
2/3	20	80.73 %	99.91 %
	30	96.14 %	99.9 %

6.1 | Newton's method

If the operator F is differentiable, then $F(x) = G(x)$, since $H(x) = 0$. In this case, the condition (C2) is not necessary and Equation (6) is reduced to

$$\delta(1 + I_h)\omega_1(\rho) - 1 = 0. \tag{13}$$

In addition, we can establish the following local result of convergence for Newton's method.

Theorem 9. *Suppose that the operator F is differentiable (viz., $F(x) = G(x)$) and the conditions (C1) and*

(N3) Let x^* be a solution of $F(x) = 0$ such that there exists the operator $[F'(x^*)]^{-1}$ and $\|[F'(x^*)]^{-1}\| \leq \delta$.
(N4) Equation (13) has at least one positive real root and we denote the smallest positive real root by r and consider $\rho_N \in \mathbb{R}_+$ such that $\rho_N < r$ with $B(x^*, \rho_N) \subset \Omega$.

If $x_0 \in B(x^*, \rho_N)$, then the sequence given by Newton's method is well defined, $x_n \in B(x^*, \rho_N)$, for all $n \geq 0$, and is convergent to a solution x^* of the equation $F(x) = 0$.

Next, we illustrate the last theorem with a nonlinear integral equations of type (7) with $\mu = 0$.

Example 10. We consider (7) with $f(s) = 0$, $a = 0$, $b = 1$, $\lambda = \frac{1}{2}$, $\mu = 0$ and the kernel $\mathcal{K}(s,t)$ as the Green function in $[0,1] \times [0,1]$. Following the same process of discretization as for Example (7), we transform the integral equation in the system

$$F(\mathbf{x}) \equiv \mathbf{x} - \frac{1}{2}A\bar{\bar{\mathbf{x}}} = 0, \qquad F : \mathbb{R}^8 \to \mathbb{R}^8,$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_8)^T$, $A = (a_{ij})_{i,j=1}^m$ and $\bar{\bar{\mathbf{x}}} = \left(x_1^2, x_2^2, \ldots, x_8^2\right)^T$. Then, for Theorem 9, we have

$$\|A\| = 0.1235\ldots, \quad \delta = \|[F'(x^*)]^{-1}\| = 1, \quad I_h = \frac{1}{2},$$

so that Equation (13) is satisfied if $\rho < 5.3953\ldots$. Therefore, the convergence ball is $B(\mathbf{x}^*, \rho_N)$ with $\rho_N < 5.3953\ldots$. Moreover, from Theorem 5, the solution is unique in the ball $\overline{B(\mathbf{x}^*, R)}$ with $R < 16.1866\ldots$.

We finish the particular case of Newton's method with a comparative study of the convergence ball. For this, we compare the convergence ball obtained by Dennis and Schnabel in their well-known study given for the local convergence of Newton's method.[10] So, we suppose that F' is Lipschitz continuous in Ω, so that

$$\text{there exists } L \geq 0 \text{ such that } \|F'(x) - F'(y)\| \leq L\|x - y\|, \text{ for all } x, y \in \Omega.$$

As a consequence, $\omega_1(z) = Lz$, $h(t) = t$ and $I_h = \int_0^1 h(t)dt = \frac{1}{2}$. In addition, Equation (13) is reduced to $\frac{3}{2}\delta L\rho - 1 = 0$, so that $r = \frac{2}{3L\delta}$ is the unique positive real root of the last equation and we can then consider ρ_N such that $\rho_N < \frac{2}{3L\delta}$.

Note that Dennis and Schnabel[10] obtain the convergence ball $B(x^*, \tilde{\rho})$ with $\tilde{\rho} < \frac{1}{2L\delta}$ under the same conditions as above, so that the convergence ball obtained by them is improved from Theorem 9. Moreover, Rheinboldt[25] obtains the same convergence ball as that obtained from Theorem 9.

t	ρ_S	R
0.0	4.4070 …	13.2210 …
0.2	4.4736 …	13.4210 …
0.4	4.5403 …	13.6210 …
0.6	4.6070 …	13.8210 …
0.8	4.6736 …	14.0210 …
1.0	4.7403 …	14.2210 …

TABLE 4 $B(\mathbf{x}^*, \rho_S)$ and $\overline{B(\mathbf{x}^*, R)}$ when $\tilde{\mathbf{x}} = t\mathbf{x}^* + (1-t)\mathbf{x}_0$, where $\mathbf{x}^* = \mathbf{0}$

6.2 | Secant-type methods

Now, we consider $G(x) = 0$ and then $F(x) = H(x)$. As a consequence, the iteration (4) is reduced to the family of secant-type methods (3). In this case, the condition (C1) is not necessary and Equation (6) is reduced to

$$\delta(\omega_2(\rho, \rho+\alpha) + \omega_2(\rho, 0)) - 1 = 0. \tag{14}$$

In addition, we can establish the following local result of convergence for the family of secant-type methods (3).

Theorem 11. *Suppose the conditions (C2), (C3), and*

(S4) Equation (14) has at least one positive real root and we denote the smallest positive real root by r and consider $\rho_S \in \mathbb{R}_+$ such that $\rho_S < r$ with $B(x^*, \rho_S) \subset \Omega$.

If $x_0 \in B(x^*, \rho_S)$ and $x_{-1} \in B(x_0, \rho_S - \eta)$ with $x_0 \neq x_{-1}$, then the sequence given by the family of secant-type methods (3) is well defined, $x_n \in B(x^*, \rho_S)$, for all $n \geq 0$, and is convergent to a solution x^* of the equation $F(x) = 0$.

Observe that we can choose $F(x) = H(x)$ if F is nondifferentiable, so that we can apply directly the family of secant-type methods (3). According to this, we consider the following example, where we see, from the point of view of the convergence ball, that the option presented in this work, $F(x) = G(x) + H(x)$, is a better option, since the convergence ball obtained is improved with respect to the option $F(x) = H(x)$.

Example 12. We consider Example 7 with

$$F(\mathbf{x}) = H(\mathbf{x}) = \mathbf{x} - \mathbf{f} - A\hat{\mathbf{x}}.$$

Then, for Theorem 11, we have $\omega_2(s,t) = \frac{1}{2}\|A\|(s+t+2)$, $\alpha = 1$, $D^{-1} = [x^*, \tilde{x}; H]^{-1}$, $\delta = 1.1382\ \ldots$, and Equation (14) is satisfied if $\rho < 3.0736\ \ldots$. Therefore, the convergence ball is $B(\mathbf{x}^*, \rho_S)$ with $\rho_S < 3.0736\ \ldots$. Note that this radius of the convergence ball is improved in Example 7, where $F(\mathbf{x})$ is decomposed as $F(\mathbf{x}) = G(\mathbf{x}) + H(\mathbf{x})$. In addition, from Theorem 5, the solution is unique in the ball $\overline{B(\mathbf{x}^*, R)}$ with $R < 11.2210\ \ldots$, which is also worse than that obtained in Example 7. As a consequence, the situation discussed in Section 2 is better than the situation discussed in this section.

We finish the example by observing Table 4. On the one hand, we obtain better convergence balls $B(\mathbf{x}^*, \rho_S)$ as the auxiliary point $\tilde{\mathbf{x}}$ is such that $\tilde{\mathbf{x}} = t\mathbf{x}^* + (1-t)\mathbf{x}_0 = (1-t)\mathbf{x}_0$, since $\mathbf{x}^* = \mathbf{0}$, and gets further away from $\mathbf{x}^*$. On the other hand, we obtain better balls of uniqueness of solution $\overline{B(\mathbf{x}^*, R)}$ as the auxiliary point $\tilde{\mathbf{x}}$ approaches $\mathbf{x}^*$.

7 | CONCLUSIONS

We usually use iterative methods to solve nonlinear operator equations of the form $F(x) = 0$. When the operator F is differentiable, Newton's method is the most used iteration to solve the equation, but, if the operator F is nondifferentiable, the most used iteration is the well-known secant method, which uses divided differences of first order instead of derivatives in its algorithm.

As the accessibility of the secant method, the set of initial guesses that guarantee the convergence of the secant method is usually a problem in the application of the method, we improve it in this work by using the technique of decomposition of the operator F, so that $F(x) = G(x) + H(x)$, where G is differentiable and H is continuous but nondifferentiable. For this, we use the family of Newton-secant-type iterative methods given in (4), which is reduced to Newton's method if $H(x) = 0$ (i.e., F is differentiable) and to the family of secant-type methods given in (3) if $G(x) = 0$. We study the accessibility of these new methods from two different points of view: First, theoretically, from the convergence balls of the methods obtained from the local study of their convergence and, second, experimentally, from a dynamic study of the methods.

CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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