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Uniqueness of unconditional basis of $H_p(\mathbb{T}) \oplus \ell_2$ and $H_p(\mathbb{T}) \oplus \mathcal{T}^{(2)}$ for $0 < p < 1$ [☆]



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ABSTRACT

Our goal in this paper is to advance the state of the art of the topic of uniqueness of unconditional basis. To that end we establish general conditions on a pair (\mathbb{X}, \mathbb{Y}) formed by a quasi-Banach space \mathbb{X} and a Banach space \mathbb{Y} which guarantee that every unconditional basis of their direct sum $\mathbb{X} \oplus \mathbb{Y}$ splits into unconditional bases of each summand. As application of our methods we obtain that, among others, the spaces $H_p(\mathbb{T}^d) \oplus \mathcal{T}^{(2)}$ and $H_p(\mathbb{T}^d) \oplus \ell_2$, for $p \in (0, 1)$ and $d \in \mathbb{N}$, have a unique unconditional basis (up to equivalence and permutation).

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Lattice techniques in quasi-Banach
spaces
Hardy spaces
Tsirelson space

1. Introduction and background

A relevant topic in Banach space theory from a structural point of view is to determine whether a given space has an unconditional basis and, in the case it does, to know whether this is the unique unconditional basis of the space. Recall that a quasi-Banach space (in particular a Banach space) \mathbb{X} with a semi-normalized unconditional basis \mathcal{B} is said to have a *unique unconditional basis* if every semi-normalized unconditional basis of \mathbb{X} is equivalent to \mathcal{B} , in which case \mathcal{B} is equivalent to all its permutations, i.e., it is symmetric.

For a Banach space it is rather unusual to have a unique unconditional basis. Indeed, it is well-known that ℓ_2 has a unique unconditional basis [28], and a classical result of Lindenstrauss and Pełczyński [30] establishes that ℓ_1 and c_0 also have a unique unconditional basis. Lindenstrauss and Zippin [32] completed the picture by showing that those three are the only Banach spaces with this property.

For Banach spaces without symmetric bases (or Banach spaces that we do not know whether they have a symmetric basis or not) it is more natural to consider the question of uniqueness of unconditional basis up to (equivalence and) permutation, UTAP for short. We say that \mathbb{X} has a UTAP unconditional basis \mathcal{B} if every semi-normalized unconditional basis of \mathbb{X} is equivalent to a permutation of \mathcal{B} . Of course, if \mathbb{X} has a symmetric basis, the notions of uniqueness of unconditional basis and uniqueness of unconditional basis up to equivalence and permutation coincide. The first movers in this direction of research were Edelstein and Wojtaszczyk, who proved that finite direct sums of c_0 , ℓ_1 and ℓ_2 have a UTAP unconditional basis [18]. Bourgain et al. embarked on a comprehensive study aimed at classifying those Banach spaces with unique unconditional basis up to permutation that culminated in 1985 with their *Memoir* [13]. They showed that the spaces $c_0(\ell_1)$, $c_0(\ell_2)$, $\ell_1(c_0)$, $\ell_1(\ell_2)$ and their complemented subspaces with an unconditional basis all have a UTAP unconditional basis, while $\ell_2(\ell_1)$ and $\ell_2(c_0)$ do not. However, the hopes of attaining a satisfactory classification were shattered when they found a nonclassical Banach space, namely the 2-convexification $\mathcal{T}^{(2)}$ of Tsirelson's space \mathcal{T} having a UTAP unconditional basis. Other significant advances in the theory were carried out by Casazza and Kalton [14,15], who proved that Tsirelson's space \mathcal{T} , certain Nakano spaces close either to ℓ_1 or c_0 , certain complemented subspaces of Orlicz sequence spaces ℓ_F , where F is a convex Orlicz function close either to the function $t \mapsto t^2$ or to the identity map, and certain infinite ℓ_1 -products and c_0 -products of spaces with a UTAP unconditional basis have a UTAP unconditional basis. The techniques they developed provided also a new approach to the uniqueness UTAP of unconditional basis in the spaces $c_0(\ell_1)$ and $\ell_1(c_0)$.

Each of these examples of Banach spaces with a UTAP unconditional basis follows one of these mutually exclusive patterns:

- (P.1) The space is close to c_0 ;
- (P.2) the space is close to ℓ_2 ;
- (P.3) the space is close to ℓ_1 ;
- (P.4) the space is a finite or infinite direct sum of spaces, each of which follows some of the previous patterns. Moreover, in the case of infinite direct sums, the way in which we sum should also follow one of those patterns.

From the comprehensive approach of Functional Analysis, Banach spaces are a particular case of quasi-Banach spaces, hence it seems only natural to transfer the problem of uniqueness of unconditional basis to this more general setting. As we will illustrate below, the situation for quasi-Banach spaces which are not Banach spaces is quite different. Kalton showed that a wide class of non-locally convex Orlicz sequence spaces, including the ℓ_p spaces for $0 < p < 1$, have a unique unconditional basis [22]. This topic was given continuity later on in a series of papers, amongst which we mention [34,29,43,6–9,2]. In particular, a wide class of non-locally convex Lorentz sequence spaces were proved to have a unique unconditional basis. As for the UTAP unconditional basis problem, important known results include the cases of finite direct sums of ℓ_p spaces for $p \in (0, 1] \cup \{2, \infty\}$ (we replace ℓ_∞ with c_0 is $p = \infty$), the mixed-norm spaces $\ell_p(\ell_2)$, $\ell_p(\ell_1)$, $\ell_1(\ell_p)$, $c_0(\ell_p)$, $\ell_p(c_0)$ for $0 < p < 1$, and the Hardy spaces $H_p(\mathbb{T}^d)$ for $0 < p < 1$ and $d \in \mathbb{N}$. These examples exhibit a pattern which generalizes (P.3), namely:

- (P.5) the Banach envelope of the space is close to ℓ_1 .

Thus, for quasi-Banach spaces that follow (P.5), the uniqueness of unconditional basis seems to be the norm rather than the exception.

Pulling the thread of pattern (P.4) suggests the following question.

Question 1.1. Let \mathbb{X} and \mathbb{Y} be quasi-Banach spaces with a UTAP unconditional basis. Does $\mathbb{X} \oplus \mathbb{Y}$ have a UTAP unconditional basis?

Since the methods used in the cited results on uniqueness of unconditional basis depend on the space, finding a positive general answer to Question 1.1 seems unlikely and remote. The authors of [1] addressed this problem and proved that if \mathbb{X} and \mathbb{Y} are quasi-Banach spaces with a UTAP unconditional basis falling either into patterns (P.5) or (P.1), then $\mathbb{X} \oplus \mathbb{Y}$ has a UTAP unconditional basis.

The expected way to take the subject further is to study what happens when we consider the direct sum of a quasi-Banach space having a UTAP unconditional basis which additionally falls either into patterns (P.5) or (P.1) (or a combination of both),

with a Banach space having a UTAP conditional basis which additionally follows pattern (P.2). In particular, the following question arises:

Question 1.2. Does $\mathbb{X} \oplus \ell_2$ or $\mathbb{X} \oplus \mathcal{T}^{(2)}$ have a UTAP unconditional basis provided that the quasi-Banach space \mathbb{X} does?

In this paper we address Question 1.2 in the case when \mathbb{X} is non-locally convex, and provide a positive answer for a wide class of spaces that includes the Hardy spaces $H_p(\mathbb{T}^d)$ for $0 < p < 1$ and $d \in \mathbb{N}$. We will derive our results from a general splitting principle for unconditional bases that we will present in Section 4 followed by some applications. Before, in Sections 2 and 3, we will develop the necessary machinery that will sustain our discussion. Specifically, in Section 2 we set up a technique for splitting unconditional bases of a direct sum of a non-locally convex quasi-Banach space with a Banach space. In Section 3 we exploit a method from [35] for splitting complemented subspaces, which combined with the notion of subprojectivity yields a sufficient condition for an operator between quasi-Banach spaces to be strictly singular.

1.1. Terminology

We use standard terminology and notation in Banach space theory as can be found, e.g., in [5]. Most of our results, however, will be established in the general setting of quasi-Banach spaces; the unfamiliar reader will find general information about quasi-Banach spaces in [27]. We next gather the notation that is more heavily used.

A *quasi-norm* on a vector space over the real or complex field \mathbb{F} , is a map $\|\cdot\|: \mathbb{X} \rightarrow [0, \infty)$ satisfying

(Q.a) $\|f\| = 0$ if and only if $f = 0$;

(Q.b) $\|\alpha f\| = |\alpha| \|f\|$ for $\alpha \in \mathbb{F}$ and $f \in \mathbb{X}$; and

(Q.c) there is a constant $\kappa \geq 1$ so that for all f and g in \mathbb{X} we have $\|f+g\| \leq \kappa(\|f\|+\|g\|)$.

If it is possible to take $\kappa = 1$ we obtain a norm. More generally, given $0 < p \leq 1$, a p -norm is a map $\|\cdot\|: \mathbb{X} \rightarrow [0, \infty)$ satisfying (Q.a), (Q.b) and

(Q.d) for all f and g in \mathbb{X} we have $\|f+g\|^p \leq \|f\|^p + \|g\|^p$.

Any p -norm is a quasi-norm. Conversely, by Aoki-Rolewicz theorem [10,37] any quasi-norm is p -convex for some $p \in (0, 1]$, i.e., there is a constant $C \geq 1$ such that

$$\left\| \sum_{j=1}^n f_j \right\|^p \leq C \sum_{j=1}^n \|f_j\|^p, \quad n \in \mathbb{N}, f_j \in \mathbb{X}.$$

Thus, any quasi-Banach space can be endowed with an equivalent p -norm.

A quasi-norm defines a metrizable vector topology on \mathbb{X} whose base of neighborhoods of zero is given by sets of the form $\{x \in \mathbb{X}: \|x\| < 1/n\}$, $n \in \mathbb{N}$. If such topology is complete we say that $(\mathbb{X}, \|\cdot\|)$ is a *quasi-Banach space*. A *p-Banach space*, $0 < p \leq 1$, is a quasi-Banach space endowed with a *p-norm*.

The symbol $\mathbb{X} \simeq \mathbb{Y}$ means that the quasi-Banach spaces \mathbb{X} and \mathbb{Y} are isomorphic. A family $(\mathbf{x}_n)_{n \in \mathcal{N}}$ in a quasi-Banach space \mathbb{X} is said to be *semi-normalized* if

$$0 < \inf_{n \in \mathcal{N}} \|\mathbf{x}_n\| \leq \sup_{n \in \mathcal{N}} \|\mathbf{x}_n\| < \infty.$$

The closed linear span of a subset V of \mathbb{X} will be denoted by $[V]$. A sequence $\mathcal{X} = (\mathbf{x}_n)_{n=1}^\infty$ in \mathbb{X} is said to be a *Schauder basic sequence* if for every $f \in [\mathbf{x}_n: n \in \mathbb{N}]$ there is a unique family $(a_n)_{n=1}^\infty$ in \mathbb{F} such that the series $\sum_{n=1}^\infty a_n \mathbf{x}_n$ converges to f . If, moreover, $[\mathbf{x}_n: n \in \mathbb{N}] = \mathbb{X}$, we say that \mathcal{X} in \mathbb{X} is a *Schauder basis* of \mathbb{X} . If \mathcal{X} is a basis of \mathbb{X} the functionals $(\mathbf{x}_n^*)_{n=1}^\infty$ in \mathbb{X}^* defined by $\mathbf{x}_n^*(f) = a_n$ whenever $f = \sum_{n=1}^\infty a_n \mathbf{x}_n$ are called the biorthogonal functionals of \mathcal{X} . The support of $f \in \mathbb{X}$ with respect to the basis \mathcal{X} is the set

$$\text{supp}(f) = \{n \in \mathbb{N}: \mathbf{x}_n^*(f) \neq 0\}.$$

A *block basic sequence* with respect to \mathcal{X} is a sequence $(\mathbf{y}_k)_{k=1}^\infty$ in $\mathbb{X} \setminus \{0\}$ such that

$$\max(\text{supp}(\mathbf{y}_k)) < \min(\text{supp}(\mathbf{y}_{k+1})), \quad k \in \mathbb{N}.$$

It will be convenient to index unconditional bases with (finite or infinite) countable sets other than \mathbb{N} . A countable family $\mathcal{B} = (\mathbf{x}_n)_{n \in \mathcal{N}}$ in \mathbb{X} is an *unconditional basic sequence* if for every $f \in [\mathbf{x}_n: n \in \mathcal{N}]$ there is a unique family $(a_n)_{n \in \mathcal{N}}$ in \mathbb{F} such that the series $\sum_{n \in \mathcal{N}} a_n \mathbf{x}_n$ converges unconditionally to f . If we additionally have $[\mathbf{x}_n: n \in \mathcal{N}] = \mathbb{X}$ then \mathcal{B} is an *unconditional basis* of \mathbb{X} .

A *sequence space* on a countable set \mathcal{N} will be a quasi-Banach lattice on \mathcal{N} for which the unit vector system $(\mathbf{e}_j)_{j \in \mathcal{N}}$ defined by $\mathbf{e}_j = (\delta_{i,j})_{i \in \mathcal{N}}$, where $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ otherwise, is an unconditional basis. We will denote by $\mathcal{E}[\mathbb{L}]$ the unit vector system of a sequence space \mathbb{L} .

If \mathcal{B} is an unconditional basis, under a suitable renorming of the space we have

$$\left\| \sum_{n \in \mathcal{N}} a_n \mathbf{x}_n \right\| \leq \left\| \sum_{n \in \mathcal{N}} b_n \mathbf{x}_n \right\|$$

provided that the vectors $\sum_{n \in \mathcal{N}} a_n \mathbf{x}_n$ and $\sum_{n \in \mathcal{N}} b_n \mathbf{x}_n$ of \mathbb{X} satisfy $|a_n| \leq |b_n|$ for all $n \in \mathcal{N}$. Hence an unconditional basis induces a lattice structure on \mathbb{X} via the identification of the vectors with the coefficients of their expansions relative to the basis, so that \mathbb{X} is lattice isomorphic to a sequence space. Because of that, we will say that an unconditional basis enjoys a certain property about lattices when its associated quasi-Banach lattice does. A quasi-Banach lattice \mathbb{L} is said to be *q-convex* (resp., *q-concave*), where $0 < q \leq$

∞ , if there is a constant $C > 0$ such that for any $m \in \mathbb{N}$ and $(f_j)_{j=1}^m$ in \mathbb{L} we have $\|f\| \leq CN$ (resp. $N \leq C\|f\|$), where

$$f = \left(\sum_{j=1}^m |f_j|^q \right)^{1/q} \quad \text{and} \quad N = \left(\sum_{j=1}^m \|f_j\|^q \right)^{1/q}.$$

The general procedure to define the element $f \in \mathbb{L}$ is described in [31, pp. 40-41]. However, when the lattice structure on \mathbb{L} is induced by an unconditional basis $(\mathbf{x}_n)_{n \in \mathcal{N}}$, if $f_j = \sum_{n \in \mathcal{N}} a_{j,n} \mathbf{x}_n$ for $1 \leq j \leq m$, the element f takes the more workable form

$$\left(\sum_{j=1}^m |f_j|^q \right)^{1/q} = \sum_{n \in \mathcal{N}} \left(\sum_{j=1}^m |a_{j,n}|^q \right)^{1/q} \mathbf{x}_n.$$

Related to lattice convexity and lattice concavity are the notions of upper and lower lattice estimates. We say that \mathbb{L} satisfies an *upper* (resp. *lower*) q -estimate if the above convexity (resp. concavity) inequalities hold in the case when $(f_j)_{j=1}^m$ is pairwise disjointly supported. Note that, in this case, $|f| = |\sum_{j=1}^m f_j|$ and so $\|f\| = \|\sum_{j=1}^m f_j\|$.

If a quasi-Banach lattice is locally convex as a quasi-Banach space, then it is 1-convex as a quasi-Banach lattice. However, despite the fact that every quasi-Banach space is q -convex for some $0 < q \leq 1$, there exist quasi-Banach lattices that are not q -convex for any q . Kalton defined in [23] a quasi-Banach lattice \mathbb{L} as being *L-convex* if there is $\varepsilon > 0$ so that whenever f and $(f_i)_{i=1}^k$ in \mathbb{L} satisfy $0 \leq f_i \leq f$ for every $i = 1, \dots, k$, and $(1 - \varepsilon)kf \leq \sum_{i=1}^k f_i$ then $\varepsilon\|f\| \leq \max_{1 \leq i \leq k} \|f_i\|$. He showed that a quasi-Banach lattice is *L-convex* if and only if it is q -convex for some $q > 0$. Most quasi-Banach lattices occurring naturally in analysis are *L-convex*. Thus, a quasi-Banach space is said to be *natural* if it is a subspace of an *L-convex* quasi-Banach lattice. Note that, in particular, any Banach space is a natural quasi-Banach space.

The property of *L-convexity* (or local convexity) allows to obtain a tight connection between upper (resp. lower) estimates and convexity (resp. concavity): if an *L-convex* lattice \mathbb{L} satisfies an upper (resp. lower) q -estimate, then it is a r -convex lattice for $0 < r < q$ (resp. r -concave lattice for $q < r < \infty$) (see [23, Theorem 1.2] and [31, Theorem 1.f.7]). Thus, the set of indices r for which \mathbb{L} is r -convex (resp. concave) is an interval with lower endpoint 0 (resp. upper endpoint ∞).

Suppose that $\mathcal{B}_x = (\mathbf{x}_n)_{n \in \mathcal{N}}$ and $\mathcal{B}_y = (\mathbf{y}_n)_{n \in \mathcal{N}}$ are (countable) families of vectors (indexed by the same set \mathcal{N}) in quasi-Banach spaces \mathbb{X} and \mathbb{Y} , respectively. We say that \mathcal{B}_x *dominates* \mathcal{B}_y if there is a bounded linear map $T: [\mathcal{B}_x] \rightarrow \mathbb{Y}$ with $T(\mathbf{x}_n) = \mathbf{y}_n$ for all $n \in \mathcal{N}$. If \mathcal{B}_x both dominates and it is dominated by \mathcal{B}_y we say that \mathcal{B}_x and \mathcal{B}_y are *equivalent*. Note that an infinite unconditional basis \mathcal{B} satisfies an upper (resp. lower) p -estimate if and only if any semi-normalized sequence disjointly supported sequence with respect to \mathcal{B} is dominated (resp. dominates) the unit vector system $\mathcal{E}[\ell_p]$ of ℓ_p .

We say that \mathcal{B}_x is *permutatively equivalent* to a family $\mathcal{B}_y = (\mathbf{y}_m)_{m \in \mathcal{M}}$ in \mathbb{Y} if there is a bijection $\pi: \mathcal{N} \rightarrow \mathcal{M}$ such that \mathcal{B}_x and $(\mathbf{y}_{\pi(n)})_{n \in \mathcal{N}}$ are equivalent.

Given families $\Lambda_i = (\lambda_{i,j})_{j \in J_i}$ for $i \in I$, we denote by $\sqcup_{i \in I} \Lambda_i$ its disjoint union, i.e.,

$$\sqcup_{i \in I} \Lambda_i = (\lambda_{i,j})_{(i,j) \in \cup_{i \in I} \{i\} \times J_i}.$$

Let $(\mathbb{X}_i)_{i \in I}$ be a finite collection of (possibly repeated) quasi-Banach spaces. The Cartesian product $\bigoplus_{i \in I} \mathbb{X}_i$ equipped with the quasi-norm

$$\|(\mathbf{x}_i)_{i \in I}\| = \sup_{i \in I} \|\mathbf{x}_i\|, \quad \mathbf{x}_i \in \mathbb{X}_i$$

is a quasi-Banach space. For $i \in F$ let $L_i: \mathbb{X}_i \rightarrow \mathbb{X}$ be the canonical “inclusion” map. Suppose that $\mathcal{B}_i = (\mathbf{x}_{i,n})_{n \in \mathcal{N}_i}$ is an unconditional basis of \mathbb{X}_i for each $i \in F$. Then the sequence

$$\bigoplus_{i \in I} \mathcal{B}_i : \sqcup_{i \in F} L_i(\mathcal{B}_i) = (L_i(\mathbf{x}_{i,j}))_{(i,j) \in \cup_{i \in F} \{i\} \times J_i}$$

is an unconditional basis of $\bigoplus_{i \in I} \mathbb{X}_i$.

A subspace \mathbb{Y} of a Banach space \mathbb{X} is said to be *complemented* in \mathbb{X} if there is a projection $P: \mathbb{X} \rightarrow \mathbb{Y}$ with $P(\mathbb{X}) = \mathbb{Y}$, in which case we say that $\text{Ker}(P)$ is a *complement* of \mathbb{Y} in \mathbb{X} . If \mathbb{Y}^c is a complement of \mathbb{Y} in \mathbb{X} , then $\mathbb{X} \simeq \mathbb{Y} \oplus \mathbb{Y}^c$ and $\mathbb{Y}^c \simeq \mathbb{Y}/\mathbb{X}$. This yields a well-known and useful lemma.

Lemma 1.3. *Let \mathbb{Y} be a complemented subspace of a quasi-Banach space \mathbb{X} . Suppose that \mathbb{U}_1 and \mathbb{U}_2 are complements of \mathbb{Y} in \mathbb{X} . Then $\mathbb{U}_1 \simeq \mathbb{U}_2$.*

An unconditional basic sequence $\mathcal{B}_y = (\mathbf{y}_m)_{m \in \mathcal{M}}$ in a quasi-Banach space \mathbb{X} is said to be *complemented* if its closed linear span $\mathbb{Y} = [\mathcal{B}_y]$ is a complemented subspace of \mathbb{X} .

A *subbasis* of an unconditional basic sequence $\mathcal{B} = (\mathbf{x}_n)_{n \in \mathcal{N}}$ is a family $(\mathbf{x}_n)_{n \in \mathcal{M}}$ for some subset \mathcal{M} of \mathcal{N} . Any subbasis of \mathcal{B} is an unconditional basic sequence which is complemented in $[\mathbf{x}_n : n \in \mathcal{N}]$. If $(\mathcal{N}_i)_{i \in F}$ is a finite partition of \mathcal{N} , and we set $\mathcal{B}_i = (\mathbf{x}_n)_{n \in \mathcal{N}_i}$ for all $i \in F$, we say that \mathcal{B} *splits into* $(\mathcal{B}_i)_{i \in F}$. In this case \mathcal{B} is permutatively equivalent to $\bigoplus_{i \in F} \mathcal{B}_i$ and so $[\mathcal{B}] \simeq \bigoplus_{i \in F} [\mathcal{B}_i]$.

The *Banach envelope* of a quasi-Banach space \mathbb{X} consists of a Banach space $\widehat{\mathbb{X}}$ together with a linear contraction $J_{\mathbb{X}}: \mathbb{X} \rightarrow \widehat{\mathbb{X}}$ satisfying the following universal property: for every Banach space \mathbb{Y} and every linear contraction $T: \mathbb{X} \rightarrow \mathbb{Y}$ there is a unique linear contraction $\widehat{T}: \widehat{\mathbb{X}} \rightarrow \mathbb{Y}$ such that $\widehat{T} \circ J_{\mathbb{X}} = T$. The dual space $\widehat{\mathbb{X}}^*$ separates the points of \mathbb{X} if and only if $J_{\mathbb{X}}$ is one-to-one. Given a basis \mathcal{B} in \mathbb{X} we put $\widehat{\mathcal{B}} := J_{\mathbb{X}}(\mathcal{B})$ for the so-called *envelope basis* of \mathcal{B} . We say that a Banach space \mathbb{Y} is isomorphic to the Banach envelope of \mathbb{X} via the map $J: \mathbb{X} \rightarrow \mathbb{Y}$ if the associated map $\widehat{J}: \widehat{\mathbb{X}} \rightarrow \mathbb{Y}$ is an isomorphism. For background on envelopes of spaces and bases we refer to [3, §10]. Here, we just record a couple of results that we will use later on.

Theorem 1.4 (see [3, Corollary 10.7]). Given quasi-Banach spaces \mathbb{X} and \mathbb{Y} , the Banach envelope of $\mathbb{X} \oplus \mathbb{Y}$ is isomorphic to $\widehat{\mathbb{X}} \oplus \widehat{\mathbb{Y}}$ via the map $(J_{\mathbb{X}}, J_{\mathbb{Y}})$.

Theorem 1.5 (see [3, Proposition 10.9]). Let \mathcal{B} be an unconditional basis of a quasi-Banach space \mathbb{X} . Then the envelope basis $\widehat{\mathcal{B}}$ of \mathcal{B} is an unconditional basis of $\widehat{\mathbb{X}}$.

Other more specific notation will be introduced in context when needed.

2. Splitting unconditional bases of a direct sum of a quasi-Banach space and a Banach space

This section is geared towards proving Lemma 2.10 below. First we introduce the necessary notions as well as some auxiliary results we will use in its proof.

Definition 2.1. We say that a finite family $(\mathbb{X}_i)_{i \in I}$ of quasi-Banach spaces is *splitting for unconditional bases* if every unconditional basis of $\bigoplus_{i \in I} \mathbb{X}_i$ splits into basic sequences $(\mathcal{B}_i)_{i \in I}$ with $[\mathcal{B}_i] \simeq \mathbb{X}_i$ for each $i \in I$.

Being able to split complemented subspaces will also be useful to us.

Definition 2.2. We say that a finite family $(\mathbb{X}_i)_{i \in I}$ of quasi-Banach spaces is *splitting for complemented subspaces* if for every complemented subspace \mathbb{Y} of $\mathbb{X} := \bigoplus_{i \in I} \mathbb{X}_i$ there are complemented subspaces \mathbb{Y}_i of \mathbb{X}_i for each $i \in I$ and an automorphism T of \mathbb{X} such that $T(\mathbb{Y}) = \bigoplus_{i \in I} \mathbb{Y}_i$.

Since ℓ_1 is the prototype of the Banach envelope of a quasi-Banach space (for instance ℓ_1 is isometrically isomorphic to the Banach envelope of ℓ_p for all $0 < p < 1$ and is also isomorphic to the Banach envelope of the Lorentz sequence space $\ell_{p,q}$ for $0 < p < 1$ and $0 < q \leq \infty$) one could conjecture that those Banach spaces “far from” ℓ_1 cannot be Banach envelopes of any non-locally convex quasi-Banach space. Kalton addressed the task of substantiating this guess in [24] and showed, for example, that ℓ_2 is not isomorphic to the Banach envelope of any non-locally convex quasi-Banach space. He also found a non-locally convex quasi-Banach space whose Banach envelope is isomorphic to c_0 . However, since Kalton also proved that c_0 is not the Banach envelope of any natural space, this example can be regarded as somewhat pathological. The following definition gives relevance to this feature of a Banach space, which will be exploited thereafter.

Definition 2.3. A Banach space \mathbb{X} will be said to be a *proper envelope* (respectively, a *proper envelope of a natural space with a basis*) if there is a non-locally convex quasi-Banach space with separating dual (resp., nonlocally convex natural quasi-Banach space with a basis) \mathbb{Y} whose Banach envelope $\widehat{\mathbb{Y}}$ is isomorphic to \mathbb{X} .

In other words, a Banach space \mathbb{X} is *not* a proper envelope (resp. \mathbb{X} is not a proper envelope of a natural space with a basis) if and only if \mathbb{X} is isomorphic to the Banach envelope of a quasi-Banach space with separating dual (resp., natural quasi-Banach space with a basis) \mathbb{Y} via a map $J: \mathbb{Y} \rightarrow \mathbb{X}$ only in the trivial case that J is an isomorphism, so that \mathbb{Y} is locally convex. In this language, Kalton's study of envelopes yields the following result.

Theorem 2.4. *Let \mathbb{U} be a Banach lattice that satisfies an upper r -estimate for some $r > 1$. Then \mathbb{U} is not the proper envelope of a natural space with a basis. If, in addition, \mathbb{U} satisfies a lower q -estimate for some $q < \infty$, then \mathbb{U} is not a proper envelope space.*

Proof. By [31, Propositions 1.d.4 and 1.f.3], the space \mathbb{U}^* has cotype q for some $q < \infty$. Then, [24, Theorem 3.4] yields the desired result. To prove the second part of the theorem, we use [31, Theorem 1.f.10] to see that \mathbb{U} has type p for some $p > 1$ followed by [24, Theorem 1.1]. \square

We will also use a couple of variations of the concept of totally incomparable spaces introduced by Rosenthal in [38], whose definitions for quasi-Banach spaces we gather next.

Definition 2.5. Two quasi-Banach spaces \mathbb{X} and \mathbb{Y} will be said to be *totally incomparable* if there is no infinite-dimensional quasi-Banach space isomorphic to both a subspace of \mathbb{X} and a subspace of \mathbb{Y} . If there is no infinite-dimensional quasi-Banach space isomorphic to both a complemented subspace of \mathbb{X} and a subspace of \mathbb{Y} , we say that the pair (\mathbb{X}, \mathbb{Y}) is *semi-complementably incomparable*. If there is no infinite-dimensional quasi-Banach space isomorphic to both a complemented subspace of \mathbb{X} and a complemented subspace of \mathbb{Y} , we say that \mathbb{X} and \mathbb{Y} are *complementably incomparable*.

Lemma 2.6. *Let \mathbb{X} be a quasi-Banach space and \mathbb{U} be a Banach space. Suppose that the pair $(\widehat{\mathbb{X}}, \mathbb{U})$ is semi-complementably incomparable (resp. $\widehat{\mathbb{X}}$ and \mathbb{U} are complementably incomparable). Then the pair (\mathbb{X}, \mathbb{U}) is semi-complementably incomparable (resp. \mathbb{X} and \mathbb{U} are complementably incomparable).*

Proof. Let $J_{\mathbb{X}}: \mathbb{X} \rightarrow \widehat{\mathbb{X}}$ be the envelope map. Suppose that \mathbb{Y} is a complemented subspace of \mathbb{X} isomorphic to a (complemented) subspace of \mathbb{U} . By Theorem 1.4, $J_{\mathbb{X}}(\mathbb{Y})$ is complemented in $\widehat{\mathbb{X}}$, and it is isomorphic to the Banach envelope of \mathbb{Y} via the map $J_{\mathbb{X}}|_{\mathbb{Y}}$. Since \mathbb{Y} is locally convex, $J_{\mathbb{X}}|_{\mathbb{Y}}$ is an isomorphic embedding. Hence, \mathbb{Y} is isomorphic to a complemented subspace of $\widehat{\mathbb{X}}$ and so $\dim(\mathbb{Y}) < \infty$. \square

Given vector spaces \mathbb{U} and \mathbb{V} with $\mathbb{U} \subseteq \mathbb{V}$ we define the *codimension* of \mathbb{U} in \mathbb{V} by

$$\text{codim}_{\mathbb{V}}(\mathbb{U}) = \dim(\mathbb{V}/\mathbb{U}).$$

If \mathbb{W} is an algebraic complement of \mathbb{U} in \mathbb{V} , then the canonical linear map from \mathbb{W} into \mathbb{V}/\mathbb{U} is a linear bijection, thus $\text{codim}_{\mathbb{V}}(\mathbb{U}) = \dim(\mathbb{W})$. It follows from the Hahn-Banach theorem that every Banach space has a great deal of hyperplanes, i.e., closed subspaces of codimension one, and it is well-known that all hyperplanes of a given Banach space are isomorphic. In contrast, there are quasi-Banach spaces with no non-zero functionals and so they contain no hyperplanes; take for instance the space $L_p([0, 1])$ for $0 < p < 1$. Aside from this ‘pathology’, hyperplanes of quasi-Banach spaces behave like those of Banach spaces. That is, if a quasi-Banach space has hyperplanes then all of them are isomorphic. To evince that this property does not depend on the Hahn-Banach theorem, we write down its proof.

Proposition 2.7. *Let \mathbb{V} be a closed finite-codimensional subspace of a quasi-Banach space \mathbb{X} . Then,*

- (i) \mathbb{V} is complemented in \mathbb{X} . In fact, any algebraic complement \mathbb{W} of \mathbb{V} is a topological complement. Moreover,
- (ii) if \mathbb{U} is a closed subspace of \mathbb{X} with $\text{codim}_{\mathbb{X}}(\mathbb{U}) = \text{codim}_{\mathbb{X}}(\mathbb{V})$, then $\mathbb{U} \simeq \mathbb{V}$.

Proof. The canonical linear bijection from \mathbb{W} to \mathbb{X}/\mathbb{V} is a topological isomorphism. Hence, if $T: \mathbb{X}/\mathbb{V} \rightarrow \mathbb{W}$ is its inverse, and $Q: \mathbb{X} \rightarrow \mathbb{X}/\mathbb{V}$ is the canonical quotient map, $T \circ Q$ is a projection onto \mathbb{W} whose kernel is \mathbb{V} . Combining the formulas

$$\begin{aligned} \text{codim}_{\mathbb{V}}(\mathbb{U} \cap \mathbb{V}) &\leq \text{codim}_{\mathbb{X}}(\mathbb{U}), \\ \text{codim}_{\mathbb{X}}(\mathbb{U} \cap \mathbb{V}) &= \text{codim}_{\mathbb{V}}(\mathbb{U} \cap \mathbb{V}) + \text{codim}_{\mathbb{X}}(\mathbb{V}) \end{aligned}$$

with the ones we get from switching the roles of \mathbb{U} and \mathbb{V} yields

$$n := \text{codim}_{\mathbb{U}}(\mathbb{U} \cap \mathbb{V}) = \text{codim}_{\mathbb{V}}(\mathbb{U} \cap \mathbb{V}) < \infty.$$

By (i), $\mathbb{V} \simeq (\mathbb{U} \cap \mathbb{V}) \oplus \mathbb{F}^n \simeq \mathbb{U}$. \square

Lemma 2.8. *Let \mathbb{X} be a quasi-Banach space. There is $d = d(\mathbb{X}) \in \mathbb{N} \cup \{0\}$ such that:*

- (i) If \mathbb{U} and \mathbb{V} are finite-dimensional quasi-Banach spaces, then $\mathbb{X} \oplus \mathbb{U} \simeq \mathbb{X} \oplus \mathbb{V}$ if and only if $\dim(\mathbb{U}) - \dim(\mathbb{V}) = jd$ for some $j \in \mathbb{Z}$.
- (ii) If \mathbb{U} and \mathbb{V} are two finite-codimensional closed subspaces of \mathbb{X} , then $\mathbb{U} \simeq \mathbb{V}$ if and only if $\text{codim}_{\mathbb{X}}(\mathbb{U}) - \text{codim}_{\mathbb{X}}(\mathbb{V}) = jd$ for some $j \in \mathbb{Z}$.

Proof. Set $\mathbb{X}[n] = \mathbb{X} \oplus \mathbb{F}^n$ for $n \in \mathbb{N} \cup \{0\}$. If the spaces $(\mathbb{X}[n])_{n=0}^{\infty}$ are mutually non-isomorphic, then (i) holds with $d = 0$. Assume that it is not the case and pick $a \in \mathbb{N} \cup \{0\}$ minimal with the property that $\mathbb{X}[a] \simeq \mathbb{X}[n]$ for some $n > a$. If $a > 0$ we would have

$$\mathbb{X}[a - 1] \oplus \mathbb{F} \simeq \mathbb{X}[a] \simeq \mathbb{X}[n - 1] \oplus \mathbb{F}.$$

From Proposition 2.7 we would get $\mathbb{X}[a-1] \simeq \mathbb{X}[n-1]$, which contradicts the minimality of a . Hence, there is $d \in \mathbb{N}$ minimal with the property that $\mathbb{X} \simeq \mathbb{X}[d]$. Given $m \in \mathbb{N} \cup \{0\}$ we have $\mathbb{X}[m] \simeq \mathbb{X}[m+d]$. Using induction we obtain that $\mathbb{X}[m] \simeq \mathbb{X}[m+jd]$ for all $j \in \mathbb{N}$.

Suppose that $0 \leq m < n$ are such that $\mathbb{X}[m] \simeq \mathbb{X}[n]$. Write $m = id + m'$ and $n = m + jd + d'$ with $i, j \in \mathbb{N} \cup \{0\}$ and $m', d' \in \mathbb{Z} \cap [0, d-1]$. Then,

$$\begin{aligned} \mathbb{X} &\simeq \mathbb{X}[(i+1)d] \simeq \mathbb{X}[m] \oplus \mathbb{F}^{d-m'} \simeq \mathbb{X}[(i+j)d + m' + d'] \oplus \mathbb{F}^{d-m'} \\ &\simeq \mathbb{X}[(i+j+1)d + d'] \simeq \mathbb{X}[d']. \end{aligned}$$

The minimality of d gives $d' = 0$, so $n - m = jd$. This proves (i).

Let \mathbb{U}, \mathbb{V} be as in (ii). Use Proposition 2.7 to pick complements \mathbb{U}^c and \mathbb{V}^c of \mathbb{U} and \mathbb{V} , respectively. By Proposition 2.7, $\mathbb{U} \simeq \mathbb{V}$ if and only if

$$\mathbb{X}_1 := \mathbb{U} \oplus \mathbb{U}^c \oplus \mathbb{V}^c \simeq \mathbb{X}_2 := \mathbb{V} \oplus \mathbb{U}^c \oplus \mathbb{V}^c.$$

Since $\mathbb{X}_1 \simeq \mathbb{X} \oplus \mathbb{V}^c$ and $\mathbb{X}_2 \simeq \mathbb{X} \oplus \mathbb{U}^c$, applying (i) yields that $\mathbb{U} \simeq \mathbb{V}$ if and only if $\dim(\mathbb{V}^c) - \dim(\mathbb{U}^c) \in d\mathbb{Z}$. \square

Remark 2.9. Most Banach spaces \mathbb{X} are isomorphic to their hyperplanes, or, in the terminology of Lemma 2.8, $d(\mathbb{X}) = 1$. In fact, Banach [11] conjectured that any Banach space should have this property. This question was solved in the negative in [20] by Gowers, who exhibited examples of Banach spaces \mathbb{X} with $d(\mathbb{X}) = 0$. Subsequently, Gowers and Maurey constructed for every d a separable Banach space \mathbb{X} with $d(\mathbb{X}) = d$ (see [21, Theorem 26 and following remarks]).

Now we are ready to state and prove the general condition for the pair (\mathbb{X}, \mathbb{U}) to be splitting for unconditional bases.

Lemma 2.10. *Let \mathbb{X} be a quasi-Banach space and \mathbb{U} be a Banach space. Suppose that:*

- (i) (\mathbb{X}, \mathbb{U}) is splitting for complemented subspaces;
- (ii) $(\widehat{\mathbb{X}}, \mathbb{U})$ is splitting for unconditional bases;
- (iii) $\widehat{\mathbb{X}}$ and \mathbb{U} are complementably incomparable; and
- (iv) either \mathbb{U} is not a proper envelope, or \mathbb{X} is natural and \mathbb{U} is not the proper envelope of a natural space with a basis.

Then the pair (\mathbb{X}, \mathbb{U}) is splitting for unconditional bases.

Proof. Let $J_{\mathbb{X}}: \mathbb{X} \rightarrow \widehat{\mathbb{X}}$ be the envelope map. Theorem 1.4 gives that $\widehat{\mathbb{X}} \oplus \mathbb{U}$ is isomorphic to the Banach envelope of $\mathbb{X} \oplus \mathbb{U}$ via the map $J = (J_{\mathbb{X}}, \text{Id}_{\mathbb{U}})$. Let $\mathcal{B} = (\mathbf{b}_n)_{n \in \mathbb{N}}$ be an unconditional basis of $\mathbb{X} \oplus \mathbb{U}$. By Theorem 1.5, $J(\mathcal{B})$ is an unconditional basis of $\widehat{\mathbb{X}} \oplus \mathbb{U}$.

Therefore we can choose a partition $(\mathcal{N}_y, \mathcal{N}_v)$ of \mathcal{N} such that, if we put $\mathcal{B}_y = (\mathbf{b}_n)_{n \in \mathcal{N}_y}$ and $\mathcal{B}_v = (\mathbf{b}_n)_{n \in \mathcal{N}_v}$, $J(\mathcal{B}_y)$ generates a space isomorphic to $\widehat{\mathbb{X}}$, and $J(\mathcal{B}_v)$ generates a space isomorphic to \mathbb{U} . Applying again Theorem 1.4 we obtain that $[J(\mathcal{B}_y)]$ is isomorphic to the Banach envelope of $[\mathcal{B}_y]$ via the map $J|_{[\mathcal{B}_y]}$, and that $[J(\mathcal{B}_v)]$ is isomorphic to the Banach envelope of $[\mathcal{B}_v]$ via the map $J|_{[\mathcal{B}_v]}$.

In the case when \mathbb{X} is natural, so is $[\mathcal{B}_v]$. Hence, since $J([\mathcal{B}_v]) \simeq \mathbb{U}$, condition (iv) gives that the map $J|_{[\mathcal{B}_v]}$ is an isomorphic embedding. In other words, \mathcal{B}_v generates a space isomorphic to \mathbb{U} .

Since $[\mathcal{B}_y]$ is complemented in $\mathbb{X} \oplus \mathbb{U}$, there exist a complemented subspace \mathbb{X}_0 of \mathbb{X} , a complemented subspace \mathbb{U}_0 of \mathbb{U} , and an automorphism T of $\mathbb{X} \oplus \mathbb{U}$ such that $T([\mathcal{B}_y]) = \mathbb{X}_0 \oplus \mathbb{U}_0$. In particular,

$$[\mathcal{B}_y] \simeq \mathbb{X}_0 \oplus \mathbb{U}_0.$$

If \mathbb{X}_0^c is a complement of \mathbb{X}_0 in \mathbb{X} and \mathbb{U}_0^c is a complement of \mathbb{U}_0 in \mathbb{U} , then $\mathbb{X}_0^c \oplus \mathbb{U}_0^c$ is a complement of $\mathbb{X}_0 \oplus \mathbb{U}_0$ in $\mathbb{X} \oplus \mathbb{U}$. Since $[\mathcal{B}_v]$ is a complement of $[\mathcal{B}_y]$ in $\mathbb{X} \oplus \mathbb{U}$, $T([\mathcal{B}_v])$ is a complement of $\mathbb{X}_0 \oplus \mathbb{U}_0$ in $\mathbb{X} \oplus \mathbb{U}$. Applying Lemma 1.3 we obtain

$$\mathbb{X}_0^c \oplus \mathbb{U}_0^c \simeq T([\mathcal{B}_v]) \simeq [\mathcal{B}_v] \simeq \mathbb{U} \simeq \mathbb{U}_0 \oplus \mathbb{U}_0^c.$$

\mathbb{U}_0 is a complemented subspace of \mathbb{U} isomorphic to a complemented subspace of $[\mathcal{B}_y]$. Since the Banach envelope of $[\mathcal{B}_y]$ is isomorphic to $\widehat{\mathbb{X}}$, applying Lemma 2.6 yields that $[\mathcal{B}_y]$ and \mathbb{U} are complementably incomparable. Thus, $\dim(\mathbb{U}_0) < \infty$. In turn, \mathbb{X}_0^c is a complemented subspace of \mathbb{X} isomorphic to a complemented subspace of \mathbb{U} . Applying again Lemma 2.6 yields $\dim(\mathbb{X}_0^c) < \infty$.

Let $d = d(\mathbb{U}) \in \mathbb{N} \cup \{0\}$ be as in Lemma 2.8. We infer that there is $j \in \mathbb{Z}$ such that

$$\dim(\mathbb{X}_0^c) - \dim(\mathbb{U}_0) = jd.$$

If $d = 0$ or $d \geq 1$, and $j = 0$ we have

$$[\mathcal{B}_y] \simeq \mathbb{X}_0 \oplus \mathbb{U}_0 \simeq \mathbb{X}_0 \oplus \mathbb{X}_0^c \simeq \mathbb{X}.$$

If $d > 0$ and $j > 0$ we pick $\mathcal{F} \subseteq \mathcal{N}_v$ with $|\mathcal{F}| = jd$, and we set

$$\mathcal{B}_f = (\mathbf{b}_n)_{n \in \mathcal{F}}, \quad \mathcal{B}_x = (\mathbf{b}_n)_{n \in \mathcal{N}_y \cup \mathcal{F}}, \quad \text{and} \quad \mathcal{B}_u = (\mathbf{b}_n)_{n \in \mathcal{N}_v \setminus \mathcal{F}}.$$

By Lemma 2.8 (ii), $[\mathcal{B}_u] \simeq \mathbb{U}$. Moreover,

$$[\mathcal{B}_x] \simeq [\mathcal{B}_y] \oplus [\mathcal{B}_f] \simeq \mathbb{X}_0 \oplus \mathbb{U}_0 \oplus [\mathcal{B}_f] \simeq \mathbb{X}_0 \oplus \mathbb{X}_0^c \simeq \mathbb{X}.$$

Finally, if $d > 0$ and $j < 0$ we pick $\mathcal{F} \subseteq \mathcal{N}_y$ with $|\mathcal{F}| = -jd$, and we set

$$\mathcal{B}_f = (\mathbf{b}_n)_{n \in \mathcal{F}}, \quad \mathcal{B}_x = (\mathbf{b}_n)_{n \in \mathcal{N}_y \setminus \mathcal{F}}, \quad \text{and} \quad \mathcal{B}_u = (\mathbf{b}_n)_{n \in \mathcal{N}_v \cup \mathcal{F}}.$$

By Lemma 2.8 (i), $[\mathcal{B}_u] \simeq [\mathcal{B}_v] \oplus [\mathcal{B}_f] \simeq \mathbb{U} \oplus [\mathcal{B}_f] \simeq \mathbb{U}$. Moreover,

$$[\mathcal{B}_x] \oplus \mathbb{U}_0 \simeq [\mathcal{B}_x] \oplus [\mathcal{B}_f] \oplus \mathbb{X}_0^c \simeq [\mathcal{B}_y] \oplus \mathbb{X}_0^c \simeq \mathbb{X}_0 \oplus \mathbb{U}_0 \oplus \mathbb{X}_0^c \simeq \mathbb{X} \oplus \mathbb{U}_0.$$

By Lemma 2.7, $[\mathcal{B}_x] \simeq \mathbb{X}$. \square

3. The role of subprojectivity

The following lemma gathers elementary connections between several concepts relevant to this paper.

Lemma 3.1. *Consider the following properties involving two quasi-Banach spaces \mathbb{X} and \mathbb{Y} :*

- (i) *Every bounded linear operator from \mathbb{X} into \mathbb{Y} is compact.*
- (ii) *\mathbb{X} and \mathbb{Y} are totally incomparable.*
- (iii) *Every bounded linear operator from \mathbb{X} into \mathbb{Y} is strictly singular.*
- (iv) *(\mathbb{X}, \mathbb{Y}) is semi-complementably incomparable.*
- (v) *\mathbb{X} and \mathbb{Y} are complementably incomparable.*

We have (i) \Rightarrow (iii), (ii) \Rightarrow (iii), (iii) \Rightarrow (iv), and (iv) \Rightarrow (v).

Proof. Only (iii) \Rightarrow (iv) deserves to be sketched. Suppose (iv) does not hold. Then, there are a projection $P: \mathbb{X} \rightarrow \mathbb{X}$ onto an infinite-dimensional subspace \mathbb{U} , and an isomorphic embedding $T: \mathbb{U} \rightarrow \mathbb{Y}$. The operator $T \circ P: \mathbb{X} \rightarrow \mathbb{Y}$ is an isomorphic embedding when restricted to \mathbb{U} . \square

We will next see that under a mild condition on \mathbb{Y} , conditions (iii) and (iv) in Lemma 3.1 are in fact equivalent.

Definition 3.2. We say that a quasi-Banach space \mathbb{Y} is *subprojective* if for every infinite dimensional subspace \mathbb{V} of \mathbb{Y} there is a further subspace $\mathbb{U} \subseteq \mathbb{V}$ which is complemented in \mathbb{Y} . If, in addition, \mathbb{U} is isomorphic to one of the members of a given set \mathcal{U} of infinite-dimensional quasi-Banach spaces, we say that \mathbb{Y} is *\mathcal{U} -subprojective* (or complementably \mathcal{U} -saturated).

Loosely speaking, the following lemma tells us that subprojectivity serves as the key that allows to pull-back complemented subspaces via non-strictly singular operators. Although the result is essentially known, for the sake of completeness we include a proof.

Lemma 3.3 (cf. [33, Corollary 2.4]). *Let \mathbb{V} be an infinite-dimensional subspace of a quasi-Banach space \mathbb{X} , let \mathcal{U} be a family of infinite-dimensional quasi-Banach spaces, and let \mathbb{Y}*

be a \mathcal{U} -subprojective quasi-Banach space. Suppose that there is a bounded linear operator $T: \mathbb{X} \rightarrow \mathbb{Y}$ such that $T|_{\mathbb{V}}$ is an isomorphic embedding. Then there is a subspace of \mathbb{V} which is complemented in \mathbb{X} and isomorphic to some member of \mathcal{U} .

Proof. Passing to a subspace we can assume that $T(\mathbb{V})$ is complemented in \mathbb{Y} and isomorphic to \mathbb{U} for some $\mathbb{U} \in \mathcal{U}$. Let $S: T(\mathbb{Y}) \rightarrow \mathbb{Y}$ be the inverse operator of $T|_{\mathbb{V}}$, and let $P: \mathbb{Y} \rightarrow T(\mathbb{V})$ be such that $P|_{T(\mathbb{V})} = \text{Id}_{T(\mathbb{V})}$. Since the map $S \circ P \circ T: \mathbb{X} \rightarrow \mathbb{V}$ is the identity on \mathbb{V} , \mathbb{V} is complemented in \mathbb{X} . \square

A Banach space \mathbb{X} admits a non-strictly singular operator into ℓ_2 if and only if ℓ_2 is isomorphic to a complemented subspace of \mathbb{X} . The following lemma provides an extension of this result.

Proposition 3.4. *Let \mathbb{X} and \mathbb{Y} be quasi-Banach spaces and \mathcal{U} be a set of infinite-dimensional quasi-Banach spaces. Suppose that no space in \mathcal{U} is isomorphic to a complemented subspace of \mathbb{X} and that \mathbb{Y} is \mathcal{U} -subprojective. Then, every operator from \mathbb{X} into \mathbb{Y} is strictly singular.*

Proof. Assume by contradiction that $T: \mathbb{X} \rightarrow \mathbb{Y}$ is an isomorphic embedding when restricted to an infinite-dimensional subspace \mathbb{V} of \mathbb{X} . By Lemma 3.3, there is an infinite-dimensional subspace of \mathbb{V} which is complemented in \mathbb{X} and isomorphic to a space from \mathcal{U} . \square

Our next result is a straightforward consequence of Proposition 3.4.

Corollary 3.5. *Let \mathbb{X} and \mathbb{Y} be quasi-Banach spaces. Suppose that (\mathbb{X}, \mathbb{Y}) is semi-complementably incomparable and that \mathbb{Y} is subprojective. Then every operator from \mathbb{X} into \mathbb{Y} is strictly singular.*

3.1. Subprojective Banach spaces

Although the definition of subprojectivity makes sense for quasi-Banach spaces, we know no example of a non-locally convex subprojective quasi-Banach space. The result of Stiles [40] that the space ℓ_p for $0 < p < 1$ is not subprojective, suggests that no non-locally convex quasi-Banach space can be subprojective. For this reason, in this paper we will keep within bounds of locally convex spaces as far as subprojectivity is concerned. The applications we will obtain will rely on the results on the subject from [33], where it is proved that subprojectivity is inherited by direct sums of Banach spaces. To be precise, we have the following.

Proposition 3.6 (cf. [33, Proposition 2.2]). *Let I be a finite set. Suppose that for each $i \in I$, \mathcal{U}_i is a set of infinite-dimensional Banach spaces and \mathbb{X}_i is a \mathcal{U}_i -subprojective Banach space. Then the space $\bigoplus_{i \in I} \mathbb{X}_i$ is \mathcal{U} -subprojective, where $\mathcal{U} = \bigcup_{i \in I} \mathcal{U}_i$.*

We emphasize that the proof of Proposition 3.6 does not carry over to quasi-Banach spaces since it is by no means clear whether the sum of two strictly singular operators is strictly singular in general. We refer the reader also to the article [33] for a list of subprojective Banach spaces. This section is aimed at adding to this list the convexifications of Tsirelson’s space and their duals. To that end, we first introduce a few lemmas.

We say that a family $(\mathbf{u}_n)_{n \in \mathcal{N}}$ in a quasi-Banach space \mathbb{X} and a family $(\mathbf{v}_n)_{n \in \mathcal{N}}$ in a quasi-Banach space \mathbb{Y} are *congruent* (in \mathbb{X} and \mathbb{Y}) if there is an isomorphism T of \mathbb{X} onto \mathbb{Y} with $T(\mathbf{u}_n) = \mathbf{v}_n$ for all $n \in \mathcal{N}$. Congruence is stronger than equivalence. We introduce it because congruence ensures that if a subspace \mathbb{U} of $[\mathbf{u}_n : n \in \mathcal{N}]$ is complemented in \mathbb{X} , then the corresponding subspace $T(\mathbb{U})$ of $[\mathbf{v}_n : n \in \mathcal{N}]$ is complemented in \mathbb{Y} .

Lemma 3.7 (cf. [12, Section 4, C2]). *Let I be a finite set. For each $i \in I$ let \mathbb{X}_i be a quasi-Banach space with a Schauder basis \mathcal{B}_i . Suppose that \mathbb{X} is an infinite-dimensional quasi-Banach space and that $J_i : \mathbb{X} \rightarrow \mathbb{X}_i$ is an isomorphic embedding for each $i \in I$. Then \mathbb{X} has a basic sequence \mathcal{B} equivalent to a block basic sequence \mathcal{B}'_i with respect to \mathcal{B}_i for all $i \in I$. Moreover, if \mathbb{X}_i is locally convex, $J_i(\mathcal{B})$ and \mathcal{B}'_i are congruent.*

Proof. There is a sequence $(f_n)_{n=1}^\infty$ in \mathbb{X} with $\sup_n \|f_n\| < \infty$ and $\inf_{n \neq m} \|f_n - f_m\| > 0$ (see [2, Lemma 2.8]). Passing to a subsequence, Cantor’s classical diagonal argument gives that $(J_i(f_n))_{n=1}^\infty$ converges coordinate-wise with respect \mathcal{B}_i for all $i \in I$. Set $\mathbf{y}_n = f_{2n-1} - f_{2n}$ for $n \in \mathbb{N}$. Since $\mathcal{B} := (\mathbf{y}_n)_{n=1}^\infty$ is semi-normalized and $J_i(\mathcal{B})$ is coordinate-wise null with respect \mathcal{B}_i for all $i \in I$, combining the gliding hump technique with the principle of small perturbations, passing to a further subsequence we obtain that $J_i(\mathcal{B})$ is equivalent (congruent if \mathbb{X}_i is locally convex) to a block basic sequence with respect to \mathcal{B}_i for all $i \in I$. \square

As a by-product, we obtain conditions under which a quasi-Banach space always contains a basic sequence. Recall that the basic sequence problem for quasi-Banach spaces was solved in the negative by Kalton in [25].

Corollary 3.8. *Let \mathbb{X} be an infinite-dimensional quasi-Banach space. If \mathbb{X} embeds in a quasi-Banach space with a Schauder basis, then \mathbb{X} contains a basic sequence.*

Lemma 3.9. *Let \mathbb{X} be a Banach space with an unconditional basis $\mathcal{B} = (\mathbf{x}_j)_{j \in \mathcal{N}}$. Let \mathbb{Y} be the subspace of \mathbb{X}^* spanned by the biorthogonal functionals $\mathcal{B}^* = (\mathbf{x}_j^*)_{j \in \mathcal{N}}$ of \mathcal{B} . Suppose that there are a partition $(\mathcal{N}_n)_{n=1}^\infty$ of \mathcal{N} into finite sets and a constant C such that $\|\sum_{n=1}^\infty f_n\| \leq C \|\sum_{n=1}^\infty g_n\|$ whenever $(f_n)_{n=1}^\infty$ and $(g_n)_{n=1}^\infty$ in \mathbb{X} satisfy $\|f_n\| \leq \|g_n\|$ and $\text{supp}(f_n) \cup \text{supp}(g_n) \subseteq \mathcal{N}_n$ for all $n \in \mathbb{N}$.*

- (i) *If $\mathcal{B}_u = (\mathbf{u}_n)_{n=1}^\infty$ and $\mathcal{B}_v = (\mathbf{v}_n)_{n=1}^\infty$ are semi-normalized sequences in \mathbb{X} with $\text{supp}(\mathbf{u}_n) \cup \text{supp}(\mathbf{v}_n) \subseteq \mathcal{N}_n$ for all $n \in \mathbb{N}$, then \mathcal{B}_u and \mathcal{B}_v are equivalent. Moreover, $[\mathcal{B}_u]$ and $[\mathcal{B}_v]$ are complemented in \mathbb{X} .*

(ii) If $\mathcal{B}_y^* = (\mathbf{y}_n^*)_{n=1}^\infty$ and $\mathcal{B}_w^* = (\mathbf{w}_n^*)_{n=1}^\infty$ are semi-normalized sequences in \mathbb{Y} with $\text{supp}(\mathbf{y}_n^*) \cup \text{supp}(\mathbf{w}_n^*) \subseteq \mathcal{N}$ for all $n \in \mathbb{N}$, then \mathcal{B}_y^* and \mathcal{B}_w^* are equivalent. Moreover, $[\mathcal{B}_y^*]$ and $[\mathcal{B}_w^*]$ are complemented in \mathbb{Y} .

Proof. Consider the following conditions on a pair $(\mathcal{B}_z, \mathcal{B}_z^*)$ formed by a sequence $\mathcal{B}_z = (z_n)_{n=1}^\infty$ in \mathbb{X} and a sequence $\mathcal{B}_z^* = (z_n^*)_{n=1}^\infty$ in \mathbb{X}^* :

(♣) $z_n^*(z_n) = 1$, $\text{supp}(z_n) \cup \text{supp}(z_n^*) \subseteq \mathcal{N}_n$ for all $n \in \mathbb{N}$, and $k[\mathcal{B}_z, \mathcal{B}_z^*] := \sup_n \max\{\|z_n\|, \|z_n^*\|\} < \infty$.

Under the assumptions in (i) (resp. in (ii)) there are sequences \mathcal{B}_u^* and \mathcal{B}_v^* in \mathbb{X}^* (resp. \mathcal{B}_y and \mathcal{B}_w in \mathbb{X}) such that the pairs $(\mathcal{B}_u, \mathcal{B}_u^*)$ and $(\mathcal{B}_v, \mathcal{B}_v^*)$ (resp. $(\mathcal{B}_y, \mathcal{B}_y^*)$ and $(\mathcal{B}_w, \mathcal{B}_w^*)$) fulfill (♣). So, it suffices to prove that if $(\mathcal{B}_u, \mathcal{B}_u^*)$ and $(\mathcal{B}_v, \mathcal{B}_v^*)$ satisfy (♣), then \mathcal{B}_u and \mathcal{B}_v are equivalent, \mathcal{B}_u^* and \mathcal{B}_v^* are equivalent, $[\mathcal{B}_u]$ and $[\mathcal{B}_v]$ are complemented in \mathbb{X} , and $[\mathcal{B}_u^*]$ and $[\mathcal{B}_v^*]$ are complemented in \mathbb{X}^* .

Set $\mathcal{B}_u^* = (\mathbf{u}_n^*)_{n=1}^\infty$, $\mathcal{B}_v^* = (\mathbf{v}_n^*)_{n=1}^\infty$, and $k = k[\mathcal{B}_u, \mathcal{B}_u^*]$. If we denote by $S_A: \mathbb{X} \rightarrow \mathbb{X}$ the coordinate projection on a set $A \subseteq \mathcal{N}$ we have

$$\|\mathbf{u}_n^*(f)\| \|\mathbf{u}_n\| \leq k^2 \|S_{\mathcal{N}_n}(f)\|, \quad f \in \mathbb{X}, \quad n \in \mathbb{N}.$$

Hence, the linear map $P: \mathbb{X} \rightarrow \mathbb{X}$ given by

$$P(f) = \sum_{n=1}^\infty \mathbf{u}_n^*(f) \mathbf{v}_n, \quad f \in \mathbb{X}$$

is well-defined and satisfies $\|P\| \leq k^2 C$. We also have $P(\mathbf{u}_n) = \mathbf{v}_n$ for all $n \in \mathbb{N}$. The dual map $P^*: \mathbb{X}^* \rightarrow \mathbb{X}^*$ is given by

$$P^*(f^*)(f) = \sum_{n=1}^\infty f^*(\mathbf{v}_n) \mathbf{u}_n^*(f), \quad f \in \mathbb{X}, \quad f^* \in \mathbb{X}^*.$$

Hence, $P^*(\mathbf{v}_n^*) = \mathbf{u}_n^*$ for all $n \in \mathbb{N}$.

The operators defined when replacing \mathcal{B}_v with \mathcal{B}_u and \mathcal{B}_u^* with \mathcal{B}_v^* yield the equivalence between \mathcal{B}_u and \mathcal{B}_v , as well as the equivalence between \mathcal{B}_u^* and \mathcal{B}_v^* . The operator defined when replacing only \mathcal{B}_v with \mathcal{B}_u yields projections from \mathbb{X} onto $[\mathcal{B}_u]$ and from \mathbb{X}^* onto $[\mathcal{B}_u^*]$. Finally, the operator defined when replacing only \mathcal{B}_u^* with \mathcal{B}_v^* yields projections from \mathbb{X} onto $[\mathcal{B}_v]$ and from \mathbb{X}^* onto $[\mathcal{B}_v^*]$. \square

Theorem 3.10. Let \mathbb{X} be a Banach space with an unconditional basis $\mathcal{B} = (\mathbf{x}_j)_{j=1}^\infty$. Let \mathbb{Y} be the subspace of \mathbb{X}^* spanned by the biorthogonal functionals $\mathcal{B}^* = (\mathbf{x}_j^*)_{j \in \mathcal{N}}$ of \mathcal{B} . Suppose that there are a constant C and an increasing sequence $(j_n)_{n=1}^\infty$ in \mathbb{N} with the following property: $\|\sum_{n=1}^\infty f_n\| \leq C \|\sum_{n=1}^\infty g_n\|$ whenever $(f_n)_{n=1}^\infty$ and $(g_n)_{n=1}^\infty$ in \mathbb{X} and

$(k_n)_{n=1}^\infty$ in \mathbb{N} satisfy $\|f_n\| \leq \|g_n\|$, $\text{supp}(f_n) \cup \text{supp}(g_n) \subseteq [k_n, k_{n+1} - 1]$, and $j_n \leq k_n$ for all $n \in \mathbb{N}$. Then, \mathbb{X} is \mathcal{U} -subprojective and \mathbb{Y} is \mathcal{U}^* -subprojective, where

$$\begin{aligned} \mathcal{U} &= \{[\mathbf{x}_{k_n} : n \in \mathbb{N}] : \forall n \in \mathbb{N}, j_n \leq k_n < k_{n+1}\}, \\ \mathcal{U}^* &= \{[\mathbf{x}_{k_n}^* : n \in \mathbb{N}] : \forall n \in \mathbb{N}, j_n \leq k_n < k_{n+1}\}. \end{aligned}$$

Proof. Pick a subspace \mathbb{U} of \mathbb{X} (resp. of \mathbb{Y}). By Lemma 3.7, passing to a subspace we can suppose that \mathbb{U} is spanned by a basic sequence, say $(\mathbf{u}_k)_{k=1}^\infty$, congruent to a block basic sequence $(\mathbf{v}_k)_{k=1}^\infty$. There is an increasing sequence $(k_n)_{n=1}^\infty$ in \mathbb{N} such that $j_n \leq \min(\text{supp}(\mathbf{v}_{k_n}))$ for all $n \in \mathbb{N}$. By Lemma 3.9, $(\mathbf{v}_{k_n})_{n=1}^\infty$ spans a complemented subspace of \mathbb{X} (resp. of \mathbb{Y}) isomorphic to $\mathbb{V} := [\mathbf{x}_{k_n} : n \in \mathbb{N}]$ (resp. $\mathbb{V} := [\mathbf{x}_{k_n}^* : n \in \mathbb{N}]$). By congruence (equivalence is not enough!), $(\mathbf{u}_{k_n})_{n=1}^\infty$ spans a complemented subspace of \mathbb{X} (resp. of \mathbb{Y}) isomorphic to \mathbb{V} . \square

It is known that Tsirelson’s space \mathcal{T} is subprojective [19, Proposition 2.4]. This fact can be derived from the properties of the lattice structure on \mathcal{T} , and this is what we will use to show the subprojectivity of their convexifications and their duals. Given $0 < r < \infty$, $\mathcal{T}^{(r)}$ denotes the quasi-Banach lattice consisting of all $f \in \mathbb{F}^\mathbb{N}$ such that $|f|^r \in \mathcal{T}$. Given $1 < s \leq \infty$, we denote by $\mathcal{T}_*^{(s)}$ the dual of the r -convexified Tsirelson’s space $\mathcal{T}^{(r)}$, where $r = s/(s - 1)$. With this terminology, $\mathcal{T}_*^{(\infty)}$ is the original Tsirelson’s space \mathcal{T}^* . Since \mathcal{T} is 1-convex and p -concave for any $p > 1$, $\mathcal{T}^{(r)}$ is r convex and p concave for any $p > r$. Consequently, $\mathcal{T}_*^{(s)}$ is s -concave and p -convex for any $p < s$.

Theorem 3.11. Let $1 \leq r < \infty$ and $1 < s \leq \infty$. Let $(\mathbf{t}_n)_{n=1}^\infty$ denote the unit vector system of $\mathcal{T}^{(r)}$, and $(\mathbf{t}_n^*)_{n=1}^\infty$ denote the unit vector system of $\mathcal{T}_*^{(s)}$. Then $\mathcal{T}^{(r)}$ is \mathcal{U} -subprojective and $\mathcal{T}_*^{(s)}$ is \mathcal{U}^* -subprojective, where,

$$\begin{aligned} \mathcal{U} &= \{[\mathbf{t}_{k_n} : n \in \mathbb{N}] : \forall n \in \mathbb{N}, j_n \leq k_n < k_{n+1}\}, \\ \mathcal{U}^* &= \{[\mathbf{t}_{k_n}^* : n \in \mathbb{N}] : \forall n \in \mathbb{N}, j_n \leq k_n < k_{n+1}\}, \end{aligned}$$

and $(j_n)_{n=1}^\infty$ is an arbitrary increasing sequence in \mathbb{N} .

Proof. It is known that the unit vector system of \mathcal{T} satisfies for any $(j_n)_{n=1}^\infty$ the assumptions in Theorem 3.10 (see [16, Corollary II.5]). Since r -convexifications inherit this property, the unit vector system of $\mathcal{T}^{(r)}$ also does. \square

We can also apply Theorem 3.10 to Nakano spaces. Given a sequence $(p_n)_{n=1}^\infty$ in $[1, \infty)$ we denote by $\ell(p_n)$ the Banach space built from the modular

$$(a_n)_{n=1}^\infty \mapsto \sum_{n=1}^\infty |a_n|^{p_n},$$

and we denote by $h(p_n)$ the separable part of $\ell(p_n)$. We have $\ell(p_n) = h(p_n)$ if and only if $\sup_n p_n < \infty$. If there is $0 < R < 1$ such that

$$\sum_{n=1}^{\infty} R^{pp_n/|p_n-p|} < \infty,$$

then $\ell(p_n) = \ell_p$ up to an equivalent norm. Since the ℓ_p spaces are particular cases of Nakano spaces, the following generalizes [36, Lemma 2], which establishes that ℓ_p is ℓ_p -subprojective.

Theorem 3.12 (cf. [39, Theorem 5.1]). *Let $(p_n)_{n=1}^{\infty}$ be a sequence in $[1, \infty)$. Suppose that there exists $\lim_n p_n = p \in [1, \infty]$. Then $h(p_n)$ is ℓ_p -subprojective (we replace ℓ_{∞} with c_0 if $p = \infty$).*

Proof. Set for each $n \in \mathbb{N}$

$$R_n = \max_{k \geq n} \frac{pp_k}{|p - p_k|}.$$

Pick an increasing sequence $(n_j)_{j=1}^{\infty}$ in \mathbb{N} with $\sum_{j=1}^{\infty} 2^{-R_{n_j}} < \infty$. Any semi-normalized block basic sequence $(\mathbf{x}_j)_{j=1}^{\infty}$ with $n_j \leq \min(\text{supp}(\mathbf{x}_j))$ for all $j \in \mathbb{N}$ is equivalent to the canonical ℓ_p -basis (see e.g. [4, Theorems 2.2 and 3.1]). Then the result follows from Theorem 3.10. \square

4. The main theorem

Our main result is Theorem 4.1 below, which establishes some easy-to-check conditions that suffice to guarantee that a pair (\mathbb{X}, \mathbb{U}) is splitting for unconditional bases, where \mathbb{X} is a quasi-Banach space and \mathbb{U} is a Banach space. Note that all the information we use about \mathbb{X} is obtained exclusively through its Banach envelope!

Theorem 4.1. *Let \mathbb{X} be a quasi-Banach space and let \mathbb{U} be a subprojective Banach space. Suppose that $\widehat{\mathbb{X}}$ and \mathbb{U} have unconditional bases which split into unconditional bases $(\mathcal{B}_j)_{j \in A}$ and $(\mathcal{B}_j)_{j \in B}$ respectively, where (A, B) is a partition of $\{1, \dots, n\}$ for some $n \in \mathbb{N}$. Suppose that for $1 \leq j \leq n$ the basis \mathcal{B}_j satisfies an upper r_j -estimate and a lower q_j -estimate, where q_j and r_j are both in $[1, \infty]$, and that*

- (i) $q_j < r_{j+1}$ for all $1 \leq j \leq n - 1$;
- (ii) $r_j > 1$ for all $j \in B$; and
- (iii) either \mathbb{X} is natural and has a basis, or $q_j < \infty$ for all $j \in B$.

Then (\mathbb{X}, \mathbb{U}) is splitting for unconditional bases. In particular, if \mathbb{X} and \mathbb{U} have a UTAP unconditional basis then $\mathbb{X} \oplus \mathbb{U}$ has a UTAP unconditional basis.

To tackle the proof Theorem 4.1 we need to see a few results. The first one is a classical theorem on strictly singular operators.

Theorem 4.2 (see [35, Theorem 4]). *Let \mathbb{X} and \mathbb{Y} be quasi-Banach spaces. Suppose that every operator from \mathbb{X} into \mathbb{Y} is strictly singular. Then (\mathbb{X}, \mathbb{Y}) is splitting for complemented subspaces.*

Now we show a result that is of interest by itself in the theory because it allows to obtain new examples of Banach spaces with a unique unconditional basis. For the ease of the reader, we record a couple of theorems which we allude to in the proof.

Theorem 4.3 (see [17, Theorem 1]). *Let \mathbb{X} and \mathbb{Y} be Banach spaces with unconditional bases \mathcal{B}_x and \mathcal{B}_y , respectively. Suppose that \mathcal{B}_x satisfies an upper r -estimate and \mathcal{B}_y satisfies a lower q -estimate, $r > q$. Then any bounded linear operator from \mathbb{X} to \mathbb{Y} is compact.*

Theorem 4.4 (see [41, Theorem 2.1]). *Let \mathbb{X} and \mathbb{Y} be Banach spaces such that every bounded linear operator from \mathbb{X} into \mathbb{Y} is compact. Then (\mathbb{X}, \mathbb{Y}) is splitting for unconditional bases.*

Theorem 4.5. *Let $(\mathbb{X}_j)_{j=1}^n$ be a finite family of Banach spaces, each of which has an unconditional basis \mathcal{B}_j . Suppose there are sequences $(q_j)_{j=1}^{n-1}$ and $(r_j)_{j=2}^n$, both in $[1, \infty]$, such that:*

- \mathcal{B}_j satisfies a lower q_j -estimate for all $1 \leq j \leq n - 1$;
- \mathcal{B}_j satisfies an upper r_j -estimate for all $2 \leq j \leq n$; and
- $q_j < r_{j+1}$ for all $1 \leq j \leq n - 1$.

Then $(\mathbb{X}_j)_{j=1}^n$ is splitting for unconditional bases.

Proof. Given $1 \leq s \leq n - 1$, the unconditional basis $\bigoplus_{j=1}^s \mathcal{B}_j$ of $\bigoplus_{j=1}^s \mathbb{X}_j$ satisfies a lower q_s -estimate. Thus the result follows by induction combining Theorem 4.3 with Theorem 4.4. \square

The last important ingredient we need to prove the main theorem is Proposition 4.7, which spells out a recipe for building mutually incomparable infinite direct sums of quasi-Banach spaces.

Lemma 4.6. *Let \mathbb{X} and \mathbb{Y} be quasi-Banach spaces with unconditional bases \mathcal{B}_x and \mathcal{B}_y , respectively. Suppose that there is $0 < q \leq \infty$ such that \mathcal{B}_x satisfies a lower q -estimate and \mathcal{B}_y satisfies an upper q -estimate. Then every sequence \mathcal{B} in $\mathbb{X} \oplus \mathbb{Y}$ disjointly supported with respect to $\mathcal{B}_x \oplus \mathcal{B}_y$ has a subsequence which is equivalent either to a sequence in \mathbb{X}*

disjointly supported with respect to \mathcal{B}_x , or to a sequence in \mathbb{Y} disjointly supported with respect to \mathcal{B}_y .

Proof. Put $\mathcal{B} = (\mathbf{u}_n, \mathbf{v}_n)_{n=1}^\infty$. Passing to a subsequence we can assume that either $\inf_n \|\mathbf{u}_n\| > 0$ or $\|\mathbf{u}_n\| \leq \varepsilon_n$ for all $n \in \mathbb{N}$, where $(\varepsilon_n)_{n=1}^\infty$ is a given sequence of positive scalars. In the former case, $\mathcal{B}_u = (\mathbf{u}_n)_{n=1}^\infty$ dominates $\mathcal{E}[\ell_q]$ so that \mathcal{B}_u also dominates \mathcal{B}_v . Hence, \mathcal{B} is equivalent to \mathcal{B}_u . In the latter case, the principle of small perturbations yields that \mathcal{B} is equivalent to \mathcal{B}_v for a suitable choice of $(\varepsilon_n)_{n=1}^\infty$. \square

Proposition 4.7. For $n \in \mathbb{N}$ let $(\mathbb{X}_j)_{j=1}^n$ be a family of quasi-Banach spaces, each of which has an unconditional basis \mathcal{B}_j . Suppose there are sequences $(q_j)_{j=1}^{n-1}$ and $(r_j)_{j=2}^n$, both in $(0, \infty]$, such that:

- $q_j < r_{j+1}$ for all $1 \leq j \leq n - 1$;
- \mathcal{B}_j satisfies a lower q_j -estimate for all $1 \leq j \leq n - 1$; and
- \mathcal{B}_j satisfies an upper r_j -estimate for all $2 \leq j \leq n$.

If (A, B) is a partition of $\{1, \dots, n\}$ then the spaces $\mathbb{X}_a := \bigoplus_{j \in A} \mathbb{X}_j$ and $\mathbb{X}_b := \bigoplus_{j \in B} \mathbb{X}_j$ are totally incomparable.

Proof. Assume by contradiction that there is an infinite-dimensional quasi-Banach space \mathbb{Y} isomorphic to a subspace of both \mathbb{X}_a and \mathbb{X}_b . By Lemma 3.7, passing to a subspace we can suppose that \mathbb{Y} has a normalized Schauder basis \mathcal{B} equivalent both to a sequence finitely disjointly supported with respect to $\bigoplus_{j \in A} \mathcal{B}_j$ and to a sequence finitely disjointly supported with respect to $\bigoplus_{j \in B} \mathcal{B}_j$. By Lemma 4.6, passing to a subbasis we obtain $i \in A$ and $k \in B$ such that \mathcal{B} is equivalent both to a sequence finitely disjointly supported with respect to \mathcal{B}_i and to a sequence finitely disjointly supported with respect to \mathcal{B}_k . Switching the roles of A and B if necessary, we assume that $i < k$. We infer that $\mathcal{E}[\ell_{r_k}]$ dominates $\mathcal{E}[\ell_{q_i}]$ so that $r_k \leq q_i$. Since $q_i < r_{i+1} \leq r_k$ we reach an absurdity. \square

We are now ready to patch together all the different pieces that play a part in the proof of our main theorem.

Completion of the Proof of Theorem 4.1. By Theorem 2.4, either \mathbb{U} is not a proper envelope, or \mathbb{X} is natural and \mathbb{U} is not the proper envelope of a natural space with a basis. By Theorem 4.5, the pair $(\widehat{\mathbb{X}}, \mathbb{U})$ is splitting for unconditional bases. By Proposition 4.7, $\widehat{\mathbb{X}}$ and \mathbb{U} are incomparable. By Lemma 2.6, \mathbb{X} and \mathbb{U} are semi-complementably incomparable. By Corollary 3.5, every operator from \mathbb{X} to \mathbb{U} is strictly singular. By Theorem 4.2, (\mathbb{X}, \mathbb{U}) is splitting for complemented subspaces. Applying Lemma 2.10 the proof is over. \square

4.1. Applications

Although we stated Theorem 4.1 in all generality that our techniques permitted, here we will only apply it in the following cases:

- $A = \{1\}$, $B = \{2\}$, and there are $1 \leq q_1 < r_2 \leq q_2 < \infty$ such that \mathcal{B}_1 is q_1 -concave, and \mathcal{B}_2 is r_2 -convex and q_2 -concave;
- $A = \{1, 3\}$, $B = \{2\}$, and there are $1 \leq q_1 < r_2 \leq q_2 < r_3$ such that \mathcal{B}_1 is q_1 -concave, \mathcal{B}_2 is r_2 -convex and q_2 -concave, and \mathcal{B}_3 is r_3 -convex.

Notwithstanding, Theorem 4.1 is crucial in order to obtain new examples of spaces with a unique unconditional basis up to a permutation. We refer the reader to [1, Corollary 6.2] for a comprehensive inventory of spaces \mathbb{X} to which Theorem 4.1 is relevant. In fact, if \mathbb{X} is a direct sum built as explained in [1, Corollary 6.2], then \mathbb{X} is an L -convex lattice; moreover $\widehat{\mathbb{X}}$ is either a q -concave lattice for any $q > 1$ or r -convex lattice for any $r < \infty$, or a direct sum of both. In light of Proposition 3.6, Theorem 3.11, and [39, Theorem 5.1], Theorem 4.1 applies to a direct sum \mathbb{U} built with r -convexifications of the Tsirelson space for $r > 1$, duals of these convexified spaces, and Nakano spaces associated to a sequence $(p_j)_{j=1}^\infty$ with $\inf_j p_j > 1$ and $\sup_j p_j < \infty$.

Because of their importance in Analysis, we single out some examples involving Hardy spaces. For the convenience of the reader we will next state a few known facts about the spaces $H_p(\mathbb{T}^d)$ that we will need in order to apply Theorem 4.1. The first unconditional bases in $H_p(\mathbb{T})$ for $0 < p < 1$ were constructed in [42]. Those bases allow a manageable expression for the norm in terms of the coefficients relative to the basis. Namely, if $(\mathbf{x}_n)_{n=0}^\infty$ is such a normalized basis then

$$\left\| \sum_{n=0}^\infty a_n \mathbf{x}_n \right\|_{H_p(\mathbb{T})} \approx \left(\int_0^1 \left(\sum_{n=0}^\infty |a_n|^2 h_n^2 \right)^{p/2} \right)^{1/p}, \quad (a_n)_{n=1}^\infty \in c_{00}, \quad (4.1)$$

where $(h_n)_{n=0}^\infty$ is the classical Haar system on $[0, 1]$ normalized with respect to the norm in $L_p([0, 1])$. Using (4.1) one can easily see that $(\mathbf{x}_n)_{n=0}^\infty$ has a block basic sequence equivalent to the unit vector basis of ℓ_2 .

Those bases allow tensor constructions of unconditional bases in $H_p(\mathbb{T}^d)$ for $d \in \mathbb{N}$ which satisfy an analogous equivalence to (4.1). Using those tensored bases, Kalton et al. [26] showed that the spaces $H_p(\mathbb{T}^d)$ and $H_p(\mathbb{T}^m)$ with $0 < p < 1$ and $d, m \in \mathbb{N}$, are isomorphic if and only if $d = m$. Then it was proved in [43] that all the spaces $H_p(\mathbb{T}^d)$ for $0 < p < 1$ and $d \in \mathbb{N}$ have a UTAP unconditional basis. Also from the d -dimensional version of (4.1) we conclude that the (unique) unconditional basis of $H_p(\mathbb{T}^d)$ has a block basic sequence equivalent to the unit vector basis in ℓ_2 . One can also show that the Banach envelope of $H_p(\mathbb{T}^d)$ is isomorphic to ℓ_1 .

Theorem 4.8. *Let \mathbb{X} be the finite direct sum of some of the spaces from the following list:*

- The Hardy space $H_p(\mathbb{T}^d)$ for $0 < p < 1$ and $d \in \mathbb{N}$;
- The Nakano space $\ell(p_n)$, where $(p_n)_{n=1}^\infty$ is a non-increasing sequence in $[1, \infty)$ with $\lim_n p_n = 1$ and $\sup_n (p_n - p_{2n}) \log(1+n) < \infty$;
- The Nakano space $h(q_n)$, where $(q_n)_{n=1}^\infty$ is a non-decreasing sequence in $[1, \infty)$ with $\lim_n q_n = \infty$ and $\sup_n (1/q_n - 1/q_{2n}) \log(1+n) < \infty$;
- Tsirelson's space \mathcal{T} ;
- the original Tsirelson's space \mathcal{T}^* .

Let \mathbb{Y} be one of the spaces ℓ_2 , $\mathcal{T}^{(2)}$, or $\mathcal{T}_*^{(2)}$. Then the space $\mathbb{X} \oplus \mathbb{Y}$ has a UTAP unconditional basis.

As the alert reader might have noticed, all known Banach spaces with a UTAP unconditional basis that follow pattern **(P.1)** are r -convex lattices for $r < \infty$, all known Banach spaces with a UTAP unconditional basis that follow pattern **(P.3)** are q -concave lattices for $q > 1$, and all known Banach spaces with a UTAP unconditional basis that follow pattern **(P.2)** are both q -convex and r -concave lattices for $q < 2 < r$. Thus Theorem 4.8 yields in particular new additions to the list of Banach spaces with a UTAP unconditional basis.

The main questions that Theorem 4.5 leave open in the spirit of the *Memoir* by Bourgain et al. [13] are whether $\ell_1(c_0) \oplus \ell_2$, $c_0(\ell_1) \oplus \ell_2$, $c_0(\ell_2) \oplus \ell_1(\ell_2)$ and $\ell_2 \oplus \mathcal{T}^{(2)}$ have a UTAP unconditional basis.

Remark 4.9. We would like to point out that trying to generalize, first [41, Theorem 2.1] and then Theorem 4.5 to quasi-Banach spaces, is a priori a feasible program to tackle Question 1.2 in the case when \mathbb{X} is non-locally convex. However, we quickly run into quasi-Banach spaces, such as $H_p(\mathbb{T})$ for $0 < p < 1$, with a UTAP unconditional basis which, despite following pattern **(P.5)**, contain a block basic sequence equivalent to the unit vector system of ℓ_2 . Thus, in particular they do not satisfy a lower q -estimate for any $q < 2$ and so we would not be able to apply the wished-for generalization of Theorem 4.5 to them. This is the reason why in this paper we drew a route to approach Question 1.2 based on the (necessarily incomplete) information that we get from the envelopes.

Remark 4.10. Although here we are mainly concerned with and motivated by pattern **(P.2)**, it is worth it noting that our methods are more general. As an example let us look at the space $H_p(\mathbb{T}^d) \oplus c_0$. It was proved in [1] that it has UTAP unconditional basis. However, the result now also easily follows from Theorem 4.1.

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