A METHOD FOR SUMMING BESSEL SERIES AND A COUPLE OF ILLUSTRATIVE EXAMPLES

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Abstract. For $\mu, \nu > -1$, we consider the Bessel series

$$U^{\mathfrak{a}}_{\mu,\nu}(x) = \frac{2^\mu \Gamma(\mu+1)}{x^\mu} \sum_{m\geq 1} \frac{a_m}{j_{m,\nu}^{\mu+1/2}} J_\mu(j_{m,\nu}x),$$

where $(j_{m,\nu})_{m\geq 1}$ are the positive zeros of J_ν and $\mathfrak{a} = (a_m)_{m\geq 1}$ is a sequence of real numbers satisfying $\sum_{m\geq 1} |a_m|/j_{m,\nu}^{\mu+1/2} < +\infty$. We propose a method for computing in a closed form the sum of the Bessel series $U^{\mathfrak{a}}_{\mu,\nu}$ assuming that for a particular value η of the parameter μ a closed expression for $U^{\mathfrak{a}}_{\eta,\nu}$ as a power series of x (not necessarily with integer exponents) is known. We illustrate the method with some examples. One of them is related to the sine coefficients of the function $1 - x^s$, $s > -1$. The closed form of the sum is then given in terms of a generalization of the Bernoulli numbers.

1. Introduction

Let J_ν be the Bessel function of order ν. For $\nu > -1$, the zeros $j_{m,\nu}$ ($m = 1, 2, \dots$) of J_ν are positive and can be ordered so that $0 < j_{m,\nu} < j_{m+1,\nu}$, $m \geq 1$ (see, for instance, [26, § 15.27, p. 483]).

For $\mu, \nu > -1$ and a sequence $\mathfrak{a} = (a_m)_{m\geq 1}$ of real numbers satisfying

$$\sum_{m\geq 1} \frac{|a_m|}{j_{m,\nu}^{\mu+1/2}} < +\infty, \tag{1}$$

we define the function

$$U^{\mathfrak{a}}_{\mu,\nu}(x) = \frac{2^\mu \Gamma(\mu+1)}{x^\mu} \sum_{m\geq 1} \frac{a_m}{j_{m,\nu}^{\mu}} J_\mu(j_{m,\nu}x), \quad x \in (0,\infty). \tag{2}$$

The functions $U^{\mathfrak{a}}_{\mu,\nu}$ were considered in [15], where we prove some properties about their real zeros and pose some conjectures on their distribution.

The problem of the explicit summation of the series (2) and other analogous ones has a somehow classical flavour and goes back to the turn of the 19th to the 20th century. Active research is still being done on the problem which reached a peak in the 1980's with the classical *Integrals and Series* by A. P. Prudnikov, Y. A.

THIS PAPER HAS BEEN PUBLISHED IN: *Proc. Amer. Math. Soc.* **150** (2022), no. 2, 763–778; doi: 10.1090/proc/15684.

2010 *Mathematics Subject Classification.* Primary 33C10; Secondary 11B68, 33C20, 42A10.

Key words and phrases. Bessel functions, Bessel series, Fourier series, Appell polynomials, Bernoulli polynomials, hypergeometric functions.

This research was partially supported by PGC2018-096504-B-C31 and PGC2018-096504-B-C32 (Ministerio de Ciencia, Innovación y Universidades), FQM-262 and US-1254600 (Junta de Andalucía), E48_20R (Gobierno de Aragón) and Feder Funds (European Union).

Brychkov and O. I. Marichev [22, § 5] and more recently in 2008 with the *Handbook of special functions* by Y. A. Brychkov [4, § 6.8].

The purpose of this paper is to propose a method for computing in a closed form the functions $U^{\mathfrak{a}}_{\mu,\nu}(x)$ assuming that for a particular value η of the parameter μ a closed expression as a power series of x (not necessarily with integer exponents) is known (see Lemma 1).

We illustrate our method with a couple of examples.

The first example (Section 2) is actually a family of examples: they have in common that generalize some of the identities in [22, §§ 5.7.19, 5.7.20, 5.7.33] or [4, §§ 6.8.7, 6.8.8, 6.8.9]. For instance, starting from the identity

$$\sum_{m\ge 1} \frac{J_0(j_{m,0}x)}{j_{m,0}^2 J_1^2(j_{m,0})} = -\frac{\log x}{2}, \quad x \in (0,1), \tag{3}$$

which is [22, § 5.7.33, (2), p. 690], using our method we get, for $\mu > -1$ and $x \in (0,1)$,

$$\sum_{m\ge 1} \frac{J_\mu(j_{m,0}x)}{j_{m,0}^{\mu+2} J_1^2(j_{m,0})} = -\frac{x^\mu}{2^{\mu+1}\Gamma(\mu+1)}\left(\log x - \frac{1}{2\mu}\left(1+\mu\gamma+\frac{\mu\Gamma'(\mu)}{\Gamma(\mu)}\right)\right) \tag{4}$$

where γ is the Euler constant; as far as we know, this identity is new.

The second example (Section 3) corresponds to the series (2) for $\nu = 1/2$ and

$$\mathfrak{a} = \left(\frac{b_m^s}{(m\pi)^{2l}}\right)_{m\ge 1}, \tag{5}$$

where l is a nonnegative integer, $s > -1$, and $2b_m^s/(m\pi)$, $m \ge 1$, are the Fourier sine coefficients of the function $1 - x^s$:

$$1 - x^s = 2\sum_{m\ge 1} \frac{b_m^s}{\pi m}\sin(\pi m x), \quad x \in (0,1). \tag{6}$$

The sequence $(b_m^s)_{m\ge 1}$ has the following explicit series expansion:

$$b_m^s = (1-(-1)^m) - \sum_{j=0}^{\infty} \frac{(-1)^j (m^2\pi^2)^{j+1}}{(2j+1)!\,(2j+s+2)}. \tag{7}$$

In order to sum explicitly the series (2), we introduce the weighted-Bernoulli numbers and polynomials (w-Bernoulli for short): for a real number $s > -1$, the w-Bernoulli numbers B_n^s, $n \ge 0$, are defined as a weighted sum of the Bernoulli numbers B_n, $n \ge 0$: $B_0^s = 1$, $B_1^s = -\frac{1}{2}$, $B_{2l+1}^s = 0$, $l \ge 1$, and

$$B_{2l+2}^s = \sum_{k=1}^{2l+2} 2^{-k}\binom{2l+2}{k}\frac{kB_{2l+2-k}}{k+s} + 2(1-2^{-2l-2})B_{2l+2}. \tag{8}$$

Then, we define the w-Bernoulli polynomials as

$$B_n^s(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}^s x^k. \tag{9}$$

Although it is far from being evident, the Bernoulli numbers and polynomials are the case $s = 1$ of the w-Bernoulli numbers and polynomials, respectively (see Theorem 4). The w-Bernoulli numbers B_{2l+2}^s, $l \ge 0$, are related to the sequence $(b_m^s/(m\pi)^{2l+2})_{m\ge 1}$ in the same way as the Bernoulli numbers B_{2l+2} are related to the sequence $(1/(m\pi)^{2l+2})_{m\ge 1}$ (see Theorem 5). Using the w-Bernoulli numbers and the method explained above, we explicitly sum in Theorem 7 the series (2) for the sequence (5) and $\nu = 1/2$.

In our opinion, the w-Bernoulli numbers and polynomials have interest by their own. In the literature, we find many generalizations of the Bernoulli numbers and polynomials, but in most of the cases they are constructed by introducing one or more parameters in the generating function for the Bernoulli polynomials (generalized Bernoulli polynomials or Nørlund polynomials [19], Apostol-Bernoulli polynomials [1, 2], hypergeometric Bernoulli polynomials [14], Bernoulli-Padé polynomials [13]; see also [12, § 24.16, (iii)] and the book [24] with many recent references) or in the operator (umbral calculus [6], Bernoulli-Dunkl polynomials [7, 8]). Our approach is quite different: the w-Bernoulli numbers and polynomials have been introduced with the purpose of summing the series (2) for the sequence (5). We complete the paper studying the asymptotic behaviour of the w-Bernoulli polynomials and with some results about their zeros (Section 4).

2. The method and the first illustrative example

For each $\mu > -1$, there exists a constant $C_\mu > 0$ such that

$$|J_\mu(x)| \le C_\mu |x|^{-1/2}, \quad \text{if } |x| \ge 1$$

(see [26, 7.21(1), p. 199] or [20, 10.7.8]). Since $j_m \sim m$ as $m \to \infty$ (see [26, § 15.53, p. 506]), the series in (2) converges uniformly on every compact set in $(0, +\infty)$ under the condition (1), so that $U^{\mathfrak{a}}_{\mu,\nu}(x)$ is well defined and continuous on $(0, +\infty)$.

The Bessel functions satisfy the identity

$$(x^{\mu+1} J_{\mu+1}(x))' = x^{\mu+1} J_\mu(x),$$

which directly gives the following one for the functions $U^{\mathfrak{a}}_{\mu,\nu}$:

$$(x^{2\mu+2} U^{\mathfrak{a}}_{\mu+1,\nu}(x))' = 2(\mu+1) x^{2\mu+1} U^{\mathfrak{a}}_{\mu,\nu}(x). \tag{10}$$

Our method is a consequence of Sonine's formula for the Bessel functions

$$J_{\mu+\nu+1}(z) = \frac{z^{\nu+1}}{2^\nu \Gamma(\nu+1)} \int_0^1 J_\mu(zs) s^{\mu+1} (1-s^2)^\nu \, ds, \tag{11}$$

valid for $\mu, \nu \in \mathbb{C}$, $\operatorname{Re}\mu > -1$, $\operatorname{Re}\nu > -1$ ([26, 12.11(1), p. 373]).

In [15, § 2], we extended Sonine's formula (11) to the functions $U^{\mathfrak{a}}_{\mu,\nu}$: for $\mu > \eta > -1$ and $\nu > -1$, and provided that

$$\sum_{m \ge 1} \frac{|a_m|}{j_{m,\nu}^{\eta+1/2}} < +\infty$$

(which is the condition required to define $U^{\mathfrak{a}}_{\eta,\nu}$ and implies (1)), we have

$$U^{\mathfrak{a}}_{\mu,\nu}(x) = 2(\mu-\eta) \binom{\mu}{\eta} \int_0^1 U^{\mathfrak{a}}_{\eta,\nu}(xs) s^{2\eta+1} (1-s^2)^{\mu-\eta-1} \, ds, \tag{12}$$

with the usual notation $\binom{a}{b} = \Gamma(a+1)/(\Gamma(b+1)\Gamma(a-b+1))$.

Our method is based in the following lemma.

Lemma 1. *Let ν, η and r_j, $j = 0, \dots, \infty$, be real numbers satisfying $\nu, \eta > -1$ and $r_j > -2\eta - 2$. Assume that*

$$U^{\mathfrak{a}}_{\eta,\nu}(x) = \sum_{j=0}^{\infty} u_j x^{r_j}, \quad x \in (0, \omega), \tag{13}$$

where $\omega \geq 1$ (possibly infinite) and the series converges absolutely. Then for any $\mu \geq \eta$ we have

$$U^{\mathfrak{a}}_{\mu,\nu}(x) = \frac{\Gamma(\mu+1)}{\Gamma(\eta+1)} \sum_{j=0}^{\infty} \frac{u_j \Gamma(\eta + r_j/2 + 1) x^{r_j}}{\Gamma(\mu + r_j/2 + 1)}, \quad x \in (0,\omega). \tag{14}$$

Moreover, if for some μ satisfying $-1 < \mu < \eta$ and $\mu + \inf\{r_j\}/2 + 1 > 0$ the series in the right hand side of (14) converges absolutely for $x \in (0,\omega)$, then the identity (14) also holds for μ.

Proof. We first assume $\mu > \eta$. Using the identity

$$\int_0^1 s^a (1-s^2)^b \, ds = \frac{1}{2(b+1)\binom{b+a/2+1/2}{b+1}}, \qquad a, b > -1,$$

for $a = 2\eta + r_j + 1$ and $b = \mu - \eta - 1$, we get from (13) and (12) that

$$\begin{aligned} U^{\mathfrak{a}}_{\mu,\nu}(x) &= 2(\mu-\eta)\binom{\mu}{\eta} \sum_{j=0}^{\infty} u_j x^{r_j} \int_0^1 s^{2\eta+r_j+1}(1-s^2)^{\mu-\eta-1}\, ds \\ &= 2(\mu-\eta)\binom{\mu}{\eta} \sum_{j=0}^{\infty} \frac{u_j x^{r_j}}{2(\mu-\eta)\binom{\mu+r_j/2}{\mu-\eta}} = \frac{\Gamma(\mu+1)}{\Gamma(\eta+1)} \sum_{j=0}^{\infty} \frac{u_j \Gamma(\eta + r_j/2+1) x^{r_j}}{\Gamma(\mu + r_j/2 + 1)}, \end{aligned}$$

for $x \in (0,\omega)$ (the interchange of the series and the integral follows from the absolute convergence of the series (13) and Lebesgue's dominated convergence theorem). This proves the lemma for $\mu \geq \eta$.

For $\mu > -1$, write now $k = 0$ if $\mu \geq \eta$ and $k = -\lfloor \mu - \eta \rfloor$ otherwise. We now prove the lemma by induction on k. The case $k = 0$ has already been proved. Since $\lfloor \mu + 1 \rfloor = k$, we have using the induction hypotheses

$$x^{2\mu+2} U^{\mathfrak{a}}_{\mu+1,\nu}(x) = \frac{\Gamma(\mu+2)}{\Gamma(\eta+1)} \sum_{j=0}^{\infty} \frac{u_j \Gamma(\eta + r_j/2 + 1) x^{r_j + 2\mu + 2}}{\Gamma(\mu + r_j/2 + 2)}, \quad x \in (0,\omega).$$

Using the identity (10) and deriving the series term by term (which the absolute convergence and the condition $\mu + \inf\{r_j\}/2 + 1 > 0$ allow), we get (14). □

The lemma can be restated as follows: if $\nu, \eta > -1$, $r_j > -\eta - 2$, $\mu \geq \eta$, and a real sequence $(\lambda_m)_{m\geq 1}$ is such that

$$\sum_{m\geq 1} \frac{|\lambda_m|}{j_{m,\nu}^{1/2}} < +\infty \quad \text{and} \quad \sum_{m\geq 1} \lambda_m J_\eta(j_{m,\nu} x) = \sum_{j=0}^{+\infty} u_j x^{r_j}, \quad x \in (0,\omega),$$

then

$$\sum_{m\geq 1} \frac{\lambda_m}{j_{m,\nu}^{\mu-\eta}} J_\mu(j_{m,\nu} x) = \sum_{j=0}^{+\infty} \frac{u_j \Gamma(\frac{\eta}{2} + \frac{r_j}{2} + 1)}{2^{\mu-\eta}\Gamma(\mu - \frac{\eta}{2} + \frac{r_j}{2} + 1)} x^{r_j + \mu - \eta}, \quad x \in (0,\omega).$$

This identity holds also if $-1 < \mu < \eta$, $\mu - \eta/2 + \inf\{r_j/2\} + 1 > 0$, and the right hand side converges absolutely for $x \in (0,\omega)$.

Using this lemma, we can extend most of the identities in [22, § 5.7.33]. For instance, consider the identity (4) of [22, p. 690]:

$$\sum_{m\geq 1} \frac{j_{m,\nu} J_\nu(j_{m,\nu} x)}{(j_{m,\nu}^2 - a^2) J_{\nu+1}(j_{m,\nu})} = \frac{J_\nu(ax)}{2 J_\nu(a)}$$

where $\nu > -1$, $x \in (0,1)$ and $a \in \mathbb{C} \setminus \{j_{m,\nu}, m \geq 1\}$. Since

$$\frac{J_\nu(ax)}{2J_\nu(a)} = \frac{1}{2J_\nu(a)} \left(\tfrac{1}{2}ax\right)^\nu \sum_{j=0}^{\infty} (-1)^j \frac{\left(\frac{1}{4}a^2x^2\right)^j}{j!\,\Gamma(\nu+j+1)},$$

Lemma 1 gives, for $\mu \geq \nu$,

$$\sum_{m\geq 1} \frac{j_{m,\nu}^{\nu-\mu+1} J_\nu(j_{m,\nu}x)}{(j_{m,\nu}^2 - a^2)J_{\nu+1}(j_{m,\nu})} = \frac{1}{2J_\nu(a)} \left(\tfrac{1}{2}ax\right)^\nu \sum_{j=0}^{\infty} (-1)^j \frac{\left(\frac{1}{4}a^2x^2\right)^j}{2^{\mu-\nu} j!\,\Gamma(\mu+j+1)}\, x^{\mu-\nu},$$

from where we get

$$\sum_{m\geq 1} \frac{j_{m,\nu}^{\nu-\mu+1} J_\mu(j_{m,\nu}x)}{(j_{m,\nu}^2 - a^2)J_{\nu+1}(j_{m,\nu})} = \frac{a^{\nu-\mu} J_\mu(ax)}{2J_\nu(a)},$$

where $\mu \geq \nu > -1$, $x \in (0,1)$ and $a \in \mathbb{C} \setminus \{\pm j_{m,\nu}, m \geq 1\}$. The cases $\mu = \nu, \nu+1$ are the identities (4) and (6) of [22, p. 690], respectively. As far as we know, the other cases are new (compare with the identity (34) of [23, p. 293]).

Something similar can be done for most of the identities in [22, § 5.7.33]. For instance, for $\tau, \mu, \nu > -1$ and $x \in [0,1]$,

$$\sum_{m\geq 1} \frac{j_{m,\nu}^{\nu-\mu-1} J_\mu(j_{m,\nu}x)}{(j_{m,\nu}^2 - a^2)J_{\nu+1}(j_{m,\nu})} = \frac{1}{2a^2} \left(\frac{J_\mu(ax)}{a^{\mu-\nu} J_\nu(a)} - \frac{\Gamma(\nu+1)x^\mu}{2^{\mu-\nu}\Gamma(\mu+1)} \right),$$

$$\sum_{m\geq 1} \frac{J_{\tau+\nu+1}(j_{m,\nu}) J_\mu(j_{m,\nu}x)}{j_{m,\nu}^{\tau-\nu+\mu+1} J_{\nu+1}^2(j_{m,\nu})} = \frac{2^{\nu-\tau-\mu-1}\Gamma(\nu+1)x^\mu}{\Gamma(\tau+1)\Gamma(\mu+1)}\, {}_2F_1\left(\begin{matrix} -\tau, \nu+1 \\ \mu+1 \end{matrix}; x^2 \right),$$

where we assume in addition $\mu > \nu - 2$ in the first series and $\tau > \nu - \mu$ in the second one. For $\mu = \nu$, these identities are (5) and (9) in [22, p. 690], respectively (notice that for $\mu = \nu$, we have in the second identity ${}_2F_1\left(\begin{smallmatrix} -\tau, \nu+1 \\ \nu+1 \end{smallmatrix}; x^2 \right) = (1-x^2)^\tau$; this is how it appears in the identity (9) in [22, p. 690]).

The identity (4) in the introduction can be proved by modifying slightly the proof of Lemma 1. Indeed, for $\nu = \eta = 0$, if we insert in (12) the function $U_{0,0}^{\mathfrak{a}}(xs) = -\log(xs)/2$ (that is, the formula (3)), we get for $\mu > 0$ and $x \in (0,1)$ that

$$U_{\mu,0}^{\mathfrak{a}}(x) = -\mu \log x \int_0^1 s(1-s^2)^{\mu-1}\,ds - \mu \int_0^1 (\log s) s(1-s^2)^{\mu-1}\,ds,$$

from where we find (4) after some computations using the well-known integral representation $\Gamma'(x)/\Gamma(x) = -\gamma + \int_0^1 (1-t^{x-1})(1-t)^{-1}\,dt$, $x > 0$. For $-1 < \mu < 0$, the proof follows by using (10).

Many of the identities in [22, §§ 5.7.19, 5.7.20] can also be extended using our method. These are a couple of examples: for $x \in (0,1)$,

$$\sum_{m\geq 1} \frac{(-1)^{m+1}(\pi m)^{2n+2-\mu} J_\mu(\pi m x)}{(\pi m)^2 - a^2} = \frac{a^{2n+1-\mu} J_\mu(ax)}{2\sin a},$$

$$\sum_{m\geq 1} (-1)^{m+1} \frac{(\pi(2m-1)/2)^{2n+1-\mu} J_\mu(\pi(2m-1)x/2)}{(\pi(2m-1)/2)^2 - a^2} = \frac{a^{2n-\mu} J_\mu(ax)}{2\cos a},$$

where $\mu > 2n-1$, $a \neq \pm m\pi$, $m \geq 1$, in the first series, and $\mu > 2n-2$, $a \neq \pm(2m-1)\pi/2$, $m \geq 1$, in the second one. For $\mu = 2n+1$, these identities are (19) and (15) in pages 679 and 681 of [22], respectively.

The proof of Lemma 1 works also in a multivariate version. To this end, we consider sets $\Omega \subseteq (0,+\infty)^k$ with the property that

$$(x_1, x_2, \dots, x_k) \in \Omega \implies \prod_{i=1}^k (0, x_i] \subseteq \Omega. \tag{15}$$

Lemma 2. *Let ν_i, η_i and $r_{i,j}$, $i = 1, \dots, k$, $j = 0, \dots, \infty$, be real numbers satisfying $\nu_i, \eta_i > -1$ and $r_{i,j} > -\eta_i - 2$. Assume that a real sequence $(\lambda_m)_{m\ge1}$ is such that*

$$\sum_{m\ge1} \frac{|\lambda_m|}{\prod_{i=1}^k j_{m,\nu_i}^{1/2}} < +\infty$$

and

$$\sum_{m\ge1} \lambda_m \prod_{i=1}^k J_{\eta_i}(j_{m,\nu_i} x_i) = \sum_{j_1,\dots,j_k=0}^{\infty} u_{j_1,\dots,j_k} \prod_{i=1}^k x_i^{r_{i,j_i}}, \quad \text{if } (x_1, x_2, \dots, x_k) \in \Omega,$$

for some set Ω with the property (15)*, where the series in the right-hand side converges absolutely. Then, for $\mu_i \ge \eta_i$ and $(x_1, x_2, \dots, x_k) \in \Omega$ we have*

$$\sum_{m\ge1} \lambda_m \prod_{i=1}^k \frac{J_{\mu_i}(j_{m,\nu_i} x_i)}{j_{m,\nu_i}^{\mu_i-\eta_i}} = \sum_{j_1,\dots,j_k=0}^{\infty} u_{j_1,\dots,j_k} \prod_{i=1}^k \frac{2^{\eta_i}\Gamma\big(\frac{\eta_i}{2} + \frac{r_{i,j_i}}{2} + 1\big) x_i^{r_{i,j_i}+\mu_i-\eta_i}}{2^{\mu_i}\Gamma\big(\mu_i - \frac{\eta_i}{2} + \frac{r_{i,j_i}}{2} + 1\big)}. \tag{16}$$

Moreover, if for μ_i, $i = 1, \dots, k$, satisfying $-1 < \mu_i < \eta_i$ and $\mu_i - \eta_i/2 + \inf_j\{r_{i,j}\}/2 + 1 > 0$, the series in the right hand side of (16) *converges absolutely for $(x_1, x_2, \dots, x_k) \in \Omega$, then the identity* (16) *also holds for μ_i, $i = 1, \dots, k$.*

Using this lemma, some of the identities in Section 5.7.29 of [22] or Sections 6.8.7, 6.8.8 and 6.8.9 of [4] can be extended. For instance, the basic identity

$$\sum_{m\ge1} \frac{(-1)^{m+1}}{(\pi m)^k} \prod_{i=1}^k \sin(\pi m x_i) = \frac{1}{2} \prod_{i=1}^k x_i,$$

holds for $|x_1| + \dots + |x_k| < 1$ (see [21, § 5.4.15, (20), p. 744]; it can be proved for $k = 1, 2$ as a Fourier sine or cosine series and then recursively by differentiation). The lemma then gives

$$\sum_{m\ge1} (-1)^{m+1} \prod_{i=1}^k \frac{J_{\mu_i}(\pi m x_i)}{(\pi m)^{\mu_i}} = \frac{1}{2} \prod_{i=1}^k \frac{x_i^{\mu_i}}{2^{\mu_i}\Gamma(\mu_i + 1)}$$

for $x_i \in (0, 1)$, $i = 1, \dots, k$, $\sum_{i=1}^k x_i < 1$ and $\mu_i > -1$. For $k = 2$ and $\mu_1 = \mu_2$, this identity is [22, § 5.7.24, (8), p. 683]; for $k = 2$ it is the case $m = 0$ of [4, § 6.8.8, (1), p. 470]; for $k = 3$, $x_2 = x_3$, $\mu_1 = 1/2$ or $\mu_1 = -1/2$, it is [4, § 6.8.9, (1) and (2), p. 471].

3. Weighted-Bernoulli numbers and polynomials

We start recalling some basic facts about the Bernoulli polynomials $B_n(x)$, $n \ge 0$, see [12]. They are defined by the generating function

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!}$$

or, equivalently,

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k, \tag{17}$$

where $B_k = B_k(0)$, $k \geq 0$, are the Bernoulli numbers. We also need the Fourier sine expansion for the Bernoulli polynomials,

$$B_{2n+1}(x) = (-1)^{n+1} \frac{2(2n+1)!}{(2\pi)^{2n+1}} \sum_{m=1}^{\infty} \frac{\sin(2\pi m x)}{m^{2n+1}}, \quad x \in [0,1]. \tag{18}$$

From the definition (8), the generating function

$$\phi(z) = \sum_{l=0}^{\infty} \frac{B_l^s}{l!} z^l$$

for the w-Bernoulli numbers can be checked without difficulty:

$$\begin{aligned}\phi(z) = \frac{z^2}{4(2+s)} \left(\frac{z}{2} + \frac{z}{e^z - 1} \right) {}_1F_2\left(\begin{matrix} 1 + s/2 \\ 3/2, 2 + s/2 \end{matrix}; z^2/16 \right) \\ - \frac{z^2}{4(1+s)} {}_1F_2\left(\begin{matrix} 1/2 + s/2 \\ 1/2, 3/2 + s/2 \end{matrix}; z^2/16 \right) + \frac{2z}{e^z - 1} - \frac{z}{e^{z/2} - 1} + 1.\end{aligned}$$

This gives for the w-Bernoulli polynomials the generating function

$$\phi(z) e^{zx} = \sum_{n=0}^{\infty} B_n^s(x) \frac{z^n}{n!}.$$

As a consequence, we conclude that the polynomials $B_n^s(x)$, $n \geq 0$, are an Appell sequence. In particular, $B_n^s(x)$ is a polynomial of degree n, and they satisfy $\frac{d}{dx} B_n^s(x) = n B_{n-1}^s(x)$ (this follows also from (9)).

The two first polynomials are $B_0^s(x) = 1$ and $B_1^s(x) = x - \frac{1}{2}$. The remaining w-Bernoulli polynomials are related to the ordinary Bernoulli polynomials by means of

$$\begin{aligned}B_{l+2}^s(x) = \sum_{j=0}^{\lfloor l/2 \rfloor} \frac{1}{4^{j+1}} \binom{l+2}{2j+2} \frac{2j+2}{2j+s+2} B_{l-2j}(x) \\ + \sum_{j=0}^{l} \binom{l+2}{j+2} \frac{(-1)^{j+1}(j+1)(j+2)}{(j+s+1)2^{j+2}} x^{l-j} \\ + 2B_{l+2}(x) - \frac{1}{2^{l+1}} B_{l+2}(2x) + x^{l+2}, \qquad l \geq 0.\end{aligned}$$

Our starting point is the sine expansion

$$x^s = 2 \sum_{m \geq 1} m\pi c_m^s \sin(\pi m x), \quad x \in (0,1),\ s > -1,$$

where the coefficients c_m^s are given by

$$c_m^s = \frac{1}{m\pi} \int_0^1 x^s \sin(\pi m x)\, dx = \sum_{j=0}^{\infty} \frac{(-m^2\pi^2)^j}{(2j+1)!\,(2j+s+2)}$$

or, in terms of a hypergeometric function,

$$c_m^s = \frac{1}{2+s}\, {}_1F_2\left(\begin{matrix} 1 + s/2 \\ 3/2, 2 + s/2 \end{matrix}; -m^2\pi^2/4 \right).$$

This and (7) give for the sequence b_m^s, $m \geq 1$, defined in (6) the identity

$$\begin{aligned} b_m^s &= (1-(-1)^m) - m^2\pi^2 c_m^s \\ &= (1-(-1)^m) - \frac{m^2\pi^2}{2+s}\,{}_1F_2\left(\begin{matrix} 1+s/2 \\ 3/2, 2+s/2 \end{matrix}; -m^2\pi^2/4\right). \end{aligned} \tag{19}$$

It is easy to see that $b_m^1 = 1$, $m \geq 1$.

Let us define now

$$q_n^s(x) = \begin{cases} 2(-1)^l \displaystyle\sum_{m\geq 1} \frac{b_m^s}{(\pi m)^n} \cos(\pi m x), & n = 2l, \\ 2(-1)^l \displaystyle\sum_{m\geq 1} \frac{b_m^s}{(\pi m)^n} \sin(\pi m x), & n = 2l+1, \end{cases} \tag{20}$$

for $n \geq 1$, $x \in [0,1]$ ($x \in (0,1)$ for $n = 1$, as we see next). Considering (6) and the cosine expansion of $x - \frac{x^{s+1}}{s+1}$ gives $q_1^s(x) = 1 - x^s$ for $x \in (0,1)$ and

$$q_2^s(x) = x - \frac{1}{2} + \frac{1}{(s+1)(s+2)} - \frac{x^{s+1}}{s+1}$$

for $x \in [0,1]$. This and termwise differentiation in (20) for $n \geq 3$ prove that $(q_n^s)'(x) = q_{n-1}^s(x)$ for $x \in [0,1]$ and $n \geq 2$. Therefore,

$$q_n^s(x) = p_{n-1}^s(x) - \frac{\Gamma(s+1)x^{s+n-1}}{\Gamma(s+n)}, \quad n \geq 1, \tag{21}$$

where p_n^s, $n \geq 0$, is a sequence of Appell polynomials in the sense that p_n^s has degree n and $(p_n^s)'(x) = p_{n-1}^s(x)$ (with $p_0^s = 1$). Observe also that

$$p_1^s(x) = x - \frac{1}{2} + \frac{1}{(s+1)(s+2)}.$$

The key ingredient for our study of the w-Bernoulli numbers and polynomials is the following relation between $p_n^s(0)$ and the w-Bernoulli numbers:

Lemma 3. *For $s > -1$ and $n \geq 0$, we have*

$$p_n^s(0) = -\frac{2^{n+1}B_{n+1}^s}{(n+1)!}. \tag{22}$$

Proof. First of all, we have $p_0^s = 1$. For $n \geq 1$, (20) and (21) give

$$p_n^s(0) = \begin{cases} 2(-1)^{l+1} \displaystyle\sum_{m\geq 1} \frac{b_m^s}{(\pi m)^{n+1}}, & n = 2l+1, \\ 0, & n = 2l. \end{cases} \tag{23}$$

Using (19), we deduce

$$p_{2l+1}^s(0) = 2(-1)^{l+1}\left(\sum_{m\geq 1} \frac{1-(-1)^m}{(\pi m)^{2l+2}} - \sum_{m\geq 1} \frac{c_m^s}{(\pi m)^{2l}}\right). \tag{24}$$

On the one hand,

$$\sum_{m\geq 1} \frac{1-(-1)^m}{(\pi m)^{2l+2}} = 2\sum_{m\geq 0} \frac{1}{(\pi(2m+1))^{2l+2}} = \frac{(-1)^l(2^{2l+2}-1)}{(2l+2)!}B_{2l+2}. \tag{25}$$

On the other hand, for $l \geq 1$ the uniform convergence of (18), together with (17), gives

$$\begin{aligned}
\sum_{m\geq 1} \frac{c_m^s}{(m\pi)^{2l}} &= \sum_{m\geq 1} \int_0^1 \frac{x^s \sin(m\pi x)}{(m\pi)^{2l+1}}\,dx = \int_0^1 x^s \sum_{m\geq 1} \frac{\sin(m\pi x)}{(m\pi)^{2l+1}}\,dx \\
&= \frac{(-1)^{l+1}2^{2l}}{(2l+1)!} \int_0^1 x^s B_{2l+1}(x/2)\,dx \\
&= \frac{(-1)^{l+1}2^{2l}}{(2l+1)!} \int_0^1 \sum_{j=0}^{2l+1} \binom{2l+1}{j} B_{2l+1-j} \frac{x^{j+s}}{2^j}\,dx \\
&= \frac{(-1)^{l+1}2^{2l}}{(2l+1)!} \sum_{j=0}^{2l+1} \binom{2l+1}{j} B_{2l+1-j} \frac{1}{2^j(j+s+1)}.
\end{aligned}$$

Taking this and (25) into (24) gives

$$\begin{aligned}
p_{2l+1}^s(0) &= -\frac{2(2^{2l+2}-1)}{(2l+2)!}B_{2l+2} - \frac{2^{2l+1}}{(2l+1)!}\sum_{j=0}^{2l+1}\binom{2l+1}{j}B_{2l+1-j}\frac{1}{2^j(j+s+1)} \\
&= -\frac{2(2^{2l+2}-1)}{(2l+2)!}B_{2l+2} - \frac{1}{(2l+2)!}\sum_{k=1}^{2l+2} 2^{2l+2-k}\binom{2l+2}{k}B_{2l+2-k}\frac{k}{s+k}
\end{aligned}$$

for $l \geq 1$. From the definition (8) we deduce that

$$p_{2l+1}^s(0) = -\frac{2^{2l+2}B_{2l+2}^s}{(2l+2)!}$$

for $l \geq 1$. Since $p_0^s = 1$, $p_1^s(x) = x - \frac{1}{2} + \frac{1}{(s+1)(s+2)}$, $p_{2l}^s(0) = 0$, $l \geq 1$, and $B_1^s = -\frac{1}{2}$, $B_2^s = \frac{1}{4} - \frac{1}{2(s+1)(s+2)}$, $B_{2l+1}^s = 0$, $l \geq 1$, we finally get (22) for every $n \geq 0$. □

We next prove that the Bernoulli numbers and polynomials are the case $s = 1$ of the w-Bernoulli numbers and polynomials, respectively.

Theorem 4. *For $s = 1$, the identities*

$$B_n^1 = B_n, \quad B_n^1(x) = B_n(x)$$

hold for $n \geq 0$.

Proof. The first identity is obviously true for n odd and $n = 0$.

For $s = 1$, we have $b_m^1 = 1$, $m \geq 1$, and then (23) and the Euler sums

$$\sum_{m\geq 1} \frac{1}{(\pi m)^{2k}} = \frac{(-1)^{k-1}2^{2k-1}}{(2k)!}B_{2k}, \quad k \geq 1,$$

give, for $n = 2l+1$,

$$p_{2l+1}^1(0) = 2(-1)^{l+1}\sum_{m\geq 1}\frac{1}{(\pi m)^{2l+2}} = -\frac{2^{2l+2}}{(2l+2)!}B_{2l+2}.$$

Thus, (22) completes the proof of the first identity.

The second identity $B_n^1(x) = B_n(x)$ is then clear from the definition (9) of the w-Bernoulli polynomials. □

The first identity in Theorem 4 implies the following identity for the Bernoulli numbers

$$(2^{2l+2}-2)B_{2l+2} = -2^{2l}\sum_{j=0}^{l}\frac{1}{4^j}\binom{2l+2}{2j+2}\frac{(2j+2)B_{2l-2j}}{(2j+3)} + (2l+1),$$

which can be deduced also from (17) and the identity $B_n(1/2) = -(1-2^{1-n})B_n$.

As we wrote in the introduction, the w-Bernoulli numbers B^s_{2l+2}, $l \geq 0$, are related to the sequence $(b^s_m/(m\pi)^{2l+2})_{m\geq 1}$ in the same way as the Bernoulli numbers B_{2l+2} are related to the sequence $(1/(m\pi)^{2l+2})_{m\geq 1}$:

Theorem 5. *For $s > -1$, let $(b^s_m)_{m\geq 1}$ be the sequence* (7)*. Then,*

$$\sum_{m\geq 1}\frac{b^s_m}{(\pi m)^{2l+2}} = \frac{(-1)^l 2^{2l+2}}{2(2l+2)!}B^s_{2l+2}.$$

Proof. It follows from (23) and (22). □

The case $s = 1$ are the celebrated Euler sums for the inverse of the even powers of the positive integers.

For the w-Bernoulli polynomials $B^s_n(x)$, we have cosine and sine expansions similar to those of the Bernoulli polynomials:

Theorem 6. *For $s > -1$, $n \geq 1$, and $x \in (0,1)$, we have*

$$\frac{x^n}{n!} - \frac{\Gamma(s+1)x^{s+n-1}}{\Gamma(s+n)} - \frac{2^n B^s_n(x/2)}{n!} = \begin{cases} 2(-1)^l \displaystyle\sum_{m\geq 1}\frac{b^s_m}{(\pi m)^n}\cos(\pi m x), & n = 2l, \\ 2(-1)^l \displaystyle\sum_{m\geq 1}\frac{b^s_m}{(\pi m)^n}\sin(\pi m x), & n = 2l+1. \end{cases}$$

Proof. Since $(p^s_n)'(x) = p^s_{n-1}(x)$, we deduce that

$$p^s_n(x) = \sum_{k=0}^{n} p^s_{n-k}(0)\frac{x^k}{k!}.$$

Using (22), we get (after easy computations)

$$p^s_n(x) = \frac{1}{(n+1)!}\left(x^{n+1} - 2^{n+1}B^s_{n+1}(x/2)\right).$$

It is then enough to use (20) and (21). □

Using the w-Bernoulli numbers and the method explained in Section 2, we explicitly sum the series (2) for the sequence (5) and $\nu = 1/2$.

Theorem 7. *For $\mu > -1$ ($\mu > -1/2$ if $l = 0$), $s > -1$, and the sequence $\mathfrak{a}$ defined in* (5)*, we have*

$$U^{\mathfrak{a}}_{\mu,1/2}(x) = \frac{(-1)^l\binom{\mu}{\frac12}}{2(2l+1)!}\left(\frac{x^{2l}}{\binom{\mu+l}{\mu-\frac12}} - \frac{(2l+1)x^{s+2l-1}}{\binom{s+2l}{2l}\binom{\mu+\frac{s}{2}+l-\frac12}{\mu-\frac12}} - \sum_{k=0}^{2l+1}\frac{\binom{2l+1}{k}B^s_{2l+1-k}x^{k-1}}{2^{k-2l-1}\binom{\mu+\frac{k}{2}-\frac12}{\mu-\frac12}}\right)$$

for $x \in (0,1)$.

Proof. It is a direct consequence of the sine expansion in Theorem 6 for $n = 2l+1$ and Lemma 1. □

4. Asymptotic behavior of the polynomials $B_n^s(x)$

It is well known that the Bernoulli polynomials satisfy

$$\lim_{n\to\infty} \frac{(-1)^{n+1}(2\pi)^{2n}}{2(2n)!} B_{2n}(x) = \cos(2\pi x),$$
$$\lim_{n\to\infty} \frac{(-1)^{n+1}(2\pi)^{2n+1}}{2(2n+1)!} B_{2n+1}(x) = \sin(2\pi x)$$

(see [10] or [17]). In this section we prove that the w-Bernoulli polynomials behave analogously. More precisely,

$$\lim_{n\to\infty} \frac{(-1)^{n+1}(2\pi)^{2n}}{2(2n)!} B_{2n}^s(x) = b_1^s \cos(2\pi x), \tag{26}$$

$$\lim_{n\to\infty} \frac{(-1)^{n+1}(2\pi)^{2n+1}}{2(2n+1)!} B_{2n+1}^s(x) = b_1^s \sin(2\pi x) \tag{27}$$

(recall that $B_n^1(x) = B_n(x)$, see Theorem 4).

For $s = 0$, we get $b_m^0 = 0$, $m \ge 0$, and $B_0^0 = 1$, $B_1^0 = -\frac{1}{2}$, and $B_n^0 = 0$, $n \ge 2$. Hence the limits (26) and (27) are simply the known case if $s = 0$. So let us now assume $s > -1$, $s \ne 0$.

We start by estimating the coefficients b_m^s given in (19), for $s > -1$ and $s \ne 0$. We use that, for $z \to +\infty$, the hypergeometric function ${}_1F_2$ satisfies

$$\begin{aligned} {}_1F_2\left(\begin{matrix} a \\ b_1, b_2 \end{matrix}; -z^2\right) &= \frac{\Gamma(b_1)\Gamma(b_2)}{2\sqrt{\pi}\,\Gamma(a)}\, z^{2\kappa} \Big(e^{-i(\kappa\pi+2z)}\big(1 + O(|z|^{-1})\big) \\ &\qquad\qquad + e^{i(\kappa\pi+2z)}\big(1 + O(|z|^{-1})\big)\Big) \\ &\quad + \frac{\Gamma(b_1)\Gamma(b_2)}{\Gamma(b_1 - a)\Gamma(b_2 - a)}\, z^{-2a}\big(1 + O(|z|^{-2})\big) \end{aligned} \tag{28}$$

with $\kappa = \frac{1}{2}(a - b_1 - b_2 + \frac{1}{2})$; see [3, 16.11.8] or [18, (2.5) and (2.8)].

From this formula, we have the following:

Lemma 8. *For $s > -1$ with $s \ne 0$, we have*

$$b_m^s = 1 + O(m^{-1}) - \frac{2^{1+s}\pi^{1/2-s}\Gamma(2 + s/2)}{(2 + s)\Gamma(1/2 - s/2)} \frac{1}{m^s}\big(1 + O(m^{-2})\big), \qquad m \to \infty.$$

Proof. If we take $z = m\pi/2$ in (28), we have

$$\begin{aligned} {}_1F_2\left(\begin{matrix} 1 + s/2 \\ 3/2, 2 + s/2 \end{matrix}; -m^2\pi^2/4\right) &= \frac{2 + s}{\pi^2 m^2}\Big((-1)^{m+1} + O(m^{-1})\Big) \\ &\quad + \frac{2^{1+s}\Gamma(2 + s/2)}{\pi^{3/2+s}\Gamma(1/2 - s/2)} \frac{1}{m^{2+s}}\Big(1 + O(m^{-2})\Big), \end{aligned}$$

and then the estimate for b_m^s follows easily using (19). ☐

We next estimate the w-Bernoulli numbers B_n^s for n even (recall that $B_n^s = 0$ if $n \ge 3$ is odd).

Theorem 9. *For $s > -1$, $s \ne 0$, we have*

$$\frac{(-1)^{n+1}(2\pi)^{2n}}{2(2n)!} B_{2n}^s = b_1^s + O\left(\frac{1}{2^{2n}}\right), \qquad n \to \infty, \tag{29}$$

where the constant implicit in the order term depends only on s.

Proof. From Theorem 5,

$$\frac{(-1)^{n+1}(2\pi)^{2n}}{2(2n)!} B_{2n}^s = \sum_{m\geq 1} \frac{b_m^s}{m^{2n}} = b_1^s + \frac{1}{2^{2n}} \sum_{m\geq 2} \frac{b_m^s}{(m/2)^{2n}} = b_1^s + O\left(\frac{1}{2^{2n}}\right).$$

Note that this is true in both cases: for $s>0$ (when $b_m^s \sim 1$) and for $-1<s<0$ (when $b_m^s \sim -m^{-s}$). ☐

In order to estimate $B_n^s(z)$, $z\in\mathbb{C}$, we will use Theorem 9, which is the case $z=0$, and the "binomial formula"

$$B_n^s(x+y) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}^s(x)\, y^k. \tag{30}$$

This kind of identity holds for every Appell sequence and can be proved directly from the generating function or from the differential relation $\frac{d}{dx}B_n^s(x) = nB_{n-1}^s(x)$.

We actually prove an stronger result than (26) and (27). To do that, we need the truncated cosine and sine functions, namely

$$T_{2n}(z) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} z^{2k}, \qquad T_{2n+1}(z) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} z^{2k+1}.$$

With this notation we have the following:

Theorem 10. *For $s>-1$, $s\neq 0$ and for n big enough, we have*

$$\left| \frac{(-1)^{n+1}(2\pi)^{2n}}{2(2n)!} B_{2n}^s(z) - b_1^s T_{2n}(2\pi z) \right| \leq \frac{C_s}{2^{2n}} e^{4\pi|z|}, \tag{31}$$

$$\left| \frac{(-1)^{n+1}(2\pi)^{2n+1}}{2(2n+1)!} B_{2n+1}^s(z) - b_1^s T_{2n+1}(2\pi z) \right| \leq \frac{C_s}{2^{2n+1}} e^{4\pi|z|}, \tag{32}$$

uniformly for z in any compact subset K of $\mathbb{C}$, where C_s denotes a constant depending only on s and K.

Proof. To prove (31), we use (30) and (29) to get

$$\begin{aligned}
\frac{(-1)^{n+1}(2\pi)^{2n}}{2(2n)!} B_{2n}^s(z) &= \frac{(-1)^{n+1}(2\pi)^{2n}}{2} \sum_{k=0}^{2n} \frac{B_{2n-k}^s}{(2n-k)!} \frac{z^k}{k!} \\
&= \frac{(-1)^{n+1}(2\pi)^{2n}}{2} B_1^s \frac{z^{2n-1}}{(2n-1)!} + \frac{(-1)^{n+1}(2\pi)^{2n}}{2} \sum_{k=0}^{n} \frac{B_{2n-2k}^s}{(2n-2k)!} \frac{z^{2k}}{(2k)!} \\
&= (-1)^n \frac{(2\pi)^{2n} z^{2n-1}}{4(2n-1)!} + \sum_{k=0}^{n} (-1)^k (-1)^{n-k+1} \frac{(2\pi)^{2n-2k} B_{2n-2k}^s}{2(2n-2k)!} \frac{(2\pi z)^{2k}}{(2k)!} \\
&= (-1)^n \frac{(2\pi)^{2n} z^{2n-1}}{4(2n-1)!} + \sum_{k=0}^{n} (-1)^k \left(b_1^s + O(2^{-2n+2k})\right) \frac{(2\pi z)^{2k}}{(2k)!} \\
&= b_1^s T_{2n}(2\pi z) + \sum_{k=0}^{n} O(2^{-2n+2k}) \frac{(2\pi z)^{2k}}{(2k)!} + (-1)^n \frac{(2\pi)^{2n} z^{2n-1}}{4(2n-1)!},
\end{aligned}$$

where the implicit constant c in $O(\cdot)$ is independent of k and n. Hence,

$$\begin{aligned}\left|\sum_{k=0}^{n} O(2^{-2n+2k})\frac{(2\pi z)^{2k}}{(2k)!} + (-1)^n \frac{(2\pi)^{2n} z^{2n-1}}{4(2n-1)!}\right| \\ \leq c\sum_{k=0}^{n} 2^{-2n+2k}\frac{(2\pi|z|)^{2k}}{(2k)!} + \frac{\pi}{2}\frac{(2\pi|z|)^{2n-1}}{(2n-1)!} \\ \leq 2^{-2n}c\sum_{k=0}^{n}\frac{(4\pi|z|)^{2k}}{(2k)!} + 2^{-2n}c'\frac{(4\pi|z|)^{2n-1}}{(2n-1)!} \\ \leq 2^{-2n}c''\sum_{k=0}^{\infty}\frac{(4\pi|z|)^{k}}{k!} = 2^{-2n}c''e^{4\pi|z|} = O\left(\frac{e^{4\pi|z|}}{2^{2n}}\right),\end{aligned}$$

again with the constants c', c'' independent of k and n. This proves (31). The proof of (32) is similar. □

As a direct consequence, we have the following.

Corollary 11. *For $s > -1$, $s \neq 0$, and any compact subset $K \subset \mathbb{C}$, the w-Bernoulli polynomials satisfy*

$$\frac{(-1)^{n+1}(2\pi)^{2n}}{2(2n)!}B^s_{2n}(z) = b^s_1\cos(2\pi z) + O\left(\frac{e^{4\pi|z|}}{2^{2n}}\right),$$
$$\frac{(-1)^{n+1}(2\pi)^{2n+1}}{2(2n+1)!}B^s_{2n+1}(z) = b^s_1\sin(2\pi z) + O\left(\frac{e^{4\pi|z|}}{2^{2n+1}}\right),$$

uniformly on K, where the implicit constant only depends on s and K.

Proof. The tail of the Taylor expansion can be estimated by

$$|\cos(z) - T_{2n}(z)| \leq \sum_{k=2n+1}^{\infty}\frac{|z|^k}{k!} \leq \frac{1}{2^{2n+1}}\sum_{k=2n+1}^{\infty}\frac{|2z|^k}{k!} \leq \frac{1}{2^{2n+1}}e^{2|z|}$$

(and similarly with $|\sin(z) - T_{2n+1}(z)|$). Then, it is enough to use Theorem 10. □

Since all the zeros of the functions $\cos(2\pi z)$ and $\sin(2\pi z)$ are real, it follows from Hurwitz's theorem [9, Chapter VII, p. 152], that the real zeros of the w-Bernoulli polynomials $B^s_n(z)$, for n even or odd, converge, when $n \to \infty$, to the real zeros of the corresponding cosine or sine functions, respectively. In addition, the complex zeros of these polynomials must converge to infinity, as $n \to \infty$. Therefore, as usual, one can be interested in the behavior of the zeros of the polynomials $B^s_n(nz)$.

Szegő showed in 1924 that, if we denote by $s_n(z) = \sum_{j=0}^{n} z^j/j!$ the nth partial sum of the exponential function e^z, the zeros of the normalized partial sum $s_n(nz)$ tend, as $n \to \infty$, to the curve $|ze^{1-z}| = 1$ in the complex plane, which is now known as Szegő curve. Similar behavior happens with the zeros of the partial sums or $\cos(z)$ and $\sin(z)$, see [25]. And, due to the convergence of Bernoulli polynomials to the trigonometric functions, again Bernoulli polynomials have this kind of behavior. In particular, the complex zeros of the Bernoulli polynomials $B_n(nz)$ tend to the curve $e^{2\pi|\operatorname{Im}(z)|} = 2\pi e|z|$ (see [11, 16, 5]).

The zero behavior of the normalized w-Bernoulli polynomials $B^s_n(nz)$ remains as a challenge. On the one hand, one can expect that the complex zeros tend to the curve $e^{2\pi|\operatorname{Im}(z)|} = 2\pi e|z|$, and the numerical experiments show that such behaviour

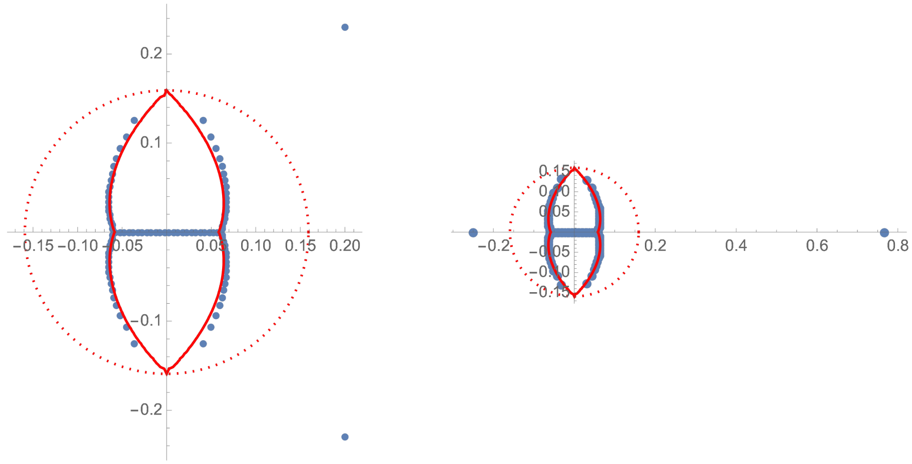

FIGURE 1. The zeros of $B_{100}^{s}(100z)$ for $s = 4$ (left) and $s = -1/2$ (right). Each dot represents a zero. The pictures show also the circle of radius $1/(2\pi)$ and the curve $e^{2\pi|\operatorname{Im}(z)|} = 2\pi e|z|$.

seems to happen. But on the other hand, a couple of spurious zeros seem to run away from the attraction of the curve, see Figure 1. In the case $s > 1$, these are a couple of conjugate zeros, that seem to have a limit (outside the curve) which depends on s. In the case $s \in (-1, 0) \cup (0, 1)$, they are a couple of real zeros whose sum is, approximately, $1/2$.

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