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On an efficient modification of the Chebyshev method

José Antonio Ezquerro | **Miguel Ángel Hernández-Verón** | **Ángel Alberto Magreñán**

Department of Mathematics and Computation, University of La Rioja, Logroño, Spain

Correspondence
Ángel Alberto Magreñán, Department of Mathematics and Computation, University of La Rioja, Logroño, La Rioja, Spain.
Email: angel-alberto.magrenan@unirioja.es

Abstract
An efficient modification of the Chebyshev method is constructed from approximating the second derivative of the operator involved by combinations of the operator in different points and it is used to locate, separate, and approximate the solutions of a Chandrasekhar integral equation from analysing its global convergence.

KEYWORDS
convergence ball, global convergence, recurrence relations, third-order iterative process

1 | INTRODUCTION

A large number of problems in applied mathematics and engineering are solved by finding the solutions of nonlinear systems of equations, differential equations, boundary value problems, integral equations, and so forth. Rarely, the solutions of these equations can be found in closed form,[1-3] so that we often follow a procedure for modeling the equations and obtain equations of the form $F(x) = 0$, which are usually solved by iterative processes that approximate their solutions. Iterative processes start from one or several initial approximations and a sequence is constructed that converges to a solution of an equation.

So, for a general situation, we consider the equation $F(x) = 0$, where F is a nonlinear operator defined on a non-empty open convex domain Ω of a Banach space X with values in a Banach space Y, and a one-point iterative process of the form

$$x_{n+1} = \Psi(x_n), \quad n \geq 0, \quad \text{for given } x_0, \tag{1}$$

where $\Psi : X \to X$, to approximate a solution of $F(x) = 0$. For this, we need to see that the solution exists and the sequence (1) converges to this solution.

As we can see in Reference 4, it is well-known that the efficiency index of an iterative method in the scalar case is $EI = q_1^{1/q_2}$, where q_1 is the order of convergence and q_2 the number of new computations of F and its derivatives per iteration, and represents a good measure of the efficiency of the iterative method.

For one-point iterative methods of order q_2, it is imposed in Reference 4 the restriction of depending explicitly on the first $q_2 - 1$ derivatives of the function involved. Moreover, for these kind of methods, we know that $q_2 = q_1$ ($q_1 \in \mathbb{N}$) and $EI = q_1^{1/q_1}$, so that the best situation is obtained for $q_1 = 3$, namely, for third-order one-point iterative methods. The best known one-point iterative methods are the Chebyshev method,[5] the Halley method,[6] and the Super-Halley method.[7] However, for nonlinear systems, third-order methods are not considered as the most favorable, rather the Newton method is, although its efficiency index $EI = 2^{1/2}$ is worse. This is due to the fact that the efficiency index does not consider other determinants.

For example, if we consider the case of solving nonlinear systems of dimension n, $F(x_1, x_2, \ldots, x_n) = 0$, where $F : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function and $F \equiv (F_1, F_2, \ldots, F_n)$ with $F_i : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, n$, it is necessary to compute the n functions F_i $(i = 1, 2, \ldots, n)$ for computing F. Moreover, for $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, the computation of F',

$$F'(\mathbf{x}) = \begin{pmatrix} (F_1)_1(\mathbf{x}) & (F_1)_2(\mathbf{x}) & \ldots & (F_1)_n(\mathbf{x}) \\ (F_2)_1(\mathbf{x}) & (F_2)_2(\mathbf{x}) & \ldots & (F_2)_n(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ (F_n)_1(\mathbf{x}) & (F_n)_2(\mathbf{x}) & \ldots & (F_n)_n(\mathbf{x}) \end{pmatrix},$$

requires the computations of the n^2 partial derivatives of first order, and the computation of F'',

$$F''(\mathbf{x}) = \left(\begin{array}{cccc|c|cccc} (F_1)_{11}(\mathbf{x}) & (F_1)_{12}(\mathbf{x}) & \ldots & (F_1)_{1n}(\mathbf{x}) & & (F_n)_{11}(\mathbf{x}) & (F_n)_{12}(\mathbf{x}) & \ldots & (F_n)_{1n}(\mathbf{x}) \\ (F_1)_{21}(\mathbf{x}) & (F_1)_{22}(\mathbf{x}) & \ldots & (F_1)_{2n}(\mathbf{x}) & & (F_n)_{21}(\mathbf{x}) & (F_n)_{22}(\mathbf{x}) & \ldots & (F_n)_{2n}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots & \ldots & \vdots & \vdots & \ddots & \vdots \\ (F_1)_{n1}(\mathbf{x}) & (F_1)_{n2}(\mathbf{x}) & \ldots & (F_1)_{nn}(\mathbf{x}) & & (F_n)_{n1}(\mathbf{x}) & (F_n)_{n2}(\mathbf{x}) & \ldots & (F_n)_{nn}(\mathbf{x}) \end{array}\right),$$

requires the computations of the $n^2(n+1)/2$ partial derivatives of second order. In addition, the application of Newton's method,

$$\begin{cases} x_0 \ \text{ given in } \ \Omega, \\ F'(x_n)\ \delta_n = -F(x_n), \quad n \geq 0, \\ x_{n+1} = x_n + \delta_n, \end{cases} \tag{2}$$

to solve the nonlinear system of n equations requires $n^2 + n$ evaluations of functions per iteration, whereas a one-point third-order method, as for example the Chebyshev method (which is possibly the most used, since its algorithm is the most simple),

$$\begin{cases} x_0 \ \text{ given in } \ \Omega, \\ F'(x_n)\ \delta_n = -F(x_n), \quad n \geq 0, \\ F'(x_n)\ \gamma_n = (-1/2)\ F''(x_n)\ \delta_n^2, \\ x_{n+1} = x_n + \delta_n + \gamma_n, \end{cases}$$

requires $n^2(n+1)/2$ evaluations of functions per iteration more than the Newton method. Therefore, it is better to use the Newton method than the Chebyshev method for solving nonlinear systems of n equations with $n \geq 2$, see Figure 1.

Another important point to bear in mind when choosing an iterative method is the number of operations (products and divisions) needed to apply it, which we define as computational cost of doing an iteration of the algorithm. So, the Newton method requires

$$(n^3 + 6n^2 - 4n)/3$$

operations to do an iteration (see (2)), whereas the Chebyshev method requires us to do the same operations plus the products $(-1/2)\ F''(x_n)\ \delta_n^2$ ($n^3 + n^2 + n$ operations) and the solution of the linear system

$$F'(x_n)\ \gamma_n = (-1/2)\ F''(x_n)\ \delta_n^2$$

($2n^2 - n$ operations). Therefore, the computational cost per iteration of the Chebyshev method is

$$\frac{4n^3 + 15n^2 - 4n}{3},$$

which is higher than that of the Newton method. As a consequence, it is clear that the application of the Newton method is a better option than the Chebyshev method for solving nonlinear system of n equations, as we can see Table 1.

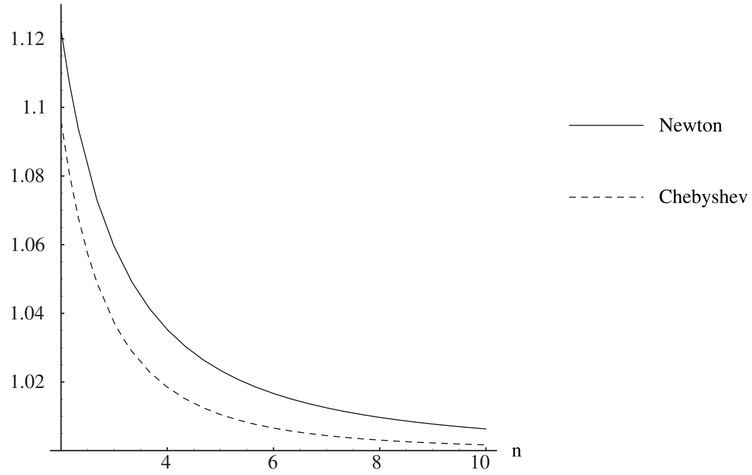

FIGURE 1 Efficiency index of the Newton and the Chebyshev methods for nonlinear systems, respectively $2^{1/(n^2+n)}$ and $3^{2/(n^3+3n^2+2n)}$

TABLE 1 Number of evaluations of functions and computational cost per iteration when the Newton and the Chebyshev method are applied to solve nonlinear systems of n equations

	The Newton method		The Chebyshev method	
n	n^2+n	$(n^3+6n^2-4n)/3$	$(n^3+3n^2+2n)/2$	$(4n^3+15n^2-4n)/3$
10	110	520	660	1820
50	2550	46,600	66,300	1,79,100
100	10,100	3,53,200	5,15,100	13,83,200

From the above-mentioned, our interest is focused on constructing iterations from a modification of the Chebyshev method which reduces the number of evaluations of functions and the computational cost. So, we consider an iterative process, as a modification of the Chebyshev method, with cubical convergence which is more efficient that the Newton and the Chebyshev methods. This iterative process is given by the following algorithm:

$$\begin{cases} x_0 \ \text{ given in } \ \Omega, \\ y_n = x_n - [F'(x_n)]^{-1}F(x_n), \\ z_n = x_n + p\,(y_n - x_n), \quad p \in (0,1]\,, \\ x_{n+1} = x_n - \dfrac{1}{p^2}\,[F'(x_n)]^{-1}\left((p^2+p-1)F(x_n) + F(z_n)\right), \quad n \geq 0, \end{cases} \tag{3}$$

which is constructed in Reference 8 from the Chebyshev method by using a slight modification of a technique developed in Reference 9 to obtain iterative processes of the form (1). The idea used in Reference 8 is to approximate the second Fréchet derivative of the operator F in the Chebyshev method by means of only combinations of F in different points, so that F'' is not used and F' is only evaluated in x_n. With this modification of the Chebyshev method, the number of evaluations of functions and the computational cost are considerably reduced, the efficiency is improved and the order of convergence is kept, see Reference 8. Notice that, for $p = 1$, the iterative process (3) correspond to the two-step frozen Newton method.[10]

The aim of this work is to justify an important feature that the iterative process (3) has relative to the qualitative study that can be carried out of a nonlinear equation: the location and separation of the solutions of the nonlinear equation. In our case, we use the Chandrasekhar integral equation to carry out this qualitative study.

The article is organized as follows. In Section 2, we obtain an efficient modification of the Chebyshev method by using a technique developed in Reference 8 that consists of approximating the second derivative of the operator involved by

TABLE 2 Number of evaluations of functions and computational cost per iteration when family (4) are applied to solve nonlinear systems of n equations

	Iterations (4)	
n	$2n^2 + n$	$(n^3 + 15n^2 - n)/3$
10	210	830
50	5050	54,150
100	20,100	3,83,300

combinations of the operator in different points and study the number of evaluations of functions, the computational cost per iteration, and the efficiency index when the method is applied to solve nonlinear systems of equations. In Section 3, we give a qualitative study of a Chandrasekhar integral equation where the solutions are located and separated. For the last, we rely on the well-known Fixed Point Theorem and use a new technique based on using auxiliary points.

Throughout the article, we denote $\overline{B(x,\varrho)} = \{y \in X : \|y - x\| \leq \varrho\}$ and $B(x,\varrho) = \{y \in X : \|y - x\| < \varrho\}$ and the set of bounded linear operators from Y to X by $\mathcal{L}(Y,X)$.

2 | PRELIMINARIES

First, from the Chebyshev method, Hernández obtains in Reference 9 the following family of third-order multipoint iterations which does not require the computation of F'':

$$\begin{cases} x_0 \ \text{ given in } \ \Omega \text{ and } p \in (0,1], \\ F'(x_n)\ \delta_n = -F(x_n), \quad n \geq 0, \\ z_n = x_n + p\ \delta_n, \\ F'(x_n)\ \hat{\gamma}_n = -\frac{1}{2p}(F'(z_n) - F'(x_n))\ \delta_n, \\ x_{n+1} = x_n + \delta_n + \hat{\gamma}_n. \end{cases} \tag{4}$$

To obtain the last family of iterations, the expression $F''(x_n)\ \delta_n^2$ of the Chebyshev method is approximated by the expression $(1/p)(F'(z_n) - F'(x_n))\ \delta_n$, so that the number of evaluations of functions and the computational cost per iteration is reduced to $2n^2 + n$ and $(n^3 + 15n^2 - n)/3$, respectively. Therefore, the choice of family (4) to solve nonlinear systems of n equations is better than the Chebyshev method, although worse than the Newton method. See Table 2 and Figure 2.

Second, by using a slight modification of the technique used in Reference 9 by Hernández, we then obtain a family of third-order iterations, which reduces even more the number of evaluations of functions and the computational cost, so that these values are close to those of the Newton method. So, our next aim is to construct new iterations from family (4) that converge when they start at the same points as the Newton method. For this, we construct some iterations from the Chebyshev method that reduce the number of necessary values of the function involved and the computational cost, while preserving cubical convergence.

To construct iterations from the Chebyshev method, we use a slight modification of the technique developed in Reference 9 to obtain family (4). The idea is now to approximate the expression $F''(x_n)\delta_n^2$ in the algorithm of Chebyshev by means of only combinations of F in different points, so that F'' is not used and F' is only evaluated in x_n. For this, we consider

$$y_n = x_n - [F'(x_n)]^{-1}F(x_n)$$

and

$$z_n = x_n + p(y_n - x_n)$$

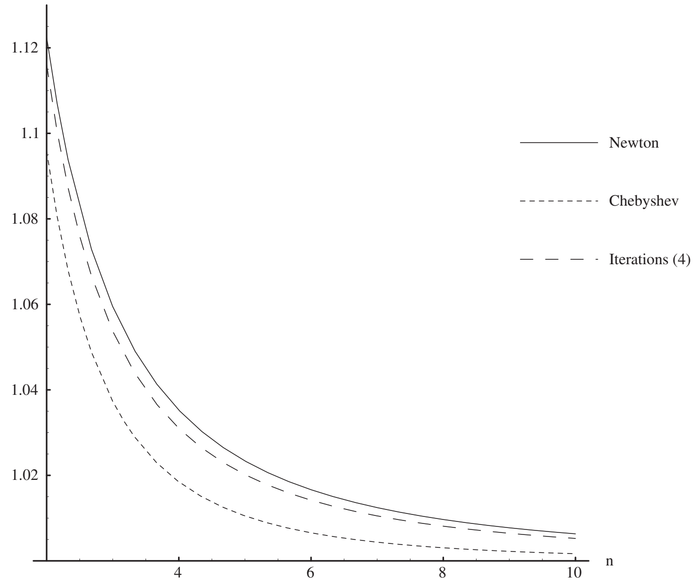

FIGURE 2 Efficiency index of the Newton and the Chebyshev methods and family (4) for nonlinear systems, respectively $2^{1/(n^2+n)}$, $3^{2/(n^3+3n^2+2n)}$, and $3^{1/(2n^2+n)}$

with $p \in (0,1]$ and Taylor's formula as follows:

$$F(z_n) = F(x_n) + pF'(x_n)(y_n - x_n) + \frac{p^2}{2}F''(x_n)(y_n - x_n)^2 + \frac{1}{2}\int_{x_n}^{z_n} F'''(x)(z_n - x)^2\, dx,$$

so that

$$F(z_n) - F(x_n) - pF'(x_n)(y_n - x_n) = \frac{p^2}{2}F''(x_n)(y_n - x_n)^2 + \frac{1}{2}\int_{x_n}^{z_n} F'''(x)(z_n - x)^2\, dx.$$

Now, as $y_n = x_n - [F'(x_n)]^{-1}F(x_n)$, we can then consider the following approximation

$$F''(x_n)(y_n - x_n)^2 \approx \frac{2}{p^2}((p-1)F(x_n) + F(z_n)),$$

and the Chebyshev method is then written as

$$\begin{cases} x_0 \ \text{ given in } \ \Omega \text{ and } p \in (0,1]\,, \\ F'(x_n)\ \delta_n = -F(x_n), \quad n \geq 0, \\ z_n = x_n + p\ \delta_n, \\ F'(x_n)\ \tilde{\gamma}_n = -\frac{1}{p^2}\left((p-1)F(x_n) + F(z_n)\right), \\ x_{n+1} = x_n + \delta_n + \tilde{\gamma}_n, \end{cases} \tag{5}$$

which can also be written as in (3).

With this modification of the Chebyshev method, we have reduced the computational cost from $n^3 + n^2 + n$ operations for computing

$$(-1/2)\ F''(x_n)\ \delta_n^2$$

to $2n$ operations for computing

$$(-1/p^2)\left((p-1)F(x_n) + F(z_n)\right),$$

which is a considerable reduction.

TABLE 3 Number of evaluations of functions and computational cost per iteration when iterations (5) are applied to solve nonlinear systems of n equations

	Iterations (5)	
n	n^2+2n	$(n^3+12n^2+2n)/3$
10	120	740
50	2600	51,700
100	10,200	3,73,400

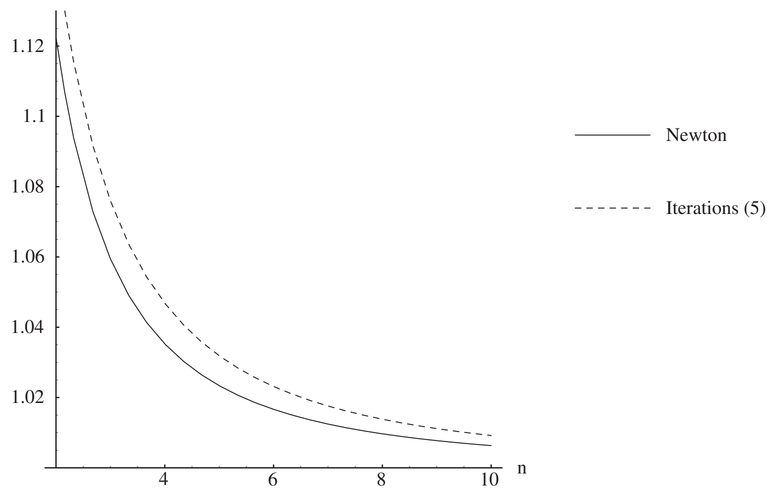

FIGURE 3 Efficiency index of the Newton method and family (5) for nonlinear systems, respectively $2^{1/(n^2+n)}$ and $3^{1/(n^2+2n)}$

Moreover, observe that the efficiency is also improved, since the number of evaluations of functions per iteration is also reduced from

$$\frac{n^3+3n^2+2n}{2}$$

to n^2+2n, see Table 3.

Observe in Figure 3 that the efficiency of the Newton method is improved by iterations of family (5), even for high values of n. As a consequence, family (5) is a better choice to solve nonlinear system $F(x_1, x_2, \dots, x_n) = 0$, since the number of computations of functions is similar.

3 | A QUALITATIVE STUDY FOR A PARTICULAR NONLINEAR EQUATION

In this section we do a qualitative study of the Chandrasekhar integral equation that consists of locating a solution and separating it from other possible solutions, along with approximate its numerical solution. So, we start considering restricted domains of existence and uniqueness of fixed points, which is based on the Fixed Point Theorem. This theorem provides global convergence for the method of successive approximations, since the method can be started at any point in the full space, but it does not allow us to locate the fixed point in a concrete domain. This is our first aim. Moreover, as the speed of convergence of the method of successive approximations is linear, our second aim is to improve it by applying the iterative process given in (3).

3.1 | Motivation

Let the Chandrasekhar equation,

$$x(s) = 1 + \frac{\varpi_0}{2}\, s\, x(s) \int_0^1 \frac{x(t)}{s+t}\, dt, \quad s \in [0,1], \tag{6}$$

arise in theory of radiative transfer;[11] where ϖ_0 is the albedo for single scattering and $x(s)$ is the unknown function which is sought in $C[0,1]$. The physical background of this equation is fairly elaborate. It was developed by Chandrasekhar[11] to solve the problem of determination of the angular distribution of the radiant flux emerging from a plane radiation field. This radiation field must be isotropic at a point, that is the distribution in independent of direction at that point. Explicit definitions of these terms may be found in the literature.[11] It is considered to be the prototype of the equation,

$$x(s) = 1 + s\, x(s) \int_0^1 \frac{\varphi(s)}{s+t}\, x(t)\, dt, \quad s \in [0,1],$$

for more general laws of scattering, where $\varphi(s)$ is an even polynomial in s with

$$\int_0^1 \varphi(s)\, ds \le \frac{1}{2}. \tag{7}$$

Integral equations of the above form also arise in many other studies.[12,13]

In general, we use numerical methods to solve it, since we cannot do it exactly. We can start locating a solution and separating it from other possible solutions. For this, we consider restricted domains of existence and uniqueness of fixed points, which is based on the well-known Fixed Point Theorem:[14]

> *If the operator $\mathcal{P} : C[0,1] \to C[0,1]$ is a contraction, then $\mathcal{P}$ has a unique fixed point in $C[0,1]$ that can be approximated from the method of successive approximations $x_{n+1} = \mathcal{P}(x_n)$, $n \ge 0$, with x_0 given in $C[0,1]$.*

Recall also that operator $\mathcal{P}$ is a contraction if $\|\mathcal{P}(x) - \mathcal{P}(y)\| < \theta \|x - y\|$ with $\theta < 1$, for all $x, y \in C[0,1]$. As operator $\mathcal{P}$ is derivable, condition $\|\mathcal{P}'(x)\| < 1$, for all $x \in C[0,1]$, is sufficient to see that $\mathcal{P}$ is a contraction.

Then, if we consider the operator $\mathcal{P} : C[0,1] \to C[0,1]$ such that

$$[\mathcal{P}(x)](s) = 1 + \frac{\varpi_0}{2}\, s\, x(s) \int_0^1 \frac{x(t)}{s+t}\, dt, \quad s \in [0,1], \tag{8}$$

it is clear that a fixed point of operator (8) is a solution of integral Equation 6. In addition, if we choose $\varpi_0 = \frac{1}{4}$, which satisfies (7), we observe that

$$\|\mathcal{P}(x) - \mathcal{P}(y)\| \le \frac{\ln 2}{8} (\|x\| + \|y\|)\, \|x - y\|, \quad \text{with } x, y \in C[0,1],$$

so that the first problem is to locate a domain that contains a fixed point of the operator $\mathcal{P}$; namely, a solution of (6) with $\varpi_0 = \frac{1}{4}$. For this, we consider the following property that the fixed points x^* of $\mathcal{P}$ have. From (8), it follows

$$\|x^*\| - 1 - \frac{\ln 2}{8} \|x^*\|^2 \le 0,$$

which is satisfied if

$$\|x^*\| \le \rho_1 = 1.1059\ \dots$$

or

$$\|x^*\| \ge \rho_2 = 10.4356\ \dots\ ,$$

so that we can consider domains of the form $\overline{B(0,\rho)}$ with $\rho \geq \rho_1$, where 0 is the zero function, and x^* is in $\overline{B(0,\rho)}$ if there exists.

In the last case, we look for uniqueness results of the fixed point in restricted domains. For this, we use the following modification of the above Fixed Point Theorem to establish a result on fixed points in a set Ω of $C[0,1]$ instead of in the full space $C[0,1]$:

> *If Ω is a convex and compact set of $C[0,1]$ and the operator $\mathcal{J} : \Omega \to \Omega$ is a contraction, then the operator $\mathcal{J}$ has a unique fixed point in Ω that can be approximated by the method of successive approximations, $x_{n+1} = \mathcal{J}(x_n)$, $n \geq 0$, from any $x_0 \in \Omega$.*

Note that the previous location of fixed points does not guarantee the existence of a fixed point of $\mathcal{P}$ in the set chosen. Taking into account this, we study when the chosen set contains a fixed point of $\mathcal{P}$ and it is unique, what allows locating and separating fixed points.

In this case, it follows that

$$\|\mathcal{P}(x) - \mathcal{P}(y)\| \leq \frac{\ln 2}{4}\rho\|x-y\|, \quad x, y \in \overline{B(0,\rho)},$$

and $\mathcal{P}$ is then a contraction operator if

$$\rho < \frac{4}{\ln 2} = 5.7707 \ldots$$

In addition, from

$$\|T(x)\| \leq 1 + \frac{\ln 2}{8}\|x\|^2,$$

we have

$$\|T(x)\| \leq 1 + \frac{\ln 2}{8}\rho^2 \leq \rho,$$

provided that $\rho \in [\rho_1, \rho_2]$.

As a consequence, we have that $\mathcal{P}(x) \in \overline{B(0,\rho)}$ and $\mathcal{P} : \overline{B(0,\rho)} \to \overline{B(0,\rho)}$ is a contraction and, from the modification of the Banach Fixed Point Theorem given previously, there exists a unique fixed point x^* of $\mathcal{P}$ in $\overline{B(0,\rho)}$ with $\rho \in [1.1059 \ldots, 5.7707 \ldots)$. So, the last procedure allow us to establish a location of a fixed point x^* of $\mathcal{P}$ and a separation of other possible fixed points given by the ball $\overline{B(0,\rho)}$, with $\rho \in [1.1059 \ldots, 5.7707 \ldots)$, for the Chandrasekhar integral equation given in (6) with $\varpi_0 = \frac{1}{4}$. In particular, the best location of the fixed point x^* is the ball $\overline{B(0, 1.1059 \ldots)}$, since it is the smallest ball of convergence that contains the fixed point. Besides, we can also obtain the best separation between x^* and other possible fixed point by choosing $\overline{B(0, 5.7707 \ldots)}$, that is the largest ball of convergence that contains x^* as the unique fixed point.

Notice that the dependence of the zero function makes the location of fixed points by balls centered on zero function is not the most appropriate. In our study we consider ball centered on a function $\tilde{x} \in C[0,1]$ different from the zero function, which leads to better locations of the fixed points.

3.2 | Location and separation of solutions

Observe that solving the Chandrasekhar equation with $\varpi_0 = \frac{1}{4}$ is equivalent to solving $F(x) = 0$, where $F : C[0,1] \to C[0,1]$ and

$$[F(x)](s) = x(s) - 1 - \frac{s}{8}\,x(s)\int_0^1 \frac{x(t)}{s+t}\,dt, \quad s \in [0,1]. \tag{9}$$

Now, we introduce the conditions under which the restricted global convergence of the iterative process (3) is obtained in balls of the form $B(\tilde{x}, R)$ (see Reference 15):

(C1) For some $\tilde{x} \in \Omega$, there exists $\tilde{\Gamma} = [F'(\tilde{x})]^{-1} \in \mathcal{L}(Y, X)$ with $\|\tilde{\Gamma}\| \leq \beta$ and $\|\tilde{\Gamma}F(\tilde{x})\| \leq \eta$.
(C2) There exists a constant $K \geq 0$ such that $\|F'(x) - F'(y)\| \leq K\|x - y\|$, for all $x, y \in \Omega$.

If we observe the condition (C2), it is easy to follow that

$$\|F'(x) - F'(\tilde{x})\| \leq \tilde{K}\|x - \tilde{x}\|, \quad \text{for all } \ x \in \Omega,$$

with $\tilde{K} \leq K$, once $\tilde{x} \in \Omega$ is fixed. Moreover, we denote

$$d = \frac{\beta}{1 - \tilde{K}\beta R} \quad \text{and} \quad e = \frac{2\eta + 2R + \tilde{K}\beta R^2}{2(1 - \tilde{K}\beta R)},$$

for $R > 0$. So, the restricted global convergence of the iterative process (3) follows now from the next result.

Theorem. *Let F be a once continuously differentiable operator defined on a non-empty open convex domain Ω of a Banach space X with values in a Banach space Y. Assume that the conditions (C1) and (C2) hold and the existence of $R > 0$ satisfying*

$$\frac{2\eta + K\beta R^2}{2(1 - \tilde{K}\beta R)} + \frac{e}{2}a_0 \leq R$$

and such that $B(\tilde{x}, R) \subset \Omega$. If conditions

$$K\beta R < 1$$

and

$$a_0 = Kde < 0.7064 \ldots$$

are satisfied, then the iterative process (3) is well-defined and converges to a solution x^* of the equation $F(x) = 0$ in the domain $\overline{B(\tilde{x}, R)}$ from every point x_0 belonging to $B(\tilde{x}, R)$. Moreover, the solution x^* of the equation $F(x) = 0$ is unique in $B(\tilde{x}, \varepsilon) \cap \Omega$, where ε is a positive root of

$$\tilde{K}\beta(R + \varepsilon) - 2 = 0. \tag{10}$$

Note that this way of analyzing the global convergence of the iterative process (3) allows locating the solution x^* in the ball $\overline{B(\tilde{x}, R)}$ and defining the ball of global convergence $\overline{B(\tilde{x}, R)}$.

Next, from the Equation 6 with $\varpi_0 = \frac{1}{4}$, we obtain $x(0) = 1$ and, as a consequence, we can select the auxiliary function $\tilde{x}(s) = 1$. Moreover,

$$[F(x_0)](s) = -\frac{s}{8}\int_0^1 \frac{dt}{s + t} = -\frac{s}{8}\ln\left(\frac{1 + s}{s}\right), \quad s \in [0, 1],$$

so that

$$\|[F(x_0)](s)\| = \frac{\ln 2}{8}.$$

Furthermore,

$$[F'(x)y](s) = y(s) - \frac{s}{8}x(s)\int_0^1 \frac{y(t)}{s + t}dt - \frac{s}{8}y(s)\int_0^1 \frac{x(t)}{s + t}dt, \quad s \in [0, 1],$$

and

$$\|I - F'(x_0)\| \leq \frac{\ln 2}{4}.$$

Now, from the Banach lemma, we have

$$\beta = \frac{4}{4-\ln 2} \approx 1.2096 \ldots \quad \text{and} \quad \eta = \frac{\ln 2}{2(4-\ln 2)} \approx 0.1048 \ldots$$

Next, it is easy to check that $K = \tilde{K} = \frac{\ln 2}{4} \approx 0.1732 \ldots$ Then, the condition

$$\frac{2\eta + K\beta R^2}{2(1-\tilde{K}\beta R)} + \frac{e}{2}a_0 \leq R$$

of the theorem is satisfied for all

$$R \in [0.1142 \ldots, 1.4518 \ldots]$$

and

$$K\beta R < 1$$

is also satisfied.

In addition, the condition

$$a_0 =< 0.7064 \ldots$$

is satisfied for

$$R \in [0.1142 \ldots, 1.3878 \ldots].$$

So, the three conditions are satisfied and, as a consequence, the iterative process (3) converges to a solution x^* of the equation $F(x) = 0$ in the domain

$$\overline{B(1, R)}$$

for

$$R \in [0.1142 \ldots, 1.3878 \ldots].$$

Therefore, the best ball of location of solution is

$$\overline{B(1, 0.1142 \ldots)}$$

and the best ball of global convergence is

$$\overline{B(1, 1.3878 \ldots)}.$$

Note that the location of the solution of (6) with $\varpi_0 = \frac{1}{4}$ improves significantly that obtained previously in Section 3.1.

3.3 | Approximation of a solution

To obtain a numerical solution of the Chandrasekhar equation given in (6) with $\varpi_0 = \frac{1}{4}$, we first discretize the problem and approach the integral by a Gauss–Legendre numerical quadrature with eight nodes

$$\int_0^1 f(t)\, dt \approx \sum_{j=1}^{8} w_j f(t_j),$$

where the nodes and weights are given in Table 4.

TABLE 4 Nodes and weights of Gauss–Legendre numerical quadrature

j	t_j	w_j
1	0.01985507175123188 …	0.050614268145188129 …
2	0.10166676129318663 …	0.111190517226687235 …
3	0.23723379504183550 …	0.156853322938943643 …
4	0.40828267875217509 …	0.181341891689180991 …
5	0.59171732124782490 …	0.181341891689180991 …
6	0.76276620495816449 …	0.156853322938943643 …
7	0.89833323870681336 …	0.111190517226687235 …
8	0.98014492824876811 …	0.050614268145188129 …

TABLE 5 Solution of the nonlinear system (11)

j	x_j^*
1	1.0101781 …
2	1.0329569 …
3	1.0547234 …
4	1.0719797 …
5	1.0844979 …
6	1.0930361 …
7	1.0984086 …
8	1.1012071 …

If we denote $x_i = x(t_i)$, $i = 1, 2, \dots, 8$, the Chandrasekhar equation is transformed into the following nonlinear system:

$$x_i = 1 + \frac{x_i}{8} \sum_{j=1}^{8} a_{ij} x_j, \quad i = 1, 2, \dots, 8, \tag{11}$$

where, $a_{ij} = \frac{t_i w_j}{t_i + t_j}$. Take into account the previous continuous study, we use the initial choice

$$x_0 = (1, 1, 1, 1, 1, 1, 1, 1)^T.$$

If we take $p = 1$, we obtain the two-step frozen Newton method and after three iterates and a tolerance of 10^{-30}, iterative process (3) with $p = 1$ converges to the numerical solution x^* of the nonlinear system of Equation 11 given in Table 5.

Notice that the efficiency index of iterations (3) is $3^{1/80} = 1.0138 \dots$, which is the best with respect to those of the Newton method ($2^{1/72} = 1.0096 \dots$), the Chebyshev method ($3^{1/360} = 1.0030 \dots$) and iterations (4) ($3^{1/136} = 1.0081 \dots$), so that the choice of iteration (3) with $p = 1$ to solve the discrete Chandrasekhar equation with $\varpi_0 = \frac{1}{4}$ is the best one.

Finally, once the solution x^* is obtained, we construct a function x_{int} (see Figures 4 and 5) by an interpolating procedure to use the values obtained from the numerical solution of the arithmetic problem $\{(t_j, x_j^*)\}_{j=1}^{8}$ and observe that the interpolated approximation x_{int} lies within the ball of existence and the ball of global convergence of solutions obtained in the above continuous study.

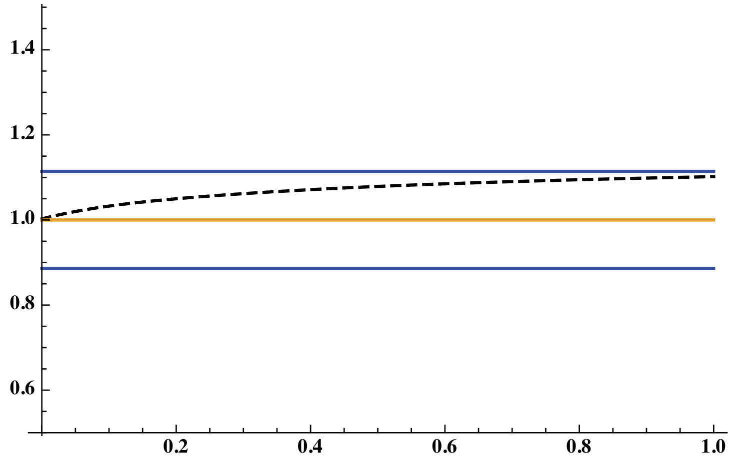

FIGURE 4 Interpolation polynomial x_{int} (dotted line) and ball of existence of solution (blue line)

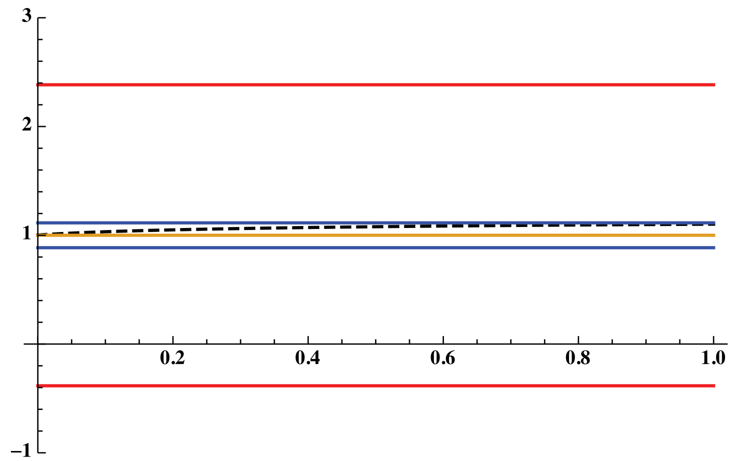

FIGURE 5 Interpolation polynomial x_{int} (dotted line), ball of existence of solution (blue line) and ball of global convergence (red line)

ORCID

Ángel Alberto Magreñán https://orcid.org/0000-0002-6991-5706

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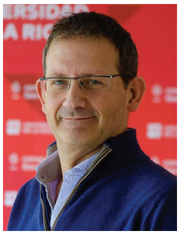

José Antonio Ezquerro received the B.S. degree from Universidad de Zaragoza and his Ph.D. degree from Universidad de Zaragoza. Currently, he is Full Professor in the Department of Mathematics and Computation, Universidad deLa Rioja, Logroño, Spain. His research interests include iterative schemes for nonlinear equations, error bounds, convergence in Banach spaces, the study of majorizing sequences and convergence domains

Miguel Ángel Hernández-Verón received the B.S. degree from Universidad de Zaragoza and his Ph.D. degree from Universidad de Zaragoza. Currently, he is Full Professor in the Department of Mathematics and Computation, Universidad deLa Rioja, Logroño, Spain. His research interests include iterative schemes for nonlinear equations, error bounds, convergence in Banach spaces, the study of majorizing sequences and convergence domains.

Ángel Alberto Magreñán received the M.S., B.S., and Ph.D. degrees in mathematics from Universidad de La Rioja, Logroño, Spain, respectively. Currently, he is Professor at Universidad de La Rioja, Logroño, Spain. His research interests include iterative schemes for nonlinear equations and numerical approximation of differential equations, real and complex dynamical study, nonlinear reconstructions, multiresolution, and wavelets algorithms.

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