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On perturbation solutions in the restricted three-body problem dynamics[☆]

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ABSTRACT

The well-known lack of existence of enough integrals to provide a closed form solution to the restricted three-body problem makes approaching it with numerical methods customary. However, useful perturbation solutions to this problem can be found in particular regions of phase space. A brief review of three different cases of interest in astrodynamics illustrates the possibilities offered by the perturbation approach in mission planning of space orbits.

1. Introduction

I never met Professor John V. Breakwell. In 1991, when I moved to Zaragoza to finish my undergraduate studies and start my PhD in the small university research group mentored by A. Deprit, Breakwell was in his last days. However, there is no doubt that his work inspired my research in different times along my career as astrodynamicist, and, in particular, in the use of perturbation methods.

For instance, reported unexpected in-track errors [1] in Brouwer's solution of the Earth's artificial satellite problem [2] were soon identified with a lack of accuracy in the initialization of the constant semi-major axis that feeds the analytical perturbation solution [3]. To solve the issue, Breakwell and Vagners [4] suggested an extremely simple and smart way of improving this value. While carrying out the transformation of the initial osculating semi-major axis to the mean one up to higher accuracy than the other orbital elements is these days part of the routine of the initialization of the constants of perturbation theories [5–8], Breakwell and Vagners' successful shortcut allows for analogous accuracy without need of increasing the order of the perturbation solution [9,10].

Other instance is Breakwell's research on gravity gradient perturbations, which gave full generality to the perturbation problem of attitude dynamics without constraining to the usual approximation of uniaxial or nearly uniaxial satellites [11,12]. Breakwell's investigations motivated further developments [13–15] which are these days in the roots of modern 6-degrees of freedom hybrid propagation programs [16].

These are just two examples of Breakwell's direct contributions to the topic of perturbation theory. But his insights were also relevant in the discussions on the critical inclination singularity in artificial satellite theory, Breakwell being among the first who pointed out that the satellite's dynamics in the vicinity of the critical inclination is in no way that of a pendulum [17]. It is also worth mentioning that, soon after Deprit's Hamiltonian perturbations method by Lie transforms appeared in print [18], Kamel showed that Deprit's fundamental recursion, in which the method relies upon, applies also to perturbations of vectorial flows. Remarkably, Breakwell is explicitly acknowledged by Kamel as his advisor in that research [19,20].

Perturbation solutions in astrodynamics were originally computed for the propagation of Earth's artificial satellite orbits [2,21]. In that case, the Kepler problem was commonly taken as the intermediary — the integrable part on which the perturbation approach rests upon— while additional forces like, for instance, non-centralities of the Earth's gravitational field, atmospheric drag, or third-body effects, were treated as small disturbances of the pure Keplerian motion. In particular, third-body perturbations enter naturally the perturbed Keplerian motion scheme like time-dependent perturbations, the third-body direction being obtained from some ephemeris file. This is the common case of Earth orbiting satellites, in which both lunar and solar perturbations are of the same order. Still, time-dependency issues are avoided in those cases in which the satellite moves with much faster mean motion than the third body. Then, the third-body position can be assumed to remain fixed in the time in which the satellite travels one orbit [7,22–25]

On the contrary, in other instances, like in the case of motion about planetary satellites, the gravitational pull of a single disturbing body dominates over the effects of other possible existing disturbing bodies or forces. Then, the restricted three-body problem provides a

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better frame to set up the perturbation approach, in which the need of dealing explicitly with time can mostly be avoided. Needless to say that, contrary to the Kepler problem, the restricted three-body problem cannot be completely reduced by integrals [26]. In this complex dynamics, the numerical approach is normally preferred in the design of mission orbits, where the computation of surfaces of section, periodic orbits, and other invariant manifolds reveal as useful tools [27–30]. While the perturbation approach still applies, it must be constrained to particular regions of phase space in which the dynamics may hinge on some simpler integrable problem without constraint to the case of perturbed Keplerian motion. In fact, most orbits of the restricted three-body problem do not resemble at all Keplerian ellipses.

In particular, three different regimes of the restricted three-body problem of interest in astrodynamics are commonly identified like amenable to the perturbation approach. Namely: a) the motion inside the sphere of influence of the primary of smaller mass, which applies to science orbits about planetary satellites and can be approached as the classical case of perturbed Keplerian motion [31–34]; b) the motion about the libration points, where the conspicuous Lyapunov and Halo orbits are found, that can be approached as a case of perturbed elliptic oscillations [35–38]; c) the co-orbital motion of the satellite and the smaller-mass primary about the bigger-mass primary [39–43]. This last case gives rise to the so-called quasi-satellite orbits, which is a particular case of the co-orbital motion with low eccentricity that can be stated in terms of perturbed harmonic oscillations.

Some peculiarities of the different perturbation solutions in the regimes a), b), and c) are briefly reviewed in what follows. In each case the distance between the massless body and the lighter-mass primary, which is conveniently placed at the origin, is notably smaller than the distance between both primaries. This fact allows us the usual expansion of the third-body potential in Legendre polynomials, which is then truncated to some power of the ratio between both distances. When this ratio is small enough —in other words, when the parallax of the heavier-mass primary is negligible— the truncation of the Legendre polynomials expansion to the first significative term can be representative of the dynamics. Additional simplifications of the restricted three-body problem in the case in which the mass of the distant primary is much higher than the mass of the primary at the origin, give rise to the celebrated Hill problem [44].[3]

In addition to its simplicity, the non-dimensionalization of the Hill problem by a proper choice of units of length and time shows that it does not depend on physical parameters [27]. This additional feature furnishes the Hill problem with a wide generality that makes it representative of the restricted dynamics under the gravitational attraction of different sun-planet, planet–satellite, or other binary systems, the particular characteristics of which are recovered after a simple rescaling of the physical units. Hence, for simplicity and greater insight, the three cases a), b), and c), are discussed in the Hill problem dynamics.

2. The Hill problem as a limit case

Commonly, the mass of the orbiter is negligible compared to celestial bodies of interest, and hence the three-body dynamics is approached in the *restricted* approximation. That is, the mass of the orbiter has no observable effects on the motion of the primaries, which, therefore, are assumed to evolve with Keplerian motion. The particular case in which the primaries evolve with circular motion gives rise to the *circular restricted* three-body problem (CRTBP), which is conveniently formulated in a rotating frame with the rotation rate of the primaries.

[3] Alternatively, Hill equations can be derived from a non-restricted problem in which two of the three involved masses are much smaller than the mass of the heavier body [45,46].

Even in the radical simplifications of the CRTBP, the resulting three degrees of freedom problem lacks of closed form solution,[4] and hence it is commonly explored with numerical tools, among which the numerical computation of periodic orbits plays a prominent role [49–51].

The CRTBP admits the Hamiltonian formulation (see [52,53], for instance)

$$\mathcal{H} = \tfrac{1}{2}\boldsymbol{R}\cdot\boldsymbol{R} - (\boldsymbol{\omega}\times\boldsymbol{r})\cdot\boldsymbol{R} + V, \tag{1}$$

where $\boldsymbol{r}$ denotes the vector from the primary of smaller mass to the massless particle, $\boldsymbol{R}$ is its conjugate momentum, and $\boldsymbol{\omega} = \boldsymbol{G}/(\boldsymbol{d}\cdot\boldsymbol{d})$, with $\boldsymbol{G}$ and $\boldsymbol{d}$ standing for the angular momentum vector of the system per mass unit and the vector from the primary of bigger mass to the other primary, respectively —which are constant in the CRTBP assumptions. The acceleration of the massless particle, which is due to the gravitational attraction of both primaries, of masses $m' > m$, as well as the formulation in a rotating frame, stems from the potential

$$V = -\frac{\mathcal{G}m'}{s} - \frac{\mathcal{G}m}{\rho} - \omega^2(1-\mu)\boldsymbol{r}\cdot\boldsymbol{d}, \tag{2}$$

in which $\mathcal{G}$ is the gravitational constant, $\rho = \|\boldsymbol{r}\|$, $s = \|\boldsymbol{d}+\boldsymbol{r}\|$, $\omega = \|\boldsymbol{\omega}\|$ is the constant rotation rate of the system, and $\mu = m/(m'+m)$. The flow stemming from Hamiltonian (1) is then obtained from the integration of Hamilton equations

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \frac{\partial\mathcal{H}}{\partial\boldsymbol{R}}, \qquad \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} = -\frac{\partial\mathcal{H}}{\partial\boldsymbol{r}}.$$

When the motion is measured relative to the smaller primary $\boldsymbol{r} = (\xi,\eta,\zeta)$ and

$$s = \sqrt{(d+\xi)^2+\eta^2+\zeta^2} = d\sqrt{1-2\epsilon(-\xi/\rho)+\epsilon^2}$$

where $\epsilon = \rho/d$. In those cases in which $\epsilon < 1$ the inverse of the distance s is customarily given by the usual expansion in Legendre polynomials $P_i(-\xi/\rho)$. Besides, in those cases in which $m \ll m'$, the bulk of the dynamics close to the primary of smaller mass is derived from the Hill problem Hamiltonian

$$\mathcal{H} = \frac{1}{2}\boldsymbol{R}\cdot\boldsymbol{R} - (\boldsymbol{\omega}\times\boldsymbol{r})\cdot\boldsymbol{R} - \frac{\mathcal{G}m}{\rho} + \frac{\omega^2}{2}\left(\rho^2 - 3\xi^2\right), \tag{3}$$

which is obtained by neglecting higher order terms from Eq. (1).

In fact, Eq. (3) does not depend on physical parameters and, when such units of length and time are chosen that ω and $\mathcal{G}m$ become the unity, can be rewritten in the form

$$\mathcal{H} = \frac{1}{2}(X^2+Y^2+Z^2) - (xY - yX) - \frac{1}{r} + \frac{1}{2}\left(r^2 - 3x^2\right). \tag{4}$$

We replaced Greek letters in Eq. (3) by corresponding ones of the Roman alphabet in Eq. (4) in order to remark the different units that are used in each of these equations. The lack of physical parameters bestows the Hill problem with a wide generality. Particularization of the results for a given system is obtained after a trivial rescaling.

In different regions of phase space, the Hill problem Hamiltonian can be arranged like a perturbation problem $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$, where $\mathcal{H}_0$ is integrable and $|\mathcal{H}_1| \ll |\mathcal{H}_0|$. In particular, when $r \ll 1$, it can be written in the form of a perturbed Keplerian problem

$$\mathcal{H}_0 = \tfrac{1}{2}(X^2+Y^2+Z^2) - 1/r \tag{5}$$

$$\mathcal{H}_1 = -(xY - yX) + \tfrac{1}{2}\left(r^2 - 3x^2\right) \tag{6}$$

which applies to the case of motion about planetary satellites. On the other hand, when the motion evolves far enough away from the origin, the Hill problem can be written in the form

$$\mathcal{H}_0 = \tfrac{1}{2}(X^2+Y^2+Z^2) - xY + yX + \tfrac{1}{2}(r^2 - 3x^2) \tag{7}$$

[4] The three-body problem admits analytical solution in the form of power series [47], yet it is widely accepted that the poor convergence of the series makes them useless at all for computational purposes [48].

$$\mathcal{H}_1 = -1/r \tag{8}$$

which applies to the case of co-orbital motion with low eccentricity. That $\mathcal{H}_0$ is integrable in this last case results from its quadratic character, which yields a linear flow as readily derived from Hamilton equations. A brief review of both astrodynamics applications is provided below, which is further complemented with a nice application to the description of the main dynamical features of the motion about the libration points.

3. Motion about planetary satellites

The preliminary design of missions to planetary satellites can take a good profit of the description of the main dynamics provided by the analytical approach. In addition, it fits quite well to the Hill problem simplifications. Indeed, as checked in Table 1 of [54], the small-mass assumption of the planetary satellite applies to most natural moons. On the other hand, the altitude of mapping satellites is commonly low, making the small parallax assumption acceptable for the dynamical model. Hence, we write the Hill problem Hamiltonian (3) in the form

$$\mathcal{H} = \mathcal{H}_{\text{Kepler}} + \mathcal{H}_{\text{Coriolis}} + \mathcal{H}_{3\text{B}}, \tag{9}$$

where the Keplerian term is

$$\mathcal{H}_{\text{Kepler}} = -\mathcal{G}m/(2a), \tag{10}$$

and a is the orbit semi-major axis. The Coriolis effect is

$$\mathcal{H}_{\text{Coriolis}} = -\mathcal{G}m/(2a)(N/n)(2\eta\cos I), \tag{11}$$

where N denotes the constant rotation rate of the planet–satellite system, $n = \sqrt{\mathcal{G}m/a^3}$ is the mean motion of the orbiter, $\eta = \sqrt{1-e^2}$ denotes the eccentricity function, with e standing for orbital eccentricity, and I is inclination. Finally, the third-body effect is [33]

$$\mathcal{H}_{3\text{B}} = \mathcal{G}m/(2a)(N/n)^2(r/a)^2\big[2 - 6(\cos h\cos\theta - \cos I\sin h\sin\theta)^2\big], \tag{12}$$

where $r = a\eta^2/(1+e\cos f)$ denotes the radius from the origin, h is the longitude of the ascending node in the rotating frame, and $\theta = f + \omega$ is the argument of the latitude, with ω denoting the argument of the periapsis and f the true anomaly. The latter, we recall, is an implicit function of the mean anomaly M through the Kepler equation.

Therefore, the ratio N/n between the rotation rate of the planet–satellite system and the orbiter's mean motion clearly scales the Hill problem Hamiltonian as the typical perturbation problem

$$\mathcal{H} = \sum_{i\geq 0}\frac{\varepsilon^i}{i!}\mathcal{H}_i, \tag{13}$$

where $\mathcal{H}_0$ is the Keplerian (10), $\mathcal{H}_1$ is the Coriolis term (11), $\mathcal{H}_2$ is the third body perturbation (12), and $\mathcal{H}_i \equiv 0$ for $i \geq 3$. Because n is not constant at this stage, the ratio N/n cannot be taken like the small parameter ε of the perturbation approach, which, on the contrary must remain formal ($\varepsilon = 1$).

Recall that all the symbols entering the perturbation Hamiltonian (13) must be taken as *functions* of some set of canonical variables due to the Hamiltonian formulation. Since we are dealing with perturbed Keplerian motion, we adhere to the tradition and rely on the set of action–angle variables of the Kepler problem —the so-called Delaunay variables. They are customarily denoted by the angles (ℓ, g, h), where $\ell = M$, $g = \omega$, $h = \Omega - Nt$, with Ω the longitude of the ascending node in the non-rotating frame, and the actions (L, G, H), where $L = \sqrt{\mu a}$ is the Delaunay action, $G = L\eta$ is the specific angular momentum, and $H = G\cos I$ is the component of the angular momentum vector in the direction orthogonal to the plane of the primaries.

The relevant dynamics of Hamiltonian (13) is better understood after removing short-period effects by means of canonical perturbation theory. That is, up to some order of the small parameter we compute a transformation from osculating to mean variables $\mathcal{T}$: $(\ell, g, h, L, G, H; \varepsilon) \mapsto (\ell', g', h', L', G', H')$ such that it converts Hamiltonian (13) into

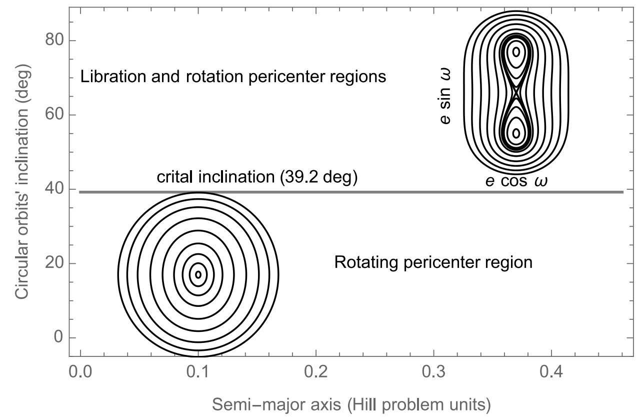

Fig. 1. Eccentricity-vector evolution about a planetary satellite, resulting from a second order truncation of a perturbation solution to the Hill problem dynamics.

$$\mathcal{H}\circ\mathcal{T} = \mathcal{K} \equiv \sum_{j=0}^{m}\frac{\varepsilon^j}{j!}\mathcal{K}_j(-, g', -, L', G', H') + \varepsilon^{m+1}\mathcal{Q}(\ell', g', h', L', G', H'). \tag{14}$$

How to find the transformation $\mathcal{T}$ is the topic of canonical perturbation theory and is profusely dealt with in the literature [53,55,56].

After neglecting effects of $\mathcal{O}(\epsilon^{m+1})$ and higher, ℓ' and h' turn into cyclic variables in the transformed Hamiltonian $\mathcal{K}$, and hence L' and H' become (formal) integrals. The preservation of L' shows the constancy of the semi-major axis $a = L'^2/\mathcal{G}m$, whereas the second integral shows the symmetry of the system about the axis perpendicular to the orbital plane of the primaries. Moreover, the latter can be written in the mean elements space like $H' = L'\sqrt{1-e^2}\cos I$, thus disclosing the coupling of the eccentricity and inclination, an effect that is generally known as the Lidov–Kozai resonance [57,58]. Replacing the pair of dynamical parameters (L', H') by the equivalent one (a, I_{circular}), where $I_{\text{circular}} = \arccos(H'/L')$ is these days customary [59,60].

Integrals decouple the flow, whose long-term dynamics can then be investigated from a reduced system in (g', G'). While this system is of just one degree of freedom, the integration of the reduced flow depends on special functions, a fact that may deprive the solution of the required insight for mission designing purposes. Alternatively, a lot of information on the evolution of the system is gained from the representation of the reduced phase as well as from the computation of particular solution, and, more specifically, the equilibria of the reduced system. To avoid singularities for circular orbits in the cylindrical map (g', G'), the reduced flow is customarily studied in the eccentricity-vector variables $(e\cos\omega, e\sin\omega)$, in which, now, $e = (1 - G'^2/L'^2)^{1/2}$ and $\omega = g'$.

After truncation of Eq. (14) to $\mathcal{O}(\varepsilon^2)$, simple computations show that circular orbits are always equilibria, whereas elliptic orbits may exist with argument of the periapsis $\pm\pi/2$. However, the latter exist only when $H'/L' \leq \sqrt{3/5}$. That is, when the inclination of the circular orbits is higher than $\approx 39°$ —the so-called critical inclination of the third body perturbation [33]. The global, reduced dynamics is then illustrated with diagrams like the one in Fig. 1, where the different phase flows are obtained from the inexpensive evaluation of the reduced Hamiltonian $\mathcal{K} = \mathcal{K}(e\cos\omega, e\sin\omega; a, I_{\text{circular}})$. Since direct and retrograde orbits are symmetric in the approximation provided by the second order truncation of the long-term Hamiltonian $\mathcal{K}$, only the former are presented in Fig. 1.

Because the reduced dynamic only shows isolated equilibria, we do not expect qualitative changes in the long-term behavior when

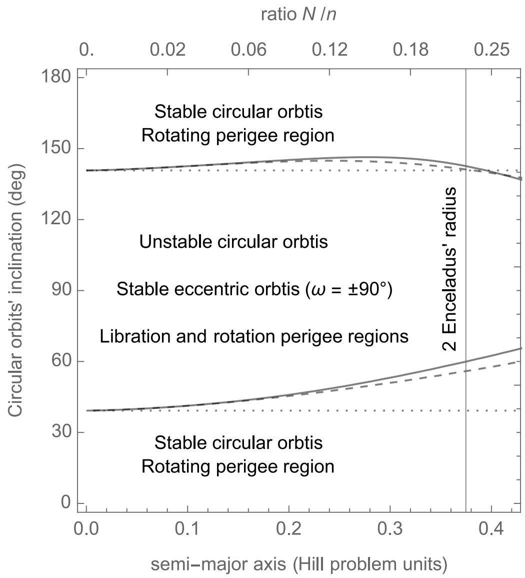

Fig. 2. Refinements of the critical inclination of the third body perturbation (dotted lines) obtained with the 4th- (dashed lines) and 6th-order truncations (full lines) of the Hill problem (after [65]).

including higher order terms in the solution. However, *quantitative* changes resulting from higher order refinements of the perturbation solution may be definitely relevant for mission designing purposes [61–63]. This is illustrated in Fig. 2, which shows that the third-body critical inclination depends on the orbit semi-major axis, rather than having a constant value, on the one hand, and that direct and retrograde orbits are not at all symmetric, on the other [64,65]. Indeed, direct orbits enjoy larger areas where circular orbits are stable than retrograde ones, cf. [66]. In particular, as clearly noted in Fig. 2, the inclination limit of 39.2° for stable circular motion predicted by the second order truncation, is increased up to approximately 60° for an orbit with semi-major axis of about 0.375 units of the Hill problem. When the Hill problem units are particularized for the dimensions of the Saturn–Enceladus system, this semi-major axis is equivalent to twice the equatorial radius of Enceladus (of about 260 km), which points to this orbital configuration like a possible placement for an Enceladus mapping mission [67,68].

This increase in the range of inclinations showing stability of almost circular direct orbits is, however, less relevant in other planet–satellite systems like in the case of the Jovian moons. For instance, a mapping orbit about Europa would have a semi-major axis of roughly 10% Europa's equatorial radius of 1565 km. When this is converted into units of the Hill problem it amounts to only 0.08, a case in which the upper limit for stability of direct circular orbits is approximately 42°, as checked in Fig. 2, that is clearly insufficient for the global coverage of this body. Therefore, one must confront unstable dynamics, in general, under which the lifetime of a mapping mission is seriously compromised by the eventual exponential increase of the eccentricity. In such cases, the classical control strategy based on the design of tours over the stable–unstable manifolds of the unstable nominal orbit [29] can also be applied to the averaged dynamics [69]. In this latter case, the proper use of the transformation from mean to osculating elements provides clear advantages in the search for long-lifetime orbits [70].

Additional effects, due, for instance, to the non-sphericity of the planetary satellite or the ellipticity of the orbits of the primaries, introduce both qualitative and quantitative modifications in the problem, and should be taken into account in the design of the science orbit. While these additional terms certainly modify the Hill problem dynamics, they can be treated as additional perturbations of the Keplerian motion and are analogously handled within the perturbation approach [54,71–79]. This is also the case of the non-sphericity of the third-body gravity, which can produce observable effects in the orbital dynamics in the vicinity of a binary asteroid system [80].

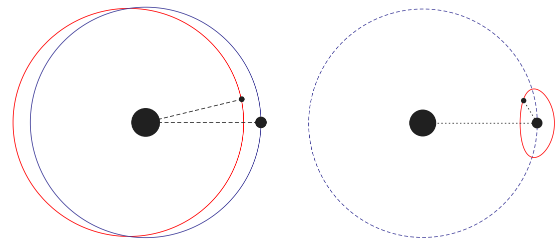

Fig. 3. 1:1 co-orbital motion in inertial (left) and rotating frame (right).

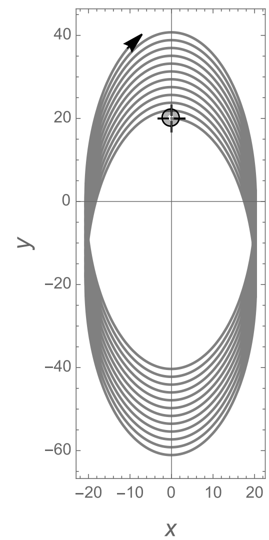

Fig. 4. Typical orbit of the integrable, quadratic part in Eq. (7) of the Hill problem Hamiltonian.

4. Quasi-satellite retrograde orbits

A particular case of co-orbital motion with low eccentricity gives rise to the so-called quasi-satellite orbits, in which the orbiter moves about the primary of smaller mass when seen in the rotating frame (see Fig. 3), but the motion takes place out of the sphere of influence of this primary. This kind of motion, which is observed in the solar system, is appealing for artificial satellites mission due to the strong stability characteristics of these distant retrograde orbits [81–85].

Except for close encounters, the interaction between the primary of smaller mass and the orbiter in co-orbital motion about the primary of bigger mass is very small. Therefore, the Hill problem Hamiltonian accepts the perturbation arrangement in Eqs. (7)–(8). In the planar case, the linear system stemming from the zeroth order Hamiltonian (7) is readily integrated in Cartesian coordinates [39,86]. The solution, which is illustrated in Fig. 4, is a drifting ellipse with eccentricity $e = \sqrt{3/4}$ —that is, the minor and major axis are in the ratio 1:2— that evolves with constant rate in the direction of the major axes. Alternatively, the integration of Eq. (7) is achieved by complete Hamiltonian reduction. The reduction is obtained by a change to epicyclic variables (ϕ, q, Φ, Q), where ϕ is the parametric phase of the reference ellipse (the eccentric anomaly of this non-Keplerian ellipse), Φ is related to the ellipse's dimension, while q and Q are related to the coordinates of its center in the directions of the major and minor axis, respectively [87–89]. This last construction of the solution immediately discloses the linear growth of both the phase ϕ and the drift q.

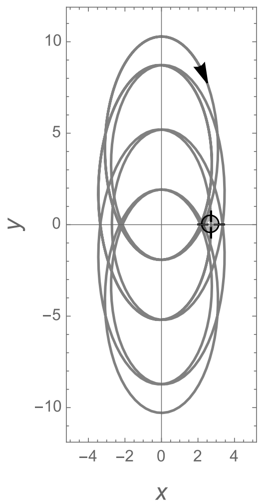

Fig. 5. Quasi-satellite, periodic orbit of the Hill problem (4).

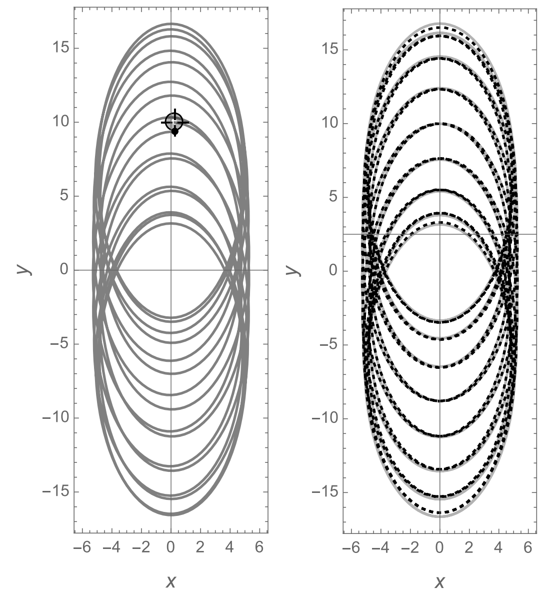

Fig. 6. Left: Non-periodic orbit of the Hill problem. The locator at $y \approx 10$ is used to remark that the orbit does not close after 16 cycles. Right: Nominal 16:1 periodic orbit (gray) superimposed to a true 16:1 periodic orbit (black, dashed) of the Hill problem (after [90]).

Depending on the energy, the nonlinear perturbation term given by the Keplerian potential (8) can turn the constant growth of q into slow oscillations of the center of the reference ellipse, in this way yielding long-period oscillations of the quasi-satellite orbit about the smaller mass primary [39,40]. This kind of trajectory has been found useful in astrodynamics applications for "orbiting" celestial bodies with very low mass, as is the case of the Martian moon Phobos [81], and is illustrated in Fig. 5.

The perturbation approach proceeds by first finding a canonical transformation $\mathcal{T}_1 : (\phi, q, \Phi, Q; \varepsilon) \mapsto (\phi', q', \Phi', Q')$ that, up to some truncation order, removes the phase ϕ' from the transformed Hamiltonian $\mathcal{H} \circ \mathcal{T}_1 = \mathcal{K}(-, q', \Phi', Q') + \mathcal{O}(\varepsilon^m)$, where the formal small parameter ε is related to dynamical considerations. Non-trivial difficulties arise in the reduction process due to the fact that elliptic integrals are unavoidably involved in the procedure [41]. This preliminary reduction discloses the nature of the long-term dynamics, and shows that, on average, the motion is basically composed of two coupled harmonic oscillations. Namely, an oscillation defining the reference ellipse (ϕ', Φ') with constant, fast frequency $\omega = \omega(\Phi')$, which is coupled with an oscillation of the center of the reference ellipse (q', Q') with slow, constant frequency $\Omega = \Omega(\Phi')$.

The basic solution can be solved analytically only for the lower orders of the perturbation approach, yet refinements of the analytical solution are obtained with the classical Lindstedt series method [88, 89]. Alternatively, we can carry out an additional reduction in order to remove the phase of the center of the reference ellipse [53,90]. Indeed, as far as $\sigma = \Omega/\omega \ll 1$, we can choose this ratio as the small parameter of the new perturbation solution. First of all, we carry out an extended harmonic transformation $\mathcal{T}_2 : (\phi', q', \Phi', Q'; \Omega(\Phi')) \mapsto (\psi, \theta, \Psi, \Theta)$. Then, we compute the transformation $\mathcal{T}_3 : (\psi, \theta, \Psi, \Theta; \sigma) \mapsto (\psi', \theta', \Psi', \Theta')$ that casts the partially reduced Hamiltonian in the form $\mathcal{K} \circ \mathcal{T}_2 \circ \mathcal{T}_3 = \mathcal{S}(-, -, \Psi', \Theta') + \mathcal{O}(\sigma^m)$. After truncation, the transformed Hamiltonian $\mathcal{S}$ is completely reduced, thus showing the constant character of the momenta Ψ' and Θ', and the linear evolution of ψ' and θ' with constant frequencies $n_\psi = \partial \mathcal{S}/\partial \Psi'$ and $n_\theta = \partial \mathcal{S}/\partial \Theta'$, respectively. The analytical solution must be complemented, of course, with the canonical transformations that allow to recover the osculating state.

The formal integrals Ψ' and Θ' of the perturbation solution are naturally used like orbit design parameters. In particular, Ψ' is directly related to the semi-major axis of the reference ellipse $a = a(\Psi')$, whereas the minimum distance of the reference ellipse to the smaller primary in the direction of the major axis can be expressed in terms of both formal integrals $y_{\min} = y_{\min}(\Psi', \Theta')$. Moreover, when the values of Ψ' and Θ' be such that the ratio $\rho = \rho(\Psi', \Theta') \equiv n_\psi / n_\theta$ is a rational number, the co-orbital motion will be periodic on average. The detailed description of the design procedure of a periodic orbit with nominal semi-major axis a and minimum distance to the primary $y_{\min}$ can be found in [53,90]. It basically conforms to the following steps.

The procedure starts from the explicit computation of $\Phi' = \Phi'(a)$ obtained from the perturbation theory. Next, $\Theta' = \Theta'(a, y_{\min})$ is likewise computed from the perturbation solution. Then, we compute the secular frequencies n_ψ and n_θ, whose ratio will not be a rational number, in general, but a real one. However, this real number can be turned into a close rational after a standard root finding procedure. This process modifies the nominal parameters but only slightly, therefore yielding the desired periodic nominal orbit. After recovering the periodic terms removed in the perturbation approach, the nominal orbit will be periodic also in the original problem, yet only up to the truncation order of the perturbation theory. If required, the periodicity can be improved through differential corrections to get a true periodic orbit in the original coordinates [91].

To illustrate the procedure, we borrow from [90] an example of a nominal orbit with design parameters $a = 10$ and $y_{\min} = 2.5$, in units of the Hill problem. For these values the ratio between the secular frequencies is $\rho = 15.8$, which does not yield periodicity. We iterate the values of the design parameters until finding a 16:1 commensurability between the secular frequencies, thus yielding a periodic orbit in the secular space. However, when we recover the periodic effects and propagate an initial osculating state in the true Hill problem dynamics, we do not find periodicity due to the truncation of the perturbation solution, as shown in the left plot Fig. 6. Still, the use of differential corrections make the initial conditions to converge fast to a true periodic orbit of the Hill problem. The agreement between the analytical solution and the desired periodic orbit is illustrated in the right plot of Fig. 6, where the analytical solution (black dots) and the true periodic orbit (gray line) are shown superimposed.

Finally, it deserves to be mentioned that, while the perturbation solution was obtained under the assumption that the libration frequency of the center of the reference ellipse is much smaller than the frequency of the orbiter on the reference ellipse, the useful case of the 1:1 resonance can also be approached advantageously with the perturbation solution. The trick is to search for a nominal orbit with $y_{\min} \approx a$.

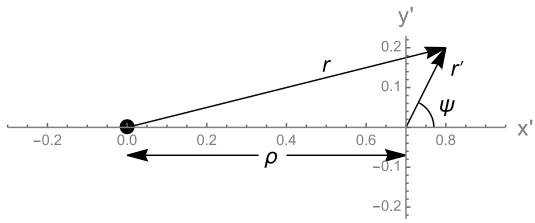

Fig. 7. Geometry of one libration point of the Hill problem. The black circle at the origin is the primary, and $\rho = 3^{-1/3} \approx 0.7$.

Then, the strong stability characteristics of the distant retrograde orbits will make the differential corrections process to readily converge to a 1:1 periodic orbit when periodicity in the orbital period ($\approx 2\pi/n_\phi$) is requested rather than in the much longer librational one ($\approx 2\pi/n_\theta$) . Particular examples of this last case can be found in [53,90].

5. The libration points dynamics

Hamilton equations of Eq. (4) immediately show that the equilibria of the Hill problem are constrained to the two symmetric collinear points $x = \pm 3^{-1/3}$, $y = z = 0$. Orbits in the vicinity of collinear points have different applications in astrodynamics, and they have been profusely studied from both the analytical and numerical point of view [92].

The first step is to translate the origin to the libration point of interest. Since they are symmetric, we only need to pay attention to one of them, and focus on the one at $(3^{-1/3}, 0, 0)$. The basic geometry is illustrated in Fig. 7, where the coordinates of the satellite referred to the libration point are denoted by $\boldsymbol{r}'$.

After reformulating the Hill problem Hamiltonian (4) in the new reference frame, for values $r'/\rho < 1$ the Keplerian potential can be expanded in Legendre polynomials, analogously as we did in the derivation of the Hill problem from the CRTBP. Thus, after some rearrangement, the Hill problem Hamiltonian is written in the form

$$\mathcal{H} = \mathcal{H}_0 - (1/\rho) \sum_{n>0} (r'/\rho)^{n+2} P_{n+2}(\cos\psi) \tag{15}$$

where $\mathcal{H}_0$ is made only of quadratic terms [93]. Therefore, the dynamics about the libration points is amenable to the perturbation approach with integrable zeroth-order term $\mathcal{H}_0$, small parameter ε proportional to the ratio r'/ρ, and perturbation terms made of monomials in Cartesian variables, as results from the nature of Legendre polynomials.

The solution to the linear dynamics stemming from the unperturbed term is obtained by the usual combination of exponentials, and shows the saddle × center × center character of the libration points. In particular, the vertical motion decouples from the planar one and is made of infinitesimal oscillations in the z-axis direction. On the contrary, the x and y components of the planar motion remain coupled, with hyperbolic and elliptic modes. However, the hyperbolic component can be removed by the proper choice of initial conditions, thus showing the existence of orbits that are infinitesimal ellipses in the (x, y)-plane. On the contrary, three dimensional periodic orbits do not exist in the linearized dynamics because the frequency of the vertical oscillations is not commensurable with that of the planar elliptic orbits.

Due to the conservative character of the Hill problem, both kinds of periodic orbits give rise to corresponding families of periodic orbits, which survive beyond the linearized dynamics [94]. On the other hand, the linearized dynamics becomes fully decoupled after a linear transformation [93,95], in this way allowing the realization of the unperturbed Hamiltonian $\mathcal{H}_0$ like the sum of the hyperbolic component and two uncoupled harmonic oscillators. This decoupling shows that there exists one manifold of the center × center type, the so-called center manifold, which is obtained after removing the hyperbolic direction by a proper choice of the initial conditions. Orbits on the center manifold are free from the exponential growth, and hence are appealing for mission design.

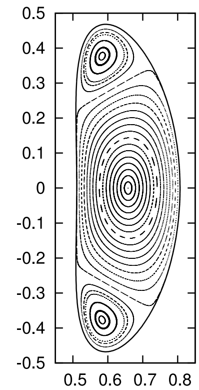

Fig. 8. Poincaré surface of section of the center manifold in the vicinity of the collinear libration points of the Hill problem. Credits: J.M. Mondelo, KePASSA 2017, ESTEC.

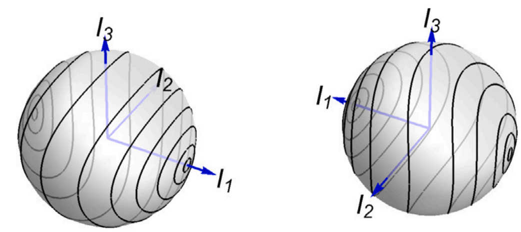

Fig. 9. Reduced phase space showing planar and vertical Lyapunov orbits like fixed points of the elliptic type (after [38]).

When the linear transformation is applied also to the perturbation term of Hamiltonian (15), the hyperbolic component is no longer decoupled. But the corresponding integral of the linearized dynamics is readily extended to the full Hamiltonian by perturbation methods [96] in a process that is customarily know as the reduction to the center manifold. After carrying out this reduction up to the desired truncation order of the small parameter, the hyperbolic direction is once more removed by the proper choice of initial conditions on the center manifold.

Since the center manifold is of just two degrees of freedom, the reduced problem can be approached with the usual tools on nonlinear dynamics, like the construction of Poincaré surfaces of section, or the computation of families of periodic orbits and other invariant manifolds. An example of a sheet of this reduced dynamics is shown in Fig. 8, which shows three fixed points of the elliptic type, corresponding to the vertical Lyapunov and the two symmetric Halo periodic orbits, surrounded by closed curves that represent invariant manifolds of quasi-periodic orbits. In this representation, the curve that bounds the section corresponds to the planar Lyapunov orbit, whose unstable manifold is also clearly identified in the figure with the curves enclosing the Halo-type regime [97].

Alternatively, for values of the energy (or, more properly, the Jacobi constant) close enough to the energy of the libration points, we can use again perturbation theory to carry out an additional reduction of the Hill problem Hamiltonian to a Hamiltonian of one degree of freedom. Traditionally, different reductions are made to deal with non-resonant and resonant motions [56,98]. However, since the frequencies of the elliptic modes of the Hill problem are quite close, 1:1 resonant orbits of the center manifold can be studied together with the planar and vertical Lyapunov periodic orbits as well as the quasi-periodic motion about them [38,93].

Indeed, after a detuning process [99], the partially reduced Hill problem Hamiltonian takes the form of a perturbed elliptic oscillator,

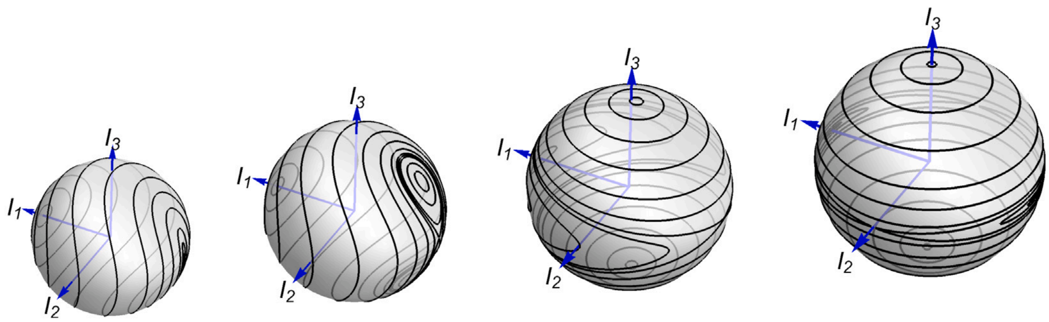

Fig. 10. Changes on the dynamics about the libration points for variations of the Jacobi constant (after [38]).

from which we remove the short-period terms, related to the elliptic anomaly, by perturbation methods to get ellipses with constant semi-major axis on average. Up to the truncation order of the new perturbation process, this additional reduction decouples the dynamics of the line of apsides and eccentricity of the ellipse from the fast motion of the satellite on the ellipse.[5] In spite of we now confront an integrable system of one degree of freedom, the solution, if found, will involve sophisticated special functions [100–102]. However, the study of the equilibria of the reduced phase space serves us to identify the relevant families of periodic orbits. The construction of local phase space diagrams, which are obtained from the inexpensive generation of contour plots of the reduced Hamiltonian, helps also in providing a global view of the long-term dynamics.

On account of the reduced phase space is the sphere [103], the use of Hopf variables [104] reveals advantageous in the description of the integrable dynamics. This is illustrated in Fig. 9 where two opposite views of the sphere are shown for a close value of the Jacobi constant to that of the libration point. The fixed points of the elliptic type $(\pm I_1, 0, 0)$ correspond to vertical $(I_1 > 0)$ and planar Lyapunov orbits $(I_1 < 0)$ [38], whereas the remaining orbits of the center manifold show quasi-periodic behavior.

The evolution of the reduced phase space for increasing values of the Hamiltonian integral (decreasing values of the Jacobi constant) is shown in Fig. 10. The increasing radius of the Hopf sphere reflects a corresponding increase in the size of the orbits with the distance to the libration point. The left plot shows how the flow on the sphere squeezes towards the meridian passing through the elliptic equilibrium corresponding to the planar Lyapunov orbit. Eventually, a pitchfork bifurcation occurs with the change of this fixed point from elliptic to hyperbolic type, and two new fixed points of the elliptic type emerge in the $I_2 = 0$ meridian plane, which are symmetric with respect to the equatorial, $I_3 = 0$ plane (second from the left plot of Fig. 10). They correspond to the bifurcation of Halo orbits from Lyapunov planar orbits. The size of the orbits continue to increase with the distance to the libration point, and the reduced flow compresses again, now towards the equator, about the fixed point corresponding to the planar Lyapunov orbit (second from the right plot of Fig. 10). At some point, a new pitchfork bifurcation from this unstable equilibrium occurs, with the corresponding change to stability (elliptic type), from which two new symmetric fixed points of the elliptic type emerge on the equatorial plane, which are symmetric with respect to the $I_2 = 0$ meridian (right plot of Fig. 10). They correspond to symmetric orbits of each of the two branches of the bridge that connects the families of planar and vertical Lyapunov orbits. Increasing values of the Hamiltonian integral produce the migration of these fixed points along the equator, until they finally collapse onto the fixed point corresponding to the vertical Lyapunov orbit, with the consequent change of the type of this fixed point from elliptic to hyperbolic (not shown in Fig. 10).

While Figs. 9 and 10 have been constructed with just a second order truncation of the perturbation solution [38], they succeed in providing the correct qualitative description of the main families of orbits, including those of the 1:1 resonant dynamics. However, the accuracy of the initial conditions provided by this truncation is clearly insufficient except in the very close proximity of the libration points. The usual differential corrections process helps in finding partner periodic orbits of the Hill problem, but commonly needs much higher order truncations to ease convergence of the corrector [93].

6. Conclusions

Perturbation techniques originally used in astronomical computations are still useful for fast orbit propagation under limited precision. On the other hand, these methods are especially well suited for the reduction of the dimension of a dynamical system without constraint to the classical averaging. The examples provided in this paper illustrate the convenience of having available the transformation from mean to osculating elements in mission designing procedures, on the one hand, as well as the importance that qualitative as well as quantitative changes introduced by higher orders may have in the selection of a nominal orbit, on the other. Therefore, current mission designing procedures may get additional benefits from using the perturbation approach in the preliminary steps.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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