Bull. Malays. Math. Sci. Soc. (2023) 46:64
https://doi.org/10.1007/s40840-022-01455-1

BULLETIN of the MALAYSIAN MATHEMATICAL SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

Some Appell–Dunkl Sequences

Judit Mínguez Ceniceros[1]

Received: 26 July 2022 / Revised: 1 December 2022 / Accepted: 26 December 2022
© The Author(s) 2023

Abstract

Some examples of Appell–Dunkl sequences are shown using determined operators. Specifically, Appell–Dunkl sequences whose generating functions are of the form $E_\alpha(xt)/(1 \pm t^m)$, where the function $E_\alpha(xt)$ is given in terms of Bessel functions. Particular cases of these examples are also generated by means of the inverse of the Dunkl operator.

Keywords Appell sequences · Dunkl transform · Appell–Dunkl sequences

Mathematics Subject Classification Primary 11B83; Secondary 33C10 · 11B68

1 Introduction

An Appell sequence $\{P_n(x)\}_{n=0}^{\infty}$ is a sequence of polynomials such that

$$P_0(x) = c \neq 0, \quad \frac{d}{dx} P_n(x) = n P_{n-1}(x),\ n \geq 1. \tag{1.1}$$

It is well known that Appell sequences may be also defined by means of a generating function

$$A(t)e^{xt} = \sum_{n=0}^{\infty} \frac{P_n(x)}{n!} t^n, \tag{1.2}$$

Communicated by Emrah Kilic.

Supported by PID2021-124332NB-C22 (Ministerio de Ciencia e Innovación).

✉ Judit Mínguez Ceniceros
judit.minguez@unirioja.es

1 Departamento de Matemáticas y Computación, Universidad de La Rioja, 26006 Logroño, Spain

Published online: 10 January 2023

where $A(t)$ is a function analytic at $t = 0$ with $A(0) \neq 0$. Typical examples of Appell polynomials are: $\{x^n\}_{n=0}^{\infty}$ with $A(t) = 1$; Hermite polynomials (see [1]), $\{\mathrm{He}_n(x)\}_{n=0}^{\infty}$, with $A(t) = e^{-t^2/2}$; Bernoulli polynomials (see [7, 11]), $\{B_n(x)\}_{n=0}^{\infty}$, with $A(t) = t/(e^t - 1)$; or Euler polynomials (see [7, 11]), $\{E_n(x)\}_{n=0}^{\infty}$, with $A(t) = 2/(e^t + 1)$. These polynomials have been widely studied in the last two centuries because they have many applications to number theory, numerical analysis, combinatorics and other areas.

In [12], they use an operator of the type

$$\mathcal{A}(\widehat{D}) = \sum_{k=0}^{\infty} \frac{c_k}{k!} \widehat{D}^k, \tag{1.3}$$

where $\widehat{D} = d/dx$ and $\{c_k\}_{k=0}^{\infty}$ is a sequence of constants in order to obtain several examples of Appell polynomials. In particular, they get Appell sequences whose generating functions are $e^{xt}/(1 \pm t^m)$. The cases $m = 1$ or $m = 2$ are also studied by means of some integrals.

If in (1.1) the derivative operator is changed by the Dunkl operator Λ_α defined by

$$\Lambda_\alpha f(x) = \frac{d}{dx} f(x) + \frac{2\alpha + 1}{2} \left(\frac{f(x) - f(-x)}{x} \right), \tag{1.4}$$

where $\alpha > -1$ is a fixed parameter (see [9, 17]), we obtain the Appell–Dunkl polynomials, $\{P_{n,\alpha}(x)\}_{n=0}^{\infty}$, that satisfy

$$P_{0,\alpha}(x) = c \neq 0, \quad \Lambda_\alpha P_{n,\alpha}(x) = \theta_{n,\alpha} P_{n-1,\alpha}(x), \tag{1.5}$$

where the multiplicative constant $\theta_{n,\alpha}$ (that will be defined later) in the place of n is used for convenience with the notation. Of course, in the case $\alpha = -1/2$, the operator Λ_α is the ordinary derivative and Appell–Dunkl sequences become the classical Appell sequences. Appell–Dunkl sequences can be also defined by means of a generating function

$$A(t) E_\alpha(xt) = \sum_{n=0}^{\infty} \frac{P_{n,\alpha}(x)}{\gamma_{n,\alpha}} t^n, \tag{1.6}$$

with $A(t)$ an analytic function at $t = 0$, $A(0) \neq 0$, and where $E_\alpha(xt)$ is an analytic function defined in terms of Bessel functions that plays the role of the exponential in the classical case (1.2), and the coefficients $\gamma_{n,\alpha}$ will be defined in the next section as a kind of factorial numbers. The generalizations of Bernoulli and Euler polynomials to the Dunkl context have been studied in [5, 6, 10, 15], and the extension of Hermite polynomials can be found in [17]. Appell–Dunkl sequences have been also considered, for instance, in [2, 3, 8].

There are not many explicit examples of Appell–Dunkl polynomials in the literature, except for those mentioned above. So, the main goal of this paper is to generate some examples of this kind of sequences using an operator analogous to (1.3) in the Dunkl context.

Using the inverse of the Dunkl operator, we are going to generate some examples of Appell–Dunkl sequences in a different way, in particular, one family where the polynomials of degree even and odd can be defined by means of determined generating functions in terms of Bessel functions. The even case provides us an example of the polynomials developed in [4]. A general study about the quadratic decomposition of Appell sequences has been studied in [14].

The structure of this paper is as follows. In Sect. 2, we see the required definitions and results of the Dunkl universe. Then, in Sect. 3 we give some families of Appell–Dunkl polynomials using the corresponding operators. Finally, in Sect. 4 some families of Appell–Dunkl polynomials, obtained also in Sect. 3, are established with the help of Dunkl primitives.

2 Dunkl definitions

For $\alpha > -1$, let J_α denote the Bessel function of order α, and for complex values of the variable z, let

$$\mathcal{I}_\alpha(z) = 2^\alpha \Gamma(\alpha+1)\,\frac{J_\alpha(iz)}{(iz)^\alpha} = \Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(z/2)^{2n}}{n!\,\Gamma(n+\alpha+1)} = {}_0F_1(\alpha+1, z^2/4)$$

(the function $\mathcal{I}_\alpha$ is a small variation of the so-called modified Bessel function of the first kind and order α, usually denoted by I_α; see [13, 16], or [18]). Let $\mathcal{G}_\alpha$ denote the function

$$\mathcal{G}_\alpha(z) = \frac{z}{2(\alpha+1)}\mathcal{I}_{\alpha+1}(z).$$

With these two functions, the following analytic function is defined

$$E_\alpha(z) = \mathcal{I}_\alpha(z) + \mathcal{G}_\alpha(z), \qquad z \in \mathbb{C}.$$

For any $\lambda \in \mathbb{C}$, we have

$$\Lambda_\alpha E_\alpha(\lambda x) = \lambda E_\alpha(\lambda x), \tag{2.1}$$

and

$$\Lambda_\alpha \mathcal{I}_\alpha(\lambda x) = \lambda \mathcal{G}_\alpha(\lambda x), \quad \Lambda_\alpha \mathcal{G}_\alpha(\lambda x) = \lambda \mathcal{I}_\alpha(\lambda x). \tag{2.2}$$

Let us note that, when $\alpha = -1/2$, we have $\Lambda_{-1/2} = d/dx$, $E_{-1/2}(\lambda x) = e^{\lambda x}$, $\mathcal{I}_\alpha(z) = \cosh z$ and $\mathcal{G}_\alpha(z) = \sinh(z)$.

From the definition of $E_\alpha(z)$, it is easy to check that

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\gamma_{n,\alpha}} \tag{2.3}$$

with

$$\gamma_{n,\alpha} = \begin{cases} 2^{2k} k!\,(\alpha+1)_k, & \text{if } n = 2k, \\ 2^{2k+1} k!\,(\alpha+1)_{k+1}, & \text{if } n = 2k+1, \end{cases} \tag{2.4}$$

and where $(a)_n$ denotes the Pochhammer symbol

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$$

(with a a nonnegative integer); of course, $\gamma_{n,-1/2} = n!$. From (2.4), we have

$$\frac{\gamma_{n,\alpha}}{\gamma_{n-1,\alpha}} = n + (\alpha + 1/2)(1 - (-1)^n) =: \theta_{n,\alpha}. \tag{2.5}$$

It also holds that

$$\mathcal{I}_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^{2k}}{\gamma_{2k,\alpha}}, \qquad \mathcal{G}_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{\gamma_{2k+1,\alpha}}. \tag{2.6}$$

We also define

$$\binom{n}{j}_\alpha = \frac{\gamma_{n,\alpha}}{\gamma_{j,\alpha}\gamma_{n-j,\alpha}}$$

that becomes the ordinary binomial numbers in the case $\alpha = -1/2$. To simplify the notation, we sometimes write $\gamma_n = \gamma_{n,\alpha}$ and $\theta_n = \theta_{n,\alpha}$.

Then, a sequence of Appell–Dunkl polynomials, $\{P_{n,\alpha}(x)\}_{n=0}^{\infty}$, may be defined as a sequence that satisfies (1.5) where $\theta_{n,\alpha}$ is given by (2.5), or as a sequence whose generating function is (1.6).

The inverse of the derivative operator, d/dx, leads to the concept of primitive of a function. It is well known that this primitive is unique except by an additive constant. In the Dunkl case, we could propose that a function F is a Dunkl primitive of f if $\Lambda_\alpha F = f$. It would require that this function, F, was unique except by an additive constant. This may be reduced to prove that $\Lambda_\alpha F = 0$, $F \in \mathcal{C}^1(\mathbb{R})$, if and only if F is a constant. If the function F is even, as $\Lambda_\alpha F = \frac{d}{dx} F$, then $\Lambda_\alpha F = 0$ implies that F is a constant. If F is an odd function and $\Lambda_\alpha F = 0$, then F should be of the form $Cx^{-2\alpha-1}$ and would not be in $\mathcal{C}^1(\mathbb{R})$ for $\alpha > -1$. So, F has to be a constant. As every function, F, can be expressed uniquely in the way $F_1 + F_2$ where F_1 is an even function and F_2 is an odd function, and the operator Λ_α transforms an even function in an odd function, and vice versa, we can define, for $\alpha > -1$, the Dunkl integral of a function f as

$$\oint f(x)\, d_\alpha x = F(x) + c,$$

where $c \in \mathbb{R}$ is a constant.

As $\Lambda_\alpha((\cdot)^{n+1})(x) = \theta_{n+1}x^n$, we have

$$\oint x^n\, d_\alpha x = \frac{x^{n+1}}{\theta_{n+1}} + c, \qquad n = 0, 1, 2, \dots,$$

and then, for a polynomial $p(x) = \sum_{k=0}^{n} a_k x^k$, its Dunkl primitive is

$$\oint p(x)\, d_\alpha x = \sum_{k=0}^{n} \frac{a_k}{\theta_{k+1}} x^{k+1} + c.$$

Let us also note that, by (1.5), the Dunkl primitive of the Appell–Dunkl polynomial $P_{n,\alpha}(x)$ is $P_{n+1,\alpha}(x)/\theta_{n+1} + c$.

$$\oint P_{n,\alpha}(x)\, d_\alpha x = \frac{P_{n+1,\alpha}(x)}{\theta_{n+1}} + c, \qquad n = 0, 1, 2, \dots.$$

It is also interesting to note that, from (2.1) with $\lambda = 1$, we can write

$$\oint E_\alpha(x)\, d_\alpha x = E_\alpha(x) + c, \tag{2.7}$$

and from (2.2)

$$\oint \mathcal{I}_\alpha(x)\, d_\alpha x = \mathcal{G}_\alpha(x) + c, \quad \oint \mathcal{G}_\alpha(x)\, d_\alpha x = \mathcal{I}_\alpha(x) + c. \tag{2.8}$$

An analogous to the formula of integration by parts in this context is proved.

Lemma 2.1 *Let f and g be two functions in $C^1(\mathbb{R})$. Then*

$$\oint \Lambda_\alpha f(x) g(x)\, d_\alpha x = f(x)g(x) + \frac{2\alpha+1}{2} \oint h(x)\, d_\alpha x - \oint f(x) \Lambda_\alpha g(x)\, d_\alpha x \tag{2.9}$$

where

$$h(x) = \frac{(f(x) - f(-x))(g(x) - g(-x))}{x}.$$

Proof Let

$$A(x) = \Lambda_\alpha f(x)g(x) = \frac{df(x)}{dx}g(x) + \frac{2\alpha+1}{2}\frac{f(x)-f(-x)}{x}g(x)$$

and

$$B(x) = f(x)\Lambda_\alpha g(x) = f(x)\frac{dg(x)}{dx} + \frac{2\alpha+1}{2}f(x)\frac{g(x)-g(-x)}{x},$$

so

$$\begin{aligned} A(x)+B(x) &= \frac{df(x)}{dx}g(x) + f(x)\frac{dg(x)}{dx} + (2\alpha+1)\frac{f(x)g(x)}{x} \\ &\quad - \frac{2\alpha+1}{2}\frac{f(-x)g(x)+f(x)g(-x)}{x} \\ &= \frac{d}{dx}\Big(f(x)g(x)\Big) + \frac{2\alpha+1}{2}\frac{f(x)g(x)-f(-x)g(-x)}{x} \\ &\quad + \frac{2\alpha+1}{2}\frac{f(x)g(x)-f(-x)g(x)-f(x)g(-x)+f(-x)g(-x)}{x} \\ &= \Lambda_\alpha(fg)(x) + \frac{2\alpha+1}{2}h(x). \end{aligned}$$

Then, applying the Dunkl integral operator

$$\oint A(x)\, d_\alpha x + \oint B(x)\, d_\alpha x = f(x)g(x) + \frac{2\alpha+1}{2}\oint h(x)\, d_\alpha x$$

and the proof is concluded. □

3 Appell–Dunkl polynomials

In this section, some sequences of Appell–Dunkl polynomials are generated with the help of determined operators which are defined as follows.

Let $\{c_k\}_{k=0}^{\infty}$ be a sequence of numbers with $c_0 \neq 0$ and $\mathcal{A}(\Lambda_\alpha)$ the operator

$$\mathcal{A}(\Lambda_\alpha) = \sum_{k=0}^{\infty}\frac{c_k}{\gamma_k}\Lambda_\alpha^k. \tag{3.1}$$

This operator can be applied to $\mathcal{C}^\infty$ functions and assuming also that the series obtained is convergent. In particular, this is always true for polynomials because the operator Λ_α applied to a polynomial of degree n generates a polynomial of degree $n-1$, so the operator (3.1) applied to polynomials has only a finite quantity on not null summands.

Now, it may be stated the main result of this paper.

Theorem 3.1 *Let $\{c_k\}_{k=0}^{\infty}$ be a sequence of numbers with $c_0 \neq 0$ and let $\mathcal{A}(\Lambda_\alpha)$ be the operator (3.1). Then the sequence of polynomials $\{P_{n,\alpha}(x)\}_{n=0}^{\infty}$ given by*

$$P_{n,\alpha}(x) = \mathcal{A}(\Lambda_\alpha)x^n = \left(\sum_{k=0}^{\infty} \frac{c_k}{\gamma_k}\Lambda_\alpha^k\right)x^n$$

is an Appell–Dunkl sequence of polynomials. Moreover, the generating function of these polynomials is $A(t)E_\alpha(xt)$ where the analytic function $A(t)$ is given by

$$A(t) = \sum_{k=0}^{\infty} \frac{c_k}{\gamma_k}t^k. \tag{3.2}$$

Proof We will see that the sequence $\{P_{n,\alpha}(x)\}_{n=0}^{\infty}$ satisfies the requirements of Definition 1.5.

$$P_{0,\alpha}(x) = \mathcal{A}(\Lambda_\alpha)x^0 = c_0 \neq 0,$$

and for $n \geq 1$,

$$\begin{aligned}\Lambda_\alpha P_{n,\alpha}(x) &= \Lambda_\alpha \left(\sum_{k=0}^{\infty} \frac{c_k}{\gamma_k}\Lambda_\alpha^k\right)x^n = \sum_{k=0}^{\infty} \frac{c_k}{\gamma_k}\Lambda_\alpha^k \Lambda_\alpha x^n \\ &= \theta_n \left(\sum_{k=0}^{\infty} \frac{c_k}{\gamma_k}\Lambda_\alpha^k\right)x^{n-1} = \theta_n P_{n-1,\alpha}(x).\end{aligned}$$

So, they are Appell–Dunkl polynomials.

Taking into account (2.3) and (3.2), the generating function, $G_\alpha(x,t)$, of these polynomials can be obtained in the following way

$$\begin{aligned}G_\alpha(x,t) &= \sum_{n=0}^{\infty} P_{n,\alpha}(x)\frac{t^n}{\gamma_n} = \sum_{n=0}^{\infty} \mathcal{A}(\Lambda_\alpha)x^n \frac{t^n}{\gamma_n} = \mathcal{A}(\Lambda_\alpha)E_\alpha(xt) \\ &= \sum_{k=0}^{\infty} \frac{c_k}{\gamma_k}\Lambda_\alpha^k E_\alpha(xt) = \sum_{k=0}^{\infty} \frac{c_k t^k}{\gamma_k}E_\alpha(xt) = A(t)E_\alpha(xt).\end{aligned} \tag{3.3}$$

□

Remark 1 By the definition of Λ_α, (1.4), it is easy to see that

$$\Lambda_\alpha^k x^n = \frac{\gamma_n}{\gamma_{n-k}}x^{n-k}.$$

So, the Appell–Dunkl polynomials can be written as

$$P_{n,\alpha}(x) = \sum_{k=0}^{n} \frac{c_k}{\gamma_k} \Lambda_\alpha^k x^n = \sum_{k=0}^{n} \frac{c_k}{\gamma_k} \frac{\gamma_n}{\gamma_{n-k}} x^{n-k} = \sum_{k=0}^{n} c_k \binom{n}{k}_\alpha x^{n-k}. \tag{3.4}$$

3.1 Generating function $E_\alpha(xt)/(1-t)$

Let $e_{n,\alpha}(x)$ be the nth truncated polynomials of $E_\alpha(x)$, that is,

$$e_{n,\alpha}(x) = \sum_{k=0}^{n} \frac{x^k}{\gamma_k}. \tag{3.5}$$

We denote

$$P_{n,\alpha}^{(1-)}(x) = \gamma_n e_{n,\alpha}(x). \tag{3.6}$$

Then it may be proved the following result.

Theorem 3.2 *The sequence $\{P_{n,\alpha}^{(1-)}(x)\}_{n=0}^{\infty}$ is an Appell–Dunkl sequence of polynomials whose generating function is*

$$G_\alpha^{(1-)}(x,t) = \frac{E_\alpha(xt)}{1-t}.$$

Moreover, these polynomials satisfy the recurrence relation

$$P_{n,\alpha}^{(1-)}(x) - \theta_n P_{n-1,\alpha}^{(1-)}(x) = x^n. \tag{3.7}$$

Proof We must simply apply Theorem 3.1 to the sequence of numbers $c_k = \gamma_k$, $k = 0, 1, 2, \ldots$ Then substituting c_k in (3.4), we have

$$P_{n,\alpha}^{(1-)}(x) = \gamma_n \sum_{k=0}^{n} \frac{x^{n-k}}{\gamma_{n-k}} = \gamma_n e_{n,\alpha}(x).$$

From (3.3), the generating function, $G_\alpha^{(1-)}(x,t)$, for $\{P_{n,\alpha}^{(1-)}(x)\}_{n=0}^{\infty}$ is given by

$$\begin{aligned} G_\alpha^{(1-)}(x,t) &= \sum_{n=0}^{\infty} P_{n,\alpha}^{(1-)}(x) \frac{t^n}{\gamma_n} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \Lambda_\alpha^k \right) \frac{x^n t^n}{\gamma_n} \\ &= \sum_{k=0}^{\infty} \Lambda_\alpha^k E_\alpha(xt) = \sum_{k=0}^{\infty} t^k E_\alpha(xt). \end{aligned}$$

Using (3.2) with $c_k = \gamma_k$, we obtain

$$\frac{E_\alpha(xt)}{1-t} = \sum_{n=0}^{\infty} P_{n,\alpha}^{(1-)}(x)\frac{t^n}{\gamma_n}.$$

In order to obtain the recurrence relation, note that

$$\left(\sum_{k=0}^{\infty} \Lambda_\alpha^k - \Lambda_\alpha \sum_{k=0}^{\infty} \Lambda_\alpha^k\right) x^n = x^n.$$

So, from the definition of $\mathcal{A}(\Lambda_\alpha)$ with $c_k = \gamma_k$

$$P_{n,\alpha}^{(1-)}(x) - \Lambda_\alpha P_{n,\alpha}^{(1-)}(x) = x^n.$$

Applying the Dunkl operator to $P_{n,\alpha}^{(1-)}(x)$, (3.7) is obtained. □

3.2 Generating function $E_\alpha(xt)/(1+t)$

We denote by $P_{n,\alpha}^{(1+)}$ the following polynomial

$$P_{n,\alpha}^{(1+)}(x) = (-1)^n \gamma_n e_{n,\alpha}(-x). \tag{3.8}$$

Theorem 3.3 *The sequence $\{P_{n,\alpha}^{(1+)}(x)\}_{n=0}^{\infty}$ is an Appell–Dunkl sequence of polynomials whose generating function is*

$$G_\alpha^{(1+)}(x,t) = \frac{E_\alpha(xt)}{1+t}.$$

Moreover, these polynomials satisfy the recurrence relation

$$P_{n,\alpha}^{(1+)}(x) + \theta_n P_{n-1,\alpha}^{(1+)}(x) = x^n. \tag{3.9}$$

Proof We apply Theorem 3.1 to the sequence of numbers $c_k = (-1)^k \gamma_k, k = 0, 1, \ldots$. Then, substituting c_k in (3.4) we have

$$P_{n,\alpha}^{(1+)}(x) = \gamma_n \sum_{k=0}^{n} (-1)^k \frac{x^{n-k}}{\gamma_{n-k}} = (-1)^n \gamma_n e_{n,\alpha}(-x),$$

where $e_{n,\alpha}(x)$ is the nth truncated of $E_\alpha(x)$ defined in (3.5).

From (3.3), the generating function, $G_\alpha^{(1+)}(x,t)$, for $\{P_{n,\alpha}^{(1+)}(x)\}_{n=0}^{\infty}$ is given by

$$G_\alpha^{(1+)}(x,t) = \textstyle\sum_{n=0}^{\infty} P_{n,\alpha}^{(1+)}(x)\frac{t^n}{\gamma_n} = \displaystyle\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}(-1)^k \Lambda_\alpha^k\right)\frac{x^n t^n}{\gamma_n} = \sum_{k=0}^{\infty}(-1)^k t^k E_\alpha(xt).$$

Using (3.2) with $c_k = (-1)^k \gamma_k$, we obtain

$$\frac{E_\alpha(xt)}{1+t} = \sum_{n=0}^{\infty} P_{n,\alpha}^{(1+)}(x) \frac{t^n}{\gamma_n}.$$

In order to obtain the recurrence relation, note that

$$\left(\sum_{k=0}^{\infty} (-1)^k \Lambda_\alpha^k + \Lambda_\alpha \sum_{k=0}^{\infty} (-1)^k \Lambda_\alpha^k \right) x^n = x^n,$$

and from the definition of $\mathcal{A}(\Lambda_\alpha)$ with $c_k = (-1)^k \gamma_k$,

$$P_{n,\alpha}^{(1+)}(x) + \Lambda_\alpha P_{n,\alpha}^{(1+)}(x) = x^n.$$

Applying the Dunkl operator to $P_{n,\alpha}^{(1+)}(x)$, (3.9) is proved. □

3.3 Generating function $E_\alpha(xt)/(1-t^m)$

We take the sequence $\{c_l\}_{l=0}^{\infty}$ as

$$c_l = \begin{cases} \gamma_l, \ l = mk, \\ 0, \ \ l \neq mk, \end{cases} \quad k = 0,\ 1, \ldots, \tag{3.10}$$

where $m \in \mathbb{N}$ is a fixed number. We denote by $P_{n,\alpha}^{(m-)}(x)$ the following polynomial

$$P_{n,\alpha}^{(m-)}(x) = \gamma_n \sum_{k=0}^{[n/m]} \frac{x^{n-mk}}{\gamma_{n-mk}},$$

where $[n/m]$ denotes the greatest integer less than or equal to n/m.

Then, we may prove the next result.

Theorem 3.4 *The sequence $\{P_{n,\alpha}^{(m-)}(x)\}_{n=0}^{\infty}$ is an Appell–Dunkl sequence of polynomials whose generating function is*

$$G_\alpha^{(m-)}(x,t) = \frac{E_\alpha(xt)}{1-t^m}.$$

Moreover, these polynomials satisfy the recurrence relation

$$P_{n,\alpha}^{(m-)}(x) - \binom{n}{m}_\alpha \gamma_m P_{n-m,\alpha}^{(m-)}(x) = x^n. \tag{3.11}$$

Proof When we apply Theorem 3.1 with the sequence (3.10), the sequence of Appell–Dunkl polynomials that appear is $P_{n,\alpha}^{(m-)}(x)$.

From (3.3), the generating function for $\{P_{n,\alpha}^{(m-)}(x)\}_{n=0}^{\infty}$ is given by

$$G_{\alpha}^{(m-)}(x,t) = \sum_{n=0}^{\infty} P_{n,\alpha}^{(m-)}(x)\frac{t^n}{\gamma_n} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\Lambda_{\alpha}^{mk}\right)\frac{x^n t^n}{\gamma_n}$$
$$= \sum_{k=0}^{\infty}\Lambda_{\alpha}^{mk}E_{\alpha}(xt) = \sum_{k=0}^{\infty} t^{mk}E_{\alpha}(xt).$$

Thus, from (3.2) with c_l as in (3.10) we deduce that

$$\frac{E_{\alpha}(xt)}{1-t^m} = \sum_{n=0}^{\infty} P_{n,\alpha}^{(m-)}(x)\frac{t^n}{\gamma_n}.$$

In order to obtain the recurrence relation, note that

$$\left(\sum_{k=0}^{\infty}\Lambda_{\alpha}^{mk} - \Lambda_{\alpha}^{m}\sum_{k=0}^{\infty}\Lambda_{\alpha}^{mk}\right)x^n = x^n,$$

and from the definition of $\mathcal{A}(\Lambda_\alpha)$ with c_l as in (3.10),

$$P_{n,\alpha}^{(m-)}(x) - \Lambda_{\alpha}^{m}P_{n,\alpha}^{(m-)}(x) = x^n.$$

Applying m times the Dunkl operator to $P_{n,\alpha}^{(m-)}(x)$, (3.11) is proved. □

3.4 Generating function $E_\alpha(xt)/(1+t^m)$

Analogously, for each $m \in \mathbb{N}$, taking $\{c_l\}_{l=0}^{\infty}$ as

$$c_l = \begin{cases} (-1)^k\gamma_l, & l = mk, \\ 0, & l \neq mk, \end{cases} \qquad k = 0, 1, \dots,$$

we denote

$$P_{n,\alpha}^{(m+)}(x) = \gamma_n \sum_{k=0}^{[n/m]}(-1)^k\frac{x^{n-mk}}{\gamma_{n-mk}},$$

where $[n/m]$ is the greatest integer less than or equal to n/m.

Then, we may prove the next result.

Theorem 3.5 *The sequence* $\{P_{n,\alpha}^{(m+)}(x)\}_{n=0}^{\infty}$ *is an Appell–Dunkl sequence of polynomials whose generating function is*

$$G_{\alpha}^{(m+)}(x,t) = \frac{E_{\alpha}(xt)}{1+t^m}.$$

Moreover, these polynomials satisfy the recurrence relation

$$P_{n,\alpha}^{(m+)}(x) + \binom{n}{m}_{\alpha} \gamma_m P_{n-m,\alpha}^{(m+)}(x) = x^n.$$

Proof The proof is analogous to Theorem 3.4. □

4 Appell–Dunkl polynomials arising from certain Dunkl primitives

In this section, we are going to use the inverse operator of Λ_α, introduced in Section 2, to obtain in a different way the Appell–Dunkl sequences of polynomials $\{P_{n,\alpha}^{(1\pm)}(x)\}_{n=0}^{\infty}$ and $\{P_{n,\alpha}^{(2-)}(x)\}_{n=0}^{\infty}$.

In the classical case (see[12]), an integral of the form

$$\int x^n e^{\pm x}\,dx$$

always results a polynomial of degree n times the corresponding exponential function $e^{\pm x}$. This is due to the formula of integration by parts. In the Dunkl case, it does not happen because in the analogous formula, (2.9) appears an extra term with the function $h(x)$. However, if in (2.9) one of the functions $f(x)$ or $g(x)$ is an even function, the function $h(x)$ is the null function.

4.1 Appell–Dunkl polynomials from $\oint x^n E_\alpha(\pm x)\,d_\alpha x$

The goal of this subsection is to obtain the sequences of polynomials $\{P_{n,\alpha}^{(1\pm)}(x)\}_{n=0}^{\infty}$ using Dunkl integrals of the form

$$\oint x^n E_\alpha(\pm x)\,d_\alpha x. \tag{4.1}$$

Theorem 4.1 *Let* $\{P_{n,\alpha}^{(1-)}(x)\}_{n=0}^{\infty}$ *and* $\{P_{n,\alpha}^{(1+)}(x)\}_{n=0}^{\infty}$ *be the sequences of polynomials defined in (3.6) and (3.8), respectively. Then,*

$$\oint x^n E_\alpha(-x)\,d_\alpha x = -E_\alpha(-x) P_{n,\alpha}^{(1-)}(x) + C_{n,\alpha}(x) + c, \tag{4.2}$$

and

$$\oint x^n E_\alpha(x)\, d_\alpha x = E_\alpha(x) P_{n,\alpha}^{(1+)}(x) + D_{n,\alpha}(x) + c, \tag{4.3}$$

where $C_{n,\alpha}(x)$ and $D_{n,\alpha}(x)$ are auxiliary functions and c is a constant, not necessarily the same at each appearance.

Proof We start taking the function $E_\alpha(-x)$ in the integral (4.1). As we can see in the introduction of this section, it is very important to know whether the functions that appear in the integral are even or odd. So, we are going to distinguish whether $n = 2k$ or $n = 2k+1$. We begin supposing that $n = 2k$ and applying (2.9) taking into account (2.7), we obtain

$$\oint x^{2k} E_\alpha(-x)\, d_\alpha x = -x^{2k} E_\alpha(-x) + \theta_{2k} \oint x^{2k-1} E_\alpha(-x)\, d_\alpha x,$$

because in this case $h(x) = 0$. If we apply again (2.9), we have that

$$h(x) = \frac{2x^{2k-1} 2\mathcal{G}_\alpha(x)}{x} = 4x^{2k-2}\mathcal{G}_\alpha(x),$$

and then,

$$\begin{aligned}\oint x^{2k} E_\alpha(-x)\, d_\alpha x = -x^{2k} E_\alpha(-x) - \frac{\gamma_{2k}}{\gamma_{2k-1}} x^{2k-1} E_\alpha(-x)\\ + \frac{\gamma_{2k}}{\gamma_{2k-2}} \oint x^{2k-2} E_\alpha(-x)\, d_\alpha x\\ + 2(2\alpha+1) \frac{\gamma_{2k}}{\gamma_{2k-1}} \oint x^{2k-2} \mathcal{G}_\alpha(x)\, d_\alpha x.\end{aligned}$$

Iterating this process $2k$ times, we can write

$$\begin{aligned}\oint x^{2k} E_\alpha(-x)\, d_\alpha x = -E_\alpha(-x) \left(x^{2k} + \frac{\gamma_{2k}}{\gamma_{2k-1}} x^{2k-1} + \cdots + \frac{\gamma_{2k}}{\gamma_1} x + \gamma_{2k} \right)\\ + 2(2\alpha+1) \oint \left(\frac{\gamma_{2k}}{\gamma_{2k-1}} x^{2k-2} + \frac{\gamma_{2k}}{\gamma_{2k-3}} x^{2k-4} + \cdots + \frac{\gamma_{2k}}{\gamma_1} \right) \mathcal{G}_\alpha(x)\, d_\alpha x.\end{aligned}$$

That is

$$\oint x^{2k} E_\alpha(-x)\, d_\alpha x = -E_\alpha(-x) P_{2k,\alpha}^{(1-)}(x) + \gamma_{2k} B_{k,\alpha}(x) + c, \tag{4.4}$$

where

$$B_{k,\alpha}(x) = 2(2\alpha+1) \oint \left(\frac{x^{2k-2}}{\gamma_{2k-1}} + \frac{x^{2k-4}}{\gamma_{2k-3}} + \cdots + \frac{1}{\gamma_1} \right) \mathcal{G}_\alpha(x)\, d_\alpha x.$$

If we suppose now that $n = 2k + 1$ and apply (2.9) taking into account (2.7) and that $h(x) = 4x^{2k}\mathcal{G}_\alpha(x)$, we have

$$\oint x^{2k+1}E_\alpha(-x)\,d_\alpha x = -x^{2k+1}E_\alpha(-x) + \frac{\gamma_{2k+1}}{\gamma_{2k}}\oint x^{2k}E_\alpha(-x)\,d_\alpha x \\ +2(2\alpha+1)\oint x^{2k}\mathcal{G}_\alpha(x)\,d_\alpha x.$$

Applying (4.4), we obtain

$$\oint x^{2k+1}E_\alpha(-x)\,d_\alpha x = -x^{2k+1}E_\alpha(-x) + \frac{\gamma_{2k+1}}{\gamma_{2k}}\Big(-E_\alpha(-x)P^{(1-)}_{2k,\alpha}(x) \\ +\gamma_{2k}B_{k,\alpha}(x) + c\Big) + 2(2\alpha+1)\oint x^{2k}\mathcal{G}_\alpha(x)\,d_\alpha x. \tag{4.5}$$

Note that

$$x^{2k+1} + \frac{\gamma_{2k+1}}{\gamma_{2k}}P^{(1-)}_{2k,\alpha}(x) = P^{(1-)}_{2k+1,\alpha}(x).$$

Then (4.5) can be rewritten as

$$\oint x^{2k+1}E_\alpha(-x)\,d_\alpha x = -E_\alpha(-x)P^{(1-)}_{2k+1,\alpha}(x) + \gamma_{2k+1}B_{k+1,\alpha}(x) + c.$$

Therefore, (4.2) is obtained with

$$C_{n,\alpha}(x) = \begin{cases}\gamma_{2k}B_{k,\alpha}(x), & n = 2k\\ \gamma_{2k+1}B_{k+1,\alpha}(x), & n = 2k+1\end{cases}.$$

Analogously, if we consider (4.1) with $E_\alpha(x)$, applying (2.9) n times, (4.3) is obtained where $D_{n,\alpha}(x)$ will be a function similar to $C_{n,\alpha}(x)$. □

4.2 Appell–Dunkl polynomials from other Dunkl integrals

In this subsection, the sequence of polynomials $\{P^{(2-)}_{n,\alpha}(x)\}_{n=0}^{\infty}$ is obtained by means of Dunkl integrals.

We define the following sequences of polynomials

$$Q_{2k,\alpha}(x) = \sum_{j=0}^{k}\frac{\gamma_{2k}}{\gamma_{2k-2j}}x^{2k-2j}, \quad R_{2k+1,\alpha}(x) = \sum_{j=0}^{k}\frac{\gamma_{2k+1}}{\gamma_{2k+1-2j}}x^{2k+1-2j}, \tag{4.6}$$

and

$$S_{n,\alpha}(x) = \begin{cases} Q_{2k,\alpha}(x), & n = 2k \\ R_{2k+1,\alpha}(x), & n = 2k+1. \end{cases} \tag{4.7}$$

Theorem 4.2 *Let $\{Q_{2k,\alpha}(x)\}_{k=0}^{\infty}$ and $\{R_{2k+1,\alpha}(x)\}_{k=0}^{\infty}$ be the sequences of polynomials defined in (4.6). Then, it holds that*

$$\oint x^{2k}\mathcal{I}_\alpha(x)\, d_\alpha x = Q_{2k,\alpha}(x)\mathcal{G}_\alpha(x) - \frac{\gamma_{2k}}{\gamma_{2k-1}} R_{2k-1,\alpha}(x)\mathcal{I}_\alpha(x) + c, \tag{4.8}$$

and

$$\oint x^{2k+1}\mathcal{G}_\alpha(x)\, d_\alpha x = R_{2k+1,\alpha}(x)\mathcal{I}_\alpha(x) - \frac{\gamma_{2k+1}}{\gamma_{2k}} Q_{2k,\alpha}(x)\mathcal{G}_\alpha(x) + c, \tag{4.9}$$

where c is a constant, not necessarily the same at each appearance.

Proof We start studying the Dunkl integral

$$\oint x^{2k}\mathcal{I}_\alpha(x)\, d_\alpha x. \tag{4.10}$$

Note that $\mathcal{I}_\alpha(x)$ and x^{2k} are both even functions. This fact makes that when we apply the formula of integration by parts, (2.9), several times to the integral (4.10), the function $h(x)$ is always the null function. Applying (2.9) twice and taking into account (2.8), we obtain

$$\begin{aligned} \oint x^{2k}\mathcal{I}_\alpha(x)\, d_\alpha x &= x^{2k}\mathcal{G}_\alpha(x) - \frac{\gamma_{2k}}{\gamma_{2k-1}} \oint x^{2k-1}\mathcal{G}_\alpha(x)\, d_\alpha x \\ &= x^{2k}\mathcal{G}_\alpha(x) - \frac{\gamma_{2k}}{\gamma_{2k-1}} x^{2k-1}\mathcal{I}_\alpha(x) + \frac{\gamma_{2k}}{\gamma_{2k-2}} \oint x^{2k-2}\mathcal{I}_\alpha(x)\, d_\alpha x. \end{aligned}$$

Iterating this method $2k$ times, we obtain (4.8).

Now, we take the following integral

$$\oint x^{2k+1}\mathcal{G}_\alpha(x)\, d_\alpha x.$$

In this case, both functions, x^{2k+1} and $\mathcal{G}_\alpha(x)$, are odd functions. So, the function $h(x)$ when we apply (2.9) is again the null function. Applying (2.9) $2k+1$ times, it holds (4.9). □

Theorem 4.3 *Let $\{S_{n,\alpha}(x)\}_{n=0}^{\infty}$ be the sequence of polynomials defined in (4.7). Then, $S_{n,\alpha}(x) = P_{n,\alpha}^{(2-)}(x)$. That is, $\{S_{n,\alpha}(x)\}_{n=0}^{\infty}$ is an Appell–Dunkl sequence of polynomials whose generating function is*

$$G_\alpha^{(2-)}(x,t) = \frac{E_\alpha(xt)}{1-t^2}. \tag{4.11}$$

Proof Note that by the definition of the polynomials, (4.6),

$$\Lambda_\alpha Q_{2k,\alpha}(x) = \theta_{2k} R_{2k-1,\alpha}(x), \qquad \Lambda_\alpha R_{2k+1,\alpha}(x) = \theta_{2k+1} Q_{2k,\alpha}(x). \tag{4.12}$$

So, the sequence $\{S_{n,\alpha}(x)\}_{n=0}^{\infty}$ is an Appell–Dunkl sequence of polynomials.

Finally, we have to obtain the generating function. If we apply twice the Dunkl operator to the polynomials $Q_{2k,\alpha}(x)$ and $R_{2k+1,\alpha}(x)$, from (4.12) we find

$$\Lambda_\alpha^2 Q_{2k,\alpha}(x) = \frac{\gamma_{2k}}{\gamma_{2k-2}} Q_{2k-2,\alpha}(x), \quad \Lambda_\alpha^2 R_{2k+1,\alpha}(x) = \frac{\gamma_{2k+1}}{\gamma_{2k-3}} R_{2k-1,\alpha}(x).$$

Now, we denote by $\mathcal{A}_2(\Lambda_\alpha)$ the operator

$$\mathcal{A}_2(\Lambda_\alpha) = \sum_{k=0}^{\infty} \Lambda_\alpha^{2k}.$$

If we apply this operator to x^{2k} instead of x^n, we obtain that

$$Q_{2k,\alpha}(x) = \mathcal{A}_2(\Lambda_\alpha)x^{2k} = \sum_{j=0}^{\infty} \Lambda_\alpha^{2j} x^{2k} = \sum_{j=0}^{k} \frac{\gamma_{2k}}{\gamma_{2k-2j}} x^{2k-2j}.$$

From (2.2),

$$\Lambda_\alpha^2 \mathcal{I}_\alpha(xt) = t^2 \mathcal{I}_\alpha(xt), \qquad \Lambda_\alpha^2 \mathcal{G}_\alpha(xt) = t^2 \mathcal{G}_\alpha(xt). \tag{4.13}$$

Then, following the same technique than in Theorem 3.1 to obtain the generating function

$$G_{\alpha,1}^{(2-)}(x,t) = \sum_{k=0}^{\infty} Q_{2k,\alpha}(x)\frac{t^{2k}}{\gamma_{2k}} = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \Lambda_\alpha^{2j} x^{2k} \frac{t^{2k}}{\gamma_{2k}}.$$

Using (2.6) and (4.13),

$$G_{\alpha,1}^{(2-)}(x,t) = \sum_{j=0}^{\infty} \Lambda_\alpha^{2j} \mathcal{I}_\alpha(xt) = \sum_{j=0}^{\infty} t^{2j} \mathcal{I}_\alpha(xt).$$

That is,

$$\frac{\mathcal{I}_\alpha(xt)}{1-t^2} = \sum_{k=0}^{\infty} Q_{2k,\alpha}(x)\frac{t^{2k}}{\gamma_{2k}}. \tag{4.14}$$

Analogously, applying the operator $\mathcal{A}_2(\Lambda_\alpha)$ to x^{2k+1}

$$R_{2k+1,\alpha}(x) = \mathcal{A}_2(\Lambda_\alpha)x^{2k+1} = \sum_{j=0}^{\infty} \Lambda_\alpha^{2j} x^{2k+1}.$$

Then, we can take the following generating function

$$G_{\alpha,2}^{(2-)}(x,t) = \sum_{k=0}^{\infty} R_{2k+1,\alpha}(x)\frac{t^{2k+1}}{\gamma_{2k+1}} = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \Lambda_\alpha^{2j} x^{2k+1}\frac{t^{2k+1}}{\gamma_{2k+1}}.$$

Using (2.6) and (4.13),

$$G_{\alpha,2}^{(2-)}(x,t) = \sum_{j=0}^{\infty} \Lambda_\alpha^{2j}\mathcal{G}_\alpha(xt) = \sum_{j=0}^{\infty} t^{2j}\mathcal{G}_\alpha(xt).$$

That is,

$$\frac{\mathcal{G}_\alpha(xt)}{1-t^2} = \sum_{k=0}^{\infty} R_{2k+1,\alpha}(x)\frac{t^{2k+1}}{\gamma_{2k+1}}. \tag{4.15}$$

By joining (4.14) and (4.15), we obtain (4.11). $\square$

Declarations

Conflict of interest We have no conflicts of interest to disclose.

Consent for publication We confirm that this work is original and has not been published elsewhere, nor it is currently under consideration for publication elsewhere.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

References

1. Abramowitz, M., Stegun, I.A.: Handbook of mathematical functions with formulas, graphs, and mathematical tables. National Bureau of Standards Applied Mathematics Series. Vol. 55. (1964)
2. Ben Cheikh, Y., Gaied, M.: Dunkl-Appell d-orthogonal polynomials. Integral Transf Spec. Funct. **18**, 581–597 (2007)
3. Bouanani, A., Khériji, L., Ihsen Tounsi, M.: Characterization of q-Dunkl Appell symmetric orthogonal q-polynomials. Expo. Math. **28**, 325–336 (2010)

4. Cholewinski, F.M.: The finite calculus associated with Bessel functions, Contemporary Mathematics, 75. American Mathematical Society, Providence, RI (1988)
5. Ciaurri, Ó., Durán, A., Pérez, M., Varona, J.L.: Bernoulli-Dunkl and Apostol-Euler-Dunkl polynomials with applications to series involving zeros of Bessel functions. J. Approx. Theory **235**, 20–45 (2018)
6. Ciaurri, Ó., Mínguez Ceniceros, J., Varona, J.L.: Bernoulli-Dunkl and Euler-Dunkl polynomials and their generalizations. Rev. R. Acad. Cienc. Exactas Fís Nat. Ser. A Mat. RACSAM **113**, 2853–2876 (2019)
7. Dilcher, K.: Bernoulli and Euler Polynomials, *NIST handbook of mathematical functions* (edited by F. W. F. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark), 587–599, National Institute of Standards and Technology, Washington, DC, and Cambridge University Press, Cambridge, 2010. Available online in http://dlmf.nist.gov/24
8. Dimovski, I.H., Hristov, V.Z.: Nonlocal operational calculi for Dunkl operators, SIGMA Symmetry Integrability Geom. Methods Appl. **5** (2009), Paper 030, 16 pp
9. Dunkl, C.F.: Differential-difference operators associated to reflection groups. Trans. Amer. Math. Soc. **311**, 167–183 (1989)
10. Durán, A., Pérez, M., Varona, J.L.: Fourier-Dunkl system of the second kind and Euler-Dunkl polynomials. J. Approx. Theory **245**, 23–39 (2019)
11. Graham, R., Knuth, D., Patashnik, O.: Concrete mathematics: a foundation for computer science, Second edition, Addison-Wesley, (1994)
12. Khan, S.A., Jagannathan, R.: On certain Appell polynomials and their generalizations based on the Tsallis q-exponential. Bull. Malays. Math. Sci. Soc. **45**, 1453–1472 (2022)
13. Lebedev, N.N.: Special Functions and their Applications. Dover, New York (1972)
14. Loureiro, A.F., Maroni, P.: Quadratic decomposition of Appell sequences. Expo. Math. **26**(2), 177–186 (2008)
15. Mínguez Ceniceros, J., Varona, J.L.: Asymptotic behavior of Bernoulli-Dunkl and Euler-Dunkl polynomials and their zeros. Funct. Approx. Comment. Math. **65**, 211–226 (2021)
16. Olver, F.W.J., Maximon, L.C.: Bessel Functions, *NIST handbook of mathematical functions* (edited by F. W. F. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark), 215–286, National Institute of Standards and Technology, Washington, DC, and Cambridge University Press, Cambridge, 2010. Available online in http://dlmf.nist.gov/10
17. Rosenblum, M.: Generalized Hermite polynomials and the Bose-like oscillator calculus. Oper. Theory Adv. Appl. **73**, 369–396 (1994)
18. Watson, G.N.: A Treatise on the Theory of Bessel Functions, 2nd edn. Cambridge Univ. Press, Cambridge (1944)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.