Asymptotic behavior of Bernoulli-Dunkl and Euler-Dunkl polynomials and their zeros

Judit Mínguez Ceniceros and Juan L. Varona

Abstract

Bernoulli-Dunkl and Euler-Dunkl polynomials have been recently introduced as an extension of Bernoulli and Euler polynomials to the Dunkl context. In this article we study the asymptotic behavior of them and prove their convergence to suitable Bessel functions, which are the functions analogous to sine and cosine in the Dunkl context. Finally, we analyze the behavior of the zeros of the Bernoulli-Dunkl and Euler-Dunkl polynomials.

Keywords: Appell-Dunkl sequences, Bernoulli-Dunkl polynomials, Euler-Dunkl polynomials, Fourier series, Fourier-Dunkl series, Bessel functions, Asymptotic behavior, Distribution of zeros.

1 Introduction

Bernoulli and Euler polynomials (see [12] for details) have many applications to number theory, numerical analysis, combinatorics, and other areas, and, as a consequence, they have been widely studied over the last two centuries. Usually, Bernoulli polynomials $B_n(x)$ and Euler polynomials $E_n(x)$ are defined by means of the generating functions

$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!}\, t^n, \qquad \frac{2e^{tx}}{e^t + 1} = \sum_{n=0}^{\infty} \frac{E_n(x)}{n!}\, t^n.$$

In [9], Dilcher studies the asymptotic behavior of these polynomials and proves that they converge to the sine and cosine functions (more details about this convergence can be seen in [22]). Namely, he proves that the following

This paper has been published as: J. Mínguez Ceniceros and J. L. Varona, Asymptotic behavior of Bernoulli-Dunkl and Euler-Dunkl polynomials and their zeros, *Funct. Approx. Comment. Math.* **65** (2021), no. 2, 211–226, `https://doi.org/10.7169/facm/1968`

Research partially supported by PGC2018-096504-B-C32 (Ministerio de Ciencia, Innovación y Universidades, Spain) and FEDER Funds (European Union).

2020 Mathematics Subject Classification: primary: 11B68; secondary: 42C10, 33C10 sequences converge uniformly on compact subsets of $\mathbb{C}$:

$$\lim_{k\to\infty}(-1)^{k-1}\frac{(2\pi)^{2k}}{2(2k)!}B_{2k}(z)=\cos(2\pi z), \tag{1}$$

$$\lim_{k\to\infty}(-1)^{k-1}\frac{(2\pi)^{2k+1}}{2(2k+1)!}B_{2k+1}(z)=\sin(2\pi z), \tag{2}$$

$$\lim_{k\to\infty}(-1)^{k}\frac{\pi^{2k+1}}{4(2k)!}E_{2k}(z)=\sin(\pi z), \tag{3}$$

$$\lim_{k\to\infty}(-1)^{k+1}\frac{\pi^{2k+2}}{4(2k+1)!}E_{2k+1}(z)=\cos(\pi z). \tag{4}$$

Additionally, applying the Hurwitz theorem, it can be deduced that the real zeros of the Bernoulli and Euler polynomials converge to the zeros of the corresponding cosine or sine functions that appear in the limits. Dilcher also proves in [11] that the zeros of these polynomials are not multiple.

In the mathematical literature, there are many kinds of generalizations of the Bernoulli (and Euler) polynomials: generalized Bernoulli polynomials or Nørlund polynomials [23, 24], Apostol-Bernoulli polynomials [1], hypergeometric Bernoulli polynomials [17, 18, 14], Bernoulli-Padé polynomials [13]; see also [12, § 24.16(iii)] and the book [27] and the references therein.

As far as this paper is concerned, since the foundational article [15] a large number of papers have been published extending the Fourier analysis to a more general context. The ordinary derivative is replaced by a differential-difference operator that depends on a constant α, and the Fourier transform is replaced by the so-called Dunkl transform. In the same way, Appell sequences of polynomials have been extended to the so called Appell-Dunkl sequences. This first happened with the Hermite polynomials in the celebrated paper [26], and recently also the Bernoulli and Euler polynomials have been defined in the Dunkl context, see [5, 6].

The aim of this paper is to study the properties analogous to (1)–(4), and the behaviour of the zeros of the polynomials, in the Dunkl context (on the real line). We use Bernoulli-Dunkl and Euler-Dunkl polynomials instead of the classical Bernoulli and Euler polynomials. To do this kind of extension, let us start recalling that, for $\alpha > -1$, the Bessel function of order α is

$$J_\alpha(x)=\left(\frac{x}{2}\right)^{\alpha}\sum_{n=0}^{\infty}\frac{(-1)^n(x/2)^{2n}}{n!\,\Gamma(\alpha+n+1)}.$$

Throughout this paper, we will use $\frac{J_\alpha(z)}{z^\alpha}$ to denote the even function

$$\frac{1}{2^\alpha}\sum_{n=0}^{\infty}\frac{(-1)^n(z/2)^{2n}}{n!\,\Gamma(\alpha+n+1)},$$

which is analytic in $\mathbb{C}$.

For the Dunkl context, instead of the exponential function e^z, we consider the following function for $\alpha > -1$,

$$E_\alpha(z) = \mathcal{I}_\alpha(z) + \frac{z}{2(\alpha+1)}\,\mathcal{I}_{\alpha+1}(z), \qquad z \in \mathbb{C},$$

where $\mathcal{I}_\alpha$ is given in terms of the Bessel function,

$$\mathcal{I}_\alpha(z) = 2^\alpha \Gamma(\alpha+1)\,\frac{J_\alpha(iz)}{(iz)^\alpha} = \Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(z/2)^{2n}}{n!\,\Gamma(n+\alpha+1)} = {}_0F_1(\alpha+1, z^2/4)$$

(the function $\mathcal{I}_\alpha$ is a small variation of the so-called modified Bessel function of the first kind and order α, usually denoted by I_α; see [19], [25] or [32]).

Following [15] for $\alpha \geq -1/2$ and [26] for $\alpha > -1$, in the real line and with the reflection group $\mathbb{Z}_2$, the Dunkl operator Λ_α is defined as

$$\Lambda_\alpha f(x) = \frac{d}{dx}f(x) + \frac{2\alpha+1}{2}\left(\frac{f(x)-f(-x)}{x}\right),$$

where f are suitable functions on $\mathbb{R}$. It is easy to check that, for any $\lambda \in \mathbb{C}$, we have

$$\Lambda_\alpha E_\alpha(\lambda x) = \lambda E_\alpha(\lambda x).$$

Let us note that, when $\alpha = -1/2$, we have $\Lambda_{-1/2} = d/dx$ and $E_{-1/2}(\lambda x) = e^{\lambda x}$.

From the definition, it is also easy to check that

$$E_\alpha(z) = \sum_{n=0}^{\infty}\frac{z^n}{\gamma_{n,\alpha}}$$

with

$$\gamma_{n,\alpha} = \begin{cases} 2^{2k}k!\,(\alpha+1)_k, & \text{if } n = 2k, \\ 2^{2k+1}k!\,(\alpha+1)_{k+1}, & \text{if } n = 2k+1, \end{cases}$$

and where $(a)_n$ denotes the Pochhammer symbol

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$$

(where a is not a negative integer); of course, $\gamma_{n,-1/2} = n!$. We also define

$$\binom{n}{j}_\alpha = \frac{\gamma_{n,\alpha}}{\gamma_{j,\alpha}\gamma_{n-j,\alpha}},$$

which becomes the ordinary binomial numbers in the case $\alpha = -1/2$.

As a generalization of the Bernoulli polynomials to the Dunkl context, Bernoulli-Dunkl polynomials $\{\mathfrak{B}_{n,\alpha}\}_n$ are defined in [5] by means of the generation function

$$\frac{E_\alpha(xt)}{\mathcal{I}_{\alpha+1}(t)} = \sum_{n=0}^{\infty}\frac{\mathfrak{B}_{n,\alpha}(x)}{\gamma_{n,\alpha}}\,t^n.$$

Similarly, Euler-Dunkl polynomials $\{\mathfrak{E}_{n,\alpha}\}_n$ were introduced in [16] by means of the generation function

$$\frac{E_\alpha(xt)}{\mathcal{I}_\alpha(t)} = \sum_{n=0}^{\infty} \frac{\mathfrak{E}_{n,\alpha}(x)}{\gamma_{n,\alpha}}\, t^n.$$

For $\alpha = -1/2$, the Bernoulli-Dunkl and the Euler-Dunkl polynomials become the classical Bernoulli and Euler polynomials, except by a change of variable to transform the interval $(0,1)$ in the interval $(-1,1)$, namely

$$\mathfrak{B}_{n,-1/2}(2x-1) = 2^n B_n(x), \qquad \mathfrak{E}_{n,-1/2}(2x-1) = 2^n E_n(x). \tag{5}$$

Many other properties of the polynomials $\mathfrak{B}_{n,\alpha}$ and $\mathfrak{E}_{n,\alpha}$ have been studied in [6]. Here, let us only mention that the derivatives of the Bernoulli and the Euler polynomials satisfy $B_n'(x) = nB_{n-1}(x)$ and $E_n'(x) = nE_{n-1}(x)$; in the case of the Bernoulli-Dunkl and the Euler-Dunkl polynomials, the role of the derivative is played by the Dunkl operator Λ_α (see the item 1 in Lemmas 2.1 and 3.1), so this is the reason to say that $\mathfrak{B}_{n,\alpha}$ and $\mathfrak{E}_{n,\alpha}$ are generalizations to the Dunkl context.

Euler's formula relates the complex exponential function with the trigonometric functions by means of

$$e^{ix} = \cos x + i \sin x.$$

If $E_\alpha(x)$ plays the role of the exponential, as

$$E_\alpha(ix) = \mathcal{I}_\alpha(ix) + i\,\frac{x\mathcal{I}_{\alpha+1}(ix)}{2(\alpha+1)} = 2^\alpha\Gamma(\alpha+1)\left(\frac{J_\alpha(x)}{x^\alpha} + ix\,\frac{J_{\alpha+1}(x)}{x^{\alpha+1}}\right),$$

the functions $x\frac{J_{\alpha+1}(x)}{x^{\alpha+1}}$ and $\frac{J_\alpha(x)}{x^\alpha}$ would correspond, in the Dunkl context, to $\sin x$ and $\cos x$, respectively (except for the constant $2^\alpha\Gamma(\alpha+1)$).

The goal of this paper is to obtain the asymptotic behavior of the Bernoulli-Dunkl and the Euler-Dunkl polynomials, and, subsequently, to study their zeros. In Theorems 1.1 and 1.2 we show the analogous formulas to (1), (2), (3) and (4) in the Dunkl context.

Theorem 1.1. *Let $\alpha > -1$. The following sequences converge uniformly on compact subsets of $\mathbb{C}$:*

$$\lim_{k\to\infty} \frac{(-1)^{k+1}(\alpha+1)s_1^{2k}}{\gamma_{2k,\alpha}}\,\frac{J_\alpha(s_1)}{s_1^\alpha}\,\mathfrak{B}_{2k,\alpha}(z) = \frac{J_\alpha(s_1z)}{(s_1z)^\alpha}, \tag{6}$$

$$\lim_{k\to\infty} \frac{(-1)^{k+1}(\alpha+1)s_1^{2k+1}}{\gamma_{2k+1,\alpha}}\,\frac{J_\alpha(s_1)}{s_1^\alpha}\,\mathfrak{B}_{2k+1,\alpha}(z) = s_1z\,\frac{J_{\alpha+1}(s_1z)}{(s_1z)^{\alpha+1}}, \tag{7}$$

where s_1 is the first positive zero of $J_{\alpha+1}(z)$.

Theorem 1.2. *Let $\alpha > -1$. The following sequences converge uniformly on compact subsets of $\mathbb{C}$:*

$$\lim_{k\to\infty} \frac{(-1)^k j_1^{2k+2}}{2\gamma_{2k,\alpha}} \frac{J_{\alpha+1}(j_1)}{j_1^{\alpha+1}} \mathfrak{E}_{2k,\alpha}(z) = \frac{J_\alpha(j_1 z)}{(j_1 z)^\alpha}, \tag{8}$$

$$\lim_{k\to\infty} \frac{(-1)^k j_1^{2k+3}}{2\gamma_{2k+1,\alpha}} \frac{J_{\alpha+1}(j_1)}{j_1^{\alpha+1}} \mathfrak{E}_{2k+1,\alpha}(z) = j_1 z \frac{J_{\alpha+1}(j_1 z)}{(j_1 z)^{\alpha+1}}, \tag{9}$$

where j_1 is the first positive zero of $J_\alpha(z)$.

Let us recall that

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin(x), \qquad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos(x)$$

so, in the case $\alpha = -1/2$, we have $s_1 = \pi$ and $j_1 = \pi/2$. Then, taking into account (5), it is easy to see that, indeed, the limits (6), (7), (8) and (9) become (1), (2), (3) and (4).

Theorems 1.1 and 1.2 will help us locate the real zeros of Bernoulli-Dunkl and Euler-Dunkl polynomials in a very direct way. With respect to complex zeros, the zero attractors of the Euler and Bernoulli polynomials have been studied in [3]; in our case, we have made some numerical experiments that show, in a graphical way, how the complex zeros of Bernoulli-Dunkl and Euler-Dunkl polynomials are distributed.

The content of the paper is organized as follows. Section 2 is devoted to study Bernoulli-Dunkl polynomials and to prove Theorem 1.1. In Section 3, we prove the results for Euler-Dunkl polynomials. In both sections we show numerical experiments. Finally, the behavior of the zeros of the polynomials are explained in Section 4.

2 Bernoulli-Dunkl polynomials

Let us start showing some basic properties, which were proved in [5].

Lemma 2.1. *The Bernoulli-Dunkl polynomials satisfy the following properties:*

1. $\Lambda_\alpha(\mathfrak{B}_{n,\alpha})(x) = \frac{\gamma_{n,\alpha}}{\gamma_{n-1,\alpha}} \mathfrak{B}_{n-1,\alpha}(x) = \big(n + (\alpha+1/2)(1-(-1)^n)\big)\mathfrak{B}_{n-1,\alpha}(x)$.
2. *$\mathfrak{B}_{2n,\alpha}(x)$ is an even polynomial, $n \ge 0$, and $\mathfrak{B}_{2n+1,\alpha}(x)$ is an odd polynomial, $n \ge 0$, which vanishes at 1 (and hence at -1) for $n \ge 1$.*
3. *They can be written by*

$$\mathfrak{B}_{n,\alpha}(x) = \sum_{j=0}^{n} \binom{n}{j}_\alpha \mathfrak{B}_{j,\alpha}(0) x^{k-j}. \tag{10}$$

As $\alpha > -1$, the Bessel function $J_{\alpha+1}(x)$ has a strictly increasing sequence of positive zeros $\{s_j\}_{j\geq 1}$, and the real function $\operatorname{Im} E_\alpha(ix) = \frac{x}{2(\alpha+1)}\mathcal{I}_{\alpha+1}(ix)$ is odd and has an infinite sequence of zeros $\{s_j\}_{j\in\mathbb{Z}}$ with $s_{-j} = -s_j$ and $s_0 = 0$. In [7], as a generalization of the traditional orthogonal system $\{e^{ijx}\}_{j\in\mathbb{Z}}$ (which corresponds to te case $\alpha = -1/2$), an orthogonal system for the Dunkl context is introduced. This system is given by

$$e_{\alpha,j}(x) = \frac{2^{\alpha/2}\Gamma(\alpha+1)^{1/2}}{|\mathcal{I}_\alpha(is_j)|}E_\alpha(is_jx), \quad j \in \mathbb{Z}\setminus\{0\},\ x \in [-1,1],$$

and

$$e_{\alpha,0}(x) = 2^{(\alpha+1)/2}\Gamma(\alpha+2)^{1/2},$$

and it is orthonormal and complete with respect to the measure $\frac{|x|^{2\alpha+1}\,dx}{2^{\alpha+1}\Gamma(\alpha+1)}$ in $[-1,1]$. A property that we will use later in this paper is

$$(-1)^j e_{\alpha,j}(0) = \frac{2^{\alpha/2}\Gamma(\alpha+1)^{1/2}}{\mathcal{I}_\alpha(is_j)}. \tag{11}$$

In a similar way to the classical Bernoulli polynomials, which have the Hurwitz expansion

$$B_n(x) = -\frac{n!}{(2\pi i)^n}\sum_{j\in\mathbb{Z}\setminus\{0\}}\frac{e^{2\pi ijx}}{j^n}, \qquad x \in [0,1],$$

the Bernoulli-Dunkl polynomials have a nice expansion on the system $\{e_{\alpha,j}(x)\}_{j\in\mathbb{Z}}$. It was shown in [5] that this expansion is the following:

Theorem 2.2. *Let $-1 < \alpha < n+1/2$ and $n \geq 2$. Then,*

$$\mathfrak{B}_{n,\alpha}(x) = \frac{-(-i)^n\gamma_{n,\alpha}}{2^{1+\alpha/2}(\alpha+1)\Gamma(\alpha+1)^{1/2}}\sum_{j\in\mathbb{Z}\setminus\{0\}}\frac{(-1)^j}{s_j^n}e_{\alpha,j}(x), \tag{12}$$

with pointwise convergence in $[-1,1]$.

As a consequence of Theorem 2.2, we can prove the following result:

Lemma 2.3. *Let $\alpha > -1$. Then, for all integer $j_0 \geq 2$,*

$$\frac{\mathfrak{B}_{2k,\alpha}(0)}{\gamma_{2k,\alpha}} = \frac{(-1)^{k+1}}{\alpha+1}\sum_{j=1}^{j_0-1}\frac{1}{s_j^{2k}\mathcal{I}_\alpha(is_j)} + O\left(\frac{1}{s_{j_0}^{2k-\alpha-1/2}}\right), \quad k \to \infty.$$

Proof. From (12) and (11) we can write

$$\frac{\mathfrak{B}_{2k,\alpha}(0)}{\gamma_{2k,\alpha}} = \frac{(-1)^{k+1}}{2(\alpha+1)}\sum_{j\in\mathbb{Z}\setminus\{0\}}\frac{1}{s_j^{2k}\mathcal{I}_\alpha(is_j)} = \frac{(-1)^{k+1}}{\alpha+1}\sum_{j=1}^{\infty}\frac{1}{s_j^{2k}\mathcal{I}_\alpha(is_j)}.$$

Then, taking the tail

$$\sum_{j=j_0}^{\infty} \frac{1}{|s_j^{2k}\mathcal{I}_\alpha(is_j)|} = \frac{1}{|s_{j_0}^{2k}|}\sum_{j=j_0}^{\infty} \frac{1}{|\mathcal{I}_\alpha(is_j)|}\left|\frac{s_{j_0}}{s_j}\right|^{2k},$$

and using the estimate (see, for instance, [16])

$$\frac{1}{|\mathcal{I}_\alpha(is_j)|} \le C_\alpha |s_j|^{\alpha+1/2},$$

where C_α is a constant that depends only on α, we have that

$$\sum_{j=j_0}^{\infty} \frac{1}{|s_j^{2k}\mathcal{I}_\alpha(is_j)|} \le \frac{C_\alpha}{|s_{j_0}|^{2k-\alpha-1/2}}\sum_{j=j_0}^{\infty} \left|\frac{s_{j_0}}{s_j}\right|^{2k-\alpha-1/2},$$

and the last series is absolutely convergent for $2k-\alpha-1/2 > 1$ because the asymptotic behavior $s_j = \pi j + O(1)$ (see [19, § 5.13], [25, § 10.21] or [32, Chapter XV]). □

We denote the section of the function $2^\alpha\Gamma(\alpha+1)\frac{J_\alpha(z)}{z^\alpha}$ as

$$T_{2k,\alpha}(z) = \sum_{j=0}^{k} \frac{(-1)^j}{\gamma_{2j,\alpha}} z^{2j},$$

and the section of $2^\alpha\Gamma(\alpha+1)z\frac{J_{\alpha+1}(z)}{z^{\alpha+1}}$ as

$$T_{2k+1,\alpha}(z) = \sum_{j=0}^{k} \frac{(-1)^j}{\gamma_{2j+1,\alpha}} z^{2j+1}.$$

Note that, if $\alpha = -1/2$, they are the partial sums of the functions $\cos z$ and $\sin z$, respectively.

Now, we prove the following result.

Theorem 2.4. *Let $\alpha > -1$. Then, for k big enough and for all $z \in \mathbb{C}$,*

$$\left|\frac{\alpha+1}{\gamma_{2k,\alpha}}\mathfrak{B}_{2k,\alpha}(z) - \frac{(-1)^{k+1}}{s_1^{2k}\mathcal{I}_\alpha(is_1)}T_{2k,\alpha}(s_1 z)\right| \le \frac{C_\alpha}{|s_2|^{2k}}E_\alpha(|s_2 z|), \tag{13}$$

$$\left|\frac{\alpha+1}{\gamma_{2k+1,\alpha}}\mathfrak{B}_{2k+1,\alpha}(z) - \frac{(-1)^{k+1}}{s_1^{2k+1}\mathcal{I}_\alpha(is_1)}T_{2k+1,\alpha}(s_1 z)\right| \le \frac{D_\alpha}{|s_2|^{2k+1}}E_\alpha(|s_2 z|), \tag{14}$$

where C_α and D_α denote constants depending only on α.

Proof. From (10), $\mathfrak{B}_{2k,\alpha}(z)$ can be expressed in terms of the coefficients $\mathfrak{B}_{j,\alpha}(0)$ and, from item 2 of Lemma 2.1, $\mathfrak{B}_{2j+1,\alpha}(0) = 0$ for every $j \ge 0$. So, we can write

$$\frac{\alpha+1}{\gamma_{2k,\alpha}}\mathfrak{B}_{2k,\alpha}(z) = \frac{\alpha+1}{\gamma_{2k,\alpha}}\sum_{j=0}^{k}\binom{2k}{2j}_\alpha \mathfrak{B}_{2j,\alpha}(0) z^{2k-2j} = (\alpha+1)\sum_{j=0}^{k}\frac{\mathfrak{B}_{2j,\alpha}(0)}{\gamma_{2j,\alpha}}\frac{z^{2k-2j}}{\gamma_{2k-2j,\alpha}}.$$

When k_0 is big enough and $k > k_0$, from Lemma 2.3 with $j_0 = 2$ we have

$$
\begin{aligned}
\frac{\alpha+1}{\gamma_{2k,\alpha}}\mathfrak{B}_{2k,\alpha}(z) &= (\alpha+1)\sum_{j=0}^{k_0}\frac{\mathfrak{B}_{2j,\alpha}(0)}{\gamma_{2j,\alpha}}\frac{z^{2k-2j}}{\gamma_{2k-2j,\alpha}} \\
&\qquad + \sum_{j=k_0+1}^{k}\left(\frac{(-1)^{j+1}}{s_1^{2j}\mathcal{I}_\alpha(is_1)} + O\left(\frac{1}{s_2^{2j-\alpha-1/2}}\right)\right)\frac{z^{2k-2j}}{\gamma_{2k-2j,\alpha}}.
\end{aligned} \tag{15}
$$

On the other hand,

$$
\begin{aligned}
\frac{1}{s_1^{2k}\mathcal{I}_\alpha(is_1)}T_{2k,\alpha}(s_1 z) &= \frac{1}{s_1^{2k}\mathcal{I}_\alpha(is_1)}\sum_{j=0}^{k}\frac{(-1)^j}{\gamma_{2j,\alpha}}s_1^{2j}z^{2j} \\
&= \frac{1}{s_1^{2k}\mathcal{I}_\alpha(is_1)}\sum_{j=0}^{k_0}\frac{(-1)^{k-j}}{\gamma_{2k-2j,\alpha}}s_1^{2k-2j}z^{2k-2j} + \sum_{j=k_0+1}^{k}\frac{(-1)^{k-j}}{s_1^{2j}\mathcal{I}_\alpha(is_1)}\frac{z^{2k-2j}}{\gamma_{2k-2j,\alpha}}.
\end{aligned} \tag{16}
$$

Taking into account (15) and (16), we obtain

$$
\begin{aligned}
&\left|\frac{\alpha+1}{\gamma_{2k,\alpha}}\mathfrak{B}_{2k,\alpha}(z) - (-1)^{k+1}\frac{1}{s_1^{2k}\mathcal{I}_\alpha(is_1)}T_{2k,\alpha}(s_1 z)\right| \\
&\le \sum_{j=0}^{k_0}\left|\frac{\alpha+1}{\gamma_{2j,\alpha}}\mathfrak{B}_{2j,\alpha}(0) - \frac{(-1)^{j+1}}{s_1^{2j}\mathcal{I}_\alpha(is_1)}\right|\frac{|z|^{2k-2j}}{\gamma_{2k-2j,\alpha}} + C_\alpha\sum_{j=k_0+1}^{k}\frac{1}{|s_2|^{2j-\alpha-1/2}}\frac{|z|^{2k-2j}}{\gamma_{2k-2j,\alpha}} \\
&\qquad \le \frac{1}{|s_2|^{2k}}\sum_{j=0}^{k_0}\left|\frac{\alpha+1}{\gamma_{2j,\alpha}}\mathfrak{B}_{2j,\alpha}(0) - \frac{(-1)^{j+1}}{s_1^{2j}\mathcal{I}_\alpha(is_1)}\right||s_2|^{2j}\frac{|s_2 z|^{2k-2j}}{\gamma_{2k-2j,\alpha}} \\
&\qquad\qquad + \frac{C_\alpha}{|s_2|^{2k-\alpha-1/2}}\sum_{j=k_0+1}^{k}\frac{|s_2 z|^{2k-2j}}{\gamma_{2k-2j,\alpha}} \le \frac{C_\alpha}{|s_2|^{2k}}E_\alpha(|s_2 z|),
\end{aligned}
$$

so we have proved (13). The proof of (14) is similar. □

The partial sums $T_{2k}(s_1 z)$ converge uniformly on compact subsets of $\mathbb{C}$ to the analytic function $\mathcal{I}_\alpha(is_1 z)$, and $T_{2k+1}(s_1 z)$ do to $\frac{s_1 z}{2(\alpha+1)}\mathcal{I}_{\alpha+1}(is_1 z)$. Thus, Theorem 1.1 is obtained as a consequence of Theorem 2.4, as follows.

Proof of Theorem 1.1. Multiplying (13) by $|(-1)^{k+1}s_1^{2k}\mathcal{I}_\alpha(is_1)|$, the right part converges to zero when $k \to \infty$ because $|s_1/s_2| < 1$. Then, we have

$$
\lim_{k\to\infty}\left|\frac{(-1)^{k+1}s_1^{2k}(\alpha+1)\mathcal{I}_\alpha(is_1)}{\gamma_{2k,\alpha}}\mathfrak{B}_{2k,\alpha}(z) - T_{2k}(s_1 z)\right| = 0,
$$

so (6) is obtained. The limit (7) can be deduced in the same way. □

We can make some numerical experiments that show, in a graphical way, the approximations given in Theorem 1.1. For convenience, let us set

$$
\mathfrak{B}^*_{2k,\alpha}(x) = \frac{(-1)^{k+1}(\alpha+1)s_1^{2k}}{\gamma_{2k,\alpha}}\frac{J_\alpha(s_1)}{s_1^\alpha}\mathfrak{B}_{2k,\alpha}(x)
$$

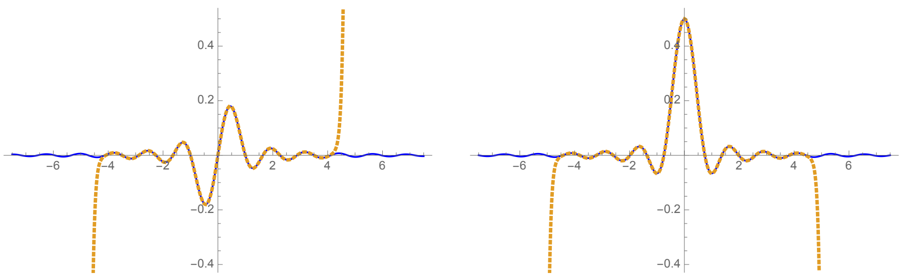

Figure 1: For $\alpha = 1$, comparison of $\mathfrak{B}^*_{45,\alpha}(x)$ and $s_1 x \frac{J_{\alpha+1}(s_1 x)}{(s_1 x)^{\alpha+1}}$ (on the left), and comparison of $\mathfrak{B}^*_{50,\alpha}(x)$ and $\frac{J_\alpha(s_1 x)}{(s_1 x)^\alpha}$ (on the right).

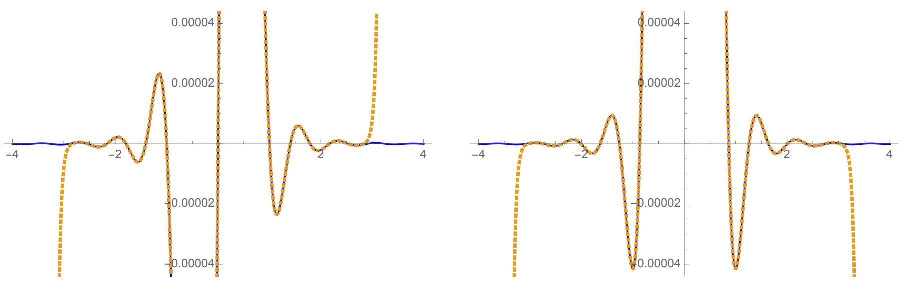

Figure 2: For $\alpha = 4$, comparison of $\mathfrak{B}^*_{45,\alpha}(x)$ and $s_1 x \frac{J_{\alpha+1}(s_1 x)}{(s_1 x)^{\alpha+1}}$ (on the left), and comparison of $\mathfrak{B}^*_{50,\alpha}(x)$ and $\frac{J_\alpha(s_1 x)}{(s_1 x)^\alpha}$ (on the right).

and

$$\mathfrak{B}^*_{2k+1,\alpha}(x) = \frac{(-1)^{k+1}(\alpha+1)s_1^{2k+1}}{\gamma_{2k+1,\alpha}} \frac{J_\alpha(s_1)}{s_1^\alpha} \mathfrak{B}_{2k+1,\alpha}(x).$$

In Figures 1 and 2, and for different values for α and n, the dashed lines correspond to the polynomials $\mathfrak{B}^*_n(x)$, and the solid line to the functions $s_1 x \frac{J_{\alpha+1}(s_1 x)}{(s_1 x)^{\alpha+1}}$ (for odd n) or $\frac{J_\alpha(s_1 x)}{(s_1 x)^\alpha}$ (for even n).

3 Euler-Dunkl polynomials

We list some of the properties of these polynomials, as stated in [16]:

Lemma 3.1. *The Euler-Dunkl polynomials satisfy the following properties:*

1. $\Lambda_\alpha(\mathfrak{E}_{n,\alpha})(x) = \frac{\gamma_{n,\alpha}}{\gamma_{n-1,\alpha}} \mathfrak{E}_{n-1,\alpha}(x) = \big(n + (\alpha + 1/2)(1 - (-1)^n)\big)\mathfrak{E}_{n-1,\alpha}(x)$.
2. $\mathfrak{E}_{2n,\alpha}(x)$ *is an even polynomial,* $n \geq 0$*, which vanishes at* 1 *(and hence at* -1*) for* $n \geq 1$*, and* $\mathfrak{E}_{2n+1,\alpha}(x)$ *is an odd polynomial,* $n \geq 0$.

3. They can be written by

$$\mathfrak{E}_{n,\alpha}(x) = \sum_{j=0}^{n} \binom{n}{j}_{\alpha} \mathfrak{E}_{j,\alpha}(0) x^{k-j}.$$

The Fourier-Dunkl orthogonal system $\{e_{\alpha,m}\}_{m\in\mathbb{Z}}$ introduced in [7] is very useful to approximate Bernoulli-Dunkl polynomials (recall Theorem 2.2), but, to approximate Euler-Dunkl polynomials, it is much better to use another orthogonal system. With this aim, the so-called Fourier-Dunkl system of the second kind is introduced in [16]. Let $\{j_m\}_{m\geq 1}$ be the zeros of the Bessel function $J_\alpha(x)$ with $\alpha > -1$, and $j_{-m} = -j_m$, and set

$$f_{\alpha,m}(x) = \frac{2^{\alpha/2+1}(\alpha+1)\Gamma(\alpha+1)^{1/2}}{|j_m \mathcal{I}_{\alpha+1}(ij_m)|} E_\alpha(ij_m x), \quad x \in [-1,1],\ m \in \mathbb{Z}\setminus\{0\}.$$

Then, $\{f_{\alpha,m}\}_{m\in\mathbb{Z}\setminus\{0\}}$ is a complete othonormal system with respect to the measure $\frac{|x|^{2\alpha+1}\,dx}{2^{\alpha+1}\Gamma(\alpha+1)}$ in $[-1,1]$. It is again useful to know the value of the orthogonal functions at $x=0$, that is,

$$f_{\alpha,m}(0) = \frac{2^{\alpha/2+1}(\alpha+1)\Gamma(\alpha+1)^{1/2}}{(-1)^{m+1}|j_m|\mathcal{I}_{\alpha+1}(ij_m)}. \tag{17}$$

With this orthogonal system, an analogous formula to (12) is obtained for Euler-Dunkl polynomials:

Theorem 3.2. *Let $-1 < \alpha < n+1/2$ and $n \geq 1$. Then,*

$$\mathfrak{E}_{n,\alpha}(z) = \frac{(-1)^{n+1} i^n \gamma_{n,\alpha}}{2^{\alpha/2}\Gamma(\alpha+1)^{1/2}} \sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{(-1)^m \operatorname{sgn}(m)}{j_m^{n+1}} f_{\alpha,m}(z), \tag{18}$$

with pointwise convergence in $[-1,1]$.

Thus, using (18), (17) and Lemma 3.1, we can give a result that is analogous to Lemma 2.3:

Lemma 3.3. *Let $\alpha > -1$. Then, for all integer $m_0 \geq 2$,*

$$\frac{\mathfrak{E}_{2k,\alpha}(0)}{\gamma_{2k,\alpha}} = (-1)^k 4(\alpha+1) \sum_{m=1}^{m_0-1} \frac{1}{j_m^{2k+2}\mathcal{I}_{\alpha+1}(ij_m)} + O\left(\frac{1}{j_{m_0}^{2k-\alpha+1/2}}\right), \quad k \to \infty.$$

Now, in the same way as in Theorem 3.4, we have the following:

Theorem 3.4. *Let $\alpha > -1$. Then, for k big enough and for all $z \in \mathbb{C}$,*

$$\left| \frac{1}{4(\alpha+1)} \frac{\mathfrak{E}_{2k,\alpha}(z)}{\gamma_{2k,\alpha}} - \frac{(-1)^k}{j_1^{2k+2}\mathcal{I}_{\alpha+1}(ij_1)} T_{2k,\alpha}(j_1 z) \right| \leq \frac{C_\alpha}{j_2^{2k}} E_\alpha(|j_2 z|),$$

$$\left| \frac{1}{4(\alpha+1)} \frac{\mathfrak{E}_{2k+1,\alpha}(z)}{\gamma_{2k+1,\alpha}} - \frac{(-1)^k}{j_1^{2k+3}\mathcal{I}_{\alpha+1}(ij_1)} T_{2k+1,\alpha}(j_1 z) \right| \leq \frac{D_\alpha}{j_2^{2k+1}} E_\alpha(|j_2 z|),$$

where C_α and D_α denote constants depending only on α.

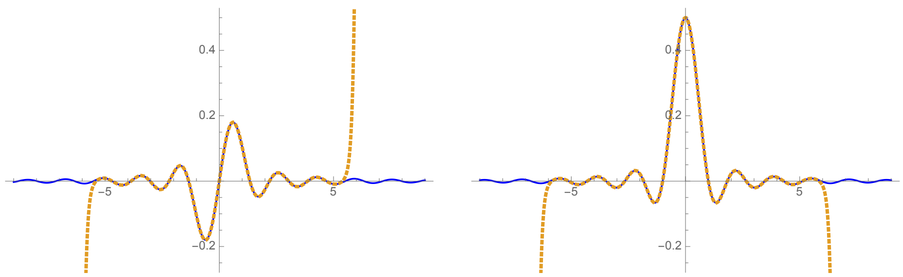

Figure 3: For $\alpha = 1$, comparison of $\mathfrak{E}^*_{45,\alpha}(x)$ and $j_1 z \frac{J_{\alpha+1}(j_1 z)}{(j_1 z)^{\alpha+1}}$ (on the left), and comparison of $\mathfrak{E}^*_{50,\alpha}(x)$ and $\frac{J_\alpha(j_1 z)}{(j_1 z)^\alpha}$ (on the right).

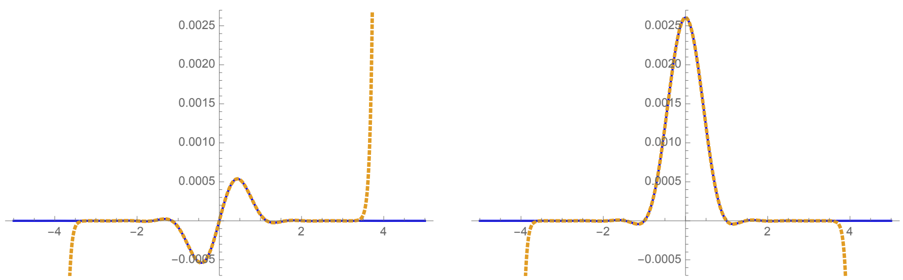

Figure 4: For $\alpha = 4$, comparison of $\mathfrak{E}^*_{45,\alpha}(x)$ and $j_1 z \frac{J_{\alpha+1}(j_1 z)}{(j_1 z)^{\alpha+1}}$ (on the left), and comparison of $\mathfrak{E}^*_{50,\alpha}(x)$ and $\frac{J_\alpha(j_1 z)}{(j_1 z)^\alpha}$ (on the right).

Using this theorem we obtain Theorem 1.2.

Again, let us make some numerical experiments to show the convergence stated in the theorem. We set

$$\mathfrak{E}^*_{2k,\alpha}(x) = \frac{(-1)^k j_1^{2k+2}}{2\gamma_{2k,\alpha}} \frac{J_{\alpha+1}(j_1)}{j_1^{\alpha+1}} \mathfrak{E}_{2k,\alpha}(x),$$

and

$$\mathfrak{E}^*_{2k+1,\alpha}(x) = \frac{(-1)^k j_1^{2k+3}}{2\gamma_{2k+1,\alpha}} \frac{J_{\alpha+1}(j_1)}{j_1^{\alpha+1}} \mathfrak{E}_{2k+1,\alpha}(x).$$

In Figures 3 and 4, and for different values for α and n, the dashed lines correspond to the polynomials $\mathfrak{E}^*_n(x)$ and the solid line to the functions $j_1 x \frac{J_{\alpha+1}(j_1 x)}{(j_1 x)^{\alpha+1}}$ (for odd n) or $\frac{J_\alpha(j_1 x)}{(j_1 x)^\alpha}$ (for even n).

4 Zeros of Bernoulli-Dunkl and Euler-Dunkl polynomials

To study the zeros of the polynomials in question, we use the Hurwitz theorem, that, under some conditions, associates the zeros of a convergent sequence of functions with that of their corresponding limit (see [8, Chapter VII, p. 152] or [21, Chapter I, p. 4]). For the sake of completeness, we quote it here:

Theorem 4.1 (Hurwitz). *Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of functions which are analytic in a region R and which converge uniformly to a function $g(z) \not\equiv 0$ in every closed subregion of R. Let ξ be an interior point of R. If ξ is a limit point of the zeros of the $g_n(z)$, then ξ is a zero of $g(z)$. Conversely, if ξ is an m-fold zero of $g(z)$, every sufficiently small neighborhood K of ξ contains exactly m zeros (counted with their multiplicities) of each $g_n(z)$, $n > N(K)$.*

The functions $\frac{J_\alpha(s_1 z)}{(s_1 z)^\alpha}$ and $s_1 z \frac{J_{\alpha+1}(s_1 z)}{(s_1 z)^{\alpha+1}}$ have only simple real zeros so, from Hurwitz's theorem, every zero of these functions is the limit of a sequence of real zeros of the Bernoulli-Dunkl polynomials. Moreover, the complex zeros of these polynomials must converge to infinity, as $n \to \infty$. (In principle, a portion of the real zeros of the polynomials could escape to infinity similarly as the complex zeros, although it is most likely not the case here.)

Analogously to the Bernoulli-Dunkl case, from Hurwitz's theorem we obtain that every zero of the functions $\frac{J_\alpha(j_1 z)}{(j_1 z)^\alpha}$ and $j_1 z \frac{J_{\alpha+1}(j_1 z)}{(j_1 z)^{\alpha+1}}$ is the limit of a sequence of real zeros of the Bernoulli-Euler polynomials. Again, the complex zeros of these polynomials must converge to infinity.

As a consequence of Hurwitz's theorem we have only obtained that the complex zeros of $\mathfrak{B}_{n,\alpha}(z)$ and $\mathfrak{E}_{n,\alpha}(z)$ must move further away from the origin as n goes to infinity, because the functions to which they converge only have real zeros. Then, to study the manner in which they go to infinity, the usual way is to *normalize* the variable by taking nz in the place of z. Historically, Szegő showed in 1924 that, if we denote $s_n(z) = \sum_{j=0}^n z^j/j!$ to be the nth partial sum of the exponential function e^z, the zeros of the normalized partial sum $s_n(nz)$ tend, as $n \to \infty$, to the curve $|ze^{1-z}| = 1$ in the complex plane, which is now known as the Szegő curve. Similar behavior happens with the zeros of the partial sums or $\cos(z)$ and $\sin(z)$, see [30].

In the classical case of Bernoulli and Euler polynomials $B_n(nz)$ and $E_n(nz)$, Boyer and Goh [3] study the limit distribution of the zeros of Euler polynomials, and they prove that their zero attractor is

$$\begin{aligned} A = \{z \in \mathbb{C} : e^{1+\pi \operatorname{Im} z}|z| = 1/\pi,\ \operatorname{Im} z > 0\} \\ \cup \{z \in \mathbb{C} : e^{1-\pi \operatorname{Im} z}|z| = 1/\pi,\ \operatorname{Im} z < 0\} \\ \cup \{x + 0 \cdot i \in \mathbb{C} : -1/(\pi e) \le x \le 1/(\pi e)\}; \end{aligned}$$

some other families of Appell polynomials have similar behaviour (see [10, 20, 4, 28]). In the recent paper [2], new explicit expressions are obtained for rescaled Appell polynomials which can be used to study the zero attractors and the asymptotics of these polynomials.

However, it does not seem easy to adapt the proofs of these results to the Dunkl context, because they depend strongly on properties of the exponential function that we do not have for $E_\alpha(z)$ (in particular, because the Dunkl translation is a far from trivial operator, see [6, 26, 29], and the relation $e^{x+y} = e^x e^y$

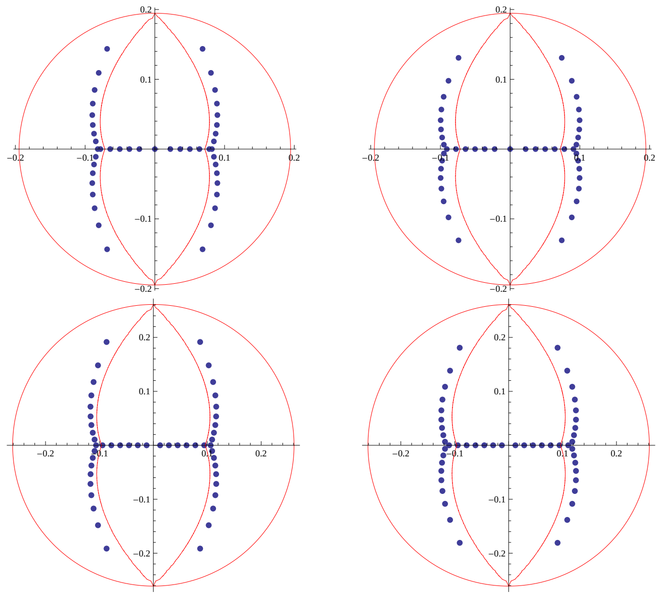

Figure 5: On the top, zeros of $T_{45,1}(45s_1z)$ (on the left) and zeros of $\mathfrak{B}_{45,1}(45z)$ (on the right). On the bottom, zeros of $T_{50,0}(50s_1z)$ (on the left) and zeros of $\mathfrak{B}_{50,0}(50z)$ (on the right).

has a much more complicated version in the Dunkl context). Instead, we have made some numerical experiments that allow us to conjecture the limit distribution of the zeros of $\mathfrak{B}_{n,\alpha}(nz)$ and $\mathfrak{E}_{n,\alpha}(nz)$.

In our case we suspect that the zeros of $\mathfrak{B}_{n,\alpha}(nz)$ are inside the circle $|z| = 1/s_1$ and have as their limit points the set

$$\begin{aligned} C = \{z \in \mathbb{C} : e^{1-s_1 \operatorname{Im} z}|z| = 1/s_1,\ \operatorname{Im} z > 0\} \\ \cup \{z \in \mathbb{C} : e^{1+s_1 \operatorname{Im} z}|z| = 1/s_1,\ \operatorname{Im} z < 0\} \\ \cup \{x + 0 \cdot i \in \mathbb{C} : -1/(s_1 e) \leq x \leq 1/(s_1 e)\}, \end{aligned} \tag{19}$$

where s_1 is the first positive zero of the function $J_{\alpha+1}(z)$. Actually, it has been proved in [31] that the zeros of the sections $T_{n,\alpha}(ns_1z)$ have the set C as their set of limit points and thus our conjecture is that the zeros of $\mathfrak{B}_{n,\alpha}(nz)$ have the same limit distribution. In Figure 5, we draw, for some values of α and n, the circle of radius $1/s_1$, the curve described in (19), the complex zeros of $T_{n,\alpha}(ns_1z)$ and the complex zeros of $\mathfrak{B}_{n,\alpha}(nz)$.

Analogously, we can observe that the zeros of the Euler-Dunkl polynomials $\mathfrak{E}_{n,\alpha}(nz)$ have as limit points the set

$$\begin{aligned} D = \{z \in \mathbb{C} : e^{1-j_1 \operatorname{Im} z}|z| = 1/j_1,\ \operatorname{Im} z > 0\} \\ \cup \{z \in \mathbb{C} : e^{1+j_1 \operatorname{Im} z}|z| = 1/j_1,\ \operatorname{Im} z < 0\} \\ \cup \{x + 0 \cdot i \in \mathbb{C} : -1/(j_1 e) \leq x \leq 1/(j_1 e)\}, \end{aligned} \tag{20}$$

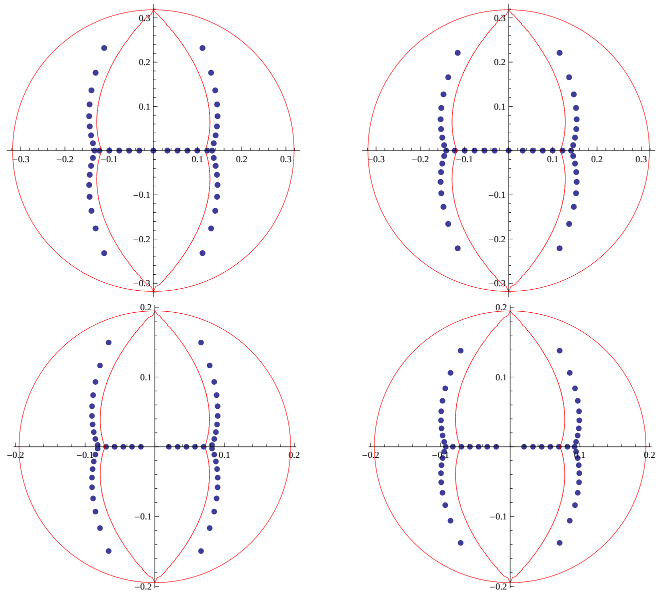

Figure 6: On the top, zeros of $T_{45,1/2}(45j_1z)$ (on the left) and zeros of $\mathfrak{E}_{45,1/2}(45z)$ (on the right). On the bottom, zeros of $T_{50,2}(50j_1z)$ (on the left) and zeros of $\mathfrak{E}_{50,2}(50z)$ (on the right).

where j_1 is the first positive zero of the function $J_\alpha(z)$. In Figure 6 we draw the circle of radius $1/j_1$, the curve (20), the complex zeros of $T_{n,\alpha}(nj_1z)$ and the complex zeros of $\mathfrak{E}_{n,\alpha}(nz)$.

References

[1] T. M. Apostol, On the Lerch zeta function, *Pacific J. Math.* **1** (1951), 161–167; addendum, *Pacific J. Math.* **2** (1952), 10.

[2] J. F. Barbero G., J. Salas and E. J. S. Villaseñor, On the asymptotics of the rescaled Appell polynomials, *Adv. in Appl. Math.* **113** (2020), 101962, 18 pp.

[3] R. P. Boyer and W. M. Y. Goh, On the zero attractor of the Euler polynomials, *Adv. in Appl. Math.* **38** (2007), 97–132.

[4] R. P. Boyer and W. M. Y. Goh, Appell polynomials and their zero attractors, *Contemp. Math.* **514** (2010), 69–96.

[5] Ó. Ciaurri, A. Durán, M. Pérez and J. L. Varona, Bernoulli-Dunkl and Apostol-Euler-Dunkl polynomials with applications to series involving zeros of Bessel functions, *J. Approx. Theory* **235** (2018), 20–45.

[6] Ó. Ciaurri, J. Mínguez Ceniceros and J. L. Varona, Bernoulli-Dunkl and Euler-Dunkl polynomials and their generalizations, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **113** (2019), 2853–2876.

[7] Ó. Ciaurri and J. L. Varona, A Whittaker-Shannon-Kotel'nikov sampling theorem related to the Dunkl transform, *Proc. Amer. Math. Soc.* **135** (2007), 2939–2947.

[8] J. B. Conway, *Functions of One Complex Variable*, 2nd ed., Graduate Texts in Mathematics, Springer-Verlag, New York-Berlin, 1978.

[9] K. Dilcher, Asymptotic behaviour of Bernoulli, Euler, and generalized Bernoulli polynomials, *J. Approx. Theory* **49** (1987), 321–330.

[10] K. Dilcher, Zeros of Bernoulli, generalized Bernoulli and Euler polynomials, *Mem. Amer. Math. Soc.* **73** (1988), No. 386.

[11] K. Dilcher, On multiple zeros of Bernoulli polynomials, *Acta Arith.* **134** (2008), 149–155.

[12] K. Dilcher, Bernoulli and Euler polynomials, *NIST Handbook of Mathematical Functions*, 587–599, National Institute of Standards and Technology, Washington, DC, and Cambridge University Press, Cambridge, 2010. Available online in `http://dlmf.nist.gov/24`

[13] K. Dilcher and L. Malloch, Arithmetic properties of Bernoulli-Padé numbers and polynomials, *J. Number Theory* **92** (2002), 330–347.

[14] D. Drissi, Characterization of Kummer hypergeometric Bernoulli polynomials and applications, *C. R. Math. Acad. Sci. Paris* **357** (2019), 743–751.

[15] C. F. Dunkl, Differential-difference operators associated to reflection groups, *Trans. Amer. Math. Soc.* **311** (1989), 167–183.

[16] A. Durán, M. Pérez and J. L. Varona, Fourier-Dunkl system of the second kind and Euler-Dunkl polynomials, *J. Approx. Theory* **245** (2019), 23–39.

[17] F. T. Howard, A sequence of numbers related to the exponential function, *Duke Math. J.* **34** (1967), 599–615.

[18] F. T. Howard, Some sequences of rational numbers related to the exponential function, *Duke Math. J.* **34** (1967), 701–716.

[19] N. N. Lebedev, *Special Functions and Their Applications*, Dover, New York, 1972.

[20] J. Mangual, Uniform convergence behavior of the Bernoulli polynomials, `https://arxiv.org/abs/math/0703452v1`

[21] M. Marden, *Geometry of Polynomials*, Amer. Math. Soc., Providence, R.I., 1996.

[22] L. M. Navas, F. J. Ruiz and J. L. Varona, The Möbius inversion formula for Fourier series applied to Bernoulli and Euler polynomials, *J. Approx. Theory* **163** (2011), 22–40.

[23] N. E. Nørlund, Mémoire sur les polynômes de Bernoulli, *Acta. Math.* **43** (1922), 121–196.

[24] N. E. Nørlund, *Vorlesungen über Differenzenrechnung*, 1st ed., Springer-Verlag, Berlin-Heidelberg, 1924.

[25] F. W. J. Olver and L. C. Maximon, Bessel functions, *NIST Handbook of Mathematical Functions*, 215–286, National Institute of Standards and Technology, Washington, DC, and Cambridge University Press, Cambridge, 2010. Available online in `http://dlmf.nist.gov/10`

[26] M. Rosenblum, Generalized Hermite polynomials and the Bose-like oscillator calculus, *Oper. Theory Adv. Appl.* **73** (1994), 369–396.

[27] H. M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier, 2012.

[28] F. Štampach, Asymptotic behavior and zeros of the Bernoulli polynomials of the second kind, *J. Approx. Theory* **262** (2021), 105517.

[29] S. Thangavelu and Y. Xu, Riesz transform and Riesz potentials for Dunkl transform, *J. Comput. Appl. Math.* **199** (2007), 181–195.

[30] R. S. Varga and A. J. Carpenter, Zeros of the partial sums of $\cos(z)$ and $\sin(z)$. I, *Numer. Algorithms* **25** (2000), no. 1-4, 363–375.

[31] A. R. Vargas, Limit curves for zeros of sections of exponential integrals, *Constr. Approx.* **40** (2014), 219–239.

[32] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge Univ. Press, Cambridge, 1944.

J. Mínguez Ceniceros
Dpto. de Matemáticas y Computación
Universidad de La Rioja
26006 Logroño, Spain
judit.minguez@unirioja.es

J. L. Varona
Dpto. de Matemáticas y Computación
Universidad de La Rioja
26006 Logroño, Spain
jvarona@unirioja.es