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# Stability Conditions for Permanent Rotations of a Heavy Gyrostat with Two Constant Rotors

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**Abstract:** In this paper, we consider the motion of an asymmetric heavy gyrostat, when its center of mass lies along one of the principal axes of inertia. We determine the possible permanent rotations and, by means of the Energy-Casimir method, we give sufficient stability conditions. We prove that there exist permanent stable rotations when the gyrostat is oriented in any direction of the space, by the action of two spinning rotors, one of them aligned along the principal axis, where the center of mass lies. We also derive necessary stability conditions that, in some cases, are the same as the sufficient ones.

**Keywords:** gyrostat rotation; stability; Energy-Casimir method

**MSC:** 70E55; 37J25

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## 1. Introduction

A gyrostat $\mathcal{G}$ is a mechanical system made up of a rigid body $\mathcal{P}$, called the platform, and other bodies $\mathcal{R}$, called the rotors, connected to the platform in such a way that the motion of the rotors does not modify the distribution of mass of the gyrostat $\mathcal{G}$. Due to this double spinning, the platform, on the one hand, and the rotors, on the other, the gyrostat is also known under the name of a dual-spin body, especially in astrodynamics, where these artifacts are widely used in spacecraft dynamics in order to stabilize their rotations; see, e.g., [1–5].

In the absence of external torques, this problem is an extension of the free rigid body motion. It seems (see [6]) that Zhukovskii [7] was the first to consider the gyrostat problem and, soon after, Volterra [8] used this model to represent the Earth's rotational motion, assuming that, in the interior of the Earth, there was a cavity filled with an inviscid, homogeneous fluid. Moreover, he obtained the solution in terms of elliptic functions. Although it is known that the problem is integrable, much work has been dedicated to it, because it depends on several parameters, such as the principal moments of inertia and the gyrostatic moments, and there is a rich choreography of bifurcations; see, for instance, [9–18], to quote only a few references. As expected, scientists have moved a step forward by adding some complexity to the problem, by considering generalized models to be of interest in practical applications, mainly in the field of astrodynamics. In this way, most of the works are devoted to the consideration of external torques or inner perturbations. For instance, some authors assume elasticity or periodic time dependence of the moments of inertia [4,5], while other authors focus on the attitude dynamics of a gyrostat rotating and moving on a circular orbit [15,19,20], or under the action of a uniform gravity field [21–29].

The paper that we present here is related to the latter problem, and it is an extension of the work previously presented by the authors [26,27]. We consider the rotation of a tri-axial gyrostat under a uniform gravity field. We assume that there are two rotors, one along the direction of a principal axis of inertia, where we assume the center of mass lies ($z$-axis), and another one along another principal axis of inertia ($x$-axis). In particular, we are interested in the existence of stable permanent rotations. When the rotors are at relative rest with respect to the platform, many works are devoted to studying the stability of particular motions, such as Staude's permanent rotations [30,31], planar motions [32,33], pendulum-like motions [34,35] or regular precessions [16,36], to mention a few. However, the action of the rotors plays an important role in the stabilization of the motion of a heavy gyrostat and it is a key point in obtaining stable solutions, mainly for its practical applications [15,18,24,37,38].

Different approaches can be followed to derive conditions on the stability of permanent rotations. The classical approach uses Lyapunov functions [17,21,22] but it is also possible to obtain insight by analyzing the invariant manifolds and their bifurcations [39]. In the case treated in this work, we use the Energy-Casimir method [40,41], as a complementary approach to derive sufficient stability conditions for the existence of stable permanent rotations. This method has been used successfully in previous works [23,26,27,42,43]. In this way, from the equations of motion—see, e.g., [6]—we focus on two families of equilibria; we denote $E_0$ and $E_1$, which constitute a generalization of those obtained in [27] and potentially cover any orientation of the gyrostat in the space. The family $E_1$ proves to be stable if $I_2$ is the largest moment of inertia or if $I_3 > I_2 > I_1$ and the angular velocity of rotation $|\omega|$ is small enough or, equivalently, if the gyrostatic moment $l_3$ is great enough. In cases not covered by the above conditions—that is to say, when $I_1$ is the largest principal moment of inertia or $I_3 > I_1 > I_2$—we can obtain stable rotations by switching $l_1$ and $l_2$, i.e., by turning on the gyrostatic moment $l_2$ and $l_1$ off.

In respect to the other family of equilibria, $E_0$, we find that it is a limit case of the other one and stability is also obtained if $|\omega|$ is small enough, although it can be extended for every value of $\omega$. Hence, in the case here considered, given a rotation axis, the action of the rotors leads to stable permanent rotations provided that the center of mass is lying on the vertical axis, which is the most frequent practical case.

The paper is organized as follows. In Section 2, we consider the equations of the motion and discuss the equilibrium solutions. Section 3 is devoted to the stability analysis, where the main results about sufficient and necessary conditions of stability are presented. Finally, conclusions are given in Section 4.

## 2. Equations of Motion and Equilibrium Solutions

We consider an asymmetric heavy gyrostat with two rotors, whose axes are aligned with the principal axes of the platform, in a uniform gravity field. It is assumed that the mass distribution of the gyrostat is not modified by the relative motion of the rotors and that the whole gyrostat rotates with a fixed point $O$, which may be different from the center of mass $G$.

We use two orthonormal reference frames centered at the fixed point $O$ (see Figure 1). On the one hand, we use the space or inertial reference frame $\mathcal{F}\{O, X, Y, Z\}$, fixed in the space, with the direction of the $Z$-axis opposite to the acceleration $\boldsymbol{g}$ of the gravity field. On the other hand, we use the body frame $\mathcal{B}\{O, x, y, z\}$, fixed with the gyrostat, so that the axes coincide with the principal axes of inertia of the gyrostat. The relative attitude between these two reference frames results from three consecutive rotations involving three angles, such as the Euler angles. Note that, as we study the permanent rotations of the gyrostat, we only need two of these angles to define the orientation of the rotating gyrostat in the inertial fixed frame $\mathcal{F}$. Let $\mathbb{I} = (I_1, I_2, I_3)$ be the inertia tensor in the body frame $\mathcal{B}$ and $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ the angular velocity of the gyrostat expressed in the body frame. Then, the angular momentum of the gyrostat, considered as a rigid body, is given by $\boldsymbol{\pi} = \mathbb{I}\boldsymbol{\omega}$. Results about the stability of permanent rotations under the action of only one rotor are given in [26,27]. The combination of these two cases, a rotor acting on the $z$ axis and another rotor on one of the other principal axes of inertia, will be considered here. In this sense, we take $l_2 = 0$ in the vector $\boldsymbol{l} = (l_1, l_2, l_3)$, the angular momentum of the rotors in the body frame. The vector $\hat{\boldsymbol{k}} = (k_1, k_2, k_3)$ is the unitary vector in the direction of the fixed $Z$ axis, expressed in the body frame $\mathcal{B}$. This vector can be expressed as

$$\hat{\boldsymbol{k}} = (\sin\varphi \sin\theta, \cos\theta, \cos\varphi \sin\theta), \tag{1}$$

where the angles $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$ give us the orientation of the gyrostat with respect to the inertial reference frame $\mathcal{F}$ (see Figure 1). If $(0, 0, z_0)$ are the coordinates of the center of mass $G$ in the body frame, the equations of motion result to be (see, e.g., [6,44])

$$\begin{aligned}
\frac{d\pi_1}{dt} &= \left(\frac{I_2 - I_3}{I_2 I_3}\right)\pi_2\pi_3 - \frac{l_3\pi_2}{I_2} + mgz_0k_2,\\
\frac{d\pi_2}{dt} &= \left(\frac{I_3 - I_1}{I_1 I_3}\right)\pi_1\pi_3 + \frac{l_3\pi_1}{I_1} - \frac{l_1\pi_3}{I_3} - mgz_0k_1,\\
\frac{d\pi_3}{dt} &= \left(\frac{I_1 - I_2}{I_1 I_2}\right)\pi_1\pi_2 + \frac{l_1\pi_2}{I_2},\\
\frac{dk_1}{dt} &= \frac{k_2\pi_3}{I_3} - \frac{k_3\pi_2}{I_2},\\
\frac{dk_2}{dt} &= \frac{k_3\pi_1}{I_1} - \frac{k_1\pi_3}{I_3},\\
\frac{dk_3}{dt} &= \frac{k_1\pi_2}{I_2} - \frac{k_2\pi_1}{I_1}.
\end{aligned} \tag{2}$$

Under these hypotheses, permanent rotations appear as the equilibrium points of Equation (2) and we have the following result.

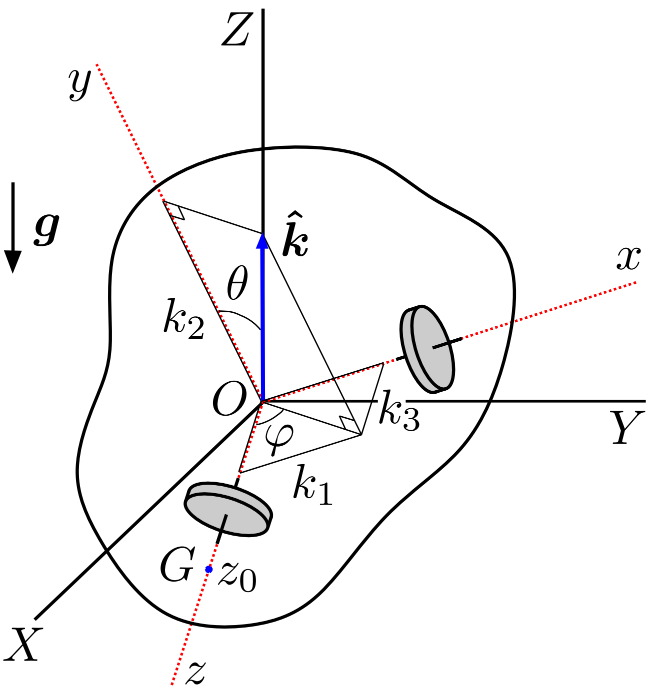


**Figure 1.** Asymmetric gyrostat and reference frames.

**Theorem 1.** *There are two families of equilibrium points. The first one is given by those points of the form*

$$E_0 \equiv (I_1\omega \sin\varphi, 0, I_3\omega \cos\varphi, \sin\varphi, 0, \cos\varphi),$$

*where* $\varphi \in [0, 2\pi)$ *and* $\omega \in \mathbb{R}$ *such that*

$$(l_3\omega - gmz_0)\sin\varphi - \omega\cos\varphi(l_1 + (I_1 - I_3)\omega\sin\varphi) = 0. \tag{3}$$

*The second one is defined by points of the form*

$$E_1 \equiv (I_1\omega\sin\varphi\sin\theta, I_2\omega\cos\theta, I_3\omega\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta, \cos\varphi\sin\theta),$$

*where* $\varphi \in [0, 2\pi)$, $\theta \in [0, \pi/2) \cup (\pi/2, \pi]$ *and* $\omega \in \mathbb{R}$ *such that*

$$l_1 + (I_1 - I_2)\omega\sin\varphi\sin\theta = 0, \qquad (I_2 - I_3)\omega^2\cos\varphi\sin\theta + gmz_0 - l_3\omega = 0. \tag{4}$$

**Proof.** In terms of the angles $\theta$ and $\varphi$, introduced in (1), the components of the angular momentum can be written as

$$\pi_1 = \omega I_1 \sin\varphi\sin\theta, \quad \pi_2 = \omega I_2\cos\theta, \quad \pi_3 = \omega I_3\cos\varphi\sin\theta, \tag{5}$$

where $\omega$ is the modulus of the angular velocity. Now, it is easy to verify that, under this parameterization, the last three equations of system (2) vanish. Thus, we have to check when the three first equations vanish, which is found to be

$$\begin{cases} \cos\theta(gmz_0 - l_3\omega + (I_2 - I_3)\omega^2\cos\varphi\sin\theta) = 0, \\ \sin\theta((gmz_0 - l_3\omega)\sin\varphi + \omega\cos\varphi(l_1 + (I_1 - I_3)\omega\sin\varphi\sin\theta)) = 0, \\ \omega\cos\theta(l_1 + (I_1 - I_2)\omega\sin\varphi\sin\theta) = 0. \end{cases} \tag{6}$$

Discarding the case $\omega = 0$, there are two kinds of solutions: those verifying $\cos\theta = 0$ and those that do not.

If $\cos\theta = 0$, the first and the third equations (6) vanish. Thus, provided that $\theta \in [0, \pi]$, $\sin\theta = 1$, and then all the equations are satisfied if and only if

$$(l_3\omega - gmz_0)\sin\varphi - \omega\cos\varphi(l_1 + (I_1 - I_3)\omega\sin\varphi) = 0.$$

If $\cos\theta \neq 0$, the third equation in (6) vanishes if

$$l_1 + (I_1 - I_2)\omega\sin\varphi\sin\theta = 0.$$

By substitution of $l_1$ in the second equation, the first two equations vanish at the same time if the condition

$$(I_2 - I_3)\omega^2\cos\varphi\sin\theta + gmz_0 - l_3\omega = 0$$

holds. □

**Remark 1.** *This result is very similar to Theorem 1 in [27], but replacing* $gmz_0$ *by* $gmz_0 - l_3\omega$*. Moreover, the family* $E_0$ *is a limit case of the family* $E_1$*. However, the two conditions (4) do not need to be satisfied at the same time, but only the linear combination (3).*

It is worth noting that $E_0$ and $E_1$ give rise to equilibrium solutions with the gyrostat oriented along any direction of the space, provided that the corresponding gyrostatic moments verify appropriate conditions, as they are (3) or (4). Our next step will be to determine under which conditions they are stable.

### 3. Stability Analysis

Let us consider the stability of the solutions in Theorem 1. It is known that (2) is a Lie–Poisson system (see [6,23]). The associated Hamiltonian function is given by

$$\mathcal{H} = \frac{1}{2}\left(\frac{\pi_1^2}{I_1} + \frac{\pi_2^2}{I_2} + \frac{\pi_3^2}{I_3}\right) + mgz_0k_3, \tag{7}$$

while the corresponding Poisson bracket is defined as

$$\{\mathcal{F},\mathcal{G}\}(\boldsymbol{\pi},\hat{\boldsymbol{k}}) = -(\boldsymbol{\pi}+\boldsymbol{l})\cdot(\nabla_\pi\mathcal{F}\times\nabla_\pi\mathcal{G}) - \hat{\boldsymbol{k}}\cdot(\nabla_\pi\mathcal{F}\times\nabla_k\mathcal{G} + \nabla_k\mathcal{F}\times\nabla_\pi\mathcal{G}). \tag{8}$$

Moreover, there are two Casimir functions:

$$C_1 \equiv k_1^2 + k_2^2 + k_3^2 = 1, \tag{9}$$

$$C_2 \equiv (\pi_1 + l_1)k_1 + \pi_2 k_2 + (\pi_3 + l_3)k_3 = p_\psi, \tag{10}$$

being $p_\psi$ the component of the total angular momentum $\boldsymbol{\pi} + \boldsymbol{l}$ along the fixed $Z$ axis. These two Casimir functions are used to define the augmented Hamiltonian given by

$$\begin{aligned}\mathcal{H}_A =& \frac{1}{2}\left(\frac{\pi_1^2}{I_1} + \frac{\pi_2^2}{I_2} + \frac{\pi_3^2}{I_3}\right) + mgz_0k_3 + ((\pi_1 + l_1)k_1 + \pi_2 k_2 \\ &+ (\pi_3 + l_3)k_3)\lambda + (k_1^2 + k_2^2 + k_3^2)\mu,\end{aligned} \tag{11}$$

where $\lambda$ and $\mu$ are suitable parameters such that the equilibrium positions are critical points of $\mathcal{H}_A$.

Under these considerations, in order to establish sufficient stability conditions, we will make use of the classical Energy-Casimir method [43,45] and, more precisely, of a generalized result given by Ortega and Ratiu [41], which reads as

**Theorem 2** (Generalized Energy-Casimir method)**.** *Let $(M, \{.,.\}, h)$ be a Poisson system, and $m \in M$ be an equilibrium of the Hamiltonian vector field $X_h$. If there is a set of conserved quantities $C_1, \ldots, C_n \in C^\infty(M)$ for which*

$$\mathbf{d}(h + C_1 + \cdots + C_n)(m) = 0,$$

*and*

$$\mathbf{d}^2(h + C_1 + \cdots + C_n)(m)\Big|_{W\times W}$$

*is definite for $W = \ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m)$, then $m$ is stable. If $W = \{0\}$, $m$ is always stable.*

Let us proceed to the application of this result. To begin with, we need to identify the space $W$, which is defined from Equations (9) and (10) as

$$W = \ker \mathbf{d}C_1 \cap \ker \mathbf{d}C_2.$$

Taking into account the parameterization (1), (5), we obtain

$$\begin{aligned}\mathbf{d}C_1 =& \sin\varphi\sin\theta d\pi_1 + \cos\theta d\pi_2 + \cos\varphi\sin\theta d\pi_3 + \\ & (l_1 + \omega I_1 \sin\varphi\sin\theta)dk_1 + \omega I_2\cos\theta dk_2 + (l_3 + \omega I_3\cos\varphi\sin\theta)dk_3, \\ \mathbf{d}C_2 =& 2\sin\varphi\sin\theta dk_1 + 2\cos\theta dk_2 + 2\cos\varphi\sin\theta dk_3.\end{aligned}$$

Solving the system $\mathbf{d}C_1 = \mathbf{d}C_2 = 0$, and identifying the six vectors of the canonic basis in $\mathbb{R}^6$, $(\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4, \hat{e}_5, \hat{e}_6)$ with $(d\pi_2, d\pi_1, d\pi_3, dk_2, dk_1, dk_3)$, we find that $W$ is generated by the following four vectors

$$\begin{cases} \boldsymbol{v}_1 = \hat{e}_1 \cos\varphi \sin\theta - \hat{e}_3 \cos\theta, \\ \boldsymbol{v}_2 = \hat{e}_2 \cos\varphi \sin\theta - \hat{e}_3 \sin\varphi \sin\theta, \\ \boldsymbol{v}_3 = \hat{e}_3(l_3 \sec\varphi \cot\theta + (I_3 - I_2)\omega \cos\theta) + \hat{e}_4 \cos\varphi \sin\theta - \hat{e}_6 \cos\theta, \\ \boldsymbol{v}_4 = \hat{e}_3(l_3 \tan\varphi - l_1 - (I_1 - I_3)\omega \sin\varphi \sin\theta) + \hat{e}_5 \cos\varphi \sin\theta - \hat{e}_6 \sin\varphi \sin\theta, \end{cases}$$

provided $\cos\varphi \sin\theta \neq 0$. Now, let us consider a vector $\boldsymbol{v}$ in $W$, expressed as

$$\boldsymbol{v} = x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + x_3\boldsymbol{v}_3 + x_4\boldsymbol{v}_4,$$

where $x_i \in \mathbb{R}$, $i = 1, \ldots, 4$. Thus, the quadratic form

$$\mathbf{d}^2(h + C_1 + \cdots + C_n)(m)\Big|_{W\times W}$$

in the variables $x_i$ is obtained from $\boldsymbol{v}^T \cdot \mathrm{Hess}(\mathcal{H}_A) \cdot \boldsymbol{v}$. In this way, we obtain

$$\mathrm{Hess}(H_A)\Big|_{W\times W} = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{12} & h_{22} & h_{23} & h_{24} \\ h_{13} & h_{23} & h_{33} & h_{34} \\ h_{14} & h_{24} & h_{34} & h_{44} \end{bmatrix}, \tag{12}$$

where

$$\begin{aligned}
h_{11} &= \frac{1}{I_2 I_3}(I_2 \cos^2\theta + I_3 \cos^2\varphi \sin^2\theta), \\
h_{12} &= \frac{1}{I_3} \sin\varphi \sin\theta \cos\theta, \\
h_{13} &= \frac{1}{I_3}\Big[(I_3\lambda + (I_2 - I_3)\omega)\cos^2\theta - l_3 \sec\varphi \cos\theta \cot\theta + I_3\lambda \cos^2\varphi \sin^2\theta\Big], \\
h_{14} &= \frac{1}{I_3}[l_1 + (I_3\lambda + (I_1 - I_3)\omega) \sin\varphi \sin\theta - l_3 \tan\varphi] \cos\theta, \\
h_{22} &= \frac{1}{I_1 I_3}(I_3 \cos^2\varphi + I_1 \sin^2\varphi) \sin^2\theta, \\
h_{23} &= \frac{1}{I_3}[(I_3\lambda + (I_2 - I_3)\omega) \sin\varphi \sin\theta - l_3 \tan\varphi] \cos\theta, \\
h_{24} &= \frac{1}{I_3}\Big[(I_3\lambda + (I_1 - I_3)\omega \sin^2\varphi) \sin\theta + (l_1 - l_3 \tan\varphi) \sin\varphi\Big] \sin\theta, \\
h_{33} &= \frac{1}{I_3}\Big[(2I_3\mu + 2(I_2 - I_3)I_3\lambda\omega + (I_2 - I_3)^2\omega^2)\cos^2\theta - \\
&\quad 2(I_3\lambda + (I_2 - I_3)\omega)l_3 \sec\varphi \cos\theta \cot\theta + l_3^2 \sec^2\varphi \cot^2\theta + 2I_3\mu \cos^2\varphi \sin^2\theta\Big], \\
h_{34} &= \frac{1}{I_3}[(l_3 \tan\varphi - l_1)l_3 \sec\varphi \cot\theta + (l_1(I_3\lambda + (I_2 - I_3)\omega) \cos\theta + \\
&\quad ((I_1 - I_3)(I_2 - I_3)\omega^2 + (I_1 + I_2 - 2I_3)I_3\omega\lambda + 2I_3\mu) \sin\varphi \sin\theta \cos\theta - \\
&\quad (2I_3(\lambda - \omega) + (I_1 + I_2)\omega)l_3 \tan\varphi \cos\theta] \\
h_{44} &= \frac{1}{I_3}\Big[2I_3\mu \cos^2\varphi \sin^2\theta + (2I_3\mu + 2(I_1 - I_3)I_3\lambda\omega + (I_1 - I_3)^2\omega^2) \sin^2\varphi \sin^2\theta + \\
&\quad 2(I_3\lambda + (I_1 - I_3)\omega)(l_1 - l_3 \tan\varphi) \sin\varphi \sin\theta + (l_1 - l_3 \tan\varphi)^2\Big].
\end{aligned} \tag{13}$$

### 3.1. Stability of the Equilibrium $E_1$

We first focus on the equilibrium $E_1$ in order to give both sufficient and necessary conditions of stability. In this way, for the sufficient conditions, we obtain the following result.

**Theorem 3.** *If $\cos\varphi\sin\theta \neq 0$, the equilibrium $E_1$ is stable if $I_2$ is the largest moment of inertia, or if $I_3 > I_2 > I_1$ and*

$$|l_3| > \max\left|2(I_2 - I_3)\omega\cos\varphi\sin\theta \pm w\sqrt{(I_3 - I_2)(I_2 - (I_2 - I_3)\cos^2\varphi\sin^2\theta)}\right| \tag{14}$$

**Proof.** First of all, it is deduced that $E_1$ is a critical point of the augmented Hamiltonian (11) if

$$\lambda = -\omega, \qquad \mu = \frac{I_2}{2}\omega^2. \tag{15}$$

Substituting this value into Equation (13), the corresponding reduced Hessian matrix is positive definite if all the principal minors are positive. In this case, if we denote the minors by $D_j$ ($j = 1, \ldots, 4$), we have

$$\begin{aligned}
D_1 &= \frac{1}{I_2 I_3}(I_2\cos^2\theta + I_3\cos^2\varphi\sin^2\theta), \\
D_2 &= \frac{\cos^2\varphi\sin^2\theta}{I_1 I_2 I_3}(I_2\cos^2\theta + (I_1\sin^2\varphi + I_3\cos^2\varphi)\sin^2\theta), \\
D_3 &= \frac{\cos^2\varphi\cos^2\theta\sin^2\theta}{I_1 I_2 I_3}B_0, \\
D_4 &= \frac{\omega^2\cos^4\varphi\cos^2\theta\sin^4\theta}{I_1 I_2 I_3}(I_2 - I_1)B_1,
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
B_0 &= (l_3 - 2(I_2 - I_3)\omega\cos\varphi\sin\theta)^2 + \omega^2(I_2 - I_3)(I_2\cos^2\theta + (I_1\sin^2\varphi + I_3\cos^2\varphi)\sin^2\theta), \\
B_1 &= (l_3 - 2(I_2 - I_3)\omega\cos\varphi\sin\theta)^2 + \omega^2(I_2 - I_3)(I_2 - (I_2 - I_3)\cos^2\varphi\sin^2\theta).
\end{aligned} \tag{17}$$

Taking into account that $D_1$ and $D_2$ are positive ($\cos\varphi\sin\theta \neq 0$), we only need to verify whether $D_3$ and $D_4$ are positive. On the one hand, it is clear that if $I_2$ is the largest moment of inertia, both $B_0$ and $B_1$ are positive, as well as $D_3$ and $D_4$. As a consequence, $E_1$ is stable.

On the other hand, if $I_2$ is not the largest moment of inertia, we distinguish two cases, depending on the sign of $I_2 - I_1$.

*Case 1.* $I_2 - I_1 > 0$. In this case, as $I_2$ is not the largest moment of inertia, it must be $I_3 > I_2 > I_1$ and the sign of $B_0$ and $B_1$ can be either positive or negative. To have stability, both of them must be positive. However,

$$B_1 - B_0 = (I_2 - I_1)(I_2 - I_3)\omega^2\sin^2\varphi\sin^2\theta \leq 0$$

and, if $B_1$ is positive, also $B_0$ is positive. Thus, it is enough to check when $B_1 > 0$. By considering $B_1$ as a second-degree polynomial in $l_3$, it is easy to verify that it has two real roots given by

$$r_{1,2;B_1} = 2(I_2 - I_3)\omega\cos\varphi\sin\theta \pm \omega\sqrt{(I_3 - I_2)(I_2 - (I_2 - I_3)\cos^2\varphi\sin^2\theta)}. \tag{18}$$

Taking into account that the coefficient of $l_3^2$ is positive, $B_1$ is positive if $|l_3| > \max|r_{1,2;B_1}|$, and the theorem is proven.

*Case 2.* $I_2 - I_1 < 0$. In this case, if $I_2 > I_3$, both $B_0$ and $B_1$ are positive and, as a consequence, $D_4 < 0$, and nothing can be said about the stability. On the contrary, if $I_2 < I_3$, we have

$$B_1 - B_0 = (I_2 - I_1)(I_2 - I_3)\omega^2 \sin^2\varphi \sin^2\theta \geq 0,$$

and $B_0$ cannot be positive if $B_1$ is negative. Thus, $D_3$ and $D_4$ cannot be positive at the same time and, again, nothing can be said about the stability. □

**Remark 2.** *In the case* $\sin\theta = 0$*, the equations of motion* (2) *reduce to*

$$(0, \pm(l_3\omega - gmz_0), \pm l_1\omega, 0, 0, 0).$$

*Then, we have an equilibrium solution if* $l_1 = 0$ *and* $l_3\omega - gmz_0 = 0$*. In this way, we recover a particular case of the equilibrium* $E_2$*, mentioned in [26], where it is established that there is stability if* $I_2$ *is the largest moment of inertia, or if* $I_2 > I_1$ *and*

$$l_3^4 > I_2(I_3 - I_2)m^2g^2z_0^2, \tag{19}$$

*in agreement with the result stated in Theorem 3. It is worth noting that the last condition, (19), is no more than a limiting case of Theorem 3. Indeed, when* $\sin\theta \to 0$*, the roots given by Equation (18) tend to the limit value*

$$r_{1,2;B_1} = \pm\omega\sqrt{(I_3 - I_2)I_2}$$

*and stability is achieved if* $|l_3| > |\omega\sqrt{(I_3 - I_2)I_2}|$*. Taking into account the relation between* $l_3$ *and* $\omega$*, we obtain the condition (19). Nonetheless, if we proceed from the very beginning, computing the principal minors of the reduced Hessian matrix in* $W \times W$ *when* $\sin\theta = 0$*, we arrive at*

$$\begin{aligned} D_1 &= \frac{1}{I_2} + \frac{I_2\omega^2}{l_3^2}, & D_2 &= \frac{l_3^2 + I_2^2\omega^2}{I_1 I_2 l_3^2}, \\ D_3 &= \frac{l_3^2 + I_2(I_2 - I_3)\omega^2}{I_1 I_2 I_3 l_3^2}, & D_4 &= \frac{(I_2 - I_1)(l_3^2 + I_2(I_2 - I_3)\omega^2)\omega^2}{I_1 I_2 I_3 l_3^2}. \end{aligned} \tag{20}$$

*From here, it follows that, if* $I_2 > I_1$ *and* $l_3^2 + I_2(I_2 - I_3)\omega^2 > 0$*, all the principal minors are positive. In this way, we recover again the conditions of stability of Theorem 3, which are the same as given in [26].*

To complete the analysis of the sufficient stability conditions for the equilibrium $E_1$, we are left with the case $\cos\varphi = 0$. In this way, we have the following result.

**Theorem 4.** *If* $\cos\varphi = 0$*, the equilibrium* $E_1$ *is stable if* $I_2$ *is the largest moment of inertia, or if* $I_3 > I_2 > I_1$ *and* $\omega$ *satisfies the inequality*

$$|gmz_0| > \sqrt{(I_3 - I_2)I_2}\,\omega^2. \tag{21}$$

**Proof.** In this case, the equilibrium $E_1$ is a critical point of the augmented Hamiltonian if Equation (15) is satisfied. Moreover, we have permanent rotations when the gyrostatic moments are given by

$$l_3 = \frac{gmz_0}{\omega}, \quad l_1 = (I_2 - I_1)\omega\sin\theta. \tag{22}$$

Now, we follow the steps for the application of Theorem 2, taking into account that $\cos\varphi = 0$. We only consider the case $\varphi = \pi/2$, as the case $\varphi = -\pi/2$ is analogous. It can be seen that if $\sin\theta \neq 0$ (the case $\sin\theta = 0$ has been already considered in Remark 2), the reduced space $W$ is generated by the vectors

$$\begin{cases} v_1 = \hat{e}_1 - \dfrac{1}{l_3}\hat{e}_6 \cos\theta, \\ v_2 = \hat{e}_2 - \dfrac{1}{l_3}\hat{e}_6 \sin\theta, \\ v_3 = \hat{e}_3, \\ v_4 = \hat{e}_4 - \hat{e}_5 \cot\theta + \dfrac{1}{l_3}\hat{e}_6 (l_1 + (I_1 - I_2)\omega \sin\theta) \cot\theta. \end{cases}$$

Now, the Hessian matrix for the augmented Hamiltonian in the reduced space $W \times W$ is given by

$$\mathrm{Hess}(\mathcal{H}_A)\Big|_{W\times W} = \begin{pmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{12} & h_{22} & h_{23} & h_{24} \\ h_{13} & h_{21} & h_{33} & h_{34} \\ h_{14} & h_{24} & h_{34} & h_{44} \end{pmatrix}, \tag{23}$$

where

$$\begin{aligned} h_{11} &= \frac{1}{I_2} + \frac{I_2\omega^4 \cos^2\theta}{g^2 m^2 z_0^2}, \\ h_{12} &= \frac{I_2\omega^4 \cos\theta \sin\theta}{g^2 m^2 z_0^2}, \quad h_{22} = \frac{1}{I_1} + \frac{I_2\omega^4 \sin^2\theta}{g^2 m^2 z_0^2}, \\ h_{13} &= \frac{\omega^2 \cos\theta}{g m z_0}, \quad h_{23} = \frac{w^2 \sin^2\theta}{g m z_0}, \quad h_{33} = \frac{1}{I_3}, \\ h_{14} &= -\omega, \quad h_{24} = \omega \cot\theta, \quad h_{34} = 0, \quad h_{44} = I_2\omega^2 \csc^2\theta. \end{aligned}$$

By means of the Sylvester criterion, the matrix is positive definite if the principal minors are all positive. For them, we obtain

$$\begin{aligned} D_1 &= \frac{1}{I_2} + \frac{I_2\omega^4 \cos^2\theta}{g^2 m^2 z_0^2}, \\ D_2 &= \frac{g^2 m^2 z_0^2 + I_2\omega^4 (I_2 \cos^2\theta + I_1 \sin^2\theta)}{I_1 I_2 g^2 m^2 z_0^2}, \\ D_3 &= \frac{B_2}{I_1 I_2 I_3 g^2 m^2 z_0^2}, \\ D_4 &= \frac{(I_2 - I_1)\omega^2 B_3 \cot^2\theta}{I_1 I_2 I_3 g^2 m^2 z_0^2}, \end{aligned} \tag{24}$$

where

$$\begin{aligned} B_2 =& g^2 m^2 z_0^2 + (I_2 - I_3)\omega^4 (I_2 \cos^2\theta + I_1 \sin^2\theta) \\ B_3 =& g^2 m^2 z_0^2 + I_2 (I_2 - I_3)\omega^4. \end{aligned} \tag{25}$$

It is clear that, if $I_2$ is the largest moment of inertia, all the principal minors are positive and the equilibrium is stable. On the contrary, if $I_2$ is not the largest moment of inertia and $I_2 > I_1$, it follows that $I_3 > I_2 > I_1$ and $D_4 > 0$ if $B_3 > 0$, or equivalently, if the inequality of the hypothesis of the theorem holds. Since

$$B_2 - B_3 = (I_2 - I_3)(I_1 - I_2)\omega^4 \sin^2\theta > 0.$$

and $B_3 > 0$, it follows that $B_2 > 0$ and then $D_3 > 0$. Therefore, the equilibrium point is stable if (21) is satisfied, as stated in the theorem.

In the case $I_1 > I_2$, as is the case in Theorem 3, $D_3$ and $D_4$ cannot be positive at the same time. On the one hand, if $I_2 > I_3$, $B_3 > 0$ and consequently $D_4 < 0$. On the other hand, if $I_2 < I_3$, it must be $B_3 < 0$. However,

$$B_2 - B_3 = (I_2 - I_3)(I_1 - I_2)\omega^4 \sin^2\theta < 0$$

and, then, $B_2$, as well as $D_3$, is negative. □

**Remark 3.** *We note that Theorem 4 is also a limit case of Theorem 3. Indeed, if* $\cos\varphi = 0$*, taking into account that* $l_3\omega = gmz_0$*, condition (14) reduces to* $|gmz_0| > \sqrt{(I_3 - I_2)I_2}\omega^2$*.*

**Remark 4.** *For the three situations not covered by Theorems 3 and 4, namely* $I_1$ *is the largest moment of inertia, and* $I_3 > I_1 > I_2$*, we can also obtain stable rotations by acting on the* $l_2$ *gyrostatic moment. Indeed, the result is analogous by considering the parameterization*

$$\begin{aligned} &k_2 = \sin\varphi\sin\theta, \quad k_1 = \cos\theta, \quad k_3 = \cos\varphi\sin\theta \\ &\pi_2 = \omega I_2\sin\varphi\sin\theta, \quad \pi_1 = \omega I_1\cos\theta, \quad \pi_3 = \omega I_3\cos\varphi\sin\theta. \end{aligned}$$

*As a consequence, it is enough to replace* $l_1$ *by* $l_2$ *and interchange* $I_1$ *and* $I_2$*.*

**Remark 5.** *It follows from Theorem 4 that, if* $I_3 > I_2 > I_1$*, there exist stable permanent rotations if* $|\omega|$ *is small enough. The same conclusion is obtained from Theorem 3, introducing the relation between* $l_3$ *and* $\omega$*, given by the second constraint in (4), into (14). Indeed, we can derive stability curves for each orientation, once the moments of inertia and the position of the center of mass are fixed. In this way, Equation (4) determines in the plane* $\omega$*–*$l_3$ *a curve of equilibria, with a stable part where sufficient conditions, given by Equation (14), are satisfied. This curve is depicted in Figure 2 for a test example with* $I_1 = 5$*,* $I_2 = 5.1$*,* $I_3 = 5.2$*,* $gmz_0 = 1$ *and a given orientation* $\theta = \varphi = \pi/3$*. The green part of the curve corresponds to stable rotations and the red one to the part where sufficient conditions are not satisfied. Figure 3 shows the evolution of a trajectory close to the equilibrium point, when the starting orientation is* $\theta = \varphi = \pi/3 + 0.012$*. In the upper panel, we observe the stable character of the equilibrium position when* $\omega = 1.1$ *is chosen in the green part of the curve, whereas the lower one shows the unstable character when* $\omega = 1.23$ *is on the red part.*

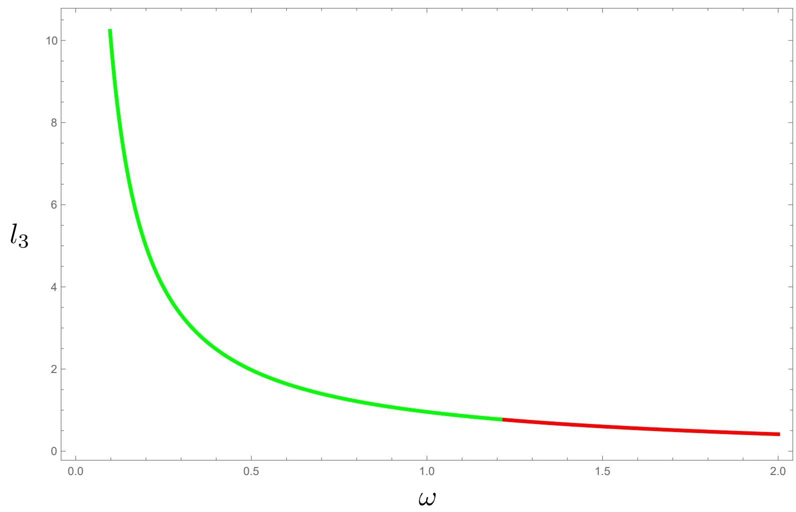


**Figure 2.** Existence of equilibrium $E_1$, in the plane $\omega$–$l_3$, for $I_1 = 5$, $I_2 = 5.1$, $I_3 = 5.2$, $gmz_0 = 1$ and a given orientation $\theta = \varphi = \pi/3$. The green part represents stable rotations whereas the red one represents the case where sufficient conditions of stability are not satisfied.

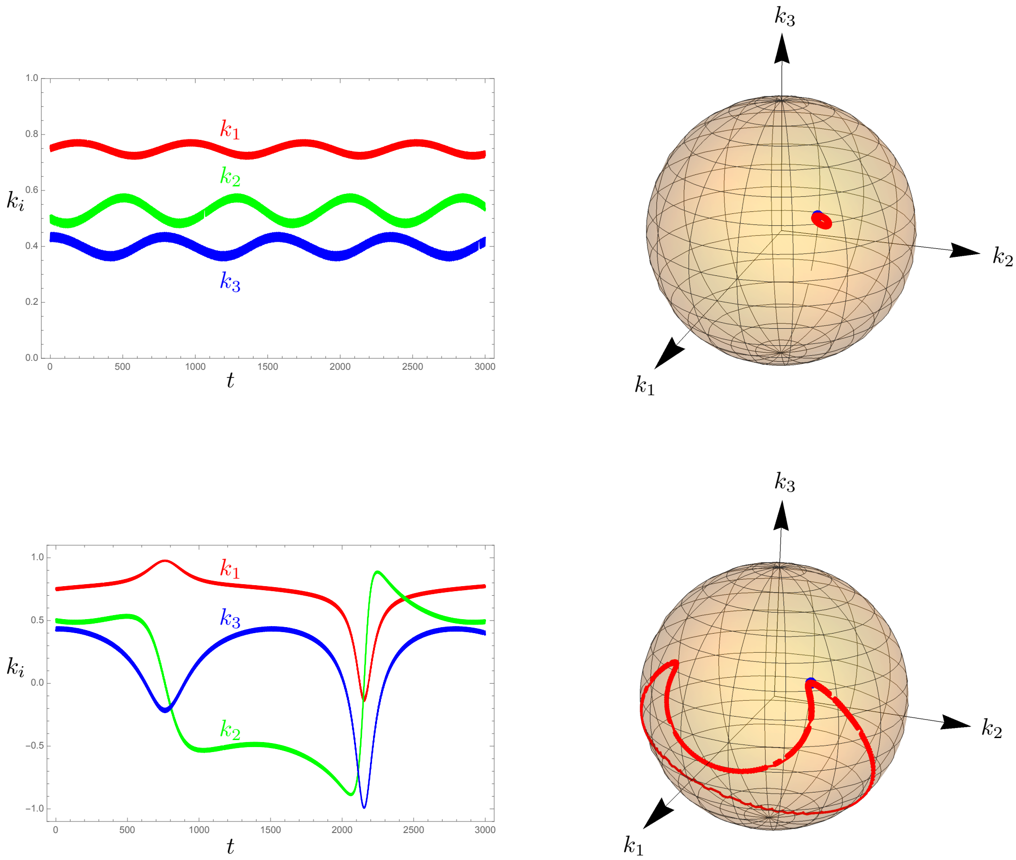


**Figure 3.** Time evolution of the orientation of the gyrostat for $I_1 = 5$, $I_2 = 5.1$, $I_3 = 5.2$, $gmz_0 = 1$ and an initial condition close to the prescribed equilibrium $E_1$, located at $\theta = \varphi = \pi/3$. The upper panel corresponds to the stable case, $\omega = 1.1$, and the lower one to $\omega = 1.23$. The blue dot represents the equilibrium position.

This behavior suggests that sufficient and necessary conditions can be the same. However, necessary conditions are readily derived by analyzing the associated linear system.

**Theorem 5.** *If the equilibrium $E_1$ is stable and $\cos\varphi \sin\theta \neq 0$, then $D_4 \geq 0$, where $D_4$ is given by (16). Analogously, if $\cos\varphi \sin\theta = 0$, $E_1$ is stable if $D_4 \geq 0$, where $D_4$ is given by (20) if $\sin\theta = 0$ and by (24) if $\cos\varphi = 0$.*

**Proof.** Stability of $E_1$ implies spectral stability, so that the real part of the eigenvalues associated with the linear system at $E_1$ cannot be greater than 0. It can be seen that the eigenvalues are the roots of a polynomial equation of the form

$$\lambda^2(\lambda^4 + b\lambda^2 + a), \tag{26}$$

where $a \cos^4\varphi \sin^4\theta = D_4$, in the case $\cos\varphi \sin\theta \neq 0$. If (26) does not have roots with a positive real part, it must be $a \geq 0$ and, consequently, $D_4 \geq 0$. The case $\cos\varphi \sin\theta = 0$ is proven in the same way. $\square$

As a consequence, sufficient and necessary conditions are the same when $I_2$ is the largest moment of inertia or $I_3 > I_2 > I_1$. For the remaining cases, further investigation is needed. However, taking into account Remark 4, by acting on the gyrostatic moment $l_2$ in these situations, we obtain a coincidence between necessary and sufficient conditions.

### 3.2. Stability of the Equilibrium $E_0$

From the results in the previous subsection, we conclude that two rotors are enough to have stable permanent rotations around an axis oriented in any direction, regardless of the values of the moments of inertia, if the center of mass lies on one of the principal axes and one of the rotors is aligned with the same axis. The only possible exception is the family $E_0$, which is a limit case of $E_1$ when $\theta = \pi/2$. Unfortunately, previous results cannot be applied because some of the principal minors are singular for $\theta = \pi/2$. This is the reason that it must be considered separately, using the condition (3). In this way, we arrive at the following result.

**Theorem 6.** *The equilibrium $E_0$ is stable if*

1. $\cos\varphi > 0$ *and* $l_3\omega > K_1$,
2. $\cos\varphi < 0$ *and* $l_3\omega < K_2$,
3. $\varphi = \pm\pi/2$ *and* $\pm l_1\omega > K_3$,

*where $K_1$ and $K_2$ are the maximum and minimum, respectively, of*

$$gmz_0 + (I_2 - I_3)\omega^2\cos\varphi,$$
$$\frac{I_1\omega^2 mgz_0 + (I_1 - I_3)\omega^2\cos^2\varphi(I_3\omega^2\cos\varphi - mgz_0\cos 2\varphi) - m^2g^2z_0^2\cos\varphi\sin^2\varphi}{\omega^2(I_1\sin^2\varphi + I_3\cos^2\varphi)},$$

*and $K_3$ is the maximum of*

$$(I_2 - I_1)\omega^2, \qquad \frac{(I_3 - I_1)I_1\omega^4 - g^2m^2z_0^2}{I_1\omega^2}.$$

**Proof.** First of all, $E_0$ is a critical point of the augmented Hamiltonian if

$$\lambda = -\omega, \quad \mu = \frac{I_3\omega^2 + (l_3\omega - gmz_0)\sec\varphi}{2}. \tag{27}$$

Substituting these values into Equation (13), we obtain that the principal minors of the reduced Hessian matrix are given by

$$\begin{aligned} D_1 &= \frac{\cos^2\varphi}{I_2}, & D_2 &= \frac{(I_3\cos^2\varphi + I_1\sin^2\varphi)\cos^2\varphi}{I_1I_2I_3}, \\ D_3 &= \frac{G_0G_1\cos^3\varphi}{I_1I_2I_3}, & D_4 &= \frac{G_1G_2\cos^2\varphi}{I_1I_2I_3\omega^2}. \end{aligned} \tag{28}$$

where

$$\begin{aligned} G_0 &= I_1\sin^2\varphi + I_3\cos^2\varphi, \\ G_1 &= l_3\omega - gmz_0 + (I_3 - I_2)\omega^2\cos\varphi, \\ G_2 &= (I_3 - I_1)I_3\omega^4\cos^3\varphi + (I_1 - (I_1 - I_3)\cos^2\varphi)l_3\omega^3 + \\ &\quad (2(I_1 - I_3)\cos^4\varphi - (I_1 - I_3)\cos^2\varphi - I_1)gmz_0\omega^2 + g^2m^2z_0^2\cos\varphi\sin^2\varphi. \end{aligned} \tag{29}$$

We note that $D_1$, $D_2$ and $G_0$ are positive, and the sufficient stability conditions reduce to $G_1\cos\varphi > 0$ and $G_1G_2 > 0$. Let us assume that $\cos\varphi > 0$; then, both $G_1$ and $G_2$ must be greater than zero. It is easy to verify that $G_{1,2} > 0$ under the hypothesis of item 1 in the theorem. Indeed, it is enough to solve $G_{1,2} > 0$ in terms of $l_3\omega$.

A similar situation appears if $\cos\varphi < 0$. In this case, both $G_1$ and $G_2$ must be negative, which immediately follows from the conditions of item 2.

We are left with the case $\cos\varphi = 0$, which corresponds to the two equilibrium positions

$$(I_1\omega, 0, 0, \pm 1, 0, 0),$$

where the plus sign stands for $\varphi = \pi/2$ and the minus sign for $\varphi = -\pi/2$. They appear if the existence condition $l_3\omega = gmz_0$ is verified. Moreover, they are critical points of the augmented Hamiltonian when

$$\lambda = -\omega, \quad \mu = \frac{1}{2}(\pm l_1 + I_1\omega)\omega.$$

The space $W$ is now generated by the vectors

$$\hat{\boldsymbol{e}}_1, \quad \hat{\boldsymbol{e}}_1 - \frac{1}{l_3}\hat{\boldsymbol{e}}_6, \quad \hat{\boldsymbol{e}}_3, \quad \hat{\boldsymbol{e}}_4,$$

and the corresponding reduced Hessian matrix reads as

$$\operatorname{Hess}(\mathcal{H}_A)\Big|_{W\times W} = \begin{pmatrix} \dfrac{1}{I_2} & 0 & 0 & -\omega \\ 0 & \dfrac{1}{I_1} + \dfrac{(\pm l_1 + I_1\omega)\omega^3}{g^2m^2z_0^2} & \dfrac{\omega^2}{gmz_0} & 0 \\ 0 & \dfrac{\omega^2}{gmz_0} & \dfrac{1}{I_3} & 0 \\ -\omega & 0 & 0 & (\pm l_1 + I_1\omega)\omega \end{pmatrix}.$$

From here, the principal minors are given by

$$\begin{aligned} D_1 &= \frac{1}{I_2}, & D_2 &= \frac{g^2m^2z_0^2 + I_1\omega^3(\pm l_1 + I_1\omega)}{I_1I_2g^2m^2z_0^2}, \\ D_3 &= \frac{I_1\omega^3(\pm l_1 + (I_1 - I_3)\omega) + g^2m^2z_0^2}{I_1I_2I_3g^2m^2z_0^2}, & D_4 &= D_3(\pm l_1 + (I_1 - I_2)\omega)\omega. \end{aligned} \tag{30}$$

We note that all the minors are positive if $l_1\omega$ satisfies the conditions in item 3. Thus, by the Sylvester criterion, the reduced Hessian matrix is positive definite and the equilibrium position is stable. □

To finish the stability analysis, we provide necessary stability conditions for the family of equilibria $E_0$, which arise from the spectral stability of the associated linear system.

**Theorem 7.** *If $E_0$ is stable, then $D_4 \geq 0$, where $D_4$ is given by (28) if $\cos\varphi \neq 0$ and by (30) if $\cos\varphi = 0$.*

**Proof.** We proceed as in Theorem 5. In this way, the characteristic polynomial associated with the equilibrium $E_0$ has also the form

$$\lambda^2(\lambda^4 + b\lambda^2 + a).$$

For the case $\cos\varphi \neq 0$, we obtain $a\cos^4\varphi = D_4$, where $D_4$ is given by (28). A necessary condition to have spectral stability is $a \geq 0$, which implies $D_4 \geq 0$. In the case $\cos\varphi = 0$, we obtain $a\omega^2 = g^2m^2z_0^2D_4$, with $D_4$ as in (30) and, again, spectral stability implies $D_4 \geq 0$. □

**Remark 6.** *Note that, in the case $D_3 > 0$, sufficient and necessary stability conditions are the same. However, it seems that necessary conditions are enough to have stable rotations. As an example, we take the case of a gyrostat with $I_1 = 5$, $I_2 = 5.1$, $I_3 = 5.2$, $mgz_0 = 1$ and $\varphi = \pi/3$. Figure 4 shows, in the plane $\omega$–$l_1$, the regions where the necessary stability conditions are satisfied, and the green color represents the region where the sufficient stability conditions are also satisfied. Besides the expected symmetry with respect to the origin, the green and red regions have in common the points $(\pm 1.443, \pm 0.12497)$. Let us take $\omega = 1.443$ and two values of $l_1$, one inside the green region ($l_1 = 0.5$) and another one inside the red one ($l_1 = -0.1$). Figure 5 shows the time evolution of the vector $\hat{\boldsymbol{k}}$, when the initial condition is slightly deviated from the equilibrium position. The* *upper panel corresponds to the case $l_1 = 0.5$, inside the green area, and the lower one to the case $l_1 = -0.1$, inside the red area. There is no significant difference between the two time histories, which constitutes evidence that necessary stability conditions are, probably, sufficient ones.*

**Remark 7.** *It is worth noting that $D_4 = 0$ when the conditions (4) vanish. Thus, the stability boundary corresponds to the bifurcation of the $E_1$ family into the $E_0$ family.*

**Remark 8.** *We stress that all the above results are also valid for a more general case. Indeed, by adding to the three first equations of the differential system (2) with circular gyroscopic forces $\boldsymbol{M}$ with components*

$$L(t, \hat{\boldsymbol{k}}, \boldsymbol{\pi})\left(\frac{k_2\pi_3}{I_3} - \frac{k_3\pi_2}{I_2}, \frac{k_3\pi_1}{I_1} - \frac{k_1\pi_3}{I_3}, \frac{k_1\pi_2}{I_2} - \frac{k_2\pi_1}{I_1}\right),$$

*where $L(t, \hat{\boldsymbol{k}}, \boldsymbol{\pi})$ is an arbitrary function, we obtain the same equilibrium solutions reported in Theorem 1. Furthermore, as proven in [46], (7), (9) and (10) remain as first integrals and the stability results are also valid for this generalized system.*

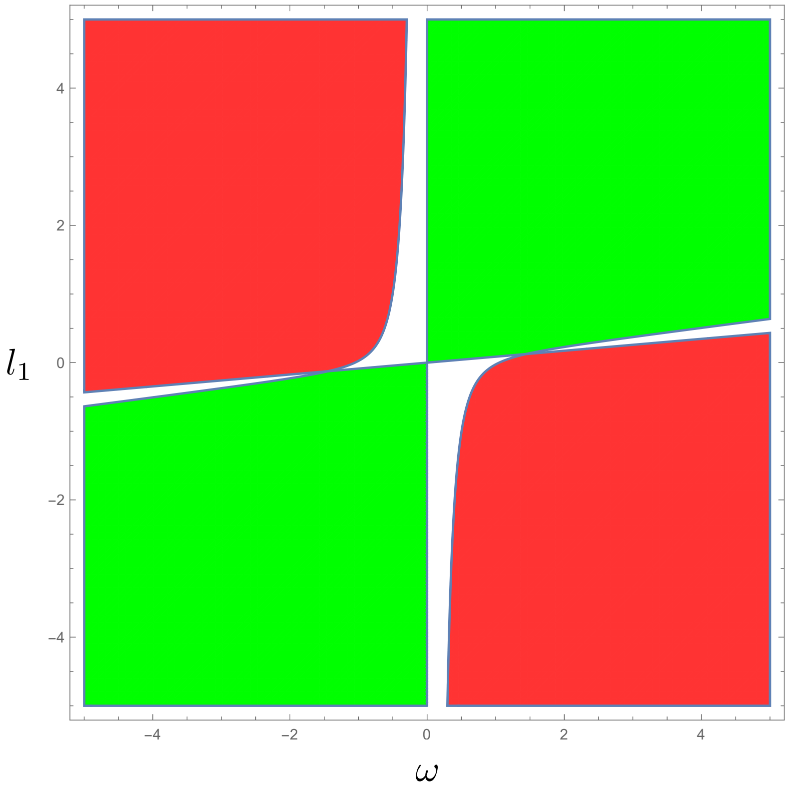


**Figure 4.** Necessary stability conditions for $E_0$ in the plane $\omega$–$l_1$, for $I_1 = 5$, $I_2 = 5.1$, $I_3 = 5.2$, $gmz_0 = 1$ and a given orientation $\varphi = \pi/3$. The green part represents both sufficient and necessary conditions, whereas the red region only represents necessary ones.

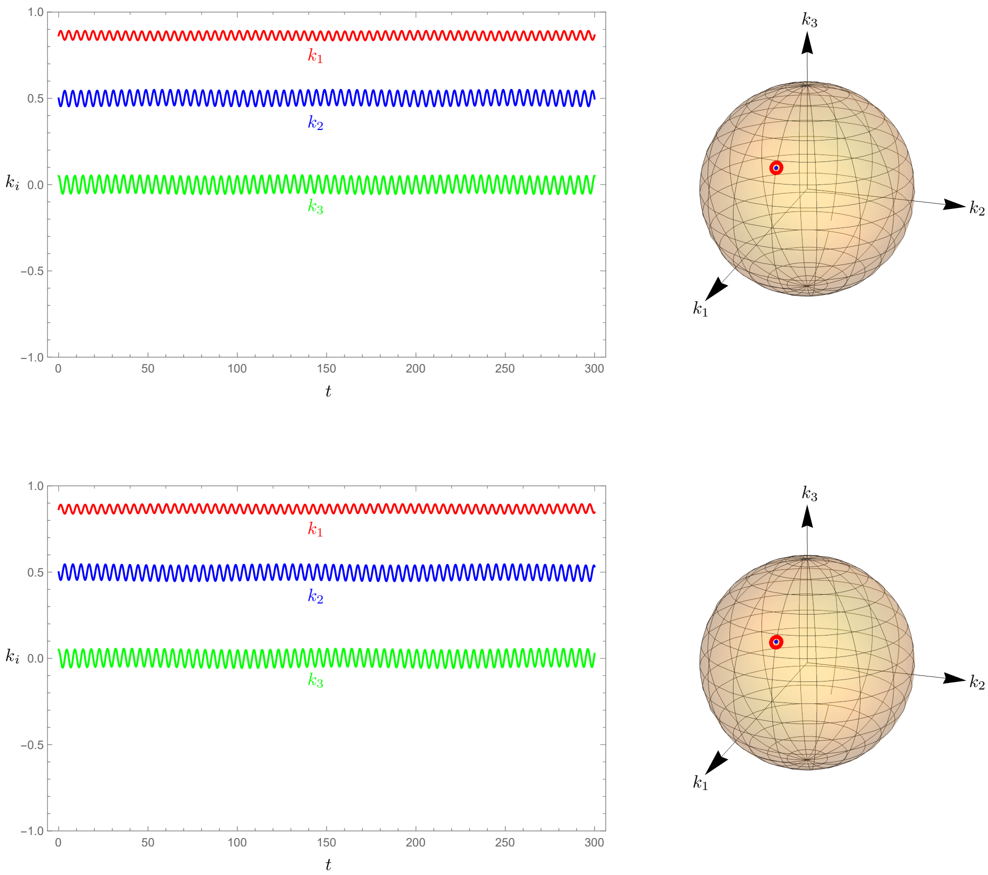


**Figure 5.** Time evolution of the orientation of the gyrostat for $I_1 = 5$, $I_2 = 5.1$, $I_3 = 5.2$, $gmz_0 = 1$ and an initial condition close to the prescribed equilibrium $E_0$, located at $\theta = \pi/2$, $\varphi = \pi/3$. The upper panel corresponds to $\omega = 1.443$ and $l_1 = 0.5$, inside the green area of Figure 4, and the lower one to $\omega = 1.443$ and $l_1 = -0.1$, in the red area. The blue dot represents the equilibrium position.

## 4. Conclusions

The main conclusion of this work is about the existence of permanent stable rotations around an axis oriented in any direction of the space by the action of two rotors, one of them aligned along the principal axis where the center of mass lies. Indeed, from the results in Section 3, it follows that, given a particular gyrostat and a concrete orientation, it is possible to find appropriate gyrostatic moments and angular velocities in such a way that the gyrostat maintains its orientation along time, even in the case of small perturbations. This is achieved if the gyrostatic moments are chosen in order that the sufficient stability conditions stated in Theorems 3, 4 and 6 hold. In the case $\cos\theta \neq 0$, the constraints (4) reduce in great manner the possible choices of $l_1$, $l_3$ and $\omega$. However, if $\cos\theta = 0$, there is an extra degree of freedom and, for every $\omega$, it is possible to obtain suitable $l_1$ and $l_3$, giving rise to stable rotations. Moreover, necessary and sufficient stability conditions match in many cases and there is evidence that necessary conditions are also sufficient ones. However, this result cannot be proven using the Energy-Casimir method.

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