



Toward an optimal theory of integration for functions taking values in quasi-Banach spaces

José L. Ansorena¹ · Glenier Bello²

Received: 21 April 2021 / Accepted: 7 March 2022 / Published online: 20 March 2022
© The Author(s) 2022

Abstract

We present a new approach to define a suitable integral for functions with values in quasi-Banach spaces. The integrals of Bochner and Riemann have deficiencies in the non-locally convex setting. The study of an integral for p -Banach spaces initiated by Vogt is neither totally satisfactory, since there are quasi-Banach spaces which are p -convex for all $0 < p < 1$, so it is not always possible to choose an optimal p to develop the integration. Our method puts the emphasis on the galb of the space, which permits a precise definition of its convexity. The integration works for all spaces of galbs known in the literature. We finish with a fundamental theorem of calculus for our integral.

Keywords Quasi-Banach space · Function quasi-norm · Integration · Galb · Tensor product

Mathematics Subject Classification 28B05 · 46G10 · 46A40 · 46A16 · 46A32 · 46E30

1 Introduction

If \mathbb{X} is a non-locally convex space, it is easy to construct a sequence of simple functions

$$s_n : [0, 1] \rightarrow \mathbb{X}, \quad s_n(t) = \sum_{m=1}^n \chi_{A_{m,n}}(t) x_{m,n},$$

J. L. Ansorena acknowledges the support of the Spanish Ministry for Science, Innovation, and Universities under Grant PGC2018-095366-B-I00 for *Análisis Vectorial, Multilineal y Aproximación*. G. Bello was supported by National Science Centre, Poland Grant UMO-2016/21/B/ST1/00241.

✉ José L. Ansorena
jose Luis.ansorena@unirioja.es

Glenier Bello
gbello@impan.pl

¹ Department of Mathematics and Computer Sciences, Universidad de La Rioja, 26004 Logroño, Spain

² Institute of Mathematics of the Polish Academy of Sciences, ul. Śniadeckich 8, 00-656 Warszawa, Poland

where $(A_{m,n})_{m=1}^n$ is a partition of the interval $[0, 1]$ for each $n \in \mathbb{N}$, and χ denotes the characteristic function, such that

$$\sup_{1 \leq m \leq n} \|x_{m,n}\| \rightarrow 0, \quad \sum_{m=1}^n \mu(A_{m,n})x_{m,n} \not\rightarrow 0,$$

as n goes to infinity, where μ denotes the Lebesgue measure (cf. [31, pp. 121–123]). Therefore, Bochner–Lebesgue integration cannot be extended to non-locally convex spaces. On the other hand, the definition of the Riemann integral extends verbatim for functions defined on an interval $[a, b]$ with values in an F -space \mathbb{X} . However, it has some problems in the non-locally convex setting. For example, Mazur and Orlicz [28] proved that the F -space \mathbb{X} is non-locally convex if and only if there is a continuous function $f : [0, 1] \rightarrow \mathbb{X}$ which is not Riemann integrable. But the main drawback is that the Riemann integral operator $\mathcal{I}_{\mathcal{R}}$, acting from the set of \mathbb{X} -valued simple functions $\mathcal{S}([a, b], \mathbb{X})$ to \mathbb{X} by

$$\mathcal{I}_{\mathcal{R}} \left(\sum_{j=1}^n x_j \chi_{[t_{j-1}, t_j)} \right) = \sum_{j=1}^n (t_j - t_{j-1})x_j,$$

is not continuous when \mathbb{X} is not locally convex (see [1, Theorem 2.3]).

An important attempt (somehow missed in the literature) to develop a theory of integration based on operators for functions with values in a quasi-Banach-space (i.e. a locally bounded F -space) was initiated by Vogt [40]. A remarkable theorem of Aoki and Rolewicz [5, 30] says that any quasi-normed space is p -convex for some $0 < p \leq 1$. The idea of Vogt was the following. Given a quasi-Banach space \mathbb{X} , let $0 < p \leq 1$ be such that \mathbb{X} is p -convex. For this fixed p , he developed a theory of integration based on an identification of tensor spaces with function spaces (see [40, Satz 4]). Among the papers that approach integration of quasi-Banach-valued functions from Vogt’s point of view we highlight [27].

The main advantage of Vogt’s integration with respect other approaches to integration in the non locally convex setting is that it provides a bounded operator from the space of integrable functions into the target quasi-Banach space. Regarding the limitations, its main drawback is that it depends heavily on the convexity parameter p chosen, and for some spaces there is no optimal choice of p . Take, for instance, the weak Lorentz space $L_{1,\infty} = L_{1,\infty}(\mathbb{R})$. This classical space, despite not being locally convex, is p -convex for any $0 < p < 1$ (see [18, (2.3) and (2.6)]).

The concept that permits a precise definition of the convexity of a space was introduced and developed by Turpin in a series of papers (cf. [36, 37]) and a monograph ([38]) in the early 1970’s. Given an F -space \mathbb{X} , its galb, denoted by $\mathcal{G}(\mathbb{X})$, is the vector space of all sequences $(a_n)_{n=1}^\infty$ of scalars such that whenever $(x_n)_{n=1}^\infty$ is a sequence in \mathbb{X} with $\lim x_n = 0$, the series $\sum_{n=1}^\infty a_n x_n$ converges in \mathbb{X} . We say that a sequence space \mathbb{Y} galbs \mathbb{X} if $\mathbb{Y} \subseteq \mathcal{G}(\mathbb{X})$. With this terminology, \mathbb{X} is p -convex if and only if $\ell_p \subseteq \mathcal{G}(\mathbb{X})$.

The galb of certain classical spaces is known. Turpin [36] computed the galb of locally bounded, non-locally convex Orlicz function spaces $L_\phi(\mu)$, where μ is either a nonatomic measure or the counting measure, and showed that the result is an Orlicz sequence space ℓ_ϕ modeled after a different Orlicz function ϕ . Hernández [13–15] continued the study initiated by Turpin and computed, in particular, the galb of certain vector-valued Orlicz spaces. The study of the convexity of Lorentz spaces took a different route. Before Turpin invented the notion of galb, Stein and Weiss [35] proved that the Orlicz sequence space $\ell \log \ell$ galbs $L_{1,\infty}$, and used this result to achieve a Fourier multiplier theorem for $L_{1,\infty}$. Sjögren [33] concluded the study by (implicitly) proving that $\mathcal{G}(L_{1,\infty}) = \ell \log \ell$. Later on, the convexity type of

Lorentz spaces $L_{1,q}$ for $0 < q < \infty$ was studied (see [10, 34]). In [9], general weighted Lorentz spaces were considered.

The geometry of spaces of galbs is quite unknown, however. Probably, the most significant advance in this direction since seminal Turpin work was made in [19]. Solving a question raised in [38], Kalton proved that if \mathbb{X} is p -convex and is not q -convex for any $q > p$, then $\mathcal{G}(\mathbb{X}) = \ell_p$.

In this paper, we use galbs to develop a theory of integration for functions taking values in quasi-Banach spaces in the spirit of Vogt that fits as well as possible the convexity of the target space. Our construction is closely related to tensor products, and to carry out it we construct topological tensor products adapted to our needs. More precisely, given a quasi-Banach space \mathbb{X} and a σ -finite measure space (Ω, Σ, μ) , for an appropriate function quasi-norm λ over \mathbb{N} (see Definition 1) we define the tensor product space $\mathbb{X} \otimes_\lambda L_1(\mu)$ so that there are bounded linear canonical maps

$$J: \mathbb{X} \otimes_\lambda L_1(\mu) \rightarrow L_1(\mu, \mathbb{X}), \quad x \otimes f \mapsto xf, \text{ and}$$

$$I: \mathbb{X} \otimes_\lambda L_1(\mu) \rightarrow \mathbb{X}, \quad x \otimes f \mapsto x \int_\Omega f \, d\mu.$$

If I factors through J , that is, there is a map \mathcal{I} (defined on the range of J) such that the diagram

$$\begin{array}{ccc} \mathbb{X} \otimes_\lambda L_1(\mu) & & \\ \downarrow J & \searrow I & \\ L_1^\lambda(\mu, \mathbb{X}) := J(\mathbb{X} \otimes_\lambda L_1(\mu)) & \xrightarrow{\mathcal{I}} & \mathbb{X} \end{array}$$

commutes, then \mathcal{I} defines a suitable integral for functions in $L_1^\lambda(\mathbb{X})$. Thus, we say that (λ, \mathbb{X}) is amenable if λ galbs \mathbb{X} (i.e., $(a_n)_{n=1}^\infty \in \mathcal{G}(\mathbb{X})$ whenever $\lambda((a_n)_{n=1}^\infty) < \infty$) and I factors through J .

There is a tight connection between the existence of the integral \mathcal{I} and the injectivity of J . In fact, we will prove that if (λ, \mathbb{X}) is amenable, then J is one-to-one (see Theorem 5). This connection leads us to study the injectivity of J . More generally, we consider the map

$$J: \mathbb{X} \otimes_\lambda L_\rho \rightarrow L_\rho(\mathbb{X})$$

associated with the quasi-Banach space \mathbb{X} , the function quasi-norm λ and a function quasi-norm ρ over (Ω, Σ, μ) , and we obtain results that generalize those previously obtained for Lebesgue spaces $L_q(\mu)$ and tensor quasi-norms in the sense of ℓ_p , $0 < p \leq q \leq \infty$ (see [40, Satz 4]).

With the terminology of this paper, Vogt proved that if \mathbb{X} is a p -Banach space (see the precise definition in the next section), $0 < p \leq 1$, then (ℓ_p, \mathbb{X}) is amenable. So, in order to exhibit the applicability of the theory of integration developed within this paper, we must exhibit new examples of amenable pairs. Since the space of galbs of the quasi-Banach space \mathbb{X} arises from a function quasi-norm on \mathbb{N} , say $\lambda_{\mathbb{X}}$, the question of whether the pair $(\lambda_{\mathbb{X}}, \mathbb{X})$ is amenable arises. For answering it, one first need to know whether the space of galbs $\mathcal{G}(\mathbb{X})$ is always 1-concave as a quasi-Banach lattice or not. See Questions 2 and 4. As long as there is no general answer to these questions, we focus on the spaces of galbs that have appeared in the literature. In Theorem 6, we prove that for all of them Question 4 has a positive answer.

Once the theory is built, the first goal should be the study of its integration properties. By construction, our integral behaves linearly and has suitable convergence properties. Hence, we finish with a fundamental theorem of calculus for our integral (see Theorem 7).

The paper is organized as follows. In Sect. 2, we introduce the terminology and notation that will be employed. The theory of function norms (i.e., the locally convex setting) has been deeply developed (cf. [6, 26]). However, even though some generalizations of results from harmonic analysis to quasi-Banach function spaces are known (see, e.g., [9, 16, 17] for recent work within this area), a systematic study in the non-locally convex setting is missing. For that reason, in Sect. 3, we do a brief survey on function quasi-norms covering the most relevant aspects, and all the results that we need. Section 4 is devoted to galbs. In Sect. 5, we briefly collect some results on tensor products. In Sect. 6, we present our main results on integration for functions taking values in quasi-Banach spaces. Finally, in Sect. 7, we give a fundamental theorem of calculus that improves [1, Theorem 5.2].

2 Terminology

We use standard terminology and notation in Banach space theory as can be found, e.g., in [3]. The unfamiliar reader will find general information about quasi-Banach spaces in [23]. We next gather the notation on quasi-Banach spaces that we will use.

A *quasi-normed* space will be a vector space over the real or complex field \mathbb{F} endowed with a *quasi-norm*, i.e., a map $\| \cdot \| : \mathbb{X} \rightarrow [0, \infty)$ satisfying

- (Q.1) $\|x\| = 0$ if and only if $x = 0$;
- (Q.2) $\|tx\| = |t|\|x\|$ for $t \in \mathbb{F}$ and $x \in \mathbb{X}$; and
- (Q.3) there is a constant $\kappa \geq 1$ so that for all x and y in \mathbb{X} we have

$$\|x + y\| \leq \kappa(\|x\| + \|y\|).$$

The smallest number κ in (Q.3) will be called the *modulus of concavity* of the quasi-norm. If it is possible to take $\kappa = 1$ we obtain a norm. A quasi-norm clearly defines a metrizable vector topology on \mathbb{X} whose base of neighborhoods of zero is given by sets of the form $\{x \in \mathbb{X} : \|x\| < 1/n\}$, $n \in \mathbb{N}$. Given $0 < p \leq 1$, a quasi-normed space is said to be *p-convex* if it has an absolutely *p-convex* neighborhood of the origin. A quasi-normed space \mathbb{X} is *p-convex* if and only if there is a constant C such that

$$\left\| \sum_{j=1}^n x_j \right\|^p \leq C \sum_{j=1}^n \|x_j\|^p, \quad n \in \mathbb{N}, x_j \in \mathbb{X}. \tag{1}$$

If, besides (Q.1) and (Q.2), (1) holds with $C = 1$ we say that $\| \cdot \|$ is a *p-norm*. Any *p-norm* is a quasi-norm with modulus of concavity at most $2^{1/p-1}$. A *p-normed* space is a quasi-normed space endowed with a *p-norm*. By the Aoki–Rolewicz theorem [5, 30] any quasi-normed space is *p-convex* for some $0 < p \leq 1$. In turn, any *p-convex* quasi-normed space can be equipped with an equivalent *p-norm*. Hence, any quasi-normed space becomes, for some $0 < p \leq 1$, a *p-normed* space under suitable renorming.

A *p-Banach* (resp. *quasi-Banach*) space is a complete *p-normed* (resp. quasi-normed) space. It is known that a *p-convex* quasi-normed space is complete if and only if for every sequence $(x_n)_{n=1}^\infty$ in \mathbb{X} such that $\sum_{n=1}^\infty \|x_n\|^p < \infty$ the series $\sum_{n=1}^\infty x_n$ converges.

A semi-quasi-norm on a vector space \mathbb{X} is a map $\| \cdot \| : \mathbb{X} \rightarrow [0, \infty)$ satisfying (Q.2) and (Q.3). A standard procedure, to which we refer as the *completion method* allow us to manufacture a quasi-Banach from a semi-quasi-norm (see e.g. [2, Sect. 2.2]).

As the Hahn–Banach Theorem depends heavily on convexity, it does not pass through general quasi-Banach spaces. In fact, there are quasi-Banach spaces as $L_p([0, 1])$ for $0 < p < 1$ whose dual space is null (see [11]). Following [23], we say that the quasi-Banach space \mathbb{X} has *point separation property* if for every $f \in \mathbb{X} \setminus \{0\}$ there is $f^* \in \mathbb{X}^*$ such that $f^*(f) \neq 0$.

Given a σ -finite measure space (Ω, Σ, μ) and a quasi-Banach space \mathbb{X} , we denote by $L_0^+(\mu)$ the set consisting of all measurable functions from Ω into $[0, \infty]$, and by $L_0(\mu, \mathbb{X})$ the vector space consisting of all measurable functions from Ω into \mathbb{X} . As usual, we identify almost everywhere (a.e. for short) coincident functions. We set $L_0(\mu) = L_0(\mu, \mathbb{F})$ and

$$\Sigma(\mu) = \{A \in \Sigma : \mu(A) < \infty\}.$$

We denote by $\mathcal{S}(\mu, \mathbb{X})$ the vector space consisting of all integrable \mathbb{X} -valued simple functions. That is,

$$\mathcal{S}(\mu, \mathbb{X}) = \text{span}(x\chi_E : E \in \Sigma(\mu), x \in \mathbb{X}).$$

We say that (Ω, Σ, μ) is *infinite-dimensional* if $\mathcal{S}(\mu) = \mathcal{S}(\mu, \mathbb{F})$ is.

An *order ideal* in $L_0(\mu)$ will be a (linear) subspace L of $L_0(\mu)$ such that the conjugate function $\bar{f} \in L$ whenever $f \in L$, and $\max\{f, g\} \in L$ whenever f and g are real-valued functions in L . A *cone* in $L_0^+(\mu)$ will be a subset C of $L_0^+(\mu)$ such that for all $f, g \in C$ and all $\alpha, \beta \geq 0$ we have $f < \infty$ a.e., $\alpha f + \beta g \in C$, and $\max\{f, g\} \in C$. It is immediate that if L is an order ideal in $L_0(\mu)$, then

$$L^+ := L \cap L_0^+(\mu)$$

is a cone in $L_0^+(\mu)$; and reciprocally, if C is a cone in $L_0^+(\mu)$, there is a unique order ideal L with $L^+ = C$. Namely,

$$L = \{f \in L_0(\mu) : |f| \leq g \text{ for some } g \in C\}.$$

Given a quasi-Banach space \mathbb{X} , we say that a quasi-Banach space \mathbb{U} is *complemented* in \mathbb{X} via a map $S : \mathbb{U} \rightarrow \mathbb{X}$ if there is a map $P : \mathbb{X} \rightarrow \mathbb{U}$ such that $P \circ S = \text{Id}_{\mathbb{U}}$.

The *unit vector system* is the sequence $(e_k)_{k=1}^\infty$ in $\mathbb{F}^\mathbb{N}$ defined by $e_k = (\delta_{k,n})_{n=1}^\infty$, where $\delta_{k,n} = 1$ if $k = n$ and $\delta_{k,n} = 0$ otherwise. A *block basis sequence* with respect to the unit vector system is a sequence $(f_k)_{k=1}^\infty$ such that

$$f_k = \sum_{n=1+n_{k-1}}^{n_k} a_n e_n, \quad k \in \mathbb{N}$$

for some sequence $(a_n)_{n=1}^\infty$ in $\mathbb{F}^\mathbb{N}$ and some increasing sequence $(n_k)_{k=0}^\infty$ of non-negative scalars with $n_0 = 0$.

3 Function quasi-norms

As mentioned in the Introduction, in contrast to the theory of function norms, there is no systematic study in the non-locally convex setting. In this section, we try to go one step forward in that direction. We begin with the basic properties of function quasi-norms. Here,

we do not impose them to satisfy a Fatou property (something that Bennet and Sharpley [6] do for function norms). We devote a subsection to the study of this property. Then we study the properties of absolute continuity and domination for function quasi-norms, as well as Minkowski-type inequalities. We also discuss the use of conditional expectation (via the notion of leveling function quasi-norms), which will be relevant for the proof of Theorem 5. We conclude the section with some comments on function quasi-norms over \mathbb{N} endowed with the counting measure, a specially important particular case.

Definition 1 A function quasi-norm over a σ -finite measure space (Ω, Σ, μ) is a mapping $\rho: L_0^+(\mu) \rightarrow [0, \infty]$ such that

- (F.1) $\rho(tf) = t\rho(f)$ for all $t \geq 0$ and $f \in L_0^+(\mu)$;
- (F.2) if $f \leq g$ a.e., then $\rho(f) \leq \rho(g)$;
- (F.3) if $E \in \Sigma(\mu)$, then $\rho(\chi_E) < \infty$;
- (F.4) for every $E \in \Sigma(\mu)$ and every $\varepsilon > 0$, there is $\delta > 0$ such that $\mu(A) \leq \varepsilon$ whenever $A \in \Sigma$ satisfies $A \subseteq E$ and $\rho(\chi_A) \leq \delta$; and
- (F.5) there is a constant κ such that $\rho(f + g) \leq \kappa(\rho(f) + \rho(g))$ for all $f, g \in L_0^+(\mu)$.

The optimal κ in (F.5) is called the *modulus of concavity* of ρ .

Notice that (F.4) implies that $\rho(\chi_E) > 0$ for all $E \in \Sigma$ with $\mu(E) > 0$.

Definition 2 A function norm is a function quasi-norm with modulus of concavity 1. More generally, given $0 < p \leq 1$, a function p -norm is a function $\rho: L_0^+(\mu) \rightarrow [0, \infty]$ which satisfies (F.1)–(F.4), and

$$(F.6) \quad \rho^p(f + g) \leq \rho^p(f) + \rho^p(g) \text{ for all } f, g \in L_0^+(\mu).$$

The inequality $a^p + b^p \leq 2^{1-p}(a + b)^p$ for all $a, b \in [0, \infty]$ and $p \in (0, 1]$ yields that any function p -norm is a function quasi-norm with modulus of concavity at most $2^{1/p-1}$.

This generalization of the notion of a function norm follows ideas from [6, 26]. Besides (F.5), the main differences between our definition and that adopted by Luxemburg and Zaanen in [26] lie in restricting ourselves to σ -finite spaces, and in imposing condition (F.3), which, on the one hand, prevents from existing non null sets E on which ρ is trivial (in the sense that if $f \in L_0^+(\mu)$ is null outside E then $\rho(f)$ is either 0 or ∞) and, on the other hand, guarantees the existence of enough functions with finite quasi-norm. Regarding the approach in [6], we point out that Bennet and Sharpley imposed a function norm to satisfy

$$(F.7) \quad \text{for every } E \in \Sigma(\mu) \text{ there is a constant } C = C_E \text{ such that}$$

$$\int_E f \, d\mu \leq C_E \rho(f), \quad f \in L_0^+(\mu).$$

The most natural examples of functions quasi-norms are L_p -quasi-norms, $0 < p < \infty$, defined by

$$f \mapsto \left(\int_{\Omega} f^p \, d\mu \right)^{1/p}, \quad f \in L_0^+(\mu).$$

To avoid introducing cumbersome notations, sometimes the symbol $L_p(\mu)$ will mean the function quasi-norm defining the space $L_p(\mu)$ instead of the space itself, and the same convention will be used for Lorentz and Orlicz spaces. Since, if μ is not purely atomic and $0 < p < 1$, $L_p(\mu)$ does not satisfy (F.7), imposing this condition to all function quasi-norms is somewhat nonsense in the non-locally convex setting. Thus we impose its natural substitute (F.4)

instead. Also, unlike Bennet and Sharpley, we do not a priori impose ρ to satisfy Fatou property (see Sect. 3.1).

Given $f \in L_0^+(\mu)$ and a function quasi-norm ρ over (Ω, Σ, μ) , we set

$$\Omega_f(s) = \{\omega \in \Omega : f(\omega) > s\} \text{ and } \rho_f(s) = \rho(\chi_{\Omega_f(s)}), \quad s \in [0, \infty).$$

Set also $\Omega_f(\infty) = \{\omega \in \Omega : f(\omega) = \infty\}$ and $\rho_f(\infty) = \rho(\chi_{\Omega_f(\infty)})$. If ρ is the function quasi-norm associated with $L_1(\mu)$, then $\mu_f := \rho_f$ is the *distribution function* of f . We say f has a finite distribution function if $\mu_f(s) < \infty$ for all $s > 0$.

Definition 3 We say that a function quasi-norm ρ is *rearrangement invariant* if every function $f \in L_0^+(\mu)$ with $\rho(f) < \infty$ has a finite distribution function, and $\rho(f) = \rho(g)$ whenever $\mu_f = \mu_g$.

The proof of the following lemma is based on the elementary inequality

$$s\rho_f(s) \leq \rho(f), \quad f \in L_0^+(\mu), \quad s \in [0, \infty].$$

Lemma 1 *Let ρ be a function quasi-norm over a σ -finite measure space (Ω, Σ, μ) .*

- (i) *If $f \in \mathcal{S}(\mu)$, then $\rho(|f|) < \infty$.*
- (ii) *If $f \in L_0^+(\mu)$ satisfies $\rho(f) < \infty$, then $f < \infty$ a.e.*
- (iii) *If $f \in L_0^+(\mu)$ satisfies $\rho(f) = 0$, then $f = 0$ a.e.*
- (iv) *Let $E \in \Sigma(\mu)$, $s > 0$, and $\varepsilon > 0$. Then there is $\delta > 0$ such that for all $f \in L_0^+(\mu)$ with $\rho(f) \leq \delta$ we have*

$$\mu(\{\omega \in E : f(\omega) > s\}) \leq \varepsilon.$$

Proof Statement (i) is clear. Now let $f \in L_0^+(\mu)$. If $\rho(f)$ is finite, then $\rho_f(\infty) = 0$ and (ii) follows. If $\rho(f) = 0$, then $\rho_f(s) = 0$ for all $s > 0$. Since $\Omega_f(0) = \bigcup_{n=1}^\infty \Omega_f(2^{-n})$, we obtain (iii). Finally, let $E \in \Sigma(\mu)$, $s > 0$, and $\varepsilon > 0$. By (F.4), there is $\tilde{\delta} > 0$ such that if $A \subseteq E$ with $\rho(\chi_A) \leq \tilde{\delta}$, then $\mu(A) \leq \varepsilon$. Take $\delta := s\tilde{\delta}$, and let $f \in L_0^+(\mu)$ with $\rho(f) \leq \delta$. Set $A := \{\omega \in E : f(\omega) > s\}$. Since $\rho(\chi_A) \leq \rho(f)/s \leq \tilde{\delta}$, we obtain (iv). □

Definition 4 A function quasi-norm ρ is said to be *p-convex* if there is a constant C such that

$$\rho^p(\sum_{j=1}^n f_j) \leq C \sum_{j=1}^n \rho^p(f_j), \quad n \in \mathbb{N}, \quad f_j \in L_0^+(\mu).$$

Proposition 1 (Aoki–Rolewicz Theorem for function quasi-norms) *Any function quasi-norm is p-convex for some $0 < p \leq 1$. Indeed, if κ is the modulus of concavity we can choose p such that $2^{1/p-1} = \kappa$.*

Proof It goes over the lines of the proof of the Aoki–Rolewicz Theorem (see e.g. [23, Lemma 1.1]). So, we omit it. □

Definition 5 Given two function quasi-norms ρ and λ over a σ -finite measure space (Ω, Σ, μ) , we say that ρ *dominates* λ if there is a constant C such that $\lambda(f) \leq C\rho(f)$ for all $f \in L_0^+(\mu)$. If ρ dominates and is dominated by λ , we say that ρ and λ are *equivalent*.

Lemma 2 *Let $0 < p \leq 1$, and let ρ be a function quasi-norm. Then ρ is equivalent to a function p-norm if and only if it is p-convex.*

Proof It is clear that any function p -norm is p -convex, and p -convexity is inherited by passing to an equivalent function quasi norm. Reciprocally, if ρ is a p -convex function quasi-norm over a σ -finite measure space (Ω, Σ, μ) , then it is immediate that the map $\lambda: L_0^+(\mu) \rightarrow [0, \infty]$ given by

$$\lambda(f) = \inf \left\{ \left(\sum_{j=1}^n \rho^p(f_j) \right)^{1/p} : n \in \mathbb{N}, f_j \in L_0^+(\mu), f = \sum_{j=1}^n f_j \right\}$$

is a function p -norm equivalent to ρ . □

Corollary 1 Any function quasi-norm is equivalent to a function p -norm for some $0 < p \leq 1$.

Proof It follows from Proposition 1 and Lemma 2. □

In light of Corollary 1, it is natural, and convenient in some situations, to restrict ourselves to function quasi-norms that are function p -norms for some p . However, we emphasize that some p -convex spaces arising naturally in Mathematical Analysis are given by a function quasi-norm that is not a p -norm. Take, for instance the 1-convex (i.e., locally convex) function space $L_{r,\infty}, r > 1$. So, when working in the general framework of non-locally convex spaces, it is convenient to know whether a given property pass to equivalent function quasi-norms.

Definition 6 Let \mathbb{X} be a quasi-Banach space, and let ρ be a function quasi-norm over a σ -finite measure space (Ω, Σ, μ) . The space

$$L_\rho(\mathbb{X}) = \{f \in L_0(\mu, \mathbb{X}) : \|f\|_\rho := \rho(\|f\|) < \infty\}.$$

endowed with the gauge $\|\cdot\|_\rho$ will be called the *vector-valued Köthe space* associated with ρ and \mathbb{X} . The space $L_\rho = L_\rho(\mathbb{F})$ will be called the *Köthe space* associated with ρ .

Note that we do not impose the functions in $L_\rho(\mathbb{X})$ to be strongly measurable. If ρ is the function quasi-norm associated to the Lebesgue space $L_p(\mu), 0 < p < \infty$, we set $L_p(\mu, \mathbb{X}) := L_\rho(\mathbb{X})$. If $A \in \Sigma$, we set $L_p(A, \mu, \mathbb{X}) := L_\rho(\mu|_A, \mathbb{X})$, where $\mu|_A$ is the restriction of μ to $\Sigma \cap \mathcal{P}(A)$. In general, if $\rho|_A$ is the function quasi-norm defined by $\rho|_A(f) = \rho(\tilde{f})$, where

$$\tilde{f}(\omega) = \begin{cases} f(\omega) & \text{if } \omega \in A, \\ 0 & \text{otherwise,} \end{cases}$$

we set $L_\rho(A, \mathbb{X}) = L_{\rho|_A}(\mathbb{X})$.

It is clear that L_ρ is an order ideal in $L_0(\mu)$. By Lemma 1 (ii), its cone is given by

$$L_\rho^+ = \{f \in L_0^+(\mu) : \rho(f) < \infty\}.$$

Lemma 3 Let ρ be a function quasi-norm over a σ -finite measure space (Ω, Σ, μ) and \mathbb{X} be a quasi-Banach space.

- (i) $L_\rho(\mathbb{X})$ is a quasi-normed space.
- (ii) $\mathcal{S}(\mu, \mathbb{X}) \subseteq L_\rho(\mathbb{X})$.
- (iii) If we endow $L_0(\mu, \mathbb{X})$ with the vector topology of the local convergence in measure, then $L_\rho(\mathbb{X}) \subseteq L_0(\mu, \mathbb{X})$ continuously.
- (iv) If \mathcal{K} is a closed subset of \mathbb{X} , then $L_\rho(\mathcal{K}) := \{f \in L_\rho(\mathbb{X}) : f(\omega) \in \mathcal{K} \text{ a.e. } \omega \in \Omega\}$ is closed in $L_\rho(\mathbb{X})$.

Proof Statements (i), (ii), and (iii) are straightforward from the very definition of function quasi-norm and Lemma 1. Now let \mathcal{K} be a closed subset of \mathbb{X} , and let x be a function in $L_\rho(\mathbb{X}) \setminus L_\rho(\mathcal{K})$ (assuming that this set is non-empty). There is $\varepsilon > 0$ and $A \subseteq \Sigma$ with $\mu(A) > 0$ such that $\|x(a) - k\| \geq \varepsilon$ for all $a \in A$ and all $k \in \mathcal{K}$. Therefore $\|x - y\|_\rho \geq \varepsilon \rho(\chi_A) > 0$ for all $y \in L_\rho(\mathcal{K})$, and we obtain (iv). \square

Lemma 4 *Let ρ be a function quasi-norm, and let \mathbb{X} be a Banach space. If a sequence $(x_n)_{n=1}^\infty$ converges to x in $L_\rho(\mathbb{X})$, then $(\|x_n\|)_{n=1}^\infty$ converges to $\|x\|$ in L_ρ .*

Proof It follows from the inequality $\| \|x_n\| - \|x\| \| \leq \|x_n - x\|$ for all $n \in \mathbb{N}$. \square

Proposition 2 *Let ρ be a function quasi-norm over a σ -finite measure space (Ω, Σ, μ) , let \mathbb{X} be a quasi-Banach space, and let $(x_n)_{n=1}^\infty$ be a sequence in $L_0(\mu, \mathbb{X})$ such that $\lim_n \|x_n - x\|_\rho = 0$ for some $x \in L_0(\mu, \mathbb{X})$. Then, there is a subsequence $(y_n)_{n=1}^\infty$ of $(x_n)_{n=1}^\infty$ such that $\lim_n y_n = x$ a.e.*

Proof Let $(A_j)_{j=1}^\infty$ be an increasing sequence of finite-measure sets such that x_n is null outside $A = \cup_{j=1}^\infty A_j$ for all $n \in \mathbb{N}$. Then $\rho(\|x\| \chi_{\Omega \setminus A}) = 0$. Therefore $x(\omega) = 0$ a.e. $\omega \in \Omega \setminus A$. By Lemma 3 (iii), for each $j \in \mathbb{N}$ there is an increasing sequence $(n_k)_{k=1}^\infty$ such that $\lim_k x_{n_k}(\omega) = x(\omega)$ a.e. $\omega \in A_j$. The Cantor diagonal technique yields a subsequence $(y_n)_{n=1}^\infty$ of $(x_n)_{n=1}^\infty$ such that $\lim_n y_n(\omega) = x(\omega)$ a.e. $\omega \in A$. \square

3.1 The Fatou property

Definition 7 Suppose that ρ is a function quasi-norm over a σ -finite measure space (Ω, Σ, μ) . We say that ρ has the *rough Fatou property* if there is a constant C such that $\rho(\lim_n f_n) \leq C \lim_n \rho(f_n)$ whenever $(f_n)_{n=1}^\infty$ is non-decreasing sequence in $L_0^+(\mu)$. If the above holds with $C = 1$ we say that ρ has the *Fatou property*. We say that ρ has the *weak Fatou property* if $\rho(\lim_n f_n) < \infty$ whenever the non-decreasing sequence $(f_n)_{n=1}^\infty$ in $L_0^+(\mu)$ satisfies $\lim_n \rho(f_n) < \infty$.

Note that Fatou property says that if $f_n \nearrow f$ then $\rho(f_n) \nearrow \rho(f)$, so it does not pass to equivalent function quasi-norms. In contrast, both rough and weak Fatou property are preserved. In fact, these two notions are equivalent.

Proposition 3 (cf. [4, Lemma]) *If ρ is a function quasi-norm with the weak Fatou property, then it also has the rough Fatou property.*

Proof Let ρ be a function quasi-norm over a σ -finite measure space (Ω, Σ, μ) . By Corollary 1, we can assume without loss of generality that it is a function p -norm for some $0 < p \leq 1$. Suppose that ρ does not have the rough Fatou property. Then, for each $k \in \mathbb{N}$ there is a non-decreasing sequence $(f_{k,n})_{n=1}^\infty$ in $L_0^+(\mu)$ with $\sup_n \rho(f_{k,n}) \leq 1$ and $\rho(\lim_n f_{k,n}) > 2^{2k/p}$. The sequence $(g_n)_{n=1}^\infty$ defined by

$$g_n = \sum_{k=1}^n 2^{-k/p} f_{k,n}, \quad n \in \mathbb{N},$$

is non-decreasing, and we have

$$2^{-k/p} f_{k,n} \leq g := \lim_n g_n, \quad k \leq n.$$

Then $\rho(g) \geq 2^{-k/p} \rho(\lim_n f_{k,n}) > 2^{k/p}$ for all $k \in \mathbb{N}$. That is, $\rho(g) = \infty$. On the other hand, since ρ is a function p -norm, $\rho^p(g_n) \leq \sum_{k=1}^n 2^{-k} \leq 1$ for all $n \in \mathbb{N}$. Therefore ρ does not have the weak Fatou property. \square

Proposition 4 (cf. [6, Theorem 1.8]) *Let λ and ρ be two function quasi-norms over the same σ -finite measure space. Suppose that ρ has the weak Fatou property. Then ρ dominates λ if and only if $L_\rho^+ \subseteq L_\lambda^+$.*

Proof The direct implication is obvious. Suppose now that ρ does not dominate λ . Then there is a sequence $(f_n)_{n=1}^\infty$ in $L_0^+(\mu)$ such that $4^n \rho(f_n) < \lambda(f_n)$ for all $n \in \mathbb{N}$. Set

$$f = \sum_{n=1}^\infty \frac{2^{-n}}{\rho(f_n)} f_n.$$

Using that ρ has the rough Fatou property (due to Proposition 3) and Proposition 1, we obtain that $\rho(f) < \infty$. Since

$$\lambda(f) \geq \sup_n \frac{2^{-n} \lambda(f_n)}{\rho(f_n)} \geq \sup_n 2^n = \infty,$$

the space L_ρ^+ is not contained in L_λ^+ . \square

Definition 8 Let $0 < p \leq 1$ and let ρ be a function quasi-norm over a σ -finite measure space (Ω, Σ, μ) . We say that ρ has the *Riesz–Fischer p -property* if for every sequence $(f_n)_{n=1}^\infty$ in $L_0^+(\mu)$ with $\sum_{n=1}^\infty \rho^p(f_n) < \infty$ we have $\rho(\sum_{n=1}^\infty f_n) < \infty$.

Lemma 5 (cf. [4, Theorem]) *Let ρ be a p -convex function quasi-norm with the weak Fatou property. Then ρ has the Riesz–Fischer p -property.*

Proof Let $(f_n)_{n=1}^\infty$ be a sequence in $L_0^+(\mu)$ with $A := \sum_{n=1}^\infty \rho^p(f_n) < \infty$. If C denotes the p -convexity constant of ρ , then

$$\rho\left(\sum_{n=1}^m f_n\right) \leq C^{1/p} \left(\sum_{n=1}^m \rho^p(f_n)\right)^{1/p} \leq C^{1/p} A^{1/p}, \quad m \in \mathbb{N}.$$

Hence $\lim_m \rho(\sum_{n=1}^m f_n) < \infty$, and therefore $\rho(\sum_{n=1}^\infty f_n) < \infty$ (since ρ has the weak Fatou property). That is, ρ has the Riesz–Fischer p -property. \square

Proposition 5 *Let ρ be a function quasi-norm over a σ -finite measure space (Ω, Σ, μ) . Given $0 < p \leq 1$, the following statements are equivalent:*

- (i) ρ has the Riesz–Fischer p -property;
- (ii) there is a constant C such that $\rho^p(\sum_{n=1}^\infty f_n) \leq C \sum_{n=1}^\infty \rho^p(f_n)$ for every sequence $(f_n)_{n=1}^\infty$ in $L_0^+(\mu)$;
- (iii) L_ρ is a p -convex quasi-Banach space.

Moreover, the optimal constant in (ii) is the p -convexity constant of L_ρ . In particular, L_ρ is a p -Banach space if and only if (ii) holds with $C = 1$.

Also, we have that any of the previous statements hold for some $0 < p \leq 1$ if and only if

- (iv) $L_\rho(\mathbb{X})$ is a quasi-Banach space for any (resp. some) nonzero quasi-Banach space \mathbb{X} .

Proof Let us see first that if (i) holds for some $0 < p \leq 1$, then (ii) also holds for the same p . We use an argument by contradiction. Fix $0 < p \leq 1$ and suppose that there is no $C > 0$ such that $\rho^p(\sum_{n=1}^\infty f_n) \leq C \sum_{n=1}^\infty \rho^p(f_n)$ for every sequence $(f_n)_{n=1}^\infty$ in $L_0^+(\mu)$. This means that for every $k \in \mathbb{N}$ there is a sequence $(f_{k,n})_{n=1}^\infty$ in $L_0^+(\mu)$ such that

$$\rho^p\left(\sum_{n=1}^\infty f_{k,n}\right) \geq k \text{ and } \sum_{n=1}^\infty \rho^p(f_{k,n}) \leq 2^{-k}.$$

Then $\sum_{(k,n) \in \mathbb{N}^2} \rho^p(f_{k,n}) \leq 1$, and also

$$\rho^p\left(\sum_{(k,n) \in \mathbb{N}^2} f_{k,n}\right) \geq \rho^p\left(\sum_{n=1}^\infty f_{k,n}\right) \geq k$$

for all $k \in \mathbb{N}$. That is, $\rho(\sum_{(k,n) \in \mathbb{N}^2} f_{k,n}) = \infty$. Hence ρ does not have the Riesz–Fischer p -property, as we wanted to prove.

Now suppose that (ii) holds for some $0 < p \leq 1$, and let \mathbb{X} be any nonzero quasi-Banach space. Let us see that $L_\rho(\mathbb{X})$ is a quasi-Banach space. By Lemma 3 (i), we already know that $L_\rho(\mathbb{X})$ is a quasi-normed space. To obtain the completeness of the space, notice that it suffices to prove that the series $\sum_{n=1}^\infty f_n$ converges in $L_\rho(\mathbb{X})$ for every sequence $(f_n)_{n=1}^\infty$ in $L_\rho(\mathbb{X})$ such that

$$\sum_{n=1}^\infty \kappa^{np} \rho^p(\|f_n\|) < \infty, \tag{2}$$

where κ is the modulus of concavity of \mathbb{X} . Using (ii) and Lemma 1 (ii), we obtain that $\sum_{n=1}^\infty \kappa^n \|f_n\|$ converges a.e. in Ω ; say it converges in $\Omega \setminus \mathcal{N}$ where $\mu(\mathcal{N}) = 0$. Set $g_n := f_n \chi_{\Omega \setminus \mathcal{N}}$. Obviously $\|g_n\| \leq \|f_n\|$, so $\rho(\|g_n\|) \leq \rho(\|f_n\|)$ for all $n \in \mathbb{N}$. Then (2) is also true if we put g_n instead of f_n .

For all $M, N \in \mathbb{N}$ with $M \geq N$, we have $\|\sum_{n=N}^M g_n\| \leq \sum_{n=N}^M \kappa^n \|g_n\|$. Since $\sum_{n=1}^\infty \kappa^n \|g_n(t)\|$ converges for all $t \in \Omega$, $(\sum_{n=1}^m g_n(t))_{m=1}^\infty$ is a Cauchy sequence in \mathbb{X} . Therefore $\sum_{n=1}^\infty g_n(t) =: f(t)$ converges for all $t \in \Omega$. Let us see that $\sum_{n=1}^\infty f_n$ converges to f in $L_\rho(\mathbb{X})$.

Notice that if a sequence $(x_n)_{n=1}^\infty$ converges to x in \mathbb{X} , since $\|x\| \leq \kappa \|x_n\| + \kappa \|x - x_n\|$, we have $\|x\| \leq \kappa \liminf_n \|x_n\|$. Recall that if two functions u, v in $L_0^+(\mu)$ are equal a.e., then $\rho(u) = \rho(v)$. Hence

$$\begin{aligned} \rho\left(\left\|f - \sum_{n=1}^m f_n\right\|\right) &= \rho\left(\left\|f - \sum_{n=1}^m g_n\right\|\right) = \rho\left(\left\|\sum_{n=m+1}^\infty g_n\right\|\right) \\ &\leq \rho\left(\kappa \liminf_{M \rightarrow \infty} \left\|\sum_{n=m+1}^M g_n\right\|\right) \leq \kappa \rho\left(\sum_{n=m+1}^\infty \kappa^n \|g_n\|\right) \\ &\leq \kappa \left(\sum_{n=m+1}^\infty \kappa^{np} \rho^p(\|g_n\|)\right)^{1/p} \xrightarrow{m \rightarrow \infty} 0. \end{aligned}$$

Therefore, $L_\rho(\mathbb{X})$ is a quasi-Banach space, as wanted.

Now suppose that $L_\rho(\mathbb{X})$ is a quasi-Banach space for some nonzero quasi-Banach space \mathbb{X} . Take a nonzero vector x in \mathbb{X} . Since obviously \mathbb{F} is isomorphic to $\{tx : t \in \mathbb{F}\}$, which is a closed subset of \mathbb{X} , it follows that L_ρ is a quasi-Banach space using Lemma 3 (iv). By the Aoki-Rolewicz theorem, L_ρ is p -convex for some $0 < p \leq 1$. Hence (iv) implies (iii).

Finally, assume that L_ρ is a p -convex (with constant C) quasi-Banach space for some $0 < p \leq 1$. Let $(f_n)_{n=1}^\infty$ be a sequence in $L_0^+(\mu)$ such that $\sum_{n=1}^\infty \rho^p(f_n) < \infty$. Since L_ρ is p -convex (with constant C), for all $M, N \in \mathbb{N}$ with $M \geq N$ we have

$$\rho \left(\sum_{n=N}^M f_n \right) \leq C^{1/p} \left(\sum_{n=N}^M \rho^p(f_n) \right)^{1/p}.$$

Therefore $(\sum_{n=1}^m f_n)_{m=1}^\infty$ is a Cauchy sequence in the quasi-Banach space $L_\rho(\mathbb{X})$, so it converges to a function f in $L_\rho(\mathbb{X})$. By Proposition 2, there is a subsequence $(\sum_{n=1}^{m_j} f_n)_{j=1}^\infty$ that converges to f a.e., say in $\Omega \setminus \mathcal{N}$ where $\mu(\mathcal{N}) = 0$. Since $(\sum_{n=1}^m f_n)_{m=1}^\infty$ is non-decreasing, it follows that it converges to f in $\Omega \setminus \mathcal{N}$. That is, $\sum_{n=1}^\infty f_n = f$ a.e., and therefore $\rho(\sum_{n=1}^\infty f_n) = \rho(f) < \infty$. Thus ρ has the Riesz–Fischer p -property, and we have proved that (iii) implies (i).

This completes the proof of the theorem. □

3.2 Absolute continuity and domination

Definition 9 Suppose that ρ is a function quasi-norm over a σ -finite measure space (Ω, Σ, μ) . We say that $f \in L_\rho^+$ is *absolutely continuous* with respect to ρ if

$$\lim_n \rho(f_n) = \rho(\lim_n f_n)$$

for every non-increasing sequence $(f_n)_{n=1}^\infty$ in $L_0^+(\mu)$ with $f_1 \leq f$. If the above holds only in the case when $\lim_n f_n = 0$, we say that f is *dominating*. We denote by L_ρ^a (resp. L_ρ^d) the set consisting of all $f \in L_0(\mu)$ such that $|f|$ is absolutely continuous (resp. dominating). We say that ρ is *absolutely continuous* (resp. *dominating*) if $L_\rho^a = L_\rho$ (resp. $L_\rho^d = L_\rho$). If $\chi_E \in L_\rho^a$ (resp. L_ρ^d) for every $E \in \Sigma(\mu)$, we say that ρ is *locally absolutely continuous* (resp. *locally dominating*).

Notice that domination is preserved under equivalence of function quasi-norms, but absolute continuity is not. Proposition 6 below yields that if the function quasi-norm is continuous (in the sense that $\lim_n \|x_n\|_\rho = \|x\|_\rho$ whenever $(x_n)_{n=1}^\infty$ and x in L_ρ satisfy $\lim_n \|x_n - x\|_\rho = 0$), then both concepts are equivalent. Notice that any function p -norm, $0 < p \leq 1$, is continuous. So, the existence of non-continuous function quasi-norms is a ‘pathology’ which only occurs in the non-locally convex setting. We must point out that, since it is by no means clear whether absolutely continuous norms are continuous, the terminology could be somewhat confusing. Notwithstanding, we prefer to use terminology similar to that it is customary within framework of function norms.

Proposition 6 *Let ρ be a function quasi-norm over a σ -finite measure space (Ω, Σ, μ) . Suppose that $f \in L_\rho^+$ is dominating. Then $\lim_n x_n = x$ in $L_\rho(\mathbb{X})$ for every quasi-Banach space \mathbb{X} and every sequence $(x_n)_{n=1}^\infty$ in $L_0(\mu, \mathbb{X})$ with $\lim_n x_n = x$ a.e. and $\|x_n\| \leq f$ a.e. for all $n \in \mathbb{N}$.*

Proof Let \mathcal{N} be a null set such that $\sup_n \|x_n(\omega)\| \leq f(\omega) < \infty$ and $\lim_n x_n(\omega) = x(\omega)$ for all $\omega \in \Omega \setminus \mathcal{N}$. Then $\|x(\omega)\| \leq \kappa f(\omega)$ for all $\omega \in \Omega \setminus \mathcal{N}$, where κ is the modulus of concavity of the quasi-norm $\|\cdot\|$. Set

$$f_n = \sup_{j \geq n} \|x_j - x\| \chi_{\Omega \setminus \mathcal{N}}, \quad n \in \mathbb{N}.$$

The sequence $(f_n)_{n=1}^\infty$ in $L_0^+(\mu)$ decreases to 0, and $\sup_n f_n \leq \kappa(\kappa + 1)f$. Consequently, $\lim_n \rho(f_n) = 0$. Since $\|x_j - x\|_\rho \leq \rho(f_n)$ whenever $j \geq n$ we are done. \square

Proposition 7 (cf. [6, Proposition 3.6]) *Let ρ be a function quasi-norm over a σ -finite measure space (Ω, Σ, μ) , and let f be a function in L_ρ^+ . Then, f is dominating if and only if*

$$\lim_n \rho(f\chi_{A_n}) = 0$$

whenever the sequence $(A_n)_{n=1}^\infty$ in Σ decreases to \emptyset .

Proof The direct implication is obvious. Conversely, suppose that $\rho(f\chi_{A_n}) \rightarrow 0$ whenever $(A_n)_{n=1}^\infty$ decreases to \emptyset . Let $(f_n)_{n=1}^\infty$ be a non-increasing sequence of functions in $L_0^+(\mu)$ such that $f_1 \leq f$ and $f_n \rightarrow 0$. Let us prove that $\rho(f_n) \rightarrow 0$. Let κ be the modulus of concavity of ρ , and fix $\varepsilon > 0$.

Assume first that $\mu(\Omega) < \infty$. Then $\rho(\chi_\Omega) < \infty$, and we can set

$$s = \frac{\varepsilon}{2\kappa\rho(\chi_\Omega)}.$$

For each $n \in \mathbb{N}$, let $B_n = \{f_n < s\} \subseteq \Omega$. It is a non-decreasing sequence in Σ whose union is Ω . Since $f_n \leq f\chi_{\Omega \setminus B_n} + s\chi_{B_n}$, we have

$$\rho(f_n) \leq \kappa\rho(f\chi_{\Omega \setminus B_n}) + \kappa s\rho(\chi_{B_n}) \leq \kappa\rho(f\chi_{\Omega \setminus B_n}) + \varepsilon/2 < \varepsilon$$

for n sufficiently large.

Now suppose that $\mu(\Omega) = \infty$. Let $(\Omega_m)_{m=1}^\infty$ be a non-decreasing sequence in $\Sigma(\mu)$ whose union is Ω . Take m such that $\kappa\rho(f\chi_{\Omega \setminus \Omega_m}) < \varepsilon/2$. Since $f_n \leq f_n\chi_{\Omega_m} + f\chi_{\Omega \setminus \Omega_m}$, using that $\mu(\Omega_m) < \infty$ and the previous case, we have

$$\rho(f_n) \leq \kappa\rho(f_n\chi_{\Omega_m}) + \kappa\rho(f\chi_{\Omega \setminus \Omega_m}) \leq \kappa\rho(f_n\chi_{\Omega_m}) + \varepsilon/2 < \varepsilon$$

for n sufficiently large. \square

Given a function quasi-norm ρ and a set $E \in \Sigma$ we define

$$\Phi[E, \rho](t) = \sup\{\rho(\chi_A) : A \in \Sigma, A \subseteq E, \mu(A) \leq t\},$$

and we set $\Phi[\rho] = \Phi[\Omega, \rho]$. Notice that the function $\Phi[E, \rho]$ is non-negative and non-decreasing. In particular, there exists the limit of $\Phi[E, \rho](t)$ when $t \rightarrow 0^+$.

Corollary 2 *A function quasi-norm ρ is locally dominating if and only if*

$$\lim_{t \rightarrow 0^+} \Phi[E, \rho](t) = 0$$

for every $E \in \Sigma(\mu)$.

Proof If $\lim_{t \rightarrow 0^+} \Phi[E, \rho](t) = 0$ for every $E \in \Sigma(\mu)$, using Proposition 7 we obtain that ρ is locally dominating. Now assume that $s := \lim_{t \rightarrow 0^+} \Phi[E, \rho](t) > 0$ for some $E \in \Sigma(\mu)$. Then there is a sequence $(A_n)_{n=1}^\infty$ of measurable subsets of E such that $\mu(A_n) \leq 1/2^n$ and $\rho(\chi_{A_n}) > s/2$ for all $n \in \mathbb{N}$. Set $B_n = \cup_{k=n}^\infty A_k$. The sequence $(B_n)_{n=1}^\infty$ decreases to a null set and $\rho(\chi_{B_n}) \geq s/2$ for all $n \in \mathbb{N}$, so χ_{B_1} is not dominating. Hence ρ is not locally dominating. \square

Definition 10 Let ρ be a function quasi-norm over a σ -finite measure space (Ω, Σ, μ) . We say $L \subseteq L_\rho$ is an *order ideal with respect to ρ* if it is an order ideal and it is closed in L_ρ .

Lemma 6 (cf. [6, Theorem 3.8]) *Let ρ be a function quasi-norm over a σ -finite measure space (Ω, Σ, μ) . Then L_ρ^d is an order ideal with respect to ρ .*

Proof It is straightforward that L_ρ^d is a subspace of $L_0(\mu)$. If a function f belongs to L_ρ^d , obviously \bar{f} also belongs to L_ρ^d . Let f and g be real-valued functions in L_ρ^d . Set $A = \{\omega \in \Omega : |f(\omega)| < |g(\omega)|\}$. Let $(h_n)_{n=1}^\infty$ be a sequence in $L_0^+(\mu)$ decreasing to 0 with $h_1 \leq \max\{|f|, |g|\}$. Since $|f|$ and $|g|$ are dominating, $h_1 \chi_A \leq |g|$, and $h_1 \chi_{\Omega \setminus A} \leq |f|$, we obtain that $\lim_n \rho(h_n \chi_A) = 0$ and $\lim_n \rho(h_n \chi_{\Omega \setminus A}) = 0$. Hence $\lim_n \rho(h_n) = 0$. Therefore $\max\{|f|, |g|\}$ is dominating. This implies that $|\max\{f, g\}|$ is also dominating, so L_ρ^d is an order ideal.

Now we prove that L_ρ^d is closed in L_ρ . Let $(f_j)_{j=1}^\infty$ be a sequence in L_ρ^d that converges in L_ρ to a function f . Let $(g_n)_{n=1}^\infty$ be a non-increasing sequence in $L_0^+(\mu)$ with $g_1 \leq |f|$ and $\lim_n g_n = 0$. Then $g_n \leq \min\{g_n, |f_j|\} + |f - f_j|$ for each $j \in \mathbb{N}$. Consequently, if κ is the modulus of concavity of ρ , we have

$$\rho(g_n) \leq \kappa \rho(\min\{g_n, |f_j|\}) + \kappa \rho(|f - f_j|).$$

Hence $\lim_n \rho(g_n) = 0$. So $|f|$ is dominating, as we wanted to prove. □

Definition 11 Let ρ be a function quasi-norm over a σ -finite measure space (Ω, Σ, μ) . We denote by L_ρ^b the closure of $\mathcal{S}(\mu)$ in L_ρ . We say that ρ is *minimal* if $L_\rho^b = L_\rho$.

Lemma 7 (cf. [6, Proposition 3.10]) *Let ρ be a function quasi-norm over a σ -finite measure space (Ω, Σ, μ) . Then L_ρ^b is an order ideal with respect to ρ . Moreover $L_\rho^{b,+}$ is the closure in L_ρ of*

$$\mathcal{C} = \{f \in L_0^+(\mu) : \|f\|_\infty < \infty, \mu_f(0) < \infty\}.$$

Proof It is obvious that L_ρ^b is an order ideal in $L_0(\mu)$, and it is closed in L_ρ by definition. Hence L_ρ^b is an order ideal with respect to ρ .

Let f be a function in \mathcal{C} , and set $E := \{0 < f < \infty\} \subseteq \Omega$. Since $\mu(E) < \infty$, we have $\rho(\chi_E) < \infty$. Fix $\varepsilon > 0$, and let $0 \leq g \leq f$ be a simple function such that $\|f - g\|_\infty < \varepsilon/\rho(\chi_E)$. Then

$$\rho(f - g) \leq \|f - g\|_\infty \rho(\chi_E) < \varepsilon.$$

This means that \mathcal{C} is contained in $L_\rho^{b,+}$. Therefore, the closure of \mathcal{C} in L_ρ is also contained in $L_\rho^{b,+}$. On the other hand, it is obvious that every non-negative simple function which is finite a.e. belongs to \mathcal{C} . So the second part of the statement follows. □

Proposition 8 (cf. [6, Theorem 3.11]) *For any function quasi-norm ρ over a σ -finite measure space (Ω, Σ, μ) we have $L_\rho^d \subseteq L_\rho^b$.*

Proof It is enough to prove that $L_\rho^{d,+} \subseteq L_\rho^{b,+}$. Let f be a function in $L_\rho^{d,+}$. Let $(A_n)_{n=1}^\infty$ be an increasing sequence in $\Sigma(\mu)$ whose union is $\{f > 0\} \subseteq \Omega$. Pick an increasing sequence $(f_j)_{j=1}^\infty$ of measurable positive simple functions with $\lim_n f_n = f$. We have $\lim_n \rho(f - f \chi_{A_n}) = 0$ and $\lim_j \rho(f \chi_{A_n} - f_j \chi_{A_n}) = 0$ for each $n \in \mathbb{N}$. Since $f_j \chi_{A_n} \in L_\rho^{b,+}$ for all $j, n \in \mathbb{N}$, we infer that $f \in L_\rho^{b,+}$. □

Corollary 3 *A function quasi-norm ρ is locally dominating if and only if $L_\rho^d = L_\rho^b$. Moreover if ρ is dominating, then ρ is minimal.*

Proof It is a straightforward consequence of Proposition 8 □

Since we could need to deal with non-continuous function quasi-norms, we give some results pointing to ensure that $\lim_n \|x_n\|_\rho = \|x\|_\rho$ under the assumption that $(x_n)_{n=1}^\infty$ converges to x .

Lemma 8 *Let ρ be a function quasi-norm over a σ -finite measure space (Ω, Σ, μ) with the Fatou property, and let $(f_n)_{n=1}^\infty$ be a sequence in $L_0^+(\mu)$. Then*

$$\rho(\liminf_n f_n) \leq \liminf_n \rho(f_n).$$

Proof Just apply Fatou property to $\inf_{k \geq n} f_k, n \in \mathbb{N}$. □

Lemma 9 *Let ρ be a function quasi-norm with the Fatou property and \mathbb{X} be a Banach space. If $x \in L_\rho(\mathbb{X})$ and $(x_n)_{n=1}^\infty \subseteq L_\rho(\mathbb{X})$ satisfy $\sup_n \|x_n\| \leq \|x\|$ and $\lim_n \rho(\|x_n - x\|) = 0$, then $\lim_n \rho(\|x_n\|) = \rho(\|x\|)$.*

Proof Since it is obvious that

$$\limsup_n \rho(\|x_n\|) \leq \rho(\|x\|),$$

we will see that $\rho(\|x\|) \leq \liminf_n \rho(\|x_n\|)$. Let $(y_n)_{n=1}^\infty$ be a subsequence of $(x_n)_{n=1}^\infty$ such that $\lim_n \rho(\|y_n\|) = \liminf_n \rho(\|x_n\|)$. Since $\lim_n \rho(\|x - y_n\|) = 0$, by Lemma 4 we have $\lim_n \rho(\|x\| - \|y_n\|) = 0$. Then Proposition 2 guarantees the existence of a subsequence $(z_n)_{n=1}^\infty$ of $(y_n)_{n=1}^\infty$ such that $\lim_n \|z_n\| = \|x\|$. Using Lemma 8 we obtain

$$\rho(\|x\|) = \rho(\lim_n \|z_n\|) \leq \liminf_n \rho(\|z_n\|) = \lim_n \rho(\|y_n\|),$$

as we wanted to prove. □

Lemma 10 *Let ρ be a function quasi-norm over a σ -finite measure space (Ω, Σ, μ) with the Fatou property, and let \mathbb{X} be a Banach space. If $x \in L_0(\mu, \mathbb{X})$ and $(x_n)_{n=1}^\infty \subseteq L_0(\mu, \mathbb{X})$ satisfy $\lim_n x_n = x$ a.e., and $\sup_n \|x_n\| \leq g$ for some $g \in L_\rho^{a,+}$, then $\lim_n \rho(\|x_n\|) = \rho(\|x\|)$.*

Proof Note that since $\lim_n x_n = x$ a.e. and \mathbb{X} is a Banach space, we have

$$\lim_n \|x_n\| = \|x\| \text{ a.e.}$$

Consider two particular cases. First, suppose that $\|x_n\| \leq \|x\|$ for all $n \in \mathbb{N}$. Obviously, $\limsup_n \rho(\|x_n\|) \leq \rho(\|x\|)$. Then, by Lemma 8, $\rho(\|x\|) \leq \liminf_n \rho(\|x_n\|)$. Second, suppose that $\|x_n\| \geq \|x\|$ for all $n \in \mathbb{N}$. Obviously, $\liminf_n \rho(\|x_n\|) \geq \rho(\|x\|)$. Set $g_n = \sup_{k \geq n} \|x_k\|$. Then $g \geq g_1$ and $(g_n)_{n=1}^\infty$ is non-increasing with $\lim_n g_n = \|x\|$ a.e. Using the absolute continuity of g , we have $\limsup_n \rho(\|x_n\|) \leq \lim \rho(g_n) = \rho(\|x\|)$. In the general case, set $g_n = \min\{\|x_n\|, \|x\|\}$ and $h_n = \max\{\|x_n\|, \|x\|\}$. Then both $(\rho(g_n))_{n=1}^\infty$ and $(\rho(h_n))_{n=1}^\infty$ converge to $\rho(\|x\|)$. Since $g_n \leq \|x_n\| \leq h_n$, the statement follows. □

Proposition 9 *Let ρ be a function quasi-norm over a σ -finite measure space (Ω, Σ, μ) with the Fatou property, and let \mathbb{X} be a Banach space. If $x \in L_0(\mu, \mathbb{X})$ and $(x_n)_{n=1}^\infty \subseteq L_0(\mu, \mathbb{X})$ satisfy $\lim_n \|x - x_n\|_\rho = 0$, and $\sup_n \|x_n\| \leq g$ for some $g \in L_\rho^{a,+}$, then $\lim_n \rho(\|x_n\|) = \rho(\|x\|)$.*

Proof It suffices to prove that any subsequence of $(x_n)_{n=1}^\infty$ has a further subsequence $(y_n)_{n=1}^\infty$ with $\lim_n \|y_n\|_\rho = \|x\|_\rho$. But this follows combining Proposition 2 with Lemma 10. □

3.3 The role of lattice convexity and Minkowski-type inequalities

Function spaces built from function quasi-norms have a lattice structure. Let ρ be a function quasi-norm over a σ -finite measure space (Ω, Σ, μ) . Given $0 < p \leq \infty$, we say that ρ is *lattice p -convex* (resp. *concave*) if L_ρ is. Equivalently, ρ is lattice p -convex (resp. concave) if and only if there is a constant C such that $G \leq CH$ (resp. $H \leq CG$) for every $n \in \mathbb{N}$ and $(f_j)_{j=1}^n$ in $L_0^+(\mu)$, where

$$G = \rho((\sum_{j=1}^n f_j^p)^{1/p}), \quad H = (\sum_{j=1}^n \rho^p(f_j))^{1/p}.$$

If the above holds for disjointly supported families, we say that ρ satisfies an *upper* (resp. *lower*) p -estimate.

If ρ is lattice p -convex, then it is \bar{p} -convex, where $\bar{p} = \min\{1, p\}$. The notions of 1-convexity and lattice 1-convexity are equivalent. This identification does not extend to $p < 1$ since there are function quasi-norms over \mathbb{N} which are lattice p -convex for no $p > 0$ (see [22]). Kalton [21] characterized quasi-Banach lattices (in particular, function quasi-norms) that are p -convex for some p as those that are L -convex. We say that a function quasi-norm is L -convex if there is $0 < \varepsilon < 1$ such that if f and $(f_j)_{j=1}^n$ in $L_0^+(\mu)$ satisfy

$$\max_{1 \leq j \leq n} f_j \leq f \quad \text{and} \quad \frac{1}{n} \sum_{j=1}^n f_j \geq (1 - \varepsilon)f,$$

then $\max_{1 \leq j \leq n} \rho(f_j) \geq \varepsilon \rho(f)$.

Given $0 < r < \infty$, the r -convexified quasi-norm $\rho^{(r)}$ is defined by

$$\rho^{(r)}(f) = \rho^{1/r}(f^r).$$

It is straightforward to check that $\rho^{(r)}$ is a function quasi-norm. If ρ has the Fatou (resp. weak Fatou) property, then $\rho^{(r)}$ does have. If ρ is p -convex (resp. concave), then $\rho^{(r)}$ is pr -convex (resp. concave). We set

$$L_\rho^{(r)} = L_{\rho^{(r)}}.$$

A question implicit in Sect. 3.2 is whether any p -convex function quasi-norm with the weak Fatou property is equivalent to a function p -norm with the Fatou property. For function norms the answer to this question is positive, and its proof relies on using the associated gauge ρ' given by

$$\rho'(f) = \sup \left\{ \int_\Omega fg \, d\mu : g \in L_0^+(\mu), \rho(g) \leq 1 \right\}.$$

In fact, we have the following analogue of [6, Theorem 2.2 of Chapter 1].

Lemma 11 *Let ρ be a function quasi-norm fulfilling (F.7). Then ρ' is a function norm with the Fatou property.*

Proof It is a routine checking. □

Theorem 1 (cf. [6] and [41, Theorem 112.2]) *Let ρ be a function norm with the weak Fatou property. Suppose that ρ satisfies (F.7). Then ρ'' is equivalent to ρ . Moreover, if ρ has the Fatou property, then $\rho'' = \rho$.*

In the non-locally convex setting, it is hopeless to try to obtain full information for ρ from the associated function norm ρ' . Nonetheless, the following is a partial positive answer to the aforementioned question.

Proposition 10 *Let $0 < p < \infty$ and let ρ be a function quasi-norm over a σ -finite measure space (Ω, Σ, μ) . Suppose that ρ is p -convex, has the weak Fatou property, and that for every $E \in \Sigma(\mu)$ there is a constant C_E such that $\int_E f^p d\mu \leq C_E \rho(f)$ for all $f \in L_0^+(\mu)$. Then ρ is equivalent to a function p -norm with the Fatou property. In fact, there is $G \subset L_0^+(\mu)$ such that ρ is equivalent to the function quasi-norm λ given by*

$$\lambda(f) = \sup_{g \in G} \left(\int_{\Omega} f^p g d\mu \right)^{1/p}.$$

Proof The function quasi-norm $\rho^{(1/p)}$ is 1-convex and, then, equivalent to a function norm σ . The properties of ρ yields that σ satisfies (F.7) and has the weak Fatou property. By Theorem 1, σ is equivalent to the function norm σ'' . Consequently, ρ is equivalent to the function quasi-norm $\lambda = \sigma''^{(p)}$. Hence, the result holds with $G = \{g \in L_0^+(\mu) : \sigma'(g) \leq 1\}$. □

Definition 12 Let ρ be a function quasi-norm over a σ -finite measure space (Ω, Σ, μ) , and let $(\Theta, \mathcal{T}, \nu)$ be another σ -finite measure space. Given $f \in L_0^+(\mu \otimes \nu)$ and $g \in L_0^+(\nu \otimes \mu)$ we set

$$\begin{aligned} \rho[1, f]: \Theta &\rightarrow [0, \infty], & \rho[1, f](\theta) &= \rho(f(\cdot, \theta)); \text{ and} \\ \rho[2, g]: \Theta &\rightarrow [0, \infty], & \rho[2, g](\theta) &= \rho(g(\theta, \cdot)). \end{aligned}$$

The following proposition is related to the results in [25, 39].

Proposition 11 *Let ρ be a locally absolutely continuous function quasi-norm over a σ -finite measure space (Ω, Σ, μ) with the Fatou property. Let $(\Theta, \mathcal{T}, \nu)$ be another σ -finite measure space. Let $f \in L_0^+(\mu \otimes \nu)$ and $g \in L_0^+(\nu \otimes \mu)$. Then $\rho[1, f]$ and $\rho[2, g]$ are measurable functions.*

Proof It suffices to prove the result for f . The Fatou property yields that if the result holds for a non-decreasing sequence $(f_n)_{n=1}^\infty$, then it also holds for $\lim_n f_n$. Consequently, we can suppose that $\mu(\Omega) < \infty$ and that f is a measurable simple function. Given a measurable simple positive function f we denote by \mathcal{M}_f the set consisting of all E in the product σ -algebra $\Sigma \otimes \mathcal{T}$ such that the result holds for $f + t\chi_E$ for every $t \geq 0$. The absolute continuity and the Fatou property yields that \mathcal{M}_f is a monotone class for any measurable simple function f . Therefore, if \mathcal{R} denotes the algebra consisting of all finite disjoint unions of measurable rectangles, the monotone class theorem yields that $\mathcal{R} \subseteq \mathcal{M}_f$ implies $\Sigma \otimes \mathcal{T} \subseteq \mathcal{M}_f$. Let \mathcal{C}_r denote the cone consisting of all positive functions measurable with respect to \mathcal{R} . Given $n \in \mathbb{N}$, let $\mathcal{C}[n]$ be the cone consisting of all measurable non-negative functions which take at most $n - 1$ different positive values. It is straightforward to check that the result holds for all functions in \mathcal{C}_r (which is clearly equal to $\mathcal{C}_r + \mathcal{C}[1]$). Suppose that the result holds for all functions in $\mathcal{C}_r + \mathcal{C}[n]$. Then $\mathcal{R} \subseteq \mathcal{M}_f$ for all $f \in \mathcal{C}_r + \mathcal{C}[n]$. Consequently, $\Sigma \otimes \mathcal{T} \subseteq \mathcal{M}_f$ for all $f \in \mathcal{C}_r + \mathcal{C}[n]$. In other words, the result holds for all functions in $\mathcal{C}_r + \mathcal{C}[n + 1]$. By induction, the result holds for every $f \in \mathcal{C} := \bigcup_{n=1}^\infty \mathcal{C}_r + \mathcal{C}[n]$. Since \mathcal{C} is the cone consisting of all measurable simple non-negative functions, we are done. □

Proposition 11 allows us to iteratively apply function quasi-noms to measurable functions defined on product spaces. A *Minkowski-type inequality* is an inequality that compares the gauges that appear when iterating in different ways.

Definition 13 Let ρ and λ be locally absolutely continuous function quasi-norms with the Fatou property over σ -finite measure spaces (Ω, Σ, μ) and $(\Theta, \mathcal{T}, \nu)$ respectively. Given $f \in L_0^+(\mu \otimes \nu)$ we set

$$(\rho, \lambda)[1, 2](f) = \rho(\lambda[2, f]), \quad (\lambda, \rho)[2, 1](f) = \lambda(\rho[1, f]).$$

We say that the pair (ρ, λ) has the *Minkowski’s integral inequality* (MII for short) *property* if there is a constant C such that

$$(\rho, \lambda)[1, 2](f) \leq C(\lambda, \rho)[2, 1](f)$$

for all $f \in L_0^+(\mu \otimes \nu)$.

The following result is obtained from the corresponding one for function norms [32]. We do not know whether a direct proof which circumvent using lattice convexity is possible.

Theorem 2 *Let ρ and λ be locally absolutely continuous L -convex function quasi-norms with the Fatou property. Then (ρ, λ) has the MII property if and only if there is $0 < p \leq \infty$ such that λ is lattice p -convex and ρ is lattice p -concave.*

Proof Pick $0 < s < \infty$ such that $\rho^{(s)}$ and $\lambda^{(s)}$ are 1-convex. Since

$$(\rho^{(s)}, \lambda^{(s)})[1, 2](f) = ((\rho, \lambda)[1, 2](f^s))^{1/s},$$

(ρ, λ) has the MII property if and only if $(\rho^{(s)}, \lambda^{(s)})$ does have. It turn, by [32, Theorems 2.3 and 2.5], $(\rho^{(s)}, \lambda^{(s)})$ has the MII property if and only if there is $q \in (0, \infty]$ such that $\lambda^{(s)}$ is lattice q -convex and $\rho^{(s)}$ is lattice q -concave. This latter condition is equivalent to the existence of $p \in (0, \infty]$ (related with q by $q = sp$) as desired. \square

Given $0 < p < \infty$ and a σ -finite measure space (Ω, Σ, μ) , the Lebesgue space $L_p(\mu)$ is absolutely continuous and lattice p -convex. Moreover, if μ is infinite-dimensional, then $L_p(\mu)$ is not lattice q -concave for any $q < p$. Consequently, we have the following.

Proposition 12 *Let $0 < p < \infty$ and ρ be a locally absolutely continuous L -convex function quasi-norm over an infinite-dimensional σ -finite measure space. Given another σ -finite measure space (Ω, Σ, μ) such that $L_0(\mu)$ is infinite-dimensional, the pair $(\rho, L_p(\mu))$ has the MII property if and only if ρ is lattice p -concave.*

Another Köthe space of interest for us is the weak Lorentz space $L_{1,\infty}(\mu)$ defined from the function quasi-norm

$$f \mapsto \sup_{s>0} s \mu_f(s) = \sup_{s>0} s \mu\{\omega \in \Omega : f(\omega) \geq s\}, \quad f \in L_0^+(\mu).$$

We will denote by $\|\cdot\|_{1,\infty}$ the quasi-norm in $L_{1,\infty}(\mu)$. We infer from the properties of the distribution function that $L_{1,\infty}(\mu)$ is continuous, has the Fatou property, and it is locally dominating. Kalton [20] proved that then $L_{1,\infty}([0, 1])$ is lattice p -convex for any $p < 1$. We emphasize that the milestone paper [21] allows to achieve this convexity result regardless the σ -finite measure space (Ω, Σ, μ) . In fact, given $0 < p < 1$, the $p^{-1/2}$ -convexified of $L_{1,\infty}(\mu)$, namely the Lorentz space $L_{p^{-1/2},\infty}$, is locally convex [18]. Therefore, by [21, Theorem 2.2], $L_{p^{-1/2},\infty}$ is lattice $p^{1/2}$ -convex. Consequently, $L_{1,\infty}(\mu)$ is lattice p -convex. Since $L_{1,\infty}(\mu)$ is not locally convex unless finite-dimensional [18], we have the following.

Theorem 3 *Let ρ be a locally absolutely continuous L -convex function quasi-norm, and let (Ω, Σ, μ) an infinite-dimensional σ -finite measure space. Then the pair $(\rho, L_{1,\infty}(\mu))$ has the MII property if and only if ρ is p -concave for some $p < 1$.*

3.4 Conditional expectation in quasi-Banach function spaces

Given a sub- σ -algebra $\Sigma_0 \subseteq \Sigma$, we denote by $L_0^+(\mu, \Sigma_0)$ the set consisting of all non-negative Σ_0 -measurable functions. Given $f \in L_0^+(\mu)$ there is a unique $g \in L_0^+(\mu, \Sigma_0)$ such that $\int_A f \, d\mu = \int_A g \, d\mu$ for all $A \in \Sigma_0$. We say that g is the *conditional expectation* of f with respect to Σ_0 , and we denote $\mathbb{E}(f, \Sigma_0) := g$.

Definition 14 Let ρ be a function quasi-norm over a σ -finite measure space (Ω, Σ, μ) . We say that ρ is *leveling* if there is a constant C such that $\rho(\mathbb{E}(f, \Sigma_0)) \leq C\rho(f)$ for every finite sub- σ -algebra Σ_0 and every $f \in L_0^+(\mu)$.

This terminology follows that used in [12]. We remark that Ellis and Halperin imposed leveling function norms to satisfy the above definition with $C = 1$. Not imposing conditional expectations to be contractive turns the notion stable under equivalence.

Given a function quasi-norm ρ , a sub- σ -algebra Σ_0 , and a quasi-Banach space \mathbb{X} , we denote by $L_\rho(\Sigma_0, \mathbb{X})$ the space consisting of all Σ_0 -measurable functions in $L_\rho(\mathbb{X})$. Note that, if $\rho|_{\Sigma_0}$ is the restriction of ρ to Σ_0 , then $L_\rho(\Sigma_0, \mathbb{X}) = L_{\rho|_{\Sigma_0}}(\mathbb{X})$. For further reference, we write down an elementary result.

Lemma 12 *Let ρ be a leveling function quasi-norm over a σ -finite measure space (Ω, Σ, μ) . Then there is a constant C such that for any finite sub- σ -algebra Σ_0 there is positive projection $T: L_\rho \rightarrow L_\rho(\Sigma_0)$ such that $\|T\| \leq C$ and $\int_A f \, d\mu = \int_A T(f) \, d\mu$ whenever $f \geq 0$ or $\int_A |f| \, d\mu < \infty$.*

Definition 15 If ρ, Σ_0 and T are as in Lemma 12, we denote $\mathbb{E}[\rho, \Sigma_0] := T$.

Lemma 13 *Leveling function quasi-norms satisfy (F.7).*

Proof Suppose that ρ is a leveling function quasi-norm over a σ -finite measure space (Ω, Σ, μ) . Given $A \in \Sigma(\mu)$ with $\mu(A) > 0$, let Σ_0 be the smallest σ -algebra containing A . For all $f \in L_0^+(\mu)$ we have

$$\int_A f \, d\mu \leq \frac{\mu(A)}{\rho(\chi_A)} \rho(\mathbb{E}(f, \Sigma_0)) \leq C \frac{\mu(A)}{\rho(\chi_A)} \rho(f).$$

□

It is known that, if $q \geq 1$, $L_q(\mu)$ has the conditional expectation property. Locally convex Lorentz and Orlicz spaces do have. More generally, we have the following. Recall that a measure space is said to be *resonant* if either is non-atomic or it consists of equi-measurable atoms.

Theorem 4 *Let ρ be a rearrangement invariant function norm over a resonant measure space. If ρ satisfies (F.7), then it is leveling.*

Proof By Calderón-Mitjagin Theorem (see [8, 29], and also [6, Theorem 2.2]), L_ρ is an interpolation space between L_1 and L_∞ . Since both L_1 and L_∞ are leveling, the result follows by interpolation. □

3.5 Function quasi-norms over \mathbb{N}

Suppose that ρ is a function quasi-norm over \mathbb{N} endowed with the counting measure. In this particular case, ρ is locally dominating, and the space of integrable simple functions is in the space c_{00} consisting of all eventually null sequences. Concerning the density of c_{00} in L_ρ we have the following.

Proposition 13 *Let ρ be a function quasi-norm over \mathbb{N} . Then ρ is not minimal if and only if ℓ_∞ is a subspace of L_ρ , in which case L_ρ has block basic sequence equivalent to the unit vector system of ℓ_∞ .*

Before tackling the proof of Proposition 13 we give an auxiliary lemma that will be used a couple of times.

Lemma 14 *Let ρ be a function quasi-norm over \mathbb{N} and let $(a_n)_{n=1}^\infty$ be a sequence in L_ρ . Then $(a_n)_{n=1}^\infty$ does not belong to L_ρ^b if and only there is an increasing sequence $(m_k)_{k=1}^\infty$ of non-negative integers such that*

$$\inf_{k \in \mathbb{N}} \rho((|a_n|)_{n=1+m_{2k-1}}^{m_{2k}}) > 0.$$

Proof Use that $(a_n)_{n=1}^\infty \in L_\rho \setminus L_\rho^b$ if and only if the series $\sum_{n=1}^\infty a_n e_n$ does not converge. \square

Proof of Proposition 13 Assume that $L_\rho^b \neq L_\rho$. By Lemma 14, there is $(a_n)_{n=1}^\infty$ in $[0, \infty)^\mathbb{N}$ such that, if

$$x_k = \sum_{n=1+m_{2k-1}}^{m_{2k}} a_n e_n, \quad k \in \mathbb{N},$$

then $\inf_k \|x_k\|_\rho > 0$ and $\sup_m \|\sum_{k=1}^m x_k\|_\rho < \infty$. So, $(x_k)_{k=1}^\infty$ is a block basic sequence as desired. \square

Corollary 4 *Let ρ be a function quasi-norm over \mathbb{N} . If ρ satisfies a lower p -estimate for some $p < \infty$, then ρ is minimal and L -convex.*

Proof Our assumptions yields that ℓ_∞ is not finitely represented in L_ρ by means of block basic sequences. Then, result follows from Proposition 13 and [21, Theorem 4.1]. \square

Notice that function quasi-norms over \mathbb{N} are closely related to unconditional bases. In fact, if ρ is a function quasi-norm over \mathbb{N} , then the unit vector system $(e_n)_{n=1}^\infty$ is an unconditional basis of L_ρ^b . Reciprocally, if $(x_n)_{n=1}^\infty$ is an unconditional basis of a quasi-Banach space \mathbb{X} , then the mapping

$$\rho((a_n)_{n=1}^\infty) = \sup \left\{ \left\| \sum_{n=1}^\infty b_n x_n \right\| : (b_n)_{n=1}^\infty \in c_{00}, \forall n \in \mathbb{N} |b_n| \leq |a_n| \right\}$$

defines a function quasi-norm over \mathbb{N} , and the linear map given by $x_n \mapsto e_n$ extends to an isomorphism from \mathbb{X} onto L_ρ^b .

4 The galb of a quasi-Banach space

In this section, we deal with function quasi-norms associated with galbs of quasi-Banach spaces.

Definition 16 A function quasi-norm over \mathbb{N} is said to be *symmetric* (or *rearrangement invariant*) if $\rho(f) = \rho(g)$ whenever $g = (b_n)_{n=1}^\infty$ is a rearrangement of $f = (a_n)_{n=1}^\infty$, i.e., there is a permutation π of \mathbb{N} such that $b_n = a_{\pi(n)}$ for all $n \in \mathbb{N}$.

The symmetry of ρ allows us to safely define $\rho(f)$ for any countable family of non-negative scalars $f = (a_j)_{j \in J}$. In the language of bases, if ρ is a symmetric function-quasi-norm, then the unit vector system is a 1-symmetric basis of L_ρ^b .

Definition 17 Given a quasi-Banach space \mathbb{X} and a sequence $f = (a_n)_{n=1}^\infty$ in $[0, \infty]^\mathbb{N}$ we define

$$\lambda_{\mathbb{X}}(f) = \sup \left\{ \left\| \sum_{n=1}^N a_n x_n \right\| : N \in \mathbb{N}, \|x_n\| \leq 1 \right\}$$

if $a_n < \infty$ for all $n \in \mathbb{N}$, and $\lambda_{\mathbb{X}}(f) = \infty$ otherwise.

Proposition 14 Let \mathbb{X} be a quasi-Banach space. Then $\lambda_{\mathbb{X}}$ is a symmetric function quasi-norm with modulus of concavity at most that of \mathbb{X} . Moreover,

- (i) $\lambda_{\mathbb{X}}$ is locally absolutely continuous.
- (ii) $\lambda_{\mathbb{X}}$ has the Fatou property.
- (iii) If \mathbb{Y} is a subspace of \mathbb{X} , then $\lambda_{\mathbb{X}}$ dominates $\lambda_{\mathbb{Y}}$.
- (iv) If \mathbb{X} and \mathbb{Y} are isomorphic, then $\lambda_{\mathbb{X}}$ and $\rho_{\mathbb{Y}}$ are equivalent.
- (v) $(\lambda_{\mathbb{X}}, \lambda_{\mathbb{X}})[1, 2]$ dominates $\lambda_{\mathbb{X}}$ (regarded as a function quasi-norm over \mathbb{N}^2).
- (vi) If \mathbb{X} is a p -Banach space, $0 < p \leq 1$, then $\lambda_{\mathbb{X}}$ is a function p -norm.
- (vii) If \mathbb{X} a p -convex quasi-Banach lattice, $0 < p \leq 1$, then $\lambda_{\mathbb{X}}$ is lattice p -convex.

Proof We will prove (vii), and we will leave the other assertions, which are reformulations of results from [38], as an exercise for the reader. Let \mathbb{X} be a p -convex quasi-Banach lattice. Recall that the lattice structure of the space allows to define the absolute value $|x|$ of any vector $x \in \mathbb{X}$ (cf. [24, Chapter 1]). Notice that ℓ_1 is a p -convex lattice, that is, we have

$$\sum_{n=1}^\infty \left(\sum_{j=1}^J |a_{n,j}|^p \right)^{1/p} \leq \left(\sum_{j \in J} \left(\sum_{n=1}^\infty |a_{n,j}| \right)^p \right)^{1/p}, \quad a_{n,j} \in \mathbb{F}.$$

Hence, the lattice defined by the quasi-norm

$$g = (x_n)_{n=1}^\infty \mapsto \left\| \sum_{n=1}^\infty |x_n| \right\|, \quad g \in \mathbb{X}^\mathbb{N},$$

is p -convex. Let C denote its p -convexity constant. Let $f_j = (a_{j,n})_{n=1}^\infty \in [0, \infty)^\mathbb{N}$, $1 \leq j \leq J$. Given $(x_n)_{n=1}^N \in B_{\mathbb{X}}^N$ we have

$$\begin{aligned} \left\| \sum_{n=1}^N \left(\sum_{j=1}^J a_{j,n}^p \right)^{1/p} x_n \right\| &\leq \left\| \sum_{n=1}^N \left(\sum_{j=1}^J (a_{j,n} |x_n|)^p \right)^{1/p} \right\| \\ &\leq C \left(\sum_{j=1}^J \left\| \sum_{n=1}^N a_{j,n} |x_n| \right\|^p \right)^{1/p} \\ &\leq C \left(\sum_{j=1}^J \lambda_{\mathbb{X}}^p(f_j) \right)^{1/p}. \end{aligned}$$

Consequently, $\lambda_{\mathbb{X}}(\sum_{j=1}^J |f_j|^p)^{1/p} \leq C(\sum_{j=1}^J \lambda_{\mathbb{X}}^p(f_j))^{1/p}$. □

Definition 18 Let \mathbb{X} be a quasi-Banach space. We denote $\mathcal{G}(\mathbb{X}) = \mathbf{L}_{\lambda_{\mathbb{X}}}$, and we say that $\mathcal{G}(\mathbb{X})$ is the *galb* of \mathbb{X} . The positive cone of $\mathcal{G}(\mathbb{X})$ will be denoted by $\mathcal{G}^+(\mathbb{X})$, and $\mathcal{G}_b(\mathbb{X})$ stands for the closure of c_{00} in $\mathcal{G}(\mathbb{X})$.

Roughly speaking, it could be said that the galb of a space is a measure of its convexity. The notion of galb was introduced and developed by Turpin, within the more general setting of “espaces vectoriels à convergence”, in a series of papers [36, 37] and a monograph [38]. In this section, we restrict ourselves to galbs of locally bounded spaces and touch only a few aspects of the theory and summarize without proofs the properties that are more relevant to our work.

Proposition 15 (see [38]) *Let \mathbb{X} be a quasi-Banach space. Then $\mathcal{G}(\mathbb{X}) \subseteq \ell_1$, and $\mathcal{G}(\mathbb{X}) = \ell_1$ if and only if \mathbb{X} is locally convex.*

Proposition 16 (see [38]) *Let \mathbb{X} be a quasi-Banach space. Then $\mathcal{G}(\mathcal{G}(\mathbb{X})) = \mathcal{G}(\mathbb{X})$.*

Proposition 17 (see [38]) *Let \mathbb{X} be a quasi-Banach space and $0 < p \leq 1$. Then \mathbb{X} is p -convex if and only if $\ell_p \subseteq \mathcal{G}(\mathbb{X})$.*

Proposition 18 (see [36]) *Let \mathbb{X} be a quasi-Banach space. Then the mapping*

$$B : \mathcal{G}(\mathbb{X}) \times c_0(\mathbb{X}) \rightarrow \mathbb{X}, \quad ((a_n)_{n=1}^\infty, (x_n)_{n=1}^\infty) \mapsto \sum_{n=1}^\infty a_n x_n$$

is well-defined, and defines a bounded bilinear map.

It is natural to wonder whether the map B defined as in Proposition 18 can be extended to a continuous bilinear map defined on $\mathcal{G}(\mathbb{X}) \times \ell_\infty(\mathbb{X})$. In fact, the authors of [23], perhaps taking for granted that the answer to this question is positive, defined a sequence $(a_n)_{n=1}^\infty$ to be in the galb of \mathbb{X} if $\sum_{n=1}^\infty a_n x_n$ converges for every bounded sequence $(x_n)_{n=1}^\infty$. If we come to think of it, we obtain the following.

Lemma 15 *Let \mathbb{X} be a quasi-Banach space and let $f = (a_n)_{n=1}^\infty \in \mathbb{F}^\mathbb{N}$. Then, $f \in \mathcal{G}_b(\mathbb{X})$ if and only if $\sum_{n=1}^\infty a_n x_n$ converges for every bounded sequence $(x_n)_{n=1}^\infty$ in \mathbb{X} .*

Proof Let G denote the set consisting of all sequences $f = (a_n)_{n=1}^\infty \in \mathbb{F}^\mathbb{N}$ such that $\sum_{n=1}^\infty a_n x_n$ converges for every bounded sequence $(x_n)_{n=1}^\infty$ in \mathbb{X} . It is routine to check that G is a closed subspace of $\mathcal{G}(\mathbb{X})$ which contains c_{00} . Consequently, $\mathcal{G}_b(\mathbb{X}) \subseteq G$. Assume that $f = (a_n)_{n=1}^\infty \in \mathcal{G}(\mathbb{X}) \setminus \mathcal{G}_b(\mathbb{X})$. Then, by Lemma 14, there are $\delta > 0$ and an increasing sequence $(m_k)_{k=1}^\infty$ of non-negative integers such that $\rho((|a_n|)_{n=1+m_{2k-1}}^{m_{2k}}) > \delta$ for all $k \in \mathbb{N}$. Consequently, there is $(x_n)_{n=1}^\infty$ in the unit ball of $\ell_\infty(\mathbb{X})$ such that

$$\left\| \sum_{n=1+m_{2k-1}}^{m_{2k}} a_n x_n \right\| \geq \delta, \quad k \in \mathbb{N}.$$

We infer that $\sum_{n=1}^\infty a_n x_n$ does not converge. □

Corollary 5 *Let \mathbb{X} be a quasi-Banach space. Then the mapping*

$$B': \mathcal{G}_b(\mathbb{X}) \times \ell_\infty(\mathbb{X}) \rightarrow \mathbb{X}, \quad ((a_n)_{n=1}^\infty, (x_n)_{n=1}^\infty) \mapsto \sum_{n=1}^\infty a_n x_n$$

is well-defined, and defines a continuous bilinear map. Moreover, if $\mathcal{G}_b(\mathbb{X}) \subsetneq G \subseteq \mathcal{G}(\mathbb{X})$, then B' can not be extended to a continuous bilinear map defined on $G \times \ell_\infty(\mathbb{X})$.

Proof It follows from Lemma 15 and, alike the proof of Proposition 18, the Open Mapping Theorem. □

In light of Corollary 5, the following question arise.

Question 1 *Is $\mathcal{G}(\mathbb{X})$ minimal for any quasi-Banach space \mathbb{X} ?*

Corollary 4 alerts us of the connection between Question 1 and the existence of lower estimates for $\lambda_{\mathbb{X}}$. Lattice concavity also plays a key role when studying galbs of vector-valued spaces.

Definition 19 We say that a symmetric function quasi-norm λ over \mathbb{N} galbs a quasi-Banach space \mathbb{X} if λ dominates $\lambda_{\mathbb{X}}$, i.e., $L_\lambda \subseteq \mathcal{G}(\mathbb{X})$. We say that λ galbs a function quasi-norm ρ if it galbs L_ρ . If λ galbs itself, we say that λ is *self-galbed*.

Remark 1 Given $0 < p \leq 1$, the function quasi-norm defining ℓ_p is self-galbed. More generally, $\lambda_{\mathbb{X}}$ is self-galbed for any quasi-Banach space \mathbb{X} (see Proposition 16).

Proposition 19 *Let ρ and λ be locally absolutely continuous L -convex function quasi-norms with the Fatou property. Suppose that λ galbs a quasi-Banach space \mathbb{X} . If there is $0 < p < \infty$ such that λ is p -concave and ρ is p -convex, then λ galbs $L_\rho(\mathbb{X})$.*

Proof By Theorem 2, the pair (λ, ρ) has the MII property for some constant C . Since λ galbs \mathbb{X} , there is a constant $K > 0$ such that λ K -dominates $\lambda_{\mathbb{X}}$. Therefore, if $(a_n)_{n=1}^\infty$ is a sequence in L_λ , and f_1, \dots, f_N belong the unit ball of $L_\rho(\mathbb{X})$, we have

$$\begin{aligned} \rho \left(\left\| \sum_{n=1}^N a_n f_n \right\| \right) &\leq \rho \left(\lambda_{\mathbb{X}} \left((a_n \|f_n\|)_{n=1}^N \right) \right) \\ &\leq K \rho \left(\lambda \left((a_n \|f_n\|)_{n=1}^N \right) \right) \\ &\leq CK \lambda \left(\rho \left((a_n \|f_n\|)_{n=1}^N \right) \right) \\ &\leq CK \lambda \left((a_n)_{n=1}^N \right) \\ &\leq CK \lambda((a_n)_{n=1}^\infty). \end{aligned}$$

Hence $(a_n)_{n=1}^\infty$ belongs the galb of $L_\rho(\mathbb{X})$. □

Proposition 19 gives, in particular, that if λ is a 1-concave function quasi-norm which galbs \mathbb{X} , then it galbs $L_1(\mu, \mathbb{X})$. As we plan to develop an integral for functions belonging to a suitable subspace of $L_1(\mu, \mathbb{X})$, the following question arises.

Question 2 Is $\mathcal{G}(\mathbb{X})$ 1-concave for any quasi-Banach space \mathbb{X} ?

Note that a positive answer to Question 2 would yield a positive answer to Question 1. To properly understand Question 2, we must go over the state-of-the-art of the theory galbs.

We point out that all known examples suggest a positive answer to Question 2. Galbs of Lorentz spaces were explored through several papers [9, 10, 33–35] within the study of convolution operators, and all computed galbs occur to be Orlicz sequence spaces modeled after a concave Orlicz function. Also, Turpin [38] proved that the galb of any locally bounded Orlicz space is an Orlicz sequence space modeled after a concave Orlicz function. Recall that an *Orlicz function* is a non-null left-continuous non-decreasing function $\varphi : [0, \infty) \rightarrow [0, \infty)$ such that $\lim_{t \rightarrow 0^+} \varphi(t) = 0$. Given an Orlicz function φ , with the convention that $\varphi(\infty) = \infty$, the gauge

$$f = (a_n)_{n=1}^\infty \mapsto \lambda_\varphi(f) = \inf \left\{ t > 0 : \sum_{n=1}^\infty \varphi \left(\frac{a_n}{t} \right) \leq 1 \right\}, \quad f \in [0, \infty]^\mathbb{N}$$

is a function quasi-norm if and only if

$$\lim_{t \rightarrow 0^+} \sup_{u \in (0,1]} \frac{\varphi(tu)}{\varphi(u)} = 0. \tag{3}$$

(see [38]), in which case λ_φ has the Fatou property. If (3) holds, the Orlicz sequence space ℓ_φ is the Köthe space associated with λ_φ . If φ is 1-concave, then $\mathcal{G}(\ell_\varphi) = \ell_\varphi$ (see [38]). We notice that convexifying an Orlicz space yields another Orlicz space. Namely, for every $0 < r < \infty$ we have $\lambda_\varphi^{(r)} = \lambda_{\varphi^{(r)}}$, where the Orlicz function $\varphi^{(r)}$ is defined by $\varphi^{(r)}(t) = \varphi(t^r)$ for all $t > 0$.

A function φ is said to be r -concave, $0 < r < \infty$, if $\varphi^{(1/r)}$ is concave.

Proposition 20 *Let φ be a r -concave Orlicz function, $0 < r < \infty$, fulfilling (3). Then λ_φ is lattice r -concave.*

Proof By a standard convexification technique we can suppose that $r = 1$. Let $(f_j)_{j=1}^J$ be a finite family consisting of non-negative sequences. We will prove that

$$H := \sum_{j=1}^J \lambda_\varphi(f_j) \leq G := \lambda_\varphi \left(\sum_{j=1}^J f_j \right).$$

To that end, it suffices to prove that if $G < \infty$ and $0 < t < H$, then, $t < G$. Assume without loss of generality that $\lambda_\varphi(f_j) > 0$ for all j . Then, pick $(t_j)_{j=1}^J$ such that $\sum_{j=1}^J t_j = t$ and $0 < t_j < \rho(f_j)$. Then, if $f_j = (a_{j,n})_{n=1}^\infty$, $a_{j,n} < \infty$ for all $n \in \mathbb{N}$, and

$$\sum_{n=1}^\infty \varphi \left(\frac{a_{j,n}}{t_j} \right) > 1, \quad j = 1, \dots, J.$$

Consequently,

$$\sum_{n=1}^{\infty} \varphi \left(\frac{\sum_{j=1}^J a_{j,n}}{t} \right) = \sum_{n=1}^{\infty} \varphi \left(\sum_{j=1}^J \frac{t_j}{t} \frac{a_{j,n}}{t_j} \right) \geq \sum_{n=1}^{\infty} \sum_{j=1}^J \frac{t_j}{t} \varphi \left(\frac{a_{j,n}}{t_j} \right) > 1.$$

Therefore, $t < G$. □

The lattice convexity of spaces of galbs is also quite unknown. It is known that if the gauge λ_φ associated with an Orlicz function φ is function quasi-norm, so that ℓ_φ is a quasi-Banach lattice, then there is $p > 0$ such that

$$\sup_{0 < u, t \leq 1} \frac{\varphi(tu)}{u^p \varphi(t)} < \infty \tag{4}$$

(see [19, Proposition 4.2]). Moreover, if (4) holds for a given p , then ℓ_φ is a p -convex lattice. Therefore, ℓ_φ is L -convex. The behavior of general spaces of galbs is unknown.

Question 3 Is $\lambda_{\mathbb{X}}$ an L -convex function quasi-norm for any quasi-Banach space \mathbb{X} ?

Note that Proposition 14 (vii) partially solves in the positive Question 3.

5 Topological tensor products built by means of symmetric function quasi-norms over \mathbb{N}

Definition 20 Let \mathbb{X} and \mathbb{Y} be quasi-Banach spaces and λ be a symmetric minimal function quasi-norm over \mathbb{N} with the Fatou property. We define

$$\| \cdot \|_{\mathbb{X} \otimes_\lambda \mathbb{Y}} : \mathbb{X} \otimes \mathbb{Y} \rightarrow [0, \infty)$$

by

$$\| \tau \|_{\mathbb{X} \otimes_\lambda \mathbb{Y}} = \inf \left\{ \lambda \left((\|x_j\| \|y_j\|)_{j=1}^n \right) : \tau = \sum_{j=1}^n x_j \otimes y_j \right\}.$$

It is clear that $\| \cdot \|_{\mathbb{X} \otimes_\lambda \mathbb{Y}}$ is a semi-quasi-norm whose modulus of concavity is at most that of λ , and that $\|x \otimes y\|_{\mathbb{X} \otimes_\lambda \mathbb{Y}} \leq C \|x\| \|y\|$ for all $x \in \mathbb{X}$ and $y \in \mathbb{Y}$, where $C = \lambda(\mathbf{e}_1)$.

Definition 21 Let \mathbb{X} and \mathbb{Y} be quasi-Banach spaces and λ be a symmetric minimal function quasi-norm with the Fatou property. The quasi-Banach space built from $\| \cdot \|_{\mathbb{X} \otimes_\lambda \mathbb{Y}}$ will be called the *topological tensor product of \mathbb{X} and \mathbb{Y} by λ* , and will be denoted by $\mathbb{X} \otimes_\lambda \mathbb{Y}$. The canonical norm-one bilinear map from $\mathbb{X} \times \mathbb{Y}$ to $\mathbb{X} \otimes_\lambda \mathbb{Y}$ given by $(x, y) \mapsto x \otimes y$ will be denoted by $T_\lambda[\mathbb{X}, \mathbb{Y}]$.

Proposition 21 Let $\mathbb{X}, \mathbb{Y}, \mathbb{U}$ and \mathbb{V} be quasi-Banach spaces, and let λ be a symmetric minimal function quasi-norm with the Fatou property.

- (i) If λ is a function p -norm, $0 < p \leq 1$, then $\mathbb{X} \otimes_\lambda \mathbb{Y}$ is a p -Banach space.
- (ii) $\mathcal{G}(\mathbf{L}_\lambda) \subseteq \mathcal{G}(\mathbb{X} \otimes_\lambda \mathbb{Y})$.
- (iii) If λ galbs \mathbb{U} , there is a constant C such that for every bounded bilinear map $B : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{U}$ there is a unique linear map $B_\lambda : \mathbb{X} \otimes_\lambda \mathbb{Y} \rightarrow \mathbb{U}$ such that $B_\lambda \circ T_\lambda[\mathbb{X}, \mathbb{Y}] = B$ and $\|B_\lambda\| \leq C \|B\|$.

- (iv) If $R: \mathbb{X} \rightarrow \mathbb{U}$ and $S: \mathbb{Y} \rightarrow \mathbb{V}$ are bounded linear operators, then there is a unique bounded linear operator $R \otimes_\lambda S: \mathbb{X} \otimes_\lambda \mathbb{Y} \rightarrow \mathbb{U} \otimes_\lambda \mathbb{V}$ such that $(R \otimes_\lambda S) \circ T_\lambda[\mathbb{X}, \mathbb{Y}] = T_\lambda[\mathbb{U}, \mathbb{V}] \circ (R, S)$.
- (v) If \mathbb{U} is complemented in \mathbb{X} through R and \mathbb{V} is complemented in \mathbb{Y} through S , then $\mathbb{U} \otimes_\lambda \mathbb{V}$ is complemented in $\mathbb{X} \otimes_\lambda \mathbb{Y}$ through $R \otimes_\lambda S$. Moreover, if \mathbb{U}^c and \mathbb{V}^c are such that $\mathbb{X} \simeq \mathbb{U} \oplus \mathbb{U}^c$ and $\mathbb{Y} \simeq \mathbb{V} \oplus \mathbb{V}^c$, then

$$\mathbb{X} \otimes_\lambda \mathbb{Y} \simeq (\mathbb{U} \otimes_\lambda \mathbb{V}) \oplus (\mathbb{U} \otimes_\lambda \mathbb{V}^c) \oplus (\mathbb{U}^c \otimes_\lambda \mathbb{V}) \oplus (\mathbb{U}^c \otimes_\lambda \mathbb{V}^c).$$

- (vi) Let ρ be a symmetric minimal function quasi-norm with the Fatou property. If ρ dominates λ , then there is a bounded linear map $I: \mathbb{X} \otimes_\rho \mathbb{Y} \rightarrow \mathbb{X} \otimes_\lambda \mathbb{Y}$ such that $I \circ T_\rho[\mathbb{X}, \mathbb{Y}] = T_\lambda[\mathbb{X}, \mathbb{Y}]$.
- (vii) There is a constant C such that if $(x_j)_{j=1}^\infty$ in \mathbb{X} and $(y_j)_{j=1}^\infty$ in \mathbb{Y} are such that

$$H = \lambda \left((\|x_j\| \|y_j\|)_{j=1}^\infty \right) < \infty, \tag{5}$$

then $\sum_{j=1}^\infty x_j \otimes y_j$ converges in $\mathbb{X} \otimes_\lambda \mathbb{Y}$ to a vector $\tau \in \mathbb{X} \otimes_\lambda \mathbb{Y}$ with $\|\tau\|_{\mathbb{X} \otimes_\lambda \mathbb{Y}} \leq CH$. Conversely, for all $\tau \in \mathbb{X} \otimes_\lambda \mathbb{Y}$ and $\varepsilon > 0$ there are $(x_n)_{n=1}^\infty$ in \mathbb{X} and $(y_n)_{n=1}^\infty$ in \mathbb{Y} such that, if

$$f := (\|x_j\| \|y_j\|)_{j=1}^\infty,$$

then $\lambda(f) \leq \varepsilon + C\|\tau\|_{\mathbb{X} \otimes_\lambda \mathbb{Y}}$ and $\tau = \sum_{j=1}^\infty x_j \otimes y_j$. Moreover, if λ is a function p -norm, $0 < p \leq 1$, we can pick $C = 1$. And, if \mathbb{X}_0 and \mathbb{Y}_0 are dense subspaces of \mathbb{X} and \mathbb{Y} respectively, we can pick $x_j \in \mathbb{X}_0$ and $y_j \in \mathbb{Y}_0$ for all $j \in \mathbb{N}$.

- (viii) If λ galbs \mathbb{X} and \mathbb{Y} is finite dimensional, then $\mathbb{X} \otimes_\lambda \mathbb{Y} \simeq \mathbb{X}^n$, where $n = \dim(\mathbb{Y})$. To be precise, if $(y_j)_{j=1}^n$ is a basis of \mathbb{Y} , the map $R: \mathbb{X}^n \rightarrow \mathbb{X} \otimes_\lambda \mathbb{Y}$ given by $(x_j)_{j=1}^n \mapsto \sum_{j=1}^n x_j \otimes y_j$ is an isomorphism.
- (ix) If λ galbs \mathbb{X} and \mathbb{Y} has the point separation property, then $\|\cdot\|_{\mathbb{X} \otimes_\lambda \mathbb{Y}}$ is a quasi-norm on $\mathbb{X} \otimes_\lambda \mathbb{Y}$.

Proof A simple computation yields (i).

Let $f = (a_k)_{k=1}^\infty \in [0, \infty)^\mathbb{N}$, and let $(\tau_k)_{k=1}^m$ in $\mathbb{X} \otimes \mathbb{Y}$ be such that $\|\tau_k\|_{\mathbb{X} \otimes_\lambda \mathbb{Y}} \leq 1$. Then, given $\varepsilon > 0$, for each $k = 1, \dots, m$ there is an expansion

$$\tau_k = \sum_{j=1}^{n_k} b_{k,j} x_{k,j} \otimes y_{k,j},$$

with $\max\{\|x_{k,j}\|, \|y_{k,j}\|\} \leq 1$ for all $(k, j) \in \mathcal{N} := \{(k, j) \in \mathbb{N}^2: 1 \leq k \leq m, 1 \leq j \leq n_k\}$ and $\lambda((b_{k,j})_{j=1}^{n_k}) \leq 1 + \varepsilon$. The expansion

$$\tau := \sum_{k=1}^m a_k \tau_k = \sum_{(k,j) \in \mathcal{N}} a_k b_{k,j} x_{k,j} \otimes y_{k,j}$$

gives

$$\|\tau\|_{\mathbb{X} \otimes_\lambda \mathbb{Y}} \leq \|(a_k b_{k,j})_{(k,j) \in \mathcal{N}}\|_\lambda \leq (1 + \varepsilon)\lambda_{L_\lambda}(f).$$

Consequently, $\lambda_{\mathbb{X} \otimes_\lambda \mathbb{Y}}(f) \leq \lambda_{L_\lambda}(f)$, and we obtain (ii).

Let us prove (iii). Let C be such that $\|\sum_{j=1}^n a_j u_j\| \leq C\lambda((a_j)_{j=1}^n)$ for all $(a_j)_{j=1}^n$ in $[0, \infty)^n$ and $(u_j)_{j=1}^n$ in the unit ball of \mathbb{U} . Given a bounded bilinear map $B: \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{U}$, let

$B_0: \mathbb{X} \otimes \mathbb{Y} \rightarrow \mathbb{U}$ be the linear map defined by $B(x \otimes y) = B(x, y)$. Given $\tau = \sum_{j=1}^n x_j \otimes y_j \in \mathbb{X} \otimes \mathbb{Y}$ we have

$$\|B_0(\tau)\| \leq C \lambda(\|B(x_j, y_j)\|_{j=1}^n) \leq C \|B\| \lambda(\|x_j\| \|y_j\|_{j=1}^n).$$

Consequently, $\|B_0(\tau)\| \leq C \|B\| \|\tau\|_{\mathbb{X} \otimes_\lambda \mathbb{Y}}$. We infer that B_0 ‘extends’ to an operator as desired.

Now we prove (iv). Let $\tau \in \mathbb{X} \otimes \mathbb{Y}$. The mere definitions of the semi-quasi-norms involved give

$$\begin{aligned} \|(R \otimes_\lambda S)\tau\|_{\mathbb{U} \otimes_\lambda \mathbb{V}} &\leq \inf \left\{ \lambda \left((\|R(x_j)\| \|S(y_j)\|)_{j=1}^n \right) : \tau = \sum_{j=1}^n x_j \otimes y_j \right\} \\ &\leq \|R\| \|S\| \|\tau\|_{\mathbb{X} \otimes_\lambda \mathbb{Y}}. \end{aligned}$$

For statement (v), it suffices to consider the case when $\mathbb{V} = \mathbb{Y}$ and $S_v = \text{Id}_\mathbb{Y}$. Let $I: \mathbb{U} \rightarrow \mathbb{X}$ and $P: \mathbb{X} \rightarrow \mathbb{U}$ be such that $P \circ I = \text{Id}_\mathbb{U}$. Then $(P \otimes_\lambda \text{Id}_\mathbb{Y}) \circ (I \otimes_\lambda \text{Id}_\mathbb{Y}) = \text{Id}_{\mathbb{U} \otimes_\lambda \mathbb{Y}}$. Let $J: \mathbb{U}^c \rightarrow \mathbb{X}$ and $Q: \mathbb{X} \rightarrow \mathbb{U}^c$ be such that $Q \circ J = \text{Id}_{\mathbb{U}^c}$ and $J \circ Q + I \circ P = \text{Id}_\mathbb{X}$. Then

$$(I \otimes_\lambda \text{Id}_\mathbb{Y}) \circ (P \otimes_\lambda \text{Id}_\mathbb{Y}) + (J \otimes_\lambda \text{Id}_\mathbb{Y}) \circ (Q \otimes_\lambda \text{Id}_\mathbb{Y}) = \text{Id}_{\mathbb{X} \otimes_\lambda \mathbb{Y}}.$$

Statement (vi) is immediate from definition.

Let us prove (vii). Assume without loss of generality that λ is function p -norm for some $0 < p \leq 1$. If (5) holds, then $\sum_{j=1}^\infty x_j \otimes y_j$ is a Cauchy series. Therefore, it converges to $\tau \in \mathbb{X} \otimes_\lambda \mathbb{Y}$. The continuity of the quasi-norm $\|\cdot\|_{\mathbb{X} \otimes_\lambda \mathbb{Y}}$ yields

$$\|\tau\|_{\mathbb{X} \otimes_\lambda \mathbb{Y}} = \lim_m \left\| \sum_{j=1}^m x_j \otimes y_j \right\|_{\mathbb{X} \otimes_\lambda \mathbb{Y}} \leq H.$$

Conversely, let $\tau \in \mathbb{X} \otimes_\lambda \mathbb{Y}$ and $\varepsilon > 0$. Assume that \mathbb{X}_0 and \mathbb{Y}_0 are dense subspaces of \mathbb{X} and \mathbb{Y} respectively. Pick $(\tau_n)_{n=1}^\infty$ in $\mathbb{X}_0 \otimes \mathbb{Y}_0$ such that $\lim_n \|\tau - \tau_n\|_{\mathbb{X} \otimes_\lambda \mathbb{Y}} = 0$, and pick a sequence $(\varepsilon_n)_{n=1}^\infty$ of positive numbers with

$$\varepsilon_1 > \|\tau\|_{\mathbb{X} \otimes_\lambda \mathbb{Y}} > \left(\sum_{n=1}^\infty \varepsilon_n^p \right)^{1/p} - \varepsilon.$$

Passing to a subsequence we can suppose that $\|\tau_n - \tau_{n-1}\|_{\mathbb{X} \otimes_\lambda \mathbb{Y}} < \varepsilon_n$ for all $n \in \mathbb{N}$, with the convention $\tau_0 = \tau$. Therefore, for all $n \in \mathbb{N}$, we can write

$$\tau_n - \tau_{n-1} = \sum_{j=1}^{j_n} x_{j,n} \otimes y_{j,n}, \quad R_n := \lambda \left((\|x_{j,n}\| \|y_{j,n}\|)_{j=1}^{j_n} \right) < \varepsilon_n.$$

Let $\mathcal{N} = \{(j, n) \in \mathbb{N}^2 : 1 \leq j \leq j_n\}$. Then

$$\lambda \left((\|x_{j,n}\| \|y_{j,n}\|)_{(j,n) \in \mathcal{N}} \right) \leq \left(\sum_{n=1}^\infty R_n^p \right)^{1/p} \leq \varepsilon + \|\tau\|_{\mathbb{X} \otimes_\lambda \mathbb{Y}}.$$

Hence, we can safely define $\tau' = \sum_{(j,n) \in \mathcal{N}} x_{j,n} \otimes y_{j,n}$, and we have

$$\tau' = \sum_{n=1}^\infty \sum_{j=1}^{j_n} x_{j,n} \otimes y_{j,n} = \sum_{n=1}^\infty (\tau_n - \tau_{n-1}) = \lim_n \tau_n = \tau.$$

Now we prove (viii). The mapping R is linear and bounded, and $R(\mathbb{X}^n)$ spans $\mathbb{X} \otimes_\lambda \mathbb{Y}$. Since λ galbs \mathbb{X} , there is a bounded linear map $S: \mathbb{X} \otimes_\lambda \mathbb{Y} \rightarrow \mathbb{X}^n$ such that $S(x \otimes y_j) = x e_j$ for all $x \in \mathbb{X}$ and $j = 1, \dots, n$. Taking into account that $S \circ R = \text{Id}_{\mathbb{X}^n}$, we are done.

Finally, let \mathbb{V} be finite-dimensional subspace of \mathbb{Y} . Since \mathbb{V} is complemented in \mathbb{Y} , $\mathbb{X} \otimes_\lambda \mathbb{V}$ is complemented in $\mathbb{X} \otimes_\lambda \mathbb{Y}$ via the canonical map. Hence, it suffices to consider the case when \mathbb{Y} is finite dimensional. In this particular case, statement (ix) follows from (viii). \square

6 Topological tensor products as spaces of functions and integrals for spaces of vector-valued functions

Let us give another approach to the proof of Proposition 21 (ix). Given quasi-Banach spaces \mathbb{X} and \mathbb{Y} , let $B: \mathbb{X} \times \mathbb{Y} \rightarrow \ell_\infty(\text{Ball}(\mathbb{Y}^*), \mathbb{X})$, where $\text{Ball}(\mathbb{Y}^*)$ denotes the unit ball of the space \mathbb{Y}^* , be defined by $B(x, y)(y^*) = y^*(y)x$. Since B is linear and bounded, if λ galbs \mathbb{X} , there is a bounded linear map $B_\lambda: \mathbb{X} \otimes_\lambda \mathbb{Y} \rightarrow \ell_\infty(\text{Ball}(\mathbb{Y}^*), \mathbb{X})$ given by $B_\lambda(x \otimes y)(y^*) = y^*(y)x$. If \mathbb{Y} has the point separation property, then B_λ is one-to-one on $\mathbb{X} \otimes \mathbb{Y}$. Consequently, no vector in $\mathbb{X} \otimes \mathbb{Y}$ is norm-zero. Note the injectivity of B_λ on $\mathbb{X} \otimes \mathbb{Y}$ does not implies the injectivity of B_λ on its closure $\mathbb{X} \otimes_\lambda \mathbb{Y}$. That is, we can not, a priori, identify vectors in $\mathbb{X} \otimes_\lambda \mathbb{Y}$ with functions defined over $\text{Ball}(\mathbb{Y}^*)$. More generally, if \mathbb{Y} embeds in \mathbb{F}^Ω for some set Ω , then $\mathbb{X} \otimes \mathbb{Y}$ embeds into \mathbb{X}^Ω , and it is natural to wonder if the character of the members of $\mathbb{X} \otimes \mathbb{Y}$ is preserved when taking the completions, that is, if we can regard the vectors in $\mathbb{X} \otimes_\lambda \mathbb{Y}$ as \mathbb{X} -valued functions defined on Ω . In this section, we address this question in the case when \mathbb{Y} is a Köthe space.

Given a quasi-Banach space \mathbb{X} and a σ -finite measure space (Ω, Σ, μ) we have a canonical linear map

$$J[\mathbb{X}, \mu]: \mathbb{X} \otimes L_0(\mu) \rightarrow L_0(\mu, \mathbb{X}), \quad x \otimes f \mapsto xf.$$

It is routine to check that $J[\mathbb{X}, \mu]$ is one-to-one. Suppose that λ is a symmetric function quasi-norm and ρ is a function quasi-norm over (Ω, Σ, μ) such that λ is p -concave and ρ is p -convex for some $0 < p < \infty$. Then λ is minimal (see Corollary 4). So, we can safely define $\mathbb{X} \otimes_\lambda L_\rho$. If, moreover, λ galbs \mathbb{X} , then λ also galbs $L_\rho(\mathbb{X})$ (see Proposition 19). Hence, if ρ has the weak Fatou property, there is a bounded linear canonical map

$$J[\rho, \mathbb{X}, \lambda]: \mathbb{X} \otimes_\lambda L_\rho \rightarrow L_\rho(\mathbb{X}), \quad x \otimes f \mapsto xf.$$

Consider the range

$$L_\rho^\lambda(\mathbb{X}) := J[\rho, \mathbb{X}, \lambda](\mathbb{X} \otimes_\lambda L_\rho)$$

of this operator endowed with the quotient topology. If $J[\rho, \mathbb{X}, \lambda]$ is one-to-one, then $L_\rho^\lambda(\mathbb{X})$ is a space isometric to $\mathbb{X} \otimes_\lambda L_\rho$ which embeds continuously into $L_\rho(\mathbb{X})$. This is our motivation to studying the injectivity of $J[\rho, \mathbb{X}, \lambda]$. Vogt [40] gave a positive answer to this question in the case when λ is the function quasi-norm associated with ℓ_p for some $0 < p \leq 1$ and ρ is the function quasi-norm associated with $L_q(\mu)$ for some $p \leq q \leq \infty$. A detailed analysis of the proof of [40, Satz 4] reveals that it depends heavily on the fact that λ is both p -convex and p -concave and ρ is both q -convex and q -concave. So, it is hopeless to try to extend this result using analogous ideas. In this paper, we use an approach based on conditional expectations.

Before going on, let us mention that if λ is the ℓ_1 -norm restricted to nonnegative sequences (and ρ and \mathbb{X} are 1-convex), then a routine computation yields that $J[\rho, \mathbb{X}, \lambda]$ is an isometric

embedding when restricted to $\mathbb{X} \otimes S(\mu)$. We infer that $J[\rho, \mathbb{X}, \lambda]$ is an isometric embedding and that $L_\rho^\lambda(\mathbb{X})$ consists of all strongly measurable functions in $L_\rho(\mathbb{X})$.

Lemma 16 *Let λ be a minimal symmetric function quasi-norm. For $i = 1, 2$, let ρ_i be a function quasi-norm with the weak Fatou property over a σ -finite measure space $(\Omega_i, \Sigma_i, \mu_i)$, and let \mathbb{X}_i be a quasi-Banach space galbed by λ . Suppose that the bounded linear operators $S: \mathbb{X}_1 \rightarrow \mathbb{X}_2, T: L_{\rho_1} \rightarrow L_{\rho_2}$ and $R: L_{\rho_1}(\mathbb{X}_1) \rightarrow L_{\rho_2}(\mathbb{X}_2)$ satisfy*

$$R(x f) = S(x) T(f), \quad x \in \mathbb{X}_1, f \in L_{\rho_1}.$$

Then, R restricts to a bounded linear map from $L_{\rho_1}^\lambda(\mathbb{X}_1) \rightarrow L_{\rho_2}^\lambda(\mathbb{X}_2)$.

Proof Our assumptions yield a commutative diagram

$$\begin{array}{ccc} \mathbb{X}_1 \otimes_\lambda L_{\rho_1} & \xrightarrow{S \otimes_\lambda T} & \mathbb{X}_2 \otimes_\lambda L_{\rho_2} \\ J[\rho_1, \mathbb{X}_1, \lambda] \downarrow & & \downarrow J[\rho_2, \mathbb{X}_2, \lambda] \\ L_{\rho_1}(\mathbb{X}_1) & \xrightarrow{R} & L_{\rho_2}(\mathbb{X}_2). \end{array}$$

We infer that R maps the range of the map $J[\rho_1, \mathbb{X}_1, \lambda]$ into the range of the map $J[\rho_2, \mathbb{X}_2, \lambda]$. That is, there is a linear map $R[\lambda]: L_{\rho_1}(\mathbb{X}_1) \rightarrow L_{\rho_2}(\mathbb{X}_2)$ such that the diagram

$$\begin{array}{ccc} \mathbb{X}_1 \otimes_\lambda L_{\rho_1} & \xrightarrow{S \otimes_\lambda T} & \mathbb{X}_2 \otimes_\lambda L_{\rho_2} \\ J[\rho_1, \mathbb{X}_1, \lambda] \downarrow & & \downarrow J[\rho_2, \mathbb{X}_2, \lambda] \\ L_{\rho_1}^\lambda(\mathbb{X}_1) & \xrightarrow{R[\lambda]} & L_{\rho_2}^\lambda(\mathbb{X}_2) \end{array}$$

commutes. Since both $L_{\rho_1}^\lambda(\mathbb{X}_1)$ and $L_{\rho_2}^\lambda(\mathbb{X}_2)$ are endowed with the quotient topology and $S \otimes_\lambda T$ is continuous, so is $R[\lambda]$. □

Let λ be a 1-concave symmetric function quasi-norm that galbs a quasi-Banach space \mathbb{X} . Let (Ω, Σ, μ) be a σ -finite measure space. If ρ is the L_1 -norm on $L_0^+(\mu)$, we denote $L_1^\lambda(\mu, \mathbb{X}) = L_\rho^\lambda(\mathbb{X})$. Given $A \in \Sigma$, we set $L_1^\lambda(A, \mu, \mathbb{X}) = L_1^\lambda(\mu|_A, \mathbb{X})$. The bounded linear operator

$$I[\mu]: L_1(\mu) \rightarrow \mathbb{F}, \quad f \mapsto \int_\Omega f \, d\mu$$

yields a bounded linear operator

$$I[\mu, \mathbb{X}, \lambda]: \mathbb{X} \otimes_\lambda L_1(\mu) \rightarrow \mathbb{X}, \quad x \otimes f \mapsto x \int_\Omega f \, d\mu.$$

Definition 22 Suppose that a 1-concave symmetric function quasi-norm λ galbs a quasi-Banach space \mathbb{X} . We say that the pair (λ, \mathbb{X}) is *amenable* if $I[\mu, \mathbb{X}, \lambda](\tau) = 0$ whenever (Ω, Σ, μ) is a σ -finite measure and the vector $\tau \in \mathbb{X} \otimes_\lambda L_1(\mu)$ satisfies $J[L_1(\mu), \mathbb{X}, \lambda](\tau) = 0$.

In other words, (λ, \mathbb{X}) is amenable if and only if for every σ -finite measure μ there is an operator

$$\mathcal{I}[\mu, \mathbb{X}, \lambda]: L_1^\lambda(\mu, \mathbb{X}) \rightarrow \mathbb{X}$$

such that the diagram

$$\begin{array}{ccc}
 \mathbb{X} \otimes_{\lambda} L_1(\mu) & & \\
 \downarrow J[L_1(\mu), \mathbb{X}, \lambda] & \searrow J[\mu, \mathbb{X}, \lambda] & \\
 L_1^{\lambda}(\mu, \mathbb{X}) & \xrightarrow{\mathcal{I}[\mu, \mathbb{X}, \lambda]} & \mathbb{X}
 \end{array}$$

commutes. The bounded linear operator $\mathcal{I}[\mu, \mathbb{X}, \lambda]$ satisfies

$$\mathcal{I}[\mu, \mathbb{X}, \lambda](x f) = x \int_{\Omega} f d\mu, \quad x \in \mathbb{X}, f \in L_1(\mu).$$

So, we must regard it as ‘integral’ for functions in $L_1^{\lambda}(\mu, \mathbb{X})$. Loosely speaking, that (λ, \mathbb{X}) is amenable means that there is an integral for functions in $L_1^{\lambda}(\mu, \mathbb{X})$.

Definition 23 Let \mathbb{X} be a quasi-Banach space. We say that a net $(T_i)_{i \in I}$ in $\mathcal{L}(\mathbb{X})$ is a *bounded approximation of the identity* if $\sup_i \|T_i\| < \infty$ and $\lim_i T_i(x) = x$ for all $x \in \mathbb{X}$. We say that \mathbb{X} has the *BAP* if it has a bounded approximation of the identity consisting of finite-rank operators.

Note that if a net $(T_i)_{i \in I}$ in $\mathcal{L}(\mathbb{X})$ is uniformly bounded then the set $\{x \in \mathbb{X} : \lim_i T_i(x) = x\}$ is closed. This yields the following elementary result.

Lemma 17 Let \mathbb{X} be a quasi-Banach space. Let $(P_i)_{i \in I}$ be a net consisting of uniformly bounded projections with $P_j \circ P_i = P_i$ if $i \leq j$ and $\cup_{i \in I} P_i(\mathbb{X})$ is dense in \mathbb{X} . Then $(P_i)_{i \in I}$ is a bounded approximation of the identity.

If ρ satisfies (F.7), then for every $A \in \Sigma(\mu)$ we have a bounded linear map

$$S[A, \rho]: L_{\rho} \rightarrow L_1(A, \mu), \quad f \mapsto f|_A.$$

Theorem 5 Let λ be a 1-concave symmetric function quasi-norm, let ρ be a leveling function quasi-norm with the weak Fatou property over a σ -finite measure space (Ω, Σ, μ) , and let \mathbb{X} be a quasi-Banach space. Suppose that (λ, \mathbb{X}) is amenable. Then $J[\rho, \mathbb{X}, \lambda]$ is one-to-one.

Proof Let $A \in \Sigma(\mu)$. By Lemma 13, ρ satisfies (F.7). Therefore, for each quasi-Banach space \mathbb{Y} there is a bounded linear operator

$$S[A, \rho, \mathbb{Y}]: L_{\rho}(\mathbb{Y}) \rightarrow L_1(A, \mu, \mathbb{Y}), \quad f \mapsto f|_A.$$

Set $S[A, \rho, \mathbb{F}] = S[A, \rho]$. It is routine to check that the diagram

$$\begin{array}{ccc}
 \mathbb{X} \otimes_{\lambda} L_{\rho} & \xrightarrow{\text{Id}_{\mathbb{X}} \otimes_{\lambda} S[A, \rho]} & \mathbb{X} \otimes_{\lambda} L_1(A, \mu) \\
 \downarrow J[\rho, \mathbb{X}, \lambda] & & \downarrow J[L_1(A, \mu), \mathbb{X}, \lambda] \\
 L_{\rho}(\mathbb{X}) & \xrightarrow{S[A, \rho, \mathbb{X}]} & L_1(A, \mu, \mathbb{X})
 \end{array}$$

commutes. Using that (λ, \mathbb{X}) is amenable we obtain the commutative diagram

$$\begin{array}{ccccc}
 \mathbb{X} \otimes_{\lambda} L_{\rho} & \xrightarrow{\text{Id}_{\mathbb{X}} \otimes_{\lambda} S[A, \rho]} & \mathbb{X} \otimes_{\lambda} L_1(A, \mu) & & (6) \\
 \downarrow J[\rho, \mathbb{X}, \lambda] & & \downarrow J[L_1(A, \mu), \mathbb{X}, \lambda] & \searrow J[\mu|_A, \mathbb{X}, \lambda] & \\
 L_{\rho}^{\lambda}(\mathbb{X}) & \xrightarrow{S[A, \rho, \mathbb{X}]} & L_1^{\lambda}(A, \mu, \mathbb{X}) & \xrightarrow{\mathcal{I}[\mu|_A, \mathbb{X}, \lambda]} & \mathbb{X}
 \end{array}$$

Suppose that $\mu(\Omega) < \infty$. Let Σ_0 be a finite sub- σ -algebra. If Σ_0 is generated by the partition $(A_j)_{j=1}^n$ of Ω consisting of nonzero measure sets, then

$$\mathbb{E}(\rho, \Sigma_0) = \sum_{j=1}^n \frac{\chi_{A_j}}{\mu(A_j)} I[\mu|_{A_j}] \circ S[A_j, \rho].$$

By Proposition 21 (viii), there is an isomorphism $S: \mathbb{X}^n \rightarrow \mathbb{X} \otimes_\lambda \mathbf{L}_\rho(\Sigma_0)$ such that

$$S((x_j)_{j=1}^n) = \sum_{j=1}^n x_j \otimes \frac{\chi_{A_j}}{\mu(A_j)}, \quad x_j \in \mathbb{X}.$$

Therefore,

$$\text{Id}_{\mathbb{X}} \otimes_\lambda \mathbb{E}(\rho, \Sigma_0) = S \circ (I[\mu|_{A_j}, \mathbb{X}, \lambda] \circ (\text{Id}_{\mathbb{X}} \otimes S[A_j, \rho]))_{j=1}^n.$$

Combining this identity with the commutative diagrams (6) associated with each set A_j yields a bounded linear map $R: \mathbf{L}_\rho^\lambda(\mathbb{X}) \rightarrow \mathbb{X} \otimes_\lambda \mathbf{L}_\rho(\Sigma_0)$ such that the diagram

$$\begin{array}{ccc} \mathbb{X} \otimes_\lambda \mathbf{L}_\rho & & \\ J[\rho, \mathbb{X}, \lambda] \downarrow & \searrow \text{Id}_{\mathbb{X}} \otimes_\lambda \mathbb{E}(\rho, \Sigma_0) & \\ \mathbf{L}_\rho^\lambda(\mathbb{X}) & \xrightarrow{R} & \mathbb{X} \otimes_\lambda \mathbf{L}_\rho(\Sigma_0) \end{array}$$

commutes. The operators $\text{Id}_{\mathbb{X}} \otimes_\lambda \mathbb{E}(\rho, \Sigma_0)$ are uniformly bounded projections. Let $(\Sigma_i)_{i \in I}$ a non-decreasing net of finite σ -algebras whose union generates Σ . By Lemma 17, $(\text{Id}_{\mathbb{X}} \otimes_\lambda \mathbb{E}(\rho, \Sigma_i))_{i \in I}$ is a bounded approximation of the identity. We infer that $J[\rho, \mathbb{X}, \lambda]$ is one-to-one, as wanted, in the particular case that $\mu(\Omega) < \infty$.

In general, let $R[A, \mathbb{X}]: \mathbf{L}_\rho(A, \mathbb{X}) \rightarrow \mathbf{L}_\rho(A, \mathbb{X})$ be the canonical projection on a set $A \in \Sigma(\mu)$. Set $R[A] = R[A, \mathbb{F}]$. Since $R[A, \mathbb{X}]$ is bounded, applying Lemma 16 yields a bounded linear operator $R[A, \mathbb{X}, \lambda]$ such that the diagram

$$\begin{array}{ccc} \mathbb{X} \otimes_\lambda \mathbf{L}_\rho & \xrightarrow{\text{Id}_{\mathbb{X}} \otimes_\lambda R[A]} & \mathbb{X} \otimes_\lambda \mathbf{L}_\rho(A) \\ J[\rho, \mathbb{X}, \lambda] \downarrow & & \downarrow J[\rho|_A, \mathbb{X}, \lambda] \\ \mathbf{L}_\rho^\lambda(\mathbb{X}) & \xrightarrow{R[A, \mathbb{X}, \lambda]} & \mathbf{L}_\rho^\lambda(A, \mathbb{X}) \end{array}$$

commutes. Let $(A_n)_{n=1}^\infty$ be a non-decreasing sequence in $\Sigma(\mu)$ whose union is Ω . By Lemma 17, $(\text{Id}_{\mathbb{X}} \otimes_\lambda R[A_n])_{n=1}^\infty$ is a bounded approximation of the identity. Since $J[\rho|_{A_n}, \mathbb{X}, \lambda]$ is one-to-one (by the previous particular case), it follows that $J[\rho, \mathbb{X}, \lambda]$ is one-to-one. \square

We emphasize that the applicability of Theorem 5 depends on the existence of amenable pairs. In the optimal situation, we would be able to choose λ to be the smallest symmetric function quasi-norm which galbs the quasi-Banach space \mathbb{X} . Thus, the following question arises.

Question 4 Let \mathbb{X} be a quasi-Banach space. Is $(\lambda_{\mathbb{X}}, \mathbb{X})$ amenable?

As long as there is no general answer to Question 4, we will focus on the spaces of galbs that have appeared in the literature. We next prove that for all of them Question 4 has a positive answer.

Theorem 6 *Let φ be a concave Orlicz function fulfilling (3). Suppose that λ_φ galbs a quasi-Banach space \mathbb{X} . Then $(\lambda_\varphi, \mathbb{X})$ is amenable.*

Proof Assume that $\varphi(1) = 1$. Assume by contradiction that there is a σ -finite measure space (Ω, Σ, μ) , a positive sequence $\alpha = (a_j)_{j=1}^\infty$ in ℓ_φ , a sequence $(f_j)_{j=1}^\infty$ in the unit ball of $L_1(\mu)$, and a sequence $(x_j)_{j=1}^\infty$ in the unit ball of \mathbb{X} such that $\sum_{j=1}^\infty a_j x_j f_j = 0$ in $L_\varphi(\mathbb{X})$ and

$$x := \sum_{j=1}^\infty a_j x_j \int_\Omega f_j d\mu \neq 0.$$

The following claim will be used a couple of times.

Claim. If $(\Omega_k)_{k=1}^\infty$ is a non-decreasing sequence in $\Sigma(\mu)$ such that $\Omega \setminus \cup_{k=1}^\infty \Omega_k$ is a null set, then $\sum_{j=1}^\infty a_j x_j \int_{\Omega_k} f_j d\mu \neq 0$ for some $k \in \mathbb{N}$.

Proof of the claim. Since $\lim_k \int_{\Omega_k} f_j d\mu = \int_\Omega f_j d\mu$ for all $j \in \mathbb{N}$ and λ_φ is dominating, we have

$$\lim_k \left\| \left(a_j \int_\Omega f_j d\mu \right)_{j=1}^\infty - \left(a_j \int_{\Omega_k} f_j d\mu \right)_{j=1}^\infty \right\|_\varphi = 0.$$

Since ℓ_φ embeds continuously in $\mathcal{G}_b(\mathbb{X})$,

$$\lim_k \left\| \sum_{j=1}^\infty a_j x_j \int_\Omega f_j d\mu - \sum_{j=1}^\infty a_j x_j \int_{\Omega_k} f_j d\mu \right\| = 0.$$

This limit readily gives our claim.

The claim allow us assume that $\mu(\Omega) < \infty$. By Proposition 21 (vii), we can assume that $f_j \in \mathcal{S}(\mu)$ for all $j \in \mathbb{N}$. Also, we can assume without lost of generality that $\lambda_\varphi(\alpha) < 1$, so that $\sum_{j=1}^\infty \varphi(a_j) < 1$. Set

$$F_m = \sum_{j=m+1}^\infty \varphi(a_j) |f_j|, \quad m \in \mathbb{N} \cup \{0\}.$$

We have $\int_\Omega F_0 d\mu < \infty$. Therefore, $F_0 < \infty$ a.e. By Severini–Egorov theorem, $\lim_m F_m = 0$ quasi-uniformly. By Proposition 2, there is an increasing sequence $(J_n)_{n=1}^\infty$ such that, if

$$G_n = \sum_{j=1}^{J_n} a_j x_j f_j, \quad n \in \mathbb{N},$$

then $\lim_n G_n = 0$ a.e. Taking into account the claim, we can assume without lost of generality that $\lim_m F_m = 0$ uniformly and that $\lim_n G_n = 0$ pointwise.

Pick $0 < \varepsilon < 1$. There is $m_0 \in \mathbb{N}$ such that $\lambda_\varphi((a_j)_{m_0+1}^\infty) < \varepsilon$, i.e.,

$$A := \sum_{j=m_0+1}^\infty \varphi\left(\frac{a_j}{\varepsilon}\right) < 1.$$

Let $m \geq m_0$ be such that

$$F_m(\omega) \leq \frac{\varepsilon(1-A)}{\mu(\Omega)}, \quad \omega \in \Omega.$$

Let Σ_0 be a finite σ -algebra such that f_j is Σ_0 -measurable for all $1 \leq j \leq m$. Let $(A_h)_{h=1}^H$ be a partition of Ω which generates Σ_0 . Pick points $\omega_h \in A_h$ for each $1 \leq h \leq H$, and set

$$g_j = f_j - \sum_{h=1}^H f_j(\omega_h)\chi_{A_h}, \quad j \in \mathbb{N}.$$

Since $g_j = 0$ for all $1 \leq j \leq m$ we have

$$\begin{aligned} x &= \lim_n \sum_{j=1}^{J_n} a_j x_j \int_{\Omega} f_j d\mu - \sum_{h=1}^H \mu(A_h) \lim_n G_n(\omega_h) \\ &= \lim_n \sum_{j=1}^{J_n} a_j x_j \int_{\Omega} g_j d\mu = \lim_n \sum_{j=m+1}^{J_n} a_j x_j \int_{\Omega} g_j d\mu. \end{aligned}$$

Notice that

$$\left\| \sum_{j=m+1}^{J_n} a_j x_j \int_{\Omega} g_j d\mu \right\| \leq \lambda_{\mathbb{X}}((a_j b_j)_{j=m+1}^{\infty}),$$

where $b_j = |\int_{\Omega} g_j d\mu|$. Recall that if a sequence $(u_n)_{n=1}^{\infty}$ converges to x in \mathbb{X} , then $\|x\| \leq \kappa \liminf \|u_n\|$, where κ is the modulus of concavity of \mathbb{X} . Therefore, since λ_{φ} galbs \mathbb{X} , we have

$$\|x\| \leq \kappa \lambda_{\mathbb{X}}((a_j b_j)_{j=m+1}^{\infty}) \leq \kappa C \lambda_{\varphi}((a_j b_j)_{j=m+1}^{\infty}),$$

for some constant $C > 0$. Now let us see that $\lambda_{\varphi}((a_j b_j)_{j=m+1}^{\infty}) \leq \varepsilon$. Using the concavity of φ and that $\varepsilon < 1$, we have

$$\begin{aligned} \sum_{j=m+1}^{\infty} \varphi\left(\frac{a_j b_j}{\varepsilon}\right) &\leq \sum_{j=m+1}^{\infty} \max\{1, b_j\} \varphi\left(\frac{a_j}{\varepsilon}\right) \\ &\leq \sum_{j=m+1}^{\infty} \left(1 + \sum_{h=1}^H |f_j(\omega_h)| \mu(A_h)\right) \varphi\left(\frac{a_j}{\varepsilon}\right) \\ &\leq \sum_{j=m+1}^{\infty} \varphi\left(\frac{a_j}{\varepsilon}\right) + \sum_{h=1}^H \sum_{j=m+1}^{\infty} \frac{1}{\varepsilon} \mu(A_h) |f_j(\omega_h)| \varphi(a_j) \\ &= \sum_{j=m+1}^{\infty} \varphi\left(\frac{a_j}{\varepsilon}\right) + \frac{1}{\varepsilon} \sum_{h=1}^H \mu(A_h) F_m(\omega_h) \\ &\leq A + \frac{1}{\varepsilon} \sum_{h=1}^H \mu(A_h) \frac{\varepsilon(1-A)}{\mu(\Omega)} = 1. \end{aligned}$$

Therefore $\|x\| \leq \kappa C \varepsilon$. Letting ε tend to 0 we arise to absurdity. □

Given a quasi-Banach space \mathbb{X} , a σ -finite measure space (Ω, Σ, μ) , a symmetric function quasi-norm λ such that (λ, \mathbb{X}) is amenable, and a function $f: \Omega \rightarrow \mathbb{X}$, we say that f is λ -integrable if $f \in L_1^{\lambda}(\mu, \mathbb{X})$, and we write

$$\int_{\Omega}^{\lambda} f d\mu = \mathcal{I}[\mu, \mathbb{X}, \lambda](f).$$

A natural question is whether $\int_{\Omega}^{\lambda} f d\mu = \mathcal{I}[\mu, \mathbb{X}, \lambda](f)$ really depends on λ . That is, do we have $\mathcal{I}[\mu, \mathbb{X}, \lambda_1](f) = \mathcal{I}[\mu, \mathbb{X}, \lambda_2](f)$ whenever (λ_1, \mathbb{X}) and (λ_2, \mathbb{X}) are amenable pairs? This question is equivalent to the following one. Given function quasi-norms ρ_1 and ρ_2 over the same σ -finite measure space (Ω, Σ, μ) we define a function quasi-norm $\rho_1 \cap \rho_2$ by

$$(\rho_1 \cap \rho_2)(f) = \inf\{\rho_1(g) + \rho_2(h) : g, h \in L_0^+(\mu), f = g + h\},$$

for each $f \in L_0^+(\mu)$. It can be proved that if ρ_1 and ρ_2 are p -concave (resp. p -convex), $0 < p < \infty$, then $\rho_1 \cap \rho_2$ is p -concave (resp. p -convex).

Question 5 Let \mathbb{X} be a quasi-Banach space, and let λ_1 and λ_2 be symmetric function quasi-norms such that (λ_1, \mathbb{X}) and (λ_2, \mathbb{X}) are amenable. Is $(\lambda_1 \cap \lambda_2, \mathbb{X})$ amenable?

Of course, a positive answer to Question 4 would yield a positive answer to Question 5.

We close this section with an example of an integrable function taking values in a Lorentz space.

Example 1 Fix $a > 1$ and define for each $t > 3$ a sequence $f(t) = (f(t, k))_{k=1}^{\infty}$ by

$$f(t, k) = \begin{cases} \frac{1}{t^2 \log^2 t \log^a(\log t)} \log(k + 1) & \text{if } k \leq t, \\ \frac{1}{t^2 \log^2 t \log^a(\log t)} \log\left(\frac{k+1}{k-\lfloor t \rfloor + 1}\right) & \text{if } k > t. \end{cases}$$

We regard f as a function from $(3, \infty)$ into the weak Lorentz space $\ell_{1,\infty}$. Recall that the space $\ell_{1,\infty}$ consists of all sequences $b = (b_k)_{k=1}^{\infty}$ whose non-increasing rearrangement $(b_k^*)_{k=1}^{\infty}$ satisfies

$$\|b\|_{1,\infty} = \sup_{k \geq 1} k b_k^* < \infty.$$

We have

$$\|f(t)\|_{1,\infty} \approx \frac{1}{t \log t \log^a(\log t)}, \quad t > 3.$$

Hence $\int_3^{\infty} \|f(t)\|_{1,\infty} dt < \infty$. However, in the lack of local convexity, this fact guarantees neither the integrability of f (see [1, Theorem 2.2]) nor that the sequence $(\int_3^{\infty} f(t, k) dt)_{k=1}^{\infty}$ belongs to $\ell_{1,\infty}$. To address this task, we must look for series expansions of the function f .

It is known [33] that the galb of $\ell_{1,\infty}$ is the space $\ell \log \ell$ consisting of all sequences $(a_n)_{n=1}^{\infty}$ such that

$$\sum_{n=1}^{\infty} |a_n| \log\left(\frac{\sum_{j=1}^{\infty} |a_j|}{|a_n|}\right) < \infty.$$

The quasi-Banach space $\ell \log \ell$ coincides with the Orlicz sequence space ℓ_{φ} , where $\varphi(t) = -t \log t$ near the origin. Since such Orlicz function can be chosen to be concave, $(\lambda_{\varphi}, \ell_{1,\infty})$ is an amenable pair by Theorem 6. We have

$$f(t, k) = \sum_{n=1}^{\infty} f_n(t) x_n(k), \quad t \in (3, \infty), k \in \mathbb{N},$$

where

$$f_n(t) = \frac{1}{t^2 \log^2 t \log^a(\log t)} \chi_{[n,\infty)}(t) \quad \text{and} \quad x_n(k) = \log\left(\frac{k-n+2}{k-n+1}\right) \chi_{[n,\infty)}(k).$$

Note that

$$\|f_n\|_1 \approx \frac{1}{n \log^2(1+n) \log^a(2+n)}, \quad n \in \mathbb{N}.$$

Then $(\|f_n\|_1)_{n=1}^\infty \in \ell \log \ell$. Since the sequence $(\|x_n\|_{1,\infty})_{n=1}^\infty$ is uniformly bounded, $f \in L_1^{\lambda_\varphi}((3, \infty), \ell_{1,\infty})$.

7 The fundamental theorem of calculus

Let \mathbb{X} be a quasi-Banach space and let λ be a symmetric function quasi-norm such that (λ, \mathbb{X}) is amenable. If $d \in \mathbb{N}$, $A \subseteq \mathbb{R}^d$ is measurable, and μ is the Lebesgue measure on A , we set $L_1^\lambda(A, \mathbb{X}) = L_1^\lambda(\mu, \mathbb{X})$ and, for $f \in L_1^\lambda(A, \mathbb{X})$, $\int_A^\lambda f(x) dx = \int_A^\lambda f d\mu$. A function $f: \mathbb{R}^d \rightarrow \mathbb{X}$ is said to be *locally λ -integrable* if $f|_A \in L_1^\lambda(A, \mathbb{X})$ for every bounded measurable $A \subseteq \mathbb{R}^d$.

Given $d \in \mathbb{N}$, we denote by \mathcal{Q} the set consisting of all d -dimensional open cubes. If $y \in \mathbb{R}^d$, the set $\mathcal{Q}[y]$ consisting of all $Q \in \mathcal{Q}$ such that $y \in Q$ is a directed set when ordered by inverse inclusion. We denote by “ $Q \in \mathcal{Q} \rightarrow y$ ” the convergence with respect to that directed set.

The following nonlocally convex version of the Lebesgue differentiation theorem for vector-valued integrals (see, e.g., [7, Proposition 5.3] for the classical locally convex version) improves [1, Theorem 5.2].

Theorem 7 *Let \mathbb{X} be a quasi-Banach space and λ be a symmetric function quasi-norm. Suppose that λ is p -concave for some $0 < p < 1$ and that (λ, \mathbb{X}) is amenable. Then, for any locally λ -integrable function $f: \mathbb{R}^d \rightarrow \mathbb{X}$,*

$$\lim_{Q \in \mathcal{Q} \rightarrow y} \frac{1}{|Q|} \int_Q^\lambda f(x) dx = f(y) \quad \text{a.e. } y \in \mathbb{R}^d. \tag{7}$$

Proof Set

$$M[\mathbb{X}, \lambda](f)(y) = \sup_{Q \in \mathcal{Q}[y]} \frac{1}{|Q|} \left\| \int_Q^\lambda f(x) dx \right\|, \quad f \in L_1^\lambda(\mathbb{R}^d, \mathbb{X}), \quad y \in \mathbb{R}^d.$$

If κ is the modulus of concavity of \mathbb{X} , the quasi-triangle inequality

$$M[\mathbb{X}, \lambda](f + g) \leq \kappa M[\mathbb{X}, \lambda](f) + \kappa M[\mathbb{X}, \lambda](g), \quad f, g \in L_1^\lambda(\mathbb{R}^d, \mathbb{X})$$

holds. Hence, the set of functions f that satisfy (7) is a vector space. Since this set contains

$$\mathcal{F} = \{x \chi_Q : x \in \mathbb{X}, \quad Q \in \mathcal{Q}\},$$

and $[\mathcal{F}] = L_1^\lambda(\mathbb{R}^d, \mathbb{X})$, it suffices to prove that the maximal function $M[\mathbb{X}, \lambda]$ is bounded from $L_1^\lambda(\mathbb{R}^d, \mathbb{X})$ into $L_{1,\infty}(\mathbb{R}^d)$.

Let M be the classical scalar-valued Hardy-Littlewood maximal function. Let $f = \sum_{j=1}^\infty x_j f_j$, where $(x_j)_{j=1}^\infty$ is in the unit ball of \mathbb{X} and $(f_j)_{j=1}^\infty$ in $L_1(\mathbb{R}^d)$ satisfies $\lambda((\|f_j\|_1)_{j=1}^\infty) < \infty$, be an expansion of $f \in L_1^\lambda(\mathbb{R}^d, \mathbb{X})$. We have

$$M[\mathbb{X}, \lambda](f) \leq \lambda((M(f_j))_{j=1}^\infty).$$

By Theorem 3, the pair $(\lambda, L_{1,\infty}(\mathbb{R}^d))$ has the MII property. Hence,

$$\|M[\mathbb{X}, \lambda](f)\|_{1,\infty} \leq \|\lambda((M(f_j))_{j=1}^\infty)\|_{1,\infty} \leq C_1 \lambda((\|M(f_j)\|_{1,\infty})_{j=1}^\infty),$$

where the constant C_1 does not depend on f . In turn, since M is bounded from $L_1(\mathbb{R}^d)$ into $L_{1,\infty}(\mathbb{R}^d)$,

$$\lambda((\|M(f_j)\|_{1,\infty})_{j=1}^\infty) \leq C_2\lambda((\|f_j\|_1)_{j=1}^\infty),$$

where the constant C_2 does not depend on f either. Consequently, if $C = C_1C_2$, $\|M[\mathbb{X}, \lambda](f)\|_{1,\infty} \leq C\|f\|_{L_1^\lambda(\mathbb{R}^d, \mathbb{X})}$ for all $f \in L_1^\lambda(\mathbb{R}^d, \mathbb{X})$. \square

We close with an application to the theory of Lipschitz functions. Derivatives and integrals are relevant tools within the study of Lipschitz maps whose target space is a Banach space (see, e.g., [3, Sect. 14]). So, the lack of an integration theory as powerful as Bochner-Lebesgue integral is a drawback to extend to quasi-Banach spaces results achieved in the locally convex setting. The theory of integration presented here allows us to obtain a partial result for functions taking values in an Orlicz space.

Theorem 8 *Let $0 < p < 1$, and let φ be a p -concave Orlicz function. Let $F = (F_n)_{n=1}^\infty : [0, 1] \rightarrow \ell_\varphi$ be a Lipschitz map such that each component $F_n : [0, 1] \rightarrow \mathbb{R}$ is monotone. Then F is almost everywhere differentiable.*

Proof Since scalar-valued Lipschitz functions are a.e. differentiable, we can safely define $f_n(t) = F'_n(t)$ a.e. $t \in [0, 1]$, and set

$$f = \sum_{n=1}^\infty f_n e_n. \tag{8}$$

We have $\|e_n\|_\varphi = 1$ and $a_n := \|f_n\|_1 = |F_n(1) - F_n(0)| \leq |F_n(1)| + |F_n(0)|$ for all $n \in \mathbb{N}$. Hence, $(a_n)_{n=1}^\infty \in \ell_\varphi$. By Theorem 6, $(\lambda_\varphi, \ell_\varphi)$ is an amenable pair, so the series expansion (8) witnesses that $f \in L_1^{\lambda_\varphi}([0, 1])$. Consequently,

$$F(t) - F(s) = \int_{[s,t]}^{\lambda_\varphi} f(s) ds, \quad 0 \leq s < t \leq 1.$$

Combining Proposition 20 with Theorem 7 gives $\lim_{t \rightarrow s} (F(t) - F(s))/(t - s) = f(s)$ a.e. $s \in [0, 1]$. \square

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Albiac, F., Ansorena, J.L.: Integration in quasi-Banach spaces and the fundamental theorem of calculus. *J. Funct. Anal.* **264**(9), 2059–2076 (2013). <https://doi.org/10.1016/j.jfa.2013.02.003>
2. Albiac, F., Ansorena, J.L., Cúth, M., Doucha, M.: Lipschitz free p -spaces for $0 < p < 1$. *Isr. J. Math.* **240**(1), 65–98 (2020). <https://doi.org/10.1007/s11856-020-2061-5>
3. Albiac, F., Kalton, N.J.: Topics in Banach Space Theory, Graduate Texts in Mathematics, vol. 233, 2nd edn. Springer, Cham (2016). <https://doi.org/10.1007/978-3-319-31557-7>. (With a foreword by Gilles Godefroy)
4. Amemiya, I.: A generalization of Riesz–Fischer’s theorem. *J. Math. Soc. Jpn.* **5**, 353–354 (1953). <https://doi.org/10.2969/jmsj/00530353>
5. Aoki, T.: Locally bounded linear topological spaces. *Proc. Imp. Acad. Tokyo* **18**, 588–594 (1942). <http://projecteuclid.org/euclid.pja/1195573733>
6. Bennett, C., Sharpley, R.: Interpolation of Operators, Pure and Applied Mathematics, vol. 129. Academic Press Inc, Boston (1988)
7. Benyamini, Y., Lindenstrauss, J.: Geometric Nonlinear Functional Analysis, Vol. 1, American Mathematical Society Colloquium Publications, vol. 48. American Mathematical Society, Providence (2000)
8. Calderón, A.P.: Spaces between L^1 and L^∞ and the theorem of Marcinkiewicz. *Stud. Math.* **26**, 273–299 (1966). <https://doi.org/10.4064/sm-26-3-301-304>
9. Carro, M., Colzani, L., Sinnamon, G.: From restricted type to strong type estimates on quasi-Banach rearrangement invariant spaces. *Stud. Math.* **182**(1), 1–27 (2007). <https://doi.org/10.4064/sm182-1-1>
10. Colzani, L., Sjögren, P.: Translation-invariant operators on Lorentz spaces $L(1, q)$ with $0 < q < 1$. *Stud. Math.* **132**(2), 101–124 (1999)
11. Day, M.M.: The spaces L^p with $0 < p < 1$. *Bull. Am. Math. Soc.* **46**, 816–823 (1940). <https://doi.org/10.1090/S0002-9904-1940-07308-2>
12. Ellis, H.W., Halperin, I.: Function spaces determined by a levelling length function. *Can. J. Math.* **5**, 576–592 (1953). <https://doi.org/10.4153/cjm-1953-065-1>
13. Hernández, F.L.: The p -convexity of Orlicz spaces. *Collect. Math.* **34**(3), 233–245 (1983)
14. Hernández, F.L.: On the galb of weighted Orlicz sequence spaces. I. *Bull. Polish Acad. Sci. Math.* **32**(3–4), 193–202 (1984)
15. Hernández, F.L.: On the galb of weighted Orlicz sequence spaces. II. *Arch. Math. (Basel)* **45**(2), 158–168 (1985). <https://doi.org/10.1007/BF01270487>
16. Ho, K.-P.: Fourier-type transforms on rearrangement-invariant quasi-Banach function spaces. *Glasg. Math. J.* **61**, 231–248 (2019). <https://doi.org/10.1017/S0017089518000186>
17. Ho, K.-P.: Linear operators, Fourier integral operators and k -plane transforms on rearrangement-invariant quasi-Banach function spaces. *Positivity* **25**, 73–96 (2021). <https://doi.org/10.1007/s11117-020-00750-0>
18. Hunt, R.A.: On $L(p, q)$ spaces. *Enseign. Math.* **2**(12), 249–276 (1966)
19. Kalton, N.J.: The convexity type of quasi-banach spaces. Unpublished (1977). <https://kaltonmemorial.missouri.edu/assets/docs/Kalton-unpubl.pdf>
20. Kalton, N.J.: Linear operators on L_p for $0 < p < 1$. *Trans. Am. Math. Soc.* **259**(2), 319–355 (1980). <https://doi.org/10.2307/1998234>
21. Kalton, N.J.: Convexity conditions for nonlocally convex lattices. *Glasg. Math. J.* **25**(2), 141–152 (1984). <https://doi.org/10.1017/S0017089500005553>
22. Kalton, N.J.: Banach envelopes of nonlocally convex spaces. *Can. J. Math.* **38**(1), 65–86 (1986). <https://doi.org/10.4153/CJM-1986-004-2>
23. Kalton, N.J., Peck, N.T., Roberts, J.W.: An F -Space Sampler. London Mathematical Society Lecture Note Series, vol. 89. Cambridge University Press, Cambridge (1984). <https://doi.org/10.1017/CBO9780511662447>
24. Lindenstrauss, J., Tzafriri, L.: Classical Banach Spaces II Function Spaces, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, vol. 97. Springer, Berlin (1979)
25. Luxemburg, W.A.J.: On the measurability of a function which occurs in a paper by AC Zaanen. *Proc. Neth. Acad. Sci. (A)* **61**, 259–265 (1958). (*Indag. Math.* **20**, 259–265)
26. Luxemburg, W.A.J., Zaanen, A.C.: Notes on Banach function spaces II. *Nederl. Akad. Wetensch. Proc. Ser. A 66 = Indag. Math.* **25**, 148–153 (1963)
27. Maurey, B.: Intégration dans les espaces p -normes. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **26**(3), 911–931 (1972)
28. Mazur, S., Orlicz, W.: Sur les espaces métriques linéaires. I. *Stud. Math.* **10**, 184–208 (1948). <https://doi.org/10.4064/sm-10-1-184-208>
29. Mitiagin, B.S.: An interpolation theorem for modular spaces. *Mat. Sb. (N.S.)* **66**(108), 473–482 (1965)

30. Rolewicz, S.: On a certain class of linear metric spaces. *Bull. Acad. Pol. Sci. Cl. XL* **5**, 471–473 (1957)
31. Rolewicz, S.: *Metric Linear Spaces. Mathematics and Its Applications (East European Series)*, vol. 20. D Reidel Publishing Co., Dordrecht (1985)
32. Schep, A.R.: Minkowski's integral inequality for function norms. In: *Operator Theory in Function Spaces and Banach Lattice. Operator Theory: Advances and Applications*, vol. 75, pp. 299–308. Birkhäuser, Basel (1995)
33. Sjögren, P.: Translation-invariant operators on weak L^1 . *J. Funct. Anal.* **89**(2), 410–427 (1990). [https://doi.org/10.1016/0022-1236\(90\)90100-Y](https://doi.org/10.1016/0022-1236(90)90100-Y)
34. Sjögren, P.: Convolutors on Lorentz spaces $L^{1,q}$ with $1 < q < \infty$. *Proc. Lond. Math. Soc. (3)* **64**(2), 397–417 (1992). <https://doi.org/10.1112/plms/s3-64.2.397>
35. Stein, E.M., Weiss, N.J.: On the convergence of Poisson integrals. *Trans. Am. Math. Soc.* **140**, 35–54 (1969). <https://doi.org/10.2307/1995121>
36. Turpin, P.: Espaces et intersections d'espaces d'Orlicz non localement convexes. *Stud. Math.* **46**, 167–195 (1973). <https://doi.org/10.4064/sm-46-2-167-195>
37. Turpin, P.: Opérateurs linéaires entre espaces d'Orlicz non localement convexes. *Stud. Math.* **46**, 153–165 (1973). <https://doi.org/10.4064/sm-46-2-153-165>
38. Turpin, P.: Convexités dans les espaces vectoriels topologiques généraux. *Dissert. Math. (Rozprawy Mat.)* **131**, 221 (1976)
39. Väth, M.: Some measurability results and applications to spaces with mixed family-norm. *Positivity* **10**, 737–753 (2006). <https://doi.org/10.1007/s11117-005-0035-9>
40. Vogt, D.: Integrationstheorie in p -normierten Räumen. *Math. Ann.* **173**, 219–232 (1967). <https://doi.org/10.1007/BF01361712>
41. Zaanen, A.C.: *Riesz Spaces, II*, North-Holland Mathematical Library, vol. 30. North-Holland Publishing Co., Amsterdam (1983)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.