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Extended Kung–Traub Methods for Solving Equations with Applications

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Abstract: Kung and Traub (1974) proposed an iterative method for solving equations defined on the real line. The convergence order four was shown using Taylor expansions, requiring the existence of the fifth derivative not in this method. However, these hypotheses limit the utilization of it to functions that are at least five times differentiable, although the methods may converge. As far as we know, no semi-local convergence has been given in this setting. Our goal is to extend the applicability of this method in both the local and semi-local convergence case and in the more general setting of Banach space valued operators. Moreover, we use our idea of recurrent functions and conditions only on the first derivative and divided difference, which appear in the method. This idea can be used to extend other high convergence multipoint and multistep methods. Numerical experiments testing the convergence criteria complement this study.

Keywords: Kung–Traub method; Banach space; convergence criterion

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1. Introduction

We consider approximating a solution x^* of equation

$$F(x) = 0, \tag{1}$$

where $F : \Omega \subset V_1 \longrightarrow V_2$ is an operator acting between Banach spaces V_1 and V_2 with $\Omega \neq \emptyset$. Kung and Traub, in [1], introduced a fourth-order iterative method for solving nonlinear equations on the real line. This method in Banach space is defined for $n = 0, 1, 2, \ldots$ by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ x_{n+1} &= y_n - [y_n, x_n; F]^{-1}F'(x_n)[y_n, x_n; F]^{-1}F(y_n). \end{aligned} \tag{2}$$

Here $[.,.;F] : \Omega \times \Omega \longrightarrow L(V_1, V_2)$ is a divided difference of order one [2]. The convergence order was obtained using Taylor expansions and hypotheses on the derivative of F of order up to five. Note that the method involves also the derivative of order one, so the assumptions on the fifth derivative reduce the applicability of the method [1,3–5].

For example: Let $V_1 = V_2 = \mathbb{R}$, $\Omega = [-0.5, 1.5]$. Define λ on Ω by

$$\lambda(t) = \begin{cases} t^3 \log t^2 + t^5 - t^4 & if\ t \neq 0 \\ 0 & if\ t = 0. \end{cases}$$

Then, we have $t_* = 1$,

$$\lambda'''(t) = 6 \log t^2 + 60t^2 - 24t + 22.$$

Obviously $\lambda'''(t)$ is not bounded on Ω. Therefore, the convergence of method (2) is not guaranteed by the analysis in [1]. In order to avoid Taylor series expansions but still obtain the fourth order of convergence for method (2), we use the computational order of convergence and the approximate computational order of convergence, which do not require more than one derivative (see Remark 1.2b).

In this paper, we introduce a majorant sequence and use our idea of recurrent functions to extend the applicability of method (2). Our analysis includes error bounds and results on uniqueness of x^* based on computable Lipschitz constants not given before in [1] and in other similar studies using Taylor series [3–13]. The advantages of the extended method include: Applications for solving nonlinear Banach space valued equations are not limited to systems of finite dimensional Euclidean space. Local convergence includes computable upper error bounds not given before. Moreover, the semi-local convergence not given before is proved. The motivation for writing this paper is the extension of the applicability of method (2), as already illustrated by the example. The novelty of the paper includes the extension of the convergence domain in both the local as well as the semi-local convergence case and the introduction of the recurrent functions proving technique, which can be used in other methods too [14–27].

The rest of the paper is set up as follows: In Section 2, we present results on majorizing sequences. Sections 3 and 4 contain the semi-local and local convergence, respectively, where in Section 5, the numerical experiments are presented. Concluding remarks are given in Section 6.

2. Majorizing Sequences

We present results on majorizing sequences.

Definition 1. *Let $\{u_n\}$ be a sequence in a Banach space. Then, a nondecreasing scalar sequence $\{m_n\}$ is called majorizing for $\{u_n\}$ if*

$$\|u_{n+1} - u_n\| \leq m_{n+1} - m_n \text{ for each } n = 0, 1, 2, \dots. \tag{3}$$

By this definition, we can use sequence $\{m_n\}$ to study the convergence of $\{u_n\}$.

Let $\eta > 0$, $\ell > 0, \ell_i > 0$, $i = 0, 1, 2, \dots, 5$ be the given parameters. Define scalar sequences $\{s_n\}$, $\{t_n\}$ for each $n = 0, 1, 2, \dots$ by $t_0 = 0, s_0 = \eta$

$$\begin{aligned} t_1 &= s_0 + \frac{\ell_0 (s_0 - t_0)^2}{2(1 - \ell_1 s_0)^2}, \\ s_{n+1} &= t_{n+1} + \frac{\alpha_{n+1}}{1 - \ell_0 t_{n+1}}, \\ t_{n+2} &= s_{n+1} + \frac{\ell \ell_4 t_{n+1} (s_{n+1} - t_{n+1})^2}{2(1 - \ell_1 (s_{n+1} + t_{n+1}))^2}, \end{aligned} \tag{4}$$

where $\alpha_{n+1} = \left(\ell_3 (t_{n+1} - t_n) + \frac{\ell_3 (1 + \ell_1 (s_n + t_n))(s_n - t_n)}{1 - \ell_0 t_n} \right) (t_{n+1} - s_n)$.

Lemma 1. *Suppose:*

$$\ell_1 \eta < 1, \tag{5}$$

for each $n = 0, 1, 2, \dots$,

$$t_{n+1} < \frac{1}{\ell_0} \tag{6}$$

and

$$s_{n+1} + t_{n+1} < \frac{1}{\ell_1}. \tag{7}$$

Then, sequences $\{s_n\}, \{t_n\}$ are nondecreasing, bounded from above by $\frac{1}{\ell}$ and as such they converge to their unique least upper bound $t^* \in [\eta, \frac{1}{\ell_0}]$. Moreover, the following hold for each $n = 0, 1, 2, \ldots$

$$t_n \leq s_n \leq t_{n+1}.$$

Proof. It follows from the definition of sequence $\{s_n\}, \{t_n\}$ and hypotheses (5)–(7). □

Remark 1. *Hypotheses (6) and (7) are verified only in special cases. That is why we introduce stronger hypotheses implying those of Lemma 1 but not necessarily vice versa.*

It is convenient for us to define sequences of functions and functions on the interval $M = [0, 1)$ for each $n = 1, 2, \ldots$ as follows:

$$\begin{aligned} f_n(t) &= \ell_5(t^{2n} + t^{2n-1})\eta + \ell_1 t^{2n-1}\eta \\ &+ \ell_1\ell_2 t^{2n-1}(t^{2n} + 2(1 + t + \ldots + t^{2n-1}))\eta^2 \\ &+ \ell_0\eta(t^{2n} + t^{2n+1} + 2(1 + t + \ldots + t^{2n-1})) - \ell_0^2\eta^2 - 1, \end{aligned}$$

$$\begin{aligned} f(t) &= a_7t^7 + a_6t^6 + a_5t^5 + a_4t^4 + a_3t^3 + a_2t^2 + a_1t + a_0, \\ f_\infty(t) &= -\left(1 - \frac{\ell_0\eta}{1-t}\right)^2 \end{aligned}$$

$$\begin{aligned} g_n(t) &= \frac{\ell\ell_4\eta^2}{2} t^{2n+2}(1 + t + \ldots t^{2n+1}) \\ &+ 2\ell_1\eta(t^{2n+2} + 2(1 + t + \ldots + t^{2n+1})) \\ &- \ell_1^2\eta^2(t^{2n+2} + 2(1 + t + \ldots + t^{2n+1}))^2 - 1, \\ g(t) &= \frac{\ell\ell_4}{2}\eta(t^7 + t^6 + t^5 + t^4 - t - 1) + 2\ell_1\eta(1+t)^2 \\ &+ \ell_1^2\eta(1 + t + t^2)(\frac{4}{1-t} + 3t^5 + 2t^4 + t^6), \end{aligned}$$

and

$$g_\infty(t) = -\left(1 - \frac{2\ell_1\eta}{1-t}\right)^2,$$

where

$$\begin{aligned} a_0 &= -(\ell_1 + \ell_5 + 2\ell_1\ell_2^2\eta), \\ a_1 &= -(\ell_5 + 2\ell_1\ell_2\eta), \\ a_2 &= \ell_0 + \ell_1 + \ell_1\ell_2\eta + \ell_5, \\ a_3 &= \ell_0 + \ell_5 + 2\ell_1\ell_2\eta, \\ a_4 &= \ell_0 + 2\ell_1\ell_2\eta, \\ a_5 &= \ell_0 + 2\ell_1\ell_2\eta, \\ a_6 &= 2\ell_1\ell_2\eta \end{aligned}$$

and

$$a_7 = \ell_1\ell_2\eta.$$

By these definitions we have

$$\begin{aligned} f(0) &= -(\ell_1 + \ell_5 + 2\ell_1\ell_2\eta) < 0, \\ f(1) &= 2(2\ell_0 + 3\ell_1\ell_2\eta) > 0, \\ g(0) &= -\frac{\ell\ell_4\eta}{2} < 0 \end{aligned}$$

and

$$g(t) \longrightarrow +\infty \text{ as } t \longrightarrow 1^-.$$

It then follows by the intermediate value theorem that functions f and g have zeros in the interval $(0,1)$. Denote the smallest such zero by b_1 and b_2, respectively. Moreover, we have for each $t \in M$

$$f_\infty(t) \le 0 \tag{8}$$

and

$$g_\infty(t) \le 0. \tag{9}$$

Furthermore, define scalar sequences $\{\gamma_n\}$ and $\{\delta_n\}$ by

$$\gamma_n = \frac{\ell_3(t_{n+1} - t_n)(1 - \ell_0 t_n) + \ell_2(1 + \ell_1(s_n + t_n))(s_n - t_n)}{(1 - \ell_0 t_n)(1 - \ell_0 t_{n+1})}$$

and

$$\delta_n = \frac{\ell\ell_4 t_{n+1}(s_{n+1} - t_{n+1})}{291 - \ell_1(t_{n+1} + s_{n+1}))^2}.$$

$$\mu_0 = \max\{\gamma_0, \delta_0\},\ \mu_1 = \min\{b_1, b_2\}. \tag{10}$$

Next, we present a second auxiliary result on majorizing sequences.

Lemma 2. *Suppose that there exists μ such that*

$$\mu_0 \le \mu \le \mu_1 < 1 - 2\ell_1\eta \tag{11}$$

and (5) holds. Then, sequences $\{s_n\}, \{t_n\}$ are well defined, nondecreasing, bounded from above by $t^{**} = \frac{\eta}{1-\mu}$, and as such they converge to their unique least upper bound $t^* \in [\eta, t^{**}]$. Moreover, the following estimates hold for each $n = 1, 2, \ldots$

$$0 \le t_{n+1} - s_n \le \mu(s_n - t_n) \le \mu^{2n+1}\eta, \tag{12}$$

$$0 \le s_n - t_n \le \mu(t_n - s_{n-1}) \le \mu^{2n}\eta, \tag{13}$$

and

$$0 \le t_n - s_n \le t_{n+1}. \tag{14}$$

Proof. Estimates (12)–(14) hold if

$$0 \le \gamma_k \le \mu, \tag{15}$$

$$0 \le \delta_k \le \mu, \tag{16}$$

and

$$t_k \le s_k \le t_{k+1}, \tag{17}$$

are true for $k = 0, 1, 2, \ldots$. Notice that by the definition of s_0, t_1 and (5), we have $s_0 \le t_1$. We also have (15)–(17), which hold for $k = 0$ by (11). Suppose that estimates (15) and (16) hold for $k = 1, 2, \ldots n$. Then, we obtain

$$\begin{aligned} s_k &\leq t_k + \mu^{2k}\eta \leq s_{k-1} + \mu^{2k-1}\eta + \mu^{2k}\eta \\ &\leq \eta + \mu\eta + \ldots + \mu^{2k}\eta \\ &= \frac{1-\mu^{2k+1}}{1-\mu}\eta < \frac{\eta}{1-\mu} = t^{**}, \end{aligned}$$

and

$$\begin{aligned} t_{k+1} &\leq s_k + \mu^{2k+1}\eta \leq t_k + \mu^{2k}\eta + \mu^{2k+1}\eta \\ &\leq \eta + \mu\eta + \ldots + \mu^{2k+1}\eta \\ &= \frac{1-\mu^{2k+2}}{1-\mu}\eta < t^{**}. \end{aligned}$$

It follows by the induction hypotheses and (17) that sequences $\{s_k\}$ and $\{t_k\}$ are nondecreasing. Estimates (15) holds if we instead show for $\ell_5 = \ell_3(1-\ell_0 t_1)$ that

$$\begin{aligned} &\ell_5(\mu^{2k+1}+\mu^{2k})\eta + \ell_1\eta\mu^{2k} + \ell_1\ell_2\eta^2\mu^{2k}\left(\frac{1-\mu^{2k+1}}{1-\mu} + \frac{1-\mu^{2k}}{1-\mu}\right) \\ &-\mu\left(1-\ell_0\left(\frac{1-\mu^{2k}}{1-\mu} + \frac{1-\mu^{2k+2}}{1-\mu}\right)\eta\right) \\ &+\ell_0^2\frac{(1-\mu^{2k})(1-\mu^{2k+2})}{(1-\mu)^2}\eta^2\Bigg) \\ \leq\ & 0 \end{aligned}$$

or

$$f_k(t) \leq 0 \text{ for } t = \mu. \tag{18}$$

We need a relationship between two consecutive functions f_k. By the definition of function f_k, we can write, in turn, by adding and subtracting f_k

$$\begin{aligned} f_{k+1}(t) &= f_k(t) + \ell_5(t^{2k+2}+t^{2k+1}-t^{2k}-t^{2k-1})\eta + \ell_1\eta(t^{2k+1}-t^{2k-1}) \\ &\quad +\ell_1\ell_2\eta^2 t^{2k-1}[t^2(1+t+\ldots+t^{2k+2}) + (1+t+\ldots+t^{2k+3})) \\ &\quad -((1+t+\ldots+t^{2k}) + (1+t+\ldots+t^{2k+-1}))] \\ &\quad +\ell_0\eta[(1+t+\ldots+t^{2k+1}) + (1+t+\ldots+t^{2k+3})) \\ &\quad -((1+t+\ldots+t^{2k-1}) + (1+t+\ldots+t^{2k+1}))] \\ &\leq f_k(t) + [\ell_5(t^3+t^2-t-1) + \ell_1(t^2-1) \\ &\quad +(t^7+2t^6+2t^5+2t^4+2t^3+t^2-2t-2))\ell_1\ell_2\eta \\ &\quad +\ell_0(t^2+t^3+t^4+t^5)]t^{2k-1}\eta \\ &= f_k(t) + f(t)t^{2k-1}\eta, \end{aligned} \tag{19}$$

where we used $t^k \leq t, k = 1,2,\ldots,$ since $t \in (0,1)$. Define $f_\infty(t) = \lim_{k\longrightarrow\infty} f_k(t)$.

Then, we can show instead of (18) that

$$f_\infty(\mu) \leq 0, \tag{20}$$

which is true by (8). Set $c_k = t^{2k+2}(1+t+\ldots+t^{2k+1})$ and $d_k = t^{2k+2} + 2(1+t+\ldots+t^{2k+1})$. As in (15), estimate (16) holds if

$$g_k(t) \leq 0 \text{ for } t = \mu. \tag{21}$$

Function $g_k(t)$ can be written as

$$g_k(t) = \frac{\ell\ell_4\eta^2}{2}t^{2k+2}c_n + 2\ell_1\eta d_n - \ell_1^2\eta^2 d_n^2 - 1.$$

Then, we again need a relationship between two consecutive functions g_k. Notice that

$$\begin{aligned} c_{k+1} - c_k &= t^{2k+4}(1 + t + \ldots + t^{2k+3}) - t^{2k+2}(1 + t + \ldots + t^{2k+1}) \\ &= t^{2k+2}(-1 - t + t^{2k+2} + t^{2k+3} + t^{2k+4} + t^{2k+5}), \end{aligned}$$

$$\begin{aligned} d_{k+1} - d_k &= (1 + t + \ldots + t^{2k+3}) + (1 + t + \ldots + t^{2k+4}) \\ &\quad -(1 + t + \ldots + t^{2k+1}) - (1 + t + \ldots + t^{2k+2}) \\ &= t^{2k+2} + 2t^{2k+3} + t^{2k+4}, \end{aligned}$$

and

$$d_{k+1} - d_k = 4(1 + t + \ldots + t^{2k+1}) + 3t^{2k+2} + 2t^{2k+3} + t^{2k+4}.$$

By adding and substracting g_k from g_{k+1} we obtain

$$\begin{aligned} g_{k+1}(t) &= g_k(t) + \frac{\ell\ell_4\eta^2}{2} t^{2k+2}(t^{2k+5} + t^{2k+4} + t^{2k+3} + t^{2k+2} - t - 1) \\ &\quad +2\ell_1\eta(t^{2k+2} + 2t^{2k+3} + t^{2k+4}) + \ell_1^2\eta^2(d_{k+1}^2 - d_k^2) \\ &\leq g_k(t) + g(t)t^{2k+2}\eta. \end{aligned}$$

Define $g_\infty(t) = \lim_{k\longrightarrow\infty} g_k(t)$. Then, we can show instead of (21) that

$$g_\infty(\mu) \leq 0,$$

which is true by (11). The induction for estimates (15)–(17) is completed. Hence, sequences $\{s_n\}, \{t_n\}$ are nondecreasing, bounded from above by t^{**} so they converge to t^*. □

3. Semi-Local Convergence

Let $U(x_0, r) = \{x \in V_1 : \|x - x_0\| < r, r > 0\}$ and $U[x_0, r] = \{x \in V_1 : \|x - x_0\| \leq r, r > 0\}$. The semi-local convergence analysis of method (2) uses conditions (H1)–(H4).

Suppose:

(H1) There exists $x_0 \in \Omega$ and $\eta \geq 0$ such that $F'(x_0)^{-1} \in L(V_2, V_1)$ and

$$\|F'(x_0)^{-1}F(x_0)\| \leq \eta.$$

(H2) For each $x \in \Omega$

$$\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \leq \ell_0\|x - x_0\|.$$

Set $\Omega_0 = U[x_0, \frac{1}{\ell_0}] \cap \Omega$.

(H3) For each $x, y \in \Omega_0$, the following holds

$$\|F'(x_0)^{-1}(F'(y) - F'(x))\| \leq \ell\|y - x\|,$$

$$\|F'(x_0)^{-1}([y, x; F] - F'(x_0))\| \leq \ell_1(\|y - x_0\| + \|x - x_0\|),$$

$$F'(x_0)^{-1}([y, x; F] - F'(x))\| \leq \ell_2\|y - x\|$$

$$\|F'(x_0)^{-1}([z, y; F] - [y, x; F])\| \leq \ell_3(\|z - y\| + \|y - x\|).$$

and

$$\|F'(x_0)^{-1}F'(x)\| \leq \ell_4\|x - x_0\|.$$

(H4) $U[x_0, t^*] \subset \Omega$.

Then, we can show the main semi-local convergence result for method (2).

Theorem 1. *Suppose that conditions (H1)–(H4) hold. Then, sequence $\{x_n\}$ generated by method (2) is well defined in $U[x_0, t^*]$, remain in $U[x_0, t^*]$ for each $n = 0, 1, 2, \ldots$ and converge to a solution $x_* \in U[x_0, t^*]$ of equation $F(x) = 0$, so that*

$$\|x_* - x_n\| \leq t^* - t_n.$$

Proof. Assertions

(A_k) $\|y_k - x_k\| \leq s_k - t_k$

(B_k) $\|x_{k+1} - y_k\| \leq t_{k+1} - s_k$

shall be proven using induction on k. It follows from the first substep of method (2) that

$$\|y_0 - x_0\| = \|F'(x_0)^{-1}F(x_0)\| \leq \eta = s_0 - t_0 = s_0 \leq t^*.$$

Hence, (A_0) is true and $y_0 \in U[x_0, t^*]$. We can write by the first substep of method (2) for $n = 0$ and (H2)

$$F(y_0) = F(y_0) - F(x_0) - F'(x_0)(y_0 - x_0),$$

so

$$\|F'(x_0)^{-1}F(y_0)\| \leq \frac{\ell_0}{2}\|y_0 - x_0\|^2 \leq \frac{\ell}{2}(s_0 - t_0)^2.$$

Next, we show the invertability of linear operator $[y_0, x_0; F]$. Indeed, we have by (H2) that

$$\begin{aligned} \|F'(x_0)^{-1}([y_0, x_0; F] - F'(x_0)) &\leq \ell_1(\|y_0 - x_0\| + \|x_0 - x_0\|) \\ &\leq \ell_1(s_0 - t_0) < 1, \end{aligned}$$

so by the Banach lemma on linear invertible operators [20], $[y_0, x_0; F]^{-1}$ exists,

$$\|[y_0, x_0; F]^{-1}F'(x_0)\| \leq \frac{1}{1 - \ell_1\|y_0 - x_0\|} \leq \frac{1}{1 - \ell_1(s_0 - t_0)} \tag{22}$$

and iterate x_1 is well defined by the second substep of method (2) for $n = 0$. We can write

$$x_1 - y_0 = -[y_0, x_0; F]^{-1}F'(x_0)[y_0, x_0; F]^{-1}F(y_0)$$

leading to

$$\begin{aligned} \|x_1 - y_0\| &\leq \|[y_0, x_0; F]^{-1}F'(x_0)\| \\ &\quad \times\|[y_0, x_0; F]^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(y_0)\| \\ &\leq \frac{\ell(s_0 - t_0)^2}{2(1 - \ell_1 s_0)^2} = t_1 - s_0, \end{aligned}$$

showing (B_0). We also obtain

$$\|x_1 - x_0\| \leq \|x_0 - y_0\| + \|y_0 - x_0\| \leq t_1 - s_0 + s_0 - t_0 = t_1 \leq t^*,$$

so $x_1 \in U[x_0, t^*]$. Suppose that (A_k) and (B_k) hold, $y_k, x_{k+1} \in U[x_0, t^*]$ and $F'(x_k)^{-1}$, $[y_k, x_k; F]^{-1}$ exist for each $k = 1, 2, \ldots$. We shall show they hold for $k = n + 1$. By the second substep of method (2), we can write, in turn

$$\begin{aligned} F(x_{n+1}) &= F(x_{n+1}) - F(y_n) - [y_n, x_n; F]F'(x_n)^{-1}[y_n, x_n; F](x_{n+1} - y_n) \\ &= ([x_{n+1}, y_n; F] - [y_n, x_n; F]F'(x_n)^{-1}[y_n, x_n; F])(x_{n+1} - y_n) \\ &= [([x_{n+1}, y_n; F] - [y_n, x_n; F]) + [y_n, x_n; F] \\ &\quad -[y_n, x_n; F]F'(x_n)^{-1}[y_n, x_n; F](x_{n+1} - y_n) \\ &= [([x_{n+1}, y_n; F] - [y_n, x_n; F]) + [y_n, x_n; F] \\ &\quad \times(I - F'(x_n)^{-1}[y_n, x_n; F])](x_{n+1} - y_n) \\ &= [([x_{n+1}, y_n; F] - [y_n, x_n; F]) + ([y_n, x_n; F] - F'(x_0) + F'(x_0)) \\ &\quad \times(F'(x_n) - [y_n, x_n; F])](x_{n+1} - y_n). \end{aligned}$$

Then, by conditions (H3) and the induction hypotheses, in turn, we obtain that

$$\begin{aligned}\|F'(x_0)^{-1}F(x_{n+1})\| &\leq [\ell_3(\|x_{n+1}-y_n\|+\|y_n-x_n\|)+(1+\ell_1(\|y_n-x_0\|+\|x_n-x_0\|))\\ &\quad \times \frac{\ell_2\|y_n-x_n\|}{1-\ell_0\|x_n-x_0\|}]\|x_{n+1}-y_n\|\\ &\leq [\ell_3(t_{n+1}-t_n)+\frac{(1+\ell_1(s_n+t_n))\ell_2(s_n-t_n)}{1-\ell_0 t_n}](t_{n+1}-s_n)\\ &= \alpha_{n+1}.\end{aligned}$$

We must show $F'(x_{n+1})$ is invertible. Indeed, we have by (H2)

$$\|F'(x_0)^{-1}(F'(x_{n+1})-F'(x_0))\| \leq \ell_0\|x_{n+1}-x_0\| \leq \ell_0 t_{n+1} < 1,$$

so

$$\|F'(x_{n+1})^{-1}F'(x_0)\| \leq \frac{1}{1-\ell_0 t_{n+1}}.$$

Hence, we obtain by method (2) and the two preceding estimates that

$$\begin{aligned}\|y_{n+1}-x_{n+1}\| &\leq \|F'(x_{n+1})^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(x_{n+1})\|\\ &\leq \frac{\alpha_{n+1}}{1-\ell_0 t_{n+1}} = s_{n+1}-t_{n+1},\end{aligned}$$

showing (A_k) for $k = n+1$. We also obtain

$$\begin{aligned}\|y_{n+1}-x_0\| &\leq \|y_{n+1}-x_{n+1}\|+\|x_{n+1}-x_0\|\\ &\leq s_{n+1}-t_{n+1}+t_{n+1}-t_0\\ &= s_{n+1} \leq t^*,\end{aligned}$$

so $y_{n+1} \in U[x_0, t^*]$. In view of the first substep of method (2), we can write

$$F(y_{n+1}) = F(y_{n+1}) - F(x_{n+1}) - F'(x_{n+1})(y_{n+1}-x_{n+1}),$$

leading to

$$\|F'(x_0)^{-1}F(y_{n+1})\| \leq \frac{\ell}{2}\|y_{n+1}-x_{n+1}\|^2 \leq \frac{\ell}{2}(s_{n+1}-t_{n+1})^2$$

so

$$\begin{aligned}\|x_{n+2}-y_{n+1}\| &\leq \|[y_{n+1},x_{n+1};F]^{-1}F'(x_0)\|^2\|F'(x_0)^{-1}F'(x_{n+1})\|\|F'(x_0)^{-1}F(y_{n+1})\|\\ &\leq \frac{\ell\ell_4 t_{n+1}(s_{n+1}-t_{n+1})^2}{2(1-\ell_1(s_{n+1}+t_{n+1}))^2}\\ &= t_{n+2}-s_{n+1},\end{aligned}$$

showing (B_k) for $k = n+1$. Moreover, we obtain

$$\begin{aligned}\|x_{n+2}-x_0\| &\leq \|x_{n+2}-y_{n+1}\|+\|y_{n+1}-x_0\|\\ &\leq t_{n+2}-s_{n+1}+s_{n+1}-t_0 = t_{n+2} \leq t^*,\end{aligned}$$

and

$$\begin{aligned}\|x_{n+1}-x_n\| &\leq \|x_{n+1}-y_n\|+\|y_n-x_n\|\\ &\leq t_{n+1}-s_n+s_n-t_n.\end{aligned}$$

Hence, we deduce $x_{n+2} \in U[x_0, t^*]$ and sequence $\{t_n\}$ is Cauchy in a Banach space V_1. Hence, it converges to some $x^* \in U[x_0, t^*]$. By letting $n \longrightarrow \infty$ in the estimate

$$\|F'(x_0)^{-1}F(x_{n+1})\| \leq \alpha_{n+1},$$

and the continuity of F, we obtain $F'(x^*) = 0$. □

Concerning the uniqueness of the solution x_* we have:

Proposition 1. *Suppose: There exists*

(i) *A simple solution x^* of equation $F(x) = 0$.*
and

(ii) *$\tilde{s} \geq t^*$ such that*

$$\ell_0(\tilde{s} + t^*) < 2.$$

Set $\Omega_1 = U[x_0, \tilde{s}] \cap \Omega$. Then, the only solution of equation $F(x) = 0$ in the region Ω_1 is x^*.

Proof. Let $\tilde{x} \in \Omega_1$ with $F(\tilde{x}) = 0$. Set $T = \int_0^1 F'(\tilde{x} + \theta(x^* - \tilde{x}))d\theta$. Then, by (H2) and (ii), we obtain

$$\begin{aligned}\|F'(x_0)^{-1}(T - F'(x_0))\| &\leq \int_0^1 \varphi_0((1-\theta)\|\tilde{x} - x_0\| + \theta\|x^* - x_0\|)d\theta \\ &\leq \frac{\ell_0}{2}(\tilde{s} + t^*) < 1,\end{aligned}$$

leading to $\tilde{x} = x^*$, where we used the identity $T(x^* - \tilde{x}) = F(x^*) - F(\tilde{x}) = 0 - 0 = 0$ and the invertability of T. □

4. Local Convergence

Let $L, L_j, j = 0,1,2,3,4$ be positive parameters. Set $S = [0, \frac{1}{L_0})$. Define function ψ_1 on the interval $S = [0, \frac{1}{L_0})$ by

$$\psi_1(t) = \frac{Lt}{2(1 - L_0 t)}.$$

Then, parameter r_1 is defined by

$$r_1 = \frac{2}{2L_0 + L} \tag{23}$$

solves equation

$$\psi_1(t) - 1 = 0.$$

Moreover, define functions q, p on interval S by

$$q(t) = L_0\psi_1(t)t - 1 \text{ and } p(t) = L_1(1 + \psi_1(t))t - 1.$$

Suppose that equations

$$q(t) = 0, \ p(t) = 0$$

have smallest solutions $r_q, r_p \in S - \{0\}$. Set $S_0 = [0, r_0)$, where $r_0 = \min\{r_q, r_p\}$. Define function ψ_2 on S_0 by

$$\begin{aligned}\psi_2(t) &= \frac{L}{2}\frac{\psi_1^2(t)t}{1 - L_0\psi_1(t)t} \\ &\quad + \frac{L_3L_4(1 + \psi_1(t))\psi_1(t)t}{(1 - L_0\psi_1(t)t)(1 - L_1(1 + \psi_1(t)t))} \\ &\quad + \frac{L_2L_4(1 + \psi_1(t))\psi_1(t)t}{(1 - L_1(1 + \psi_1(t)t))^2}.\end{aligned}$$

Suppose that equation

$$\psi_2(t) = 0$$

has the smallest solution $r_2 \in S_0 - \{0\}$. We shall prove that

$$r = \min\{r_i\}, \ i = 1,2 \tag{24}$$

is a convergence radius for method (2). Set $S_1 = [0, r)$. By these definitions, we have that for each $t \in S_1$

$$0 \le L_0 t < 1, \tag{25}$$

$$0 \le q(t) < 1, \tag{26}$$

$$0 \le p(t) < 1, \tag{27}$$

and

$$0 \le \psi_i(t) < 1. \tag{28}$$

As in the semi-local convergence case we develop the following conditions (C1)–(C4). Suppose:

(C1) x^* is a simple solution of equation $F(x) = 0$.

(C2) For each $x \in \Omega$

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \le L_0\|x - x_0\|.$$

Set $\Omega_1 = U(x^*, \frac{1}{L_0}) \cap \Omega$.

(C3) For each $x, y \in \Omega_1$

$$\|F'(x^*)^{-1}(F'(y) - F'(x))\| \le L\|y - x\|,$$

$$\|F'(x^*)^{-1}([y, x; F] - F'(x^*))\| \le L_1(\|y - x^*\| + \|x - x^*\|),$$

$$\|F'(x^*)^{-1}([y, x; F] - F'(x))\| \le L_2\|y - x\|$$

$$\|F'(x^*)^{-1}([z, y; F] - F'(y))\| \le L_3\|y - x\|.$$

and

$$\|F'(x^*)^{-1}F'(x)\| \le L_4\|x - x^*\|.$$

(C4) $U[x^*, r] \subset \Omega$.

Then, we can show the local convergence result for method (2).

Theorem 2. *Under conditions (C1)–(C4) further suppose that $x_0 \in U(x^*, r) - \{x^*\}$. Then, sequence $\{x_n\}$ generated by method (2) is well defined in $U(x^*, r)$, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \ldots$ and converges to x^* so that*

$$\|y_n - x^*\| \le \psi_1(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\| < r \tag{29}$$

and

$$|x_{n+1} - x^*\| \le \psi_2(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\|. \tag{30}$$

Proof. Let $z \in U(x^*, r) - \{x^*\}$. Using (C1), (C2), (24) and (25) we obtain

$$\|F'(x^*)^{-1}(F'(z) - F'(x^*))\| \le L_0\|z - x^*\| \le L_0 r < 1,$$

so $F'(z)$ is invertible with

$$\|F'(z)^{-1}F'(x^*)\| \le \frac{1}{1 - L_0\|z - x^*\|}. \tag{31}$$

Iterate y_0 is well defined from (31) for $z = x_0$ and the first substep of method (2). Using (24), (28) (for $i = 1$), (31) (for $z = x_0$) and (C3), we obtain

$$\begin{aligned}\|y_0-x^*\| &\leq \|F'(x_0)^{-1}F'(x^*)\| \\ &\quad\times\|\int_0^1 F'(x^*)^{-1}(F'(x_0+\theta(x_0-x^*))-F'(x_0))d\theta(x_0-x^*)\| \\ &\leq \frac{\bar{L}\|x_0-x^*\|^2}{2(1-L_0\|x_0-x^*\|)} \\ &\leq \psi_1(\|x_0-x^*\|)\|x_0-x^*\| \\ &\leq \|x_0-x^*\|<r, \end{aligned} \tag{32}$$

showing (29) for $n=0$ and $y_0\in U(x^*,r)$. Next, we shall show that $[u,v;F]^{-1}\in L(V_2,V_1)$ for $u,v\in U(x^*,r)$. Indeed, by (24), (26), (C3) and (32) we have

$$\begin{aligned}\|F'(x^*)^{-1}([y_0,x_0;F]-F'(x^*))\| &\leq L_1(\|y_0-x^*\|+\|x_0-x^*\|) \\ &\leq L_1(\psi_1(\|x_0-x^*\|)+1)\|x_0-x^*\| \\ &= p(\|x_0-x^*\|)\leq p(r)<1,\end{aligned}$$

so

$$\|[y_0,x_0;F]^{-1}F'(x^*)\|\leq\frac{1}{1-p(\|x_0-x^*\|)}. \tag{33}$$

We also have that (11) holds for $z=y_0$. Hence, iterate x_1 is well defined by the second substep of method (2). Then, we can write in turn that

$$\begin{aligned}x_1-x^* &= y_0-x_*-F'(y_0)^{-1}F(x_0) \\ &\quad+(F'(y_0)^{-1}-[y_0,x_0;F]^{-1}F'(x_0)[y_0,x_0;F]^{-1})F(y_0).\end{aligned}$$

However, we obtain

$$\begin{aligned}&F'(y_0)^{-1}-[y_0,x_0;F]^{-1}F'(x_0)[y_0,x_0;F]^{-1} \\ =\ &F'(y_0)^{-1}(I-F'(y_0)[y_0,x_0;F]^{-1}F'(x_0)[y_0,x_0;F]^{-1}) \\ =\ &F'(y_0)^{-1}([y_0,x_0;F]-F'(y_0)+F'(y_0) \\ &-F'(y_0)[y_0,x_0;F]^{-1}F'(x_0))[y_0,x_0;F]^{-1}\end{aligned}$$

and

$$\begin{aligned}&[y_0,x_0;F]-F'(y_0)+F'(y_0)-F'(y_0)[y_0,x_0;F]^{-1}F'(x_0) \\ =\ &([y_0,x_0;F]-F'(y_0)) \\ &+F'(y_0)[y_0,x_0;F]^{-1}([y_0,x_0;F]-F'(x_0)),\end{aligned}$$

so

$$\begin{aligned}x_1-x^* &= y_0-x^*-F'(y_0)^{-1}F(y_0) \\ &\quad+F'(y_0)^{-1}([y_0,x_0;F]-F'(y_0))[y_0,x_0;F]^{-1}F(y_0) \\ &\quad+[y_0,x_0;F]^{-1}([y_0,x_0;F]-F'(x_0))[y_0,x_0;F]^{-1}F(y_0),\end{aligned} \tag{34}$$

In view of (24), (28) (for $i=2$), (31) (for $z=x_0,y_0$) and (32)–(34), we obtain, in turn,

$$\begin{aligned}\|x_1-x^*\| &\leq \frac{L\|y_0-x^*\|^2}{2(1-L_0\|y_0-x^*\|)} \\ &\quad+\frac{L_3\|y_0-x_0\|L_4\|y_0-x^*\|}{(1-L_0\|y_0-x^*\|)(1-L_1(\|y_0-x^*\|+\|x_0-x^*\|))} \\ &\quad+\frac{L_2\|y_0-x_0\|L_4\|y_0-x^*\|}{(1-L_1(\|y_0-x^*\|+\|x_0-x^*\|))} \\ &\leq \psi_2(\|x_0-x^*\|)\|x_0-x^*\|\leq\|x_0-x^*\|,\end{aligned} \tag{35}$$

showing (30) for $n = 0$ and $x_1 \in U(x^*, r)$. Moreover, exchange x_0, y_0, x_1 by x_j, y_j, x_{j+1}, in the preceding calculations, respectively, to complete the induction for estimates (29) and (30). Furthermore, from the estimate

$$\|x_{j+1} - x^*\| \le c\|x_j - x^*\|, \tag{36}$$

where $c = \psi_2(\|x_0 - x^*\|) \in [0, 1)$, we conclude $\lim_{j \longrightarrow \infty} x_j = x^*$ and $x_{j+1} \in U(x^*, r)$. □

Remark 2. *(a) The value r_1 was given by us in [6] for the radius of convergence for Newton's method. It then follows from (24) that*

$$r \le r_1. \tag{37}$$

Hence, the radius of convergence r for method (2) cannot be larger than Newton's. Notice that the radius of convergence given independently by Rheinboldt [7] and Traub [8] is $\rho = \frac{2}{3K}$, where K is the Lipschitz constant on Ω. We also have $\rho \le r$, since $L_0 \le K$ and $L \le K$.
(b) We compute the computational order of convergence (COC) defined by

$$COC = \ln\left(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}\right) / \ln\left(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|}\right)$$

or the approximate computational order of convergence (ACOC)

$$ACOC = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) / \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right).$$

Then, we obtain in practice the convergence order and avoid the existence of the higher order Fréchet derivatives for operator F.

Next, we present a uniqueness of the solution result.

Proposition 2. *Suppose:*
(a) There exists a simple solution x^* of equation $F(x) = 0$
(b) There exists $\bar{r} \ge r$ such that

$$\bar{r} < \frac{2}{L_0}. \tag{38}$$

Set $\Omega_2 = \Omega \cap U[x^*, \bar{r}]$. Then, the only solution of equation $F(x) = 0$ in the region Ω_1 is x^*.

Proof. Let $b \in \Omega_2$ with $F(b) = 0$. Set $Q = \int_0^1 F'(x^* + \theta(b - x^*))d\theta$. Then, using (C1) and (38), we obtain

$$\begin{aligned} \|F'(x^*)^{-1}(Q - F'(x^*))\| &\le \int_0^1 L_0\theta\|b - x^*\|d\theta \\ &\le \frac{L_0}{2}\bar{r} < 1, \end{aligned}$$

leading to $b = x^*$, since $Q^{-1} \in L(V_2, V_1)$ and $A(b - x^*) = F(b) - F(x^*) = 0 - 0 = 0$. □

5. Numerical Experiments

We provide some examples, with $[x, y; F] = \int_0^1 F'(y + \theta(x - y))d\theta$.

Example 1. *Define function*

$$\psi(x) = b_0 x + b_1 + b_2 \sin b_3 x,\ x_0 = 0,$$

where b_j, $j = 0, 1, 2, 3$ are parameters. Then, clearly for b_3 large and b_2 small, $\frac{\ell_0}{\ell}$ can be small (arbitrarily). Notice that $\frac{\ell_0}{\ell} \longrightarrow 0$.

Example 2. *Consider $V_1 = V_2 = C[0,1]$, $\Omega = U[0,1]$ and $Q : \Omega \longrightarrow V_2$ defined by*

$$Q(\psi)(x) = \varphi(x) - 5\int_0^1 x\theta\psi(\theta)^3 d\theta. \tag{39}$$

We obtain

$$Q'(\psi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\psi(\theta)^2\xi(\theta)d\theta, \text{ for each } \xi \in D.$$

Then, since $x^* = 0$, conditions (C1)–(C4) are verified for $L_0 = 7.5, L = L_4 = K = 15$, $L_1 = \frac{L_0}{2}, L_2 = L_3 = \frac{L}{2}$. Then, the radii are:

$$r = r_1 = 0.066667,\ r_2 = 0.109818, \text{and}\ \rho = \frac{2}{3K} = 0.0667.$$

Example 3. *Consider the motion system*

$$G_1'(x) = e^x,\ G_2'(y) = (e-1)y + 1,\ G_3'(z) = 1$$

with $G_1(0) = G_2(0) = G_3(0) = 0$. Let $G = (G_1, G_2, G_3)$. Let $V_1 = V_2 = \mathbb{R}^3$, $\Omega = \bar{U}(0,1), x^* = (0,0,0)^T$. Define function G on Ω for $w = (x,y,z)^T$ by

$$G(w) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

Then, we obtain

$$G'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence, conditions (C1)–(C4) are verified for $L_0 = (e-1), L = e^{\frac{1}{e-1}} = L_4, L_1 = \frac{L_0}{2}$, $L_2 = L_3 = \frac{L}{2}$, $K = e$. Then, the radii are:

$$r = r_1 = 0.382692,\ r_2 = 0.417923 \text{ and } \rho = \frac{2}{3K} = 0.2453.$$

Example 4. *Let V_1, V_2 and Ω be as in the Example 2. It is well-known that the boundary value problem [2]*

$$\varphi(0) = 0, (1) = 1,$$

$$\varphi'' = -\varphi - \sigma\varphi^2$$

can be given as a Hammerstein-like nonlinear integral equation

$$\varphi(s) = s + \int_0^1 M(s,t)(\varphi^3(t) + \sigma\varphi^2(t))dt$$

where σ is a parameter. Then, define $F : \Omega \longrightarrow V_2$ by

$$[F(x)](s) = x(s) - s - \int_0^1 M(s,t)(x^3(t) + \sigma x^2(t))dt.$$

Choose $x_0(s) = s$ and $\Omega = U(x_0, \rho_0)$. Then, clearly $U(x_0, \rho_0) \subset U(0, \rho_0 + 1)$, since $\|x_0\| = 1$. Suppose $2\sigma < 5$. Then, conditions (H1)–(H4) are verified for

$$\ell_0 = \frac{2\sigma + 3\rho_0 + 6}{8},\ \ell = \frac{\sigma + 6\rho_0 + 3}{4},$$

$\ell_1 = \frac{\ell_0}{2}, \ell_2 = \frac{\ell}{2}, \ell_3 = \frac{\ell}{2}$ and $\mu = \frac{1+\sigma}{5-2\sigma}$. Notice that $\ell_0 < \ell_1$.

In general the radius of convergence decreases, when the order increases. However, notice that in the local convergence Examples 2 and 3, the radii for the fourth-order method (2) compare favorably to the ones given in [7,8] for Newton's (see r and ρ).

6. Conclusions

The Kung–Traub method was revisited, and its applicability was extended in both the semi-local and local convergence case from the real to the Banach space setting. Our analysis includes error bounds and uniqueness on x^* information not available before and under weak conditions. This idea is very general and can be used to extend the applicability of other methods.

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