Fourier series for coherent pairs of Jacobi measures

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ABSTRACT
Let (μ_1, μ_2) be a coherent pair of Jacobi measures. In most cases, we obtain convergence and boundedness of the Fourier series for Sobolev polynomials with respect to this kind of measures.

KEYWORDS
Sobolev orthogonal polynomials; Fourier expansions; Jacobi polynomials; Coherent pair.

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1. Introduction

Let μ_1, μ_2 be positive Borel measures supported on $\mathbb{R}$ and consider the Sobolev inner product

$$(f,g)_S = \int_{\mathbb{R}} f(x)g(x)\,d\mu_1(x) + \lambda \int_{\mathbb{R}} f'(x)g'(x)\,d\mu_2(x), \quad \lambda > 0. \tag{1}$$

Let $\{P_n\}_n$ and $\{T_n\}_n$ denote some orthogonal polynomial sequences with respect to μ_1 and μ_2, respectively. In [5], it is introduced the concept of coherent pair of measures in this way: the pair (μ_1, μ_2) is a coherent pair of measures if there exist nonzero constants A_n and B_n such that

$$T_n = A_n P'_{n+1} + B_n P'_n, \quad n \geq 1.$$

This kind of measures has turned out to be very important in the research of Sobolev orthogonal polynomials. In [10], a complete classification of all coherent pairs was given. More precisely, (μ_1, μ_2) is a coherent pair of measures if at least one of the two measures is a Jacobi or a Laguerre measure. The main target of this paper is to study the convergence and uniform boundedness of the Fourier series in terms of orthonormal polynomials associated with the inner Sobolev product (1) where (μ_1, μ_2)

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forms a coherent pair of measure. In this work, we are going to consider pairs where one of the measures will be a Jacobi measure

$$d\mu_{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}\,dx, \quad x \in [-1,1],$$

with $\alpha > -1$, $\beta > 0$. It may happen two possibilities:

i) (μ_1, μ_2) is a coherent pair of measures of Jacobi type I if $\mu_2 = \mu_{\alpha,\beta}$ and depending on α, μ_1 would be
   a) If $\alpha > 0$, $d\mu_1(x) = (\xi - x)(1-x)^{\alpha-1}(1+x)^{\beta-1}\,dx$ with $\xi \geq 1$.
   b) If $\alpha = 0$, $d\mu_1(x) = (1+x)^{\beta-1}\,dx + M\delta(1)$, with $M \geq 0$.
   c) If $-1 < \alpha < 0$, $d\mu_1(x) = (1-x)^{\alpha}(1+x)^{\beta-1}\,dx$.

ii) (μ_1, μ_2) is a coherent pair of measures of Jacobi type II if $\mu_1 = \mu_{\alpha,\beta-1}$ and

$$d\mu_2(x) = \frac{1}{\xi - x}(1-x)^{\alpha+1}(1+x)^{\beta}\,dx + M\delta(\xi), \quad \xi \geq 1,\ M \geq 0.$$

Notice that for $\xi = 1$ and $M = 0$ the coherent pair is of type I.

It has been proved in [10] that all coherent pairs where μ_2 is a Jacobi measure on $(-1,1)$ are of the above-mentioned form, or can be transformed to one of them by the transformation $x \to -x$.

Given $1 \leq p < \infty$ and μ a positive Borel measure supported on $[-1,1]$. We will write $L^p(\mu)$ as the space of all measurable functions on $[-1,1]$ for which

$$\|f\|_{L^p(\mu)} = \left(\int_{-1}^{1} |f(x)|^p\,d\mu(x)\right)^{1/p} < \infty.$$

Let (μ_1, μ_2) be a coherent pair of measures of Jacobi type, we define the space $W^p_{1,2}$, for $1 \leq p < \infty$, as the space of measurable functions f defined on $[-1,1]$ such that there exists f' almost everywhere and

$$\|f\|^p_{W^p_{1,2}} = \|f\|^p_{L^p(\mu_1)} + \lambda\|f'\|^p_{L^p(\mu_2)} < \infty.$$

Let $\{R_n\}_n$ be the sequence of orthonormal polynomials with respect to (1). Let $G_n f$ be the n-th partial sum given by

$$G_n f(x) = \sum_{k=0}^{n} e_k(f) R_k(x), \quad e_k(f) = (R_k, f)_S.$$

The main result of this paper provide us characterizations of the uniform boundedness of the operator G_n for Jacobi coherent pairs.

Theorem 1.1. *Let $\alpha > -1$, $\beta > 0$ and (μ_1, μ_2) be a coherent pair of Jacobi measures where $M = 0$ for type II. Let $f \in W^p_{1,2}$ be with $1 < p < \infty$. Then*

$$\|G_n f\|_{W^p_{1,2}} \leq C\|f\|_{W^p_{1,2}}, \tag{2}$$

with a constant C independent of n and f, if and only if *i) for type I,*

$$\left|(\alpha+1)\left(\frac{1}{p}-\frac{1}{2}\right)\right|<\frac{1}{4}, \quad \left|(\beta+1)\left(\frac{1}{p}-\frac{1}{2}\right)\right|<\frac{1}{4}. \tag{3}$$

ii) for type II,

$$\left|(\alpha+2)\left(\frac{1}{p}-\frac{1}{2}\right)\right|<\frac{1}{4}, \quad \left|(\beta+1)\left(\frac{1}{p}-\frac{1}{2}\right)\right|<\frac{1}{4}. \tag{4}$$

If the space $W_{1,2}$ was complete and the polynomials formed a dense class, the uniform boundedness of G_n would be equivalent to the convergence in $W^p_{1,2}$. The next result gives us both requirements.

Theorem 1.2. *Let $\alpha > -1$, $\beta > 0$. Let (μ_1, μ_2) be a coherent pair of Jacobi measures with $M = 0$ for the type II. Then*

i) the set of polynomials is dense in the space $W^p_{1,2}$;
ii) $W^p_{1,2}$ is a complete space.

So, Theorem 1.1 and Theorem 1.2 imply the following corollary.

Corollary 1.3. *Let $\alpha > -1$, $\beta > 0$ and (μ_1, μ_2) be a coherent pair of Jacobi measures where $M = 0$ for type II. Let $f \in W^p_{1,2}$ be with $1 < p < \infty$. Then*

$$\lim_{n\to\infty} \|G_n f - f\|_{W^p_{1,2}} = 0,$$

if and only if

i) for type I, (3) *holds.*
ii) for type II, (4) *holds.*

As we can read in [8] Sobolev orthogonal polynomials with respect to a Sobolev inner product

$$(f, g) = \sum_{k=0}^{m} \int_{\mathbb{R}} f^{(k)} g^{(k)}\, d\mu_k,$$

have been widely studied, providing us with many applications and publications. The lack of Christoffel-Darboux formula for Sobolev orthogonal polynomials, except for particular cases, deprives an important tool for studying convergence and summability of Fourier orthogonal expansions. In the last years, several papers have advanced in this topic. In [13], the author presents some results on linear summation methods for Fourier series in orthonormal polynomials of discrete Sobolev spaces. In [1], convergence and uniform boundedness of Fourier series are obtained in this context ($\mu_0 = \mu_{\alpha,\alpha} + M(\delta_1 + \delta_{-1})$, $\mu_1 = N(\delta_1 + \delta_{-1})$ and $\mu_k = 0$, $k \geq 2$). More recently, we can find papers as [15,16] which study more properties of Fourier series in discrete Sobolev spaces.

In the continuous case (all the measures have continuous support), necessary conditions of convergence are obtained in [3,4,7] for particular cases. Approximation by polynomials is analyzed in [18] in the case $\mu_k = \mu_{\alpha,\beta}$. In [2,6] the expansions for

Jacobi-Sobolev polynomials with $\mu_k = \mu_{\alpha+k,\beta+k}$ are studied obtaining a complete characterization of the uniform boundedness and the convergence of the partial sum operators for the Fourier series in [2]. In this paper, we add to this topic by studying Fourier series of ortonormal polynomials with respect to an inner Sobolev product where the measures form a coherent pair of Jacobi measures.

This paper is structured as follows: in Section 2, we present the necessary definitions and results concerning the Jacobi polynomials. Section 3 is devoted to give some auxiliary results for Jacobi-Sobolev polynomials of type I and to prove i) of Theorem 1.1. Analogously, Section 4 is devoted to those of type II and proving ii) of Theorem 1.1. Finally, Theorem 1.2 is proved in Section 5.

2. Jacobi polynomials

Let $\{P_n^{(\alpha,\beta)}\}_n$ be the sequence of Jacobi polynomials for arbitrary α, β, defined by the Rodrigues formula, which are orthogonal with respect to the measure $d\mu_{\alpha,\beta}(x)$,

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\left(\frac{d}{dx}\right)^n ((1-x)^{n+\alpha}(1+x)^{n+\beta}).$$

We denote $\tau_n^{(\alpha,\beta)}$ the leading coefficient of $P_n^{(\alpha,\beta)}$ for $n \geq 2$ that is given by

$$\tau_n^{(\alpha,\beta)} = 2^{-n}\binom{2n+\alpha+\beta}{n}. \tag{5}$$

For $\alpha > -1$ and $\beta > -1$, we have

$$\begin{aligned}(d_n^{(\alpha,\beta)})^2 := \|P_n^{(\alpha,\beta)}(x)\|^2_{L^2(\mu_{\alpha,\beta})} &= \int_{-1}^{1}(P_n^{(\alpha,\beta)}(x))^2(1-x)^\alpha(1+x)^\beta\,dx \\ &= \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)} \approx \frac{2^{\alpha+\beta}}{n}. \end{aligned}\tag{6}$$

For arbitrary α and β the following relations are satisfied, as we can see in [11],

$$\frac{d}{dx}P_n^{(\alpha,\beta)}(x) = \frac{1}{2}(n+\alpha+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}(x) \tag{7}$$

and

$$P_n^{(\alpha-1,\beta-1)}(x) = \frac{n+\alpha+\beta-1}{2n+\alpha+\beta-1}P_n^{(\alpha,\beta-1)}(x) - \frac{n+\beta-1}{2n+\alpha+\beta-1}P_{n-1}^{(\alpha,\beta-1)}(x). \tag{8}$$

For $\alpha > -1$ it holds

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n} = \frac{n^\alpha}{\Gamma(\alpha+1)}\left(1+O\left(\frac{1}{n}\right)\right). \tag{9}$$

Let $\{B_n^{(\alpha,\beta)}(x)\}_n$ be the sequence of orthonormal Jacobi polynomials. Taking $\tau =$

$\max\{\alpha,\, \beta\} > -1$, the following equivalence is well-known (see [17], Exercise 91),

$$\|B_n^{(\alpha,\beta)}\|_{L^p(\mu_{\alpha,\beta})} \simeq \begin{cases} 1, & 2\tau > p\tau - 2 + p/2, \\ (\log n)^{1/p}, & 2\tau = p\tau - 2 + p/2, \\ n^{\tau+1/2-2(\tau+1)/p}, & 2\tau < p\tau - 2 + p/2. \end{cases} \tag{10}$$

Let $\mathbb{S}_n^{(\alpha,\beta)}(f)$ be the n-th partial sum of Fourier expansion in terms of orthonormal Jacobi polynomials,

$$\mathbb{S}_n^{(\alpha,\beta)}(f) = \sum_{k=0}^{n} b_k^{(\alpha,\beta)}(f) B_k^{(\alpha,\beta)}(x),$$

where

$$b_k^{(\alpha,\beta)}(f) = \int_{-1}^{1} f(y) B_k^{(\alpha,\beta)}(y)\, d\mu_{\alpha,\beta}(y).$$

In [12], Muckenhoupt provides us with the following theorem that will be very useful to prove our main result, Theorem 1.1.

Theorem 2.1. *Let $\alpha,\, \beta > -1$ and $1 < p < \infty$. There exists a constant C, independent of n and f, such that*

$$\|\mathbb{S}_n^{(\alpha,\beta)} f\|_{L^p(\mu_{\alpha,\beta})} \leq C \|f\|_{L^p(\mu_{\alpha,\beta})}$$

if and only if

$$\left|(\alpha+1)\left(\frac{1}{p} - \frac{1}{2}\right)\right| < \frac{1}{4}, \quad \left|(\beta+1)\left(\frac{1}{p} - \frac{1}{2}\right)\right| < \frac{1}{4}. \tag{11}$$

We define the operators

$$T_{d,l}^{(\alpha,\beta)(\gamma,\delta)} g(x) = \sum_{k=3}^{\infty} \frac{C}{k} b_{k-l}^{(\gamma,\delta)}(g) B_{k-d}^{(\alpha,\beta)}(x), \quad d,\, l = 0,\, 1.$$

With the same arguments of Proposition 3 of [2] we can prove

Proposition 2.2. *For $\alpha,\, \beta,\, \gamma,\, \delta > -1$ and $1 < p < \infty$, it is verified that*

$$\|T_{d,l}^{(\alpha,\beta)(\gamma,\delta)} g\|_{L^p(\mu_{\alpha,\beta})} \leq C \|g\|_{L^p(\mu_{\gamma,\delta})},$$

for each $g \in L^p(\mu_{\gamma,\delta})$.

3. Jacobi-Sobolev orthogonal polynomials type I

3.1. Auxiliary results

Let (μ_1, μ_2) be a coherent pair of Jacobi type I. Let $\{Q_n\}_n$ be the corresponding sequence of orthogonal polynomials with respect to (1), such that for $n \geq 2$ we choose the leading coefficient of $Q_n(x)$ equals to the leading coefficient of $P_n^{(\alpha-1,\beta-1)}(x)$. In [11] the authors studied the asymptotics of these polynomials and proved the following relation:

$$P_n^{(\alpha-1,\beta-1)}(x) = Q_n(x) - a_{n-1}Q_{n-1}(x), \quad n \geq 3, \quad a_n = O(1/n^2). \tag{12}$$

From (8) and (9) we can prove

$$P_n^{(0,\beta-1)}(1) = 1, \quad P_n^{(-1,\beta-1)}(1) = 0. \tag{13}$$

If we denote $q_n := (Q_n, Q_n)_S^{1/2}$, (12) can be written in terms of orthonormal polynomials as

$$\frac{d_n^{(\alpha-1,\beta-1)}}{q_n} B_n^{(\alpha-1,\beta-1)}(x) = R_n(x) - a_{n-1}\frac{q_{n-1}}{q_n} R_{n-1}(x), \quad n \geq 3. \tag{14}$$

Lemma 3.1. *For $n \geq 3$ the Fourier coefficients $e_n(f) = (R_n, f)_S$ can be expressed as*

a) If $\alpha > 0$,

$$\begin{aligned} e_n(f) = {} & \frac{d_n^{(\alpha-1,\beta-1)}}{q_n} b_n^{(\alpha-1,\beta-1)}((\xi - \cdot)f) \\ & + \frac{\lambda}{2}(n+\alpha+\beta-1)\frac{d_{n-1}^{(\alpha,\beta)}}{q_n} b_{n-1}^{(\alpha,\beta)}(f') + a_{n-1}\frac{q_{n-1}}{q_n} e_{n-1}(f). \end{aligned} \tag{15}$$

b), c) If $-1 < \alpha \leq 0$,

$$\begin{aligned} e_n(f) = {} & \frac{n+\alpha+\beta-1}{2n+\alpha+\beta-1} \frac{d_n^{(\alpha,\beta-1)}}{q_n} b_n^{(\alpha,\beta-1)}(f) \\ & - \frac{n+\beta-1}{2n+\alpha+\beta-1} \frac{d_{n-1}^{(\alpha,\beta-1)}}{q_n} b_{n-1}^{(\alpha,\beta-1)}(f) \\ & + \frac{\lambda}{2}(n+\alpha+\beta-1)\frac{d_{n-1}^{(\alpha,\beta)}}{q_n} b_{n-1}^{(\alpha,\beta)}(f') + a_{n-1}\frac{q_{n-1}}{q_n} e_{n-1}(f). \end{aligned} \tag{16}$$

Proof. Using (14) for the case a) and (14), (8) and (13) for the cases b) and c). □

Using Theorem 2 of [9], we can obtain the asymptotic for q_n.

Lemma 3.2. *Let $-1 < \alpha$ and $\beta > 0$ then*

$$g_1(n) + \frac{\lambda}{4}(n+\alpha+\beta-1)^2(d_{n-1}^{(\alpha,\beta)})^2 \le q_n^2 \le g_2(n) + \frac{\lambda}{4}(n+\alpha+\beta-1)^2(d_{n-1}^{(\alpha,\beta)})^2, \quad (17)$$

where $g_1(n) \approx C/n$ and $g_2(n) \approx C/n$.

Lemma 3.3. *Let $\alpha > -1$, $\beta > 0$ and $\tau = \max\{\alpha, \beta\}$. Then*

$$\|R_n\|_{W^p_{1,2}} \le C \begin{cases} 1, & 2\tau > p\tau - 2 + p/2, \\ (\log n)^{1/p}, & 2\tau = p\tau - 2 + p/2, \\ n^{\tau+1/2-2(\tau+1)/p}, & 2\tau < p\tau - 2 + p/2. \end{cases} \quad (18)$$

Proof. If $\alpha > 0$, we use (12) to prove

$$\|Q_n\|_{L^p(\mu_1)} \le C\|P_n^{(\alpha-1,\beta-1)}\|_{L^p(\mu_1)}, \quad \|Q'_n\|_{L^p(\mu_2)} \le Cn\|P_{n-1}^{(\alpha,\beta)}\|_{L^p(\mu_2)}.$$

Then

$$\|Q_n\|_{W^p_{1,2}} \le C(\|P_n^{(\alpha-1,\beta-1)}\|_{L^p(\mu_{\alpha-1,\beta-1})} + \lambda n\|P_{n-1}^{(\alpha,\beta)}\|_{L^p(\mu_{\alpha,\beta})})).$$

Note that if $\alpha = 0$, from (12) and (13) we have that

$$|Q_n(1)| \le \frac{C}{n^2}. \quad (19)$$

If $-1 < \alpha \le 0$, using (12), (13), (19) and (8) we have

$$\|Q_n\|_{L^p(\mu_1)} \le C\|P_n^{(\alpha,\beta-1)}\|_{L^p(\mu_{\alpha,\beta-1})}, \quad \|Q'_n\|_{L^p(\mu_2)} \le Cn\|P_{n-1}^{(\alpha,\beta)}\|_{L^p(\mu_2)}.$$

Then

$$\|Q_n\|_{W^p_{1,2}} \le C(\|P_n^{(\alpha,\beta-1)}\|_{L^p(\mu_{\alpha,\beta-1})} + \lambda n\|P_{n-1}^{(\alpha,\beta)}\|_{L^p(\mu_{\alpha,\beta})}).$$

It is well-known that

$$\|P_{n-1}^{(\alpha,\beta)}\|_{L^p(\mu_{\alpha,\beta})} \simeq \|P_n^{(\alpha,\beta)}\|_{L^p(\mu_{\alpha,\beta})},$$

and using the equivalences (10) and (6), we have

$$\left.\begin{array}{l} \|P_n^{(\alpha-1,\beta-1)}\|_{L^p(\mu_{\alpha-1,\beta-1})} \\ \|P_n^{(\alpha,\beta-1)}\|_{L^p(\mu_{\alpha,\beta-1})} \end{array}\right\} \le Cn\|P_n^{(\alpha,\beta)}\|_{L^p(\mu_{\alpha,\beta})}.$$

Then, for $\alpha > -1$ we can write

$$\|Q_n\|_{W^p_{1,2}} \le C(n\|P_n^{(\alpha,\beta)}\|_{L^p(\mu_{\alpha,\beta})} + \lambda n\|P_n^{(\alpha,\beta)}\|_{L^p(\mu_{\alpha,\beta})}) \le Cn\|P_n^{(\alpha,\beta)}\|_{L^p(\mu_{\alpha,\beta})}.$$

Taking into account (17) and (6)

$$q_n \geq Cn^{1/2},$$

then

$$\|R_n\|_{W^p_{1,2}} = \frac{\|Q_n\|_{W^p_{1,2}}}{q_n} \leq C\frac{n\|P_n^{(\alpha,\beta)}\|_{L^p(\mu_{\alpha,\beta})}}{n^{1/2}} \leq C\|B_n^{(\alpha,\beta)}\|_{L^p(\mu_{\alpha,\beta})}.$$

Thus, from (10) we have proved the result. □

3.2. Proof of Theorem 1.1, i)

3.2.1. Sufficient conditions.

First of all, notice that applying Hölder inequality and (10) we have that for $g \in L^p(\mu_{\alpha,\beta})$

$$|b_k^{(\alpha,\beta)}(g)| \leq C\|g\|_{L^p(\mu_{\alpha,\beta})}\|B_k^{(\alpha,\beta)}\|_{L^q(\mu_{\alpha,\beta})} \leq C\|g\|_{L^p(\mu_{\alpha,\beta})}, \tag{20}$$

where q is the conjugate of p $(1/p+1/q=1)$. The last inequality is true if (3) holds. On the other hand, for $g \in W^p_{1,2}$, from Hölder inequality and (18) we have that if (3) is satisfied

$$\begin{aligned} |e_k(g)| &\leq C\left(\|g\|_{L^p(\mu_1)}\|R_k\|_{L^q(\mu_1)} + \lambda\|g'\|_{L^p(\mu_2)}\|R_k'\|_{L^q(\mu_2)}\right) \\ &\leq C\|g\|_{W^p_{1,2}}\|R_k\|_{W^q_{1,2}} \leq C\|g\|_{W^p_{1,2}}. \end{aligned} \tag{21}$$

We are going to start proving Theorem 1.1 for the type a), i.e., $\alpha > 0$. Taking

$$\begin{aligned}
M_0(x) :=& G_2 f(x),\\
M_1(x) :=& \sum_{k=3}^{n} \frac{(d_k^{(\alpha-1,\beta-1)})^2}{q_k^2} b_k^{(\alpha-1,\beta-1)}((\xi-\cdot)f)B_k^{(\alpha-1,\beta-1)}(x),\\
M_2(x) :=& \frac{\lambda}{2}\sum_{k=3}^{n} \frac{(k+\alpha+\beta-1)d_{k-1}^{(\alpha,\beta)}d_k^{(\alpha-1,\beta-1)}}{q_k^2} b_{k-1}^{(\alpha,\beta)}(f')B_k^{(\alpha-1,\beta-1)}(x),\\
M_3(x) :=& \sum_{k=3}^{n} \frac{a_{k-1}d_k^{(\alpha-1,\beta-1)}q_{k-1}}{q_k^2} e_{k-1}(f)B_k^{(\alpha-1,\beta-1)}(x),\\
M_4(x) :=& \sum_{k=3}^{n} \frac{a_{k-1}d_k^{(\alpha-1,\beta-1)}q_{k-1}}{q_k^2} b_k^{(\alpha-1,\beta-1)}((\xi-\cdot)f)R_{k-1}(x),\\
M_5(x) :=& \frac{\lambda}{2}\sum_{k=3}^{n} \frac{(k+\alpha+\beta-1)d_{k-1}^{(\alpha,\beta)}a_{k-1}q_{k-1}}{q_k^2} b_{k-1}^{(\alpha,\beta)}(f')R_{k-1}(x),\\
M_6(x) :=& \sum_{k=3}^{n} \frac{a_{k-1}^2q_{k-1}^2}{q_k^2} e_{k-1}(f)R_{k-1}(x).
\end{aligned}$$

Using (14) and (15) we can write the n-th partial sum as

$$G_n f(x) = M_0(x) + M_1(x) + M_2(x) + M_3(x) + M_4(x) + M_5(x) + M_6(x).$$

So, we are going to bound the norms of each $M_i(x)$, $i = 0, \dots, 6$. Using Minkowski inequality, (21) and (18), we obtain that

$$\|M_0\|^p_{W^p_{1,2}} \le C\|f\|^p_{W^p_{1,2}}. \tag{22}$$

We study now

$$M_1(x) = \sum_{k=3}^{n} \frac{(d_k^{(\alpha-1,\beta-1)})^2}{q_k^2} b_k^{(\alpha-1,\beta-1)}((\xi-\cdot)f) B_k^{(\alpha-1,\beta-1)}(x).$$

In this case,

$$\left| \frac{(d_k^{(\alpha-1,\beta-1)})^2}{q_k^2} \right| \le \frac{C}{k^2}.$$

So, from Minkowski inequality, (20), (10) and (3)

$$\begin{aligned} &\|M_1\|_{L^p(\mu_1)} \\ &\quad \le \sum_{k=3}^{n} \frac{C}{k^2} \left(\int_{-1}^{1} \left(|b_k^{(\alpha-1,\beta-1)}((\xi-\cdot)f)||B_k^{(\alpha-1,\beta-1)}(x)| \right)^p d\mu_1(x) \right)^{1/p} \\ &\quad \le C\|(\xi-\cdot)f\|_{L^p(\mu_{\alpha-1,\beta-1})} \sum_{k=3}^{n} \frac{C}{k^2} \|B_k^{(\alpha-1,\beta-1)}\|_{L^p(\mu_1)} \le C\|f\|_{L^p(\mu_1)}. \end{aligned} \tag{23}$$

The derivative of $M_1(x)$ is

$$M_1'(x) = \frac{1}{2} \sum_{k=3}^{n} \frac{(k+\alpha+\beta-1) d_k^{(\alpha-1,\beta-1)} d_{k-1}^{(\alpha,\beta)}}{q_k^2} b_k^{(\alpha-1,\beta-1)}((\xi-\cdot)f) B_{k-1}^{(\alpha,\beta)}(x).$$

Let $h(k) = \frac{\lambda}{4}(k+\alpha+\beta-1)^2 (d_{k-1}^{(\alpha,\beta)})^2$, then from Lemma 3.2

$$\frac{1}{g_2(k)+h(k)} \le \frac{1}{q_k^2} \le \frac{1}{g_1(k)+h(k)}.$$

Thus

$$\left| \frac{1}{q_k^2} - \frac{1}{h(k)} \right| \le \frac{g_2(k)}{h(k)(g_1(k)+h(k))} \simeq \frac{C}{k^3}. \tag{24}$$

So, we can rewrite $M_1'(x)$ as

$$M_1'(x) = M_{1,1}'(x) + M_{1,2}'(x) := \sum_{k=3}^{n} A_k b_k^{(\alpha-1,\beta-1)}((\xi-\cdot)f)B_{k-1}^{(\alpha,\beta)}(x) + \sum_{k=3}^{n} \frac{C}{k} b_k^{(\alpha-1,\beta-1)}((\xi-\cdot)f)B_{k-1}^{(\alpha,\beta)}(x),$$

where $A_k = O(1/k^2)$. For the first summnand, applying again (20) and (10)

$$\|M_{1,1}'\|_{L^p(\mu_2)} \le C\|f\|_{L^p(\mu_1)}, \tag{25}$$

if (3) holds. For the second term, from Proposition 2.2 and Theorem 2.1

$$\begin{aligned}\|M_{1,2}'\|^p_{L^p(\mu_2)} &= \int_{-1}^{1} |T_{-1,0}^{(\alpha,\beta)(\alpha-1,\beta-1)}(\mathbb{S}_n^{(\alpha-1,\beta-1)}((\xi-\cdot)f))|^p (1-x)^\alpha (1+x)^\beta \, dx \\ &\le C\int_{-1}^{1} |\mathbb{S}_n^{(\alpha-1,\beta-1)}((\xi-\cdot)f)|^p (1-x)^{\alpha-1}(1+x)^{\beta-1}\, dx \le C\|f\|^p_{L^p(\mu_1)},\end{aligned} \tag{26}$$

if (3) holds. Thus, from (23), (25) and (26)

$$\|M_1\|^p_{W^p_{1,2}} \le C\|f\|^p_{L^p(\mu_1)}. \tag{27}$$

For $M_2(x)$,

$$M_2(x) = \frac{\lambda}{2}\sum_{k=3}^{n} \frac{(k+\alpha+\beta-1)d_{k-1}^{(\alpha,\beta)}d_k^{(\alpha-1,\beta-1)}}{q_k^2} b_{k-1}^{(\alpha,\beta)}(f')B_k^{(\alpha-1,\beta-1)}(x),$$

note that the coefficients satisfy

$$\left|\frac{\lambda}{2}\frac{(k+\alpha+\beta-1)d_{k-1}^{(\alpha,\beta)}d_k^{(\alpha-1,\beta-1)}}{q_k^2}\right| \le \frac{C}{k}.$$

Then using (24) we can write

$$M_2(x) = M_{2,1}(x) + M_{2,2}(x) := \sum_{k=3}^{n} A_k b_{k-1}^{(\alpha,\beta)}(f')B_k^{(\alpha-1,\beta-1)}(x) + \sum_{k=3}^{n} \frac{C}{k} b_{k-1}^{(\alpha,\beta)}(f')B_k^{(\alpha-1,\beta-1)}(x),$$

where $A_k = O(1/k^2)$. For the first term of the sum from (20) and (10)

$$\|M_{2,1}\|^p_{L^p(\mu_1)} \le C\|f'\|^p_{L^p(\mu_2)}, \tag{28}$$

if (3) is satisfied. For the second term, from Proposition 2.2 and Theorem 2.1

$$\begin{aligned}\|M_{2,2}\|^p_{L^p(\mu_1)} &= \int_{-1}^{1} |T_{0,-1}^{(\alpha-1,\beta-1)(\alpha,\beta)}\mathbb{S}_n^{(\alpha,\beta)}(f')|^p \, d\mu_1(x) \\ &\leq (\xi+1)\int_{-1}^{1} |T_{0,-1}^{(\alpha-1,\beta-1)(\alpha,\beta)}\mathbb{S}_n^{(\alpha,\beta)}(f')|^p \, d\mu_{\alpha-1,\beta-1}(x) \\ &\leq C\int_{-1}^{1} |\mathbb{S}_n^{(\alpha,\beta)}(f')|^p \, d\mu_{\alpha,\beta}(x) \leq C\|f'\|^p_{L^p(\mu_2)}. \end{aligned} \tag{29}$$

So, from (28) and (29) we have

$$\|M_2\|^p_{L^p(\mu_1)} \leq C\|f'\|^p_{L^p(\mu_2)}. \tag{30}$$

Derivating $M_2(x)$

$$M_2'(x) = \frac{\lambda}{4}\sum_{k=3}^{n} \frac{(k+\alpha+\beta-1)^2(d_{k-1}^{(\alpha,\beta)})^2}{q_k^2} b_{k-1}^{(\alpha,\beta)}(f')B_{k-1}^{(\alpha,\beta)}(x),$$

and writing the coefficients like

$$\frac{1}{q_k^2} = \frac{1}{q_k^2} - \frac{1}{h(k)} + \frac{1}{h(k)},$$

it is easy to see that

$$M_2'(x) = \sum_{k=3}^{n} A_k b_{k-1}^{(\alpha,\beta)}(f')B_{k-1}^{(\alpha,\beta)}(x) + \frac{\lambda}{4}\mathbb{S}_{n-1}^{(\alpha,\beta)}(f') - \frac{\lambda}{4}\mathbb{S}_2^{(\alpha,\beta)}(f'),$$

with $A_k = O(1/k^2)$. So, from (20), (10) and Theorem 2.1, we may prove

$$\|M_2'\|^p_{L^p(\mu_2)} \leq C\|f'\|^p_{L^p(\mu_2)}, \tag{31}$$

if (3) holds, and from (30) and (31)

$$\|M_2\|^p_{W^p_{1,2}} \leq C\|f'\|^p_{L^p(\mu_2)}. \tag{32}$$

For $M_3(x)$ the treatment is anologous.

$$M_3(x) = \sum_{k=3}^{n} \frac{a_{k-1}d_k^{(\alpha-1,\beta-1)}q_{k-1}}{q_k^2} e_{k-1}(f)B_k^{(\alpha-1,\beta-1)}(x),$$

and the coefficients satisfy

$$\left|\frac{a_{k-1}d_k^{(\alpha-1,\beta-1)}q_{k-1}}{q_k^2}\right| \leq \frac{C}{k^3}.$$

Then from Minkowski inequality, (21) and (10) we have that

$$\|M_3\|_{L^p(\mu_1)} \le \sum_{k=3}^{n} \frac{C}{k^3} \left(\int_{-1}^{1} \left(|e_{k-1}(f) B_k^{(\alpha-1,\beta-1)}(x)| \right)^p \, d\mu_1(x) \right)^{1/p}$$
$$\le C\|f\|_{W^p_{1,2}} \sum_{k=3}^{n} \frac{C}{k^3} \|B_k^{(\alpha-1,\beta-1)}\|_{L^p(\mu_1)} \le C\|f\|_{W^p_{1,2}}.$$

Analogously the result is obtained for the derivative of M_3. Since,

$$\|M_3\|^p_{W^p_{1,2}} \le C\|f\|^p_{W^p_{1,2}}, \tag{33}$$

when (3) is true. We take now $M_4(x)$.

$$M_4(x) = \sum_{k=3}^{n} \frac{a_{k-1} d_k^{(\alpha-1,\beta-1)} q_{k-1}}{q_k^2} b_k^{(\alpha-1,\beta-1)}((\xi - \cdot)f) R_{k-1}(x).$$

Observe that

$$\left| \frac{a_{k-1} d_k^{(\alpha-1,\beta-1)} q_{k-1}}{q_k^2} \right| \le \frac{C}{k^3}.$$

Therefore from (20) and (18), we obtain

$$\|M_4\|^p_{W^p_{1,2}} \le C\|f\|^p_{L^p(\mu_1)}, \tag{34}$$

if (3) holds. For $M_5(x)$,

$$M_5(x) = \frac{\lambda}{2} \sum_{k=3}^{n} \frac{(k+\alpha+\beta-1) d_{k-1}^{(\alpha,\beta)} a_{k-1} q_{k-1}}{q_k^2} b_{k-1}^{(\alpha,\beta)}(f') R_{k-1}(x),$$

the coefficients satisfy

$$\left| \frac{(k+\alpha+\beta-1) d_{k-1}^{(\alpha,\beta)} a_{k-1} q_{k-1}}{q_k^2} \right| \le \frac{C}{k^2},$$

so we obtain the desired bound with (20) and (18).

$$\|M_5\|^p_{W^p_{1,2}} \le C\|f'\|^p_{L^p(\mu_2)}, \tag{35}$$

if (3) holds. Finally, for $M_6(x)$

$$M_6(x) = \sum_{k=3}^{n} \frac{a_{k-1}^2 q_{k-1}^2}{q_k^2} e_{k-1}(f) R_{k-1}(x),$$

and taking into account that

$$\left|\frac{a_{k-1}^2 q_{k-1}^2}{q_k^2}\right| \leq \frac{C}{k^4},$$

we easily obtain

$$\|M_6\|_{W_{1,2}^p}^p \leq C\|f\|_{W_{1,2}^p}^p. \tag{36}$$

Clearly, joining (22), (27), (32), (33), (34), (35) and (36), we have proved the direct implication for $\alpha > 0$.

Now, we are going to study the cases b) and c), that is, $-1 < \alpha \leq 0$. Taking

$$\begin{aligned}
L_0(x) :=& G_2 f(x),\\
L_1(x) :=& \sum_{k=3}^{n} \frac{(k+\alpha+\beta-1)d_k^{(\alpha,\beta-1)}}{q_k^2(2k+\alpha+\beta-1)} b_k^{(\alpha,\beta-1)}(f) P_k^{(\alpha-1,\beta-1)}(x),\\
L_2(x) :=& -\sum_{k=3}^{n} \frac{(k+\beta-1)d_{k-1}^{(\alpha,\beta-1)}}{q_k^2(2k+\alpha+\beta-1)} b_{k-1}^{(\alpha,\beta-1)}(f) P_k^{(\alpha-1,\beta-1)}(x),\\
L_3(x) :=& \frac{\lambda}{2}\sum_{k=3}^{n} \frac{(k+\alpha+\beta-1)d_{k-1}^{(\alpha,\beta)}}{q_k^2} b_{k-1}^{(\alpha,\beta)}(f') P_k^{(\alpha-1,\beta-1)}(x),\\
L_4(x) :=& \sum_{k=3}^{n} \frac{a_{k-1}q_{k-1}}{q_k^2} e_{k-1}(f) P_k^{(\alpha-1,\beta-1)}(x),\\
L_5(x) :=& \sum_{k=3}^{n} \frac{(k+\alpha+\beta-1)d_k^{(\alpha,\beta-1)}a_{k-1}q_{k-1}}{q_k^2(2k+\alpha+\beta-1)} b_k^{(\alpha,\beta-1)}(f) R_{k-1}(x),\\
L_6(x) :=& -\sum_{k=3}^{n} \frac{(k+\beta-1)d_{k-1}^{(\alpha,\beta-1)}a_{k-1}q_{k-1}}{q_k^2(2k+\alpha+\beta-1)} b_{k-1}^{(\alpha,\beta-1)}(f) R_{k-1}(x),\\
L_7(x) :=& \frac{\lambda}{2}\sum_{k=3}^{n} \frac{(k+\alpha+\beta-1)d_{k-1}^{(\alpha,\beta)}a_{k-1}q_{k-1}}{q_k^2} b_{k-1}^{(\alpha,\beta)}(f') R_{k-1}(x),\\
L_8(x) :=& \sum_{k=3}^{n} \frac{a_{k-1}^2 q_{k-1}}{q_k^2} e_{k-1}(f) R_{k-1}(x).
\end{aligned}$$

From (12) and (16) the n-partial sum can be written as

$$G_n f(x) = L_0(x) + L_1(x) + \cdots + L_8(x).$$

Using Minkowski inequality, (21) and (18), we obtain that

$$\|L_0\|_{W_{1,2}^p}^p \leq C\|f\|_{W_{1,2}^p}^p. \tag{37}$$

First of all, note that $L_1(1) = 0$ if $\alpha = 0$ because of (13). Now we apply (8) to take

$$L_1(x) = L_{1,1}(x) + L_{1,2}(x) := \sum_{k=3}^{n} \frac{(d_k^{(\alpha,\beta-1)})^2}{q_k^2} \left(\frac{k+\alpha+\beta-1}{2k+\alpha+\beta-1}\right)^2 b_k^{(\alpha,\beta-1)}(f) B_k^{(\alpha,\beta-1)}(x) - \sum_{k=3}^{n} \frac{d_k^{(\alpha,\beta-1)} d_{k-1}^{(\alpha,\beta-1)}}{q_k^2} \frac{(k+\alpha+\beta-1)(k+\beta-1)}{(2k+\alpha+\beta-1)^2} b_k^{(\alpha,\beta-1)}(f) B_k^{(\alpha,\beta-1)}(x).$$

In both cases, applying (20) and (10) we have

$$\|L_1\|^p_{L^p(\mu_1)} \leq C \|f\|^p_{L^p(\mu_1)}, \tag{38}$$

if (3) is satisfied. When we derive $L_1(x)$

$$L_1'(x) = \sum_{k=3}^{n} \frac{(k+\alpha+\beta-1)^2 d_k^{(\alpha,\beta-1)} d_{k-1}^{(\alpha,\beta)}}{2q_k^2(2k+\alpha+\beta-1)} b_k^{(\alpha,\beta-1)}(f) B_{k-1}^{(\alpha,\beta)}(x).$$

Analogously to $M_1'(x)$, the derivative of $L_1(x)$ becomes

$$L_1'(x) = L_{1,1}'(x) + L_{1,2}'(x) := \sum_{k=3}^{n} A_k b_k^{(\alpha,\beta-1)}(f) B_{k-1}^{(\alpha,\beta)}(x) + \sum_{k=3}^{n} \frac{C}{k} b_k^{(\alpha,\beta-1)}(f) B_{k-1}^{(\alpha,\beta)}(x), \quad A_k = O(1/k^2).$$

From (20) and (10), if (3) holds, we obtain

$$\|L_{1,1}'\|^p_{L^p(\mu_2)} \leq C \|f\|^p_{L^p(\mu_{\alpha,\beta-1})} \leq C \|f\|^p_{L^p(\mu_1)}.$$

From Proposition 2.2 and Theorem 2.1 we have

$$\|L_{1,2}'\|^p_{L^p(\mu_2)} = \int_{-1}^{1} |T_{-1,0}^{(\alpha,\beta)(\alpha,\beta-1)} \mathbb{S}_n^{(\alpha,\beta-1)}(f)|^p d\mu_{\alpha,\beta}(x) \leq C \int_{-1}^{1} |\mathbb{S}_n^{(\alpha,\beta-1)}(f)|^p d\mu_{\alpha,\beta-1}(x) \leq C \|f\|^p_{L^p(\mu_1)}.$$

So, we have proved that

$$\|L_1\|^p_{W^p_{1,2}} \leq C \|f\|^p_{L^p(\mu_1)}. \tag{39}$$

The boundedness for $L_2(x)$ is totally similar to $L_1(x)$. For

$$L_3(x) = \frac{\lambda}{2} \sum_{k=3}^{n} \frac{(k+\alpha+\beta-1) d_{k-1}^{(\alpha,\beta)}}{q_k^2} b_{k-1}^{(\alpha,\beta)}(f') P_k^{(\alpha-1,\beta-1)}(x).$$

Firstly, note that $L_3(1) = 0$ when $\alpha = 0$. Taking into account (8) we obtain that

$$L_3(x) = \sum_{k=3}^{n} A_k b_{k-1}^{(\alpha,\beta)}(f') B_k^{(\alpha,\beta-1)}(x) + \sum_{k=3}^{n} D_k b_{k-1}^{(\alpha,\beta)}(f') B_{k-1}^{(\alpha,\beta)}(x) + \sum_{k=3}^{n} \frac{C}{k} b_{k-1}^{(\alpha,\beta)}(f') B_k^{(\alpha,\beta-1)}(x) + \sum_{k=3}^{n} \frac{C}{k} b_{k-1}^{(\alpha,\beta)}(f') B_{k-1}^{(\alpha,\beta-1)}(x),$$

where A_k, $D_k = O(1/k^2)$. So, (20) and (10) are applied in the two first terms, and Proposition 2.2 and Theorem 2.1 in the two last terms obtaining

$$\|L_3\|_{L^p(\mu_1)}^p \leq C \|f'\|_{L^p(\mu_2)}^p. \tag{40}$$

Making the derivative we have

$$L_3'(x) = \frac{\lambda}{4} \sum_{k=3}^{n} A_k b_{k-1}^{(\alpha,\beta)}(f') B_{k-1}^{(\alpha,\beta)}(x)| + \mathbb{S}_{n-1}^{(\alpha,\beta)}(f') - \mathbb{S}_2^{(\alpha,\beta)}(f'),$$

where $A_k = O(1/k^3)$. Then from (20), (10), Theorem 2.1 and (40) we can write

$$\|L_3\|_{W_{1,2}^p} \leq C \|f'\|_{L^p(\mu_2)}^p. \tag{41}$$

For $L_4(x)$

$$L_4(x) = \sum_{k=3}^{n} \frac{d_k^{(\alpha-1,\beta-1)} a_{k-1} q_{k-1}}{q_k^2} e_{k-1}(f) B_k^{(\alpha-1,\beta-1)}(x),$$

as

$$\left| \frac{d_k^{(\alpha-1,\beta-1)} a_{k-1} q_{k-1}}{q_k^2} \right| \leq \frac{C}{k^3},$$

from (21) and (10), we obtain

$$\|L_4\|_{W_{1,2}^p}^p \leq C \|f\|_{W_{1,2}^p}^p. \tag{42}$$

With the same arguments,

$$\|L_i\|_{W_{1,2}^p}^p \leq C \|f\|_{W_{1,2}^p}^p, \quad i = 5, \ldots, 8. \tag{43}$$

Finally from (37), (39), (41), (42) and (43) we have the desired result.

3.2.2. Necessary conditions.

If (2) holds, it is clear that

$$\|e_k(f) R_k\|_{W_{1,2}^p} = \|G_n f - G_{n-1} f\|_{W_{1,2}^p} \leq C \|f\|_{W_{1,2}^p}. \tag{44}$$

Consider the linear functionals on $W^p_{1,2}$

$$T_n(f) = e_k(f) \|R_k\|_{W^p_{1,2}}.$$

Then, for every $f \in W^p_{1,2}$, $\sup_n |T_n(f)| < \infty$ holds. Because of $W^p_{1,2}$ is a complete space, as we will prove in Theorem 1.2, the Banach-Steinhaus theorem implies $\sup_n \|T_n\| < \infty$. On the other hand, by duality we have

$$\|T_n\| = \|R_k\|_{W^p_{1,2}} \|R_k\|_{W^q_{1,2}},$$

where q is the conjugate of p. Therefore,

$$\sup_n \|R_k\|_{W^p_{1,2}} \|R_k\|_{W^q_{1,2}} < \infty. \tag{45}$$

And from (18), (3) holds.

4. Jacobi-Sobolev orthogonal polynomials type II

4.1. Auxiliary results

Let (μ_1, μ_2) be a coherent pair of Jacobi type II with $\xi > 1$ and $M = 0$. Recall that in this case, $\alpha > -1$, $\beta > 0$ and

$$d\mu_1(x) = (1-x)^\alpha (1+x)^{\beta-1} dx, \quad d\mu_2(x) = \frac{1}{\xi - x}(1-x)^{\alpha+1}(1+x)^\beta dx, \quad \xi > 1.$$

Let $\{Q_n\}_n$ be the corresponding sequence of orthogonal polynomials with respect to (1), such that for $n \geq 2$ we choose the leading coefficient of $Q_n(x)$ equals to the leading coefficient of $P_n^{(\alpha-1,\beta-1)}(x)$. Let $\{T_n\}_n$ be the sequence of orthogonal polynomials with respect to $d\mu_2$, with leading coefficients equal to the leading coefficients $\tau_n^{(\alpha,\beta)}$ of $P_n^{(\alpha,\beta)}$. In [11], the authors proved the following lemma:

Lemma 4.1. *There exist positive constants c_n such that*

$$T_n(x) = \frac{n+\alpha+\beta+1}{2n+\alpha+\beta+1} P_n^{(\alpha+1,\beta)}(x) - \frac{n+\beta}{2n+\alpha+\beta+1} c_n P_{n-1}^{(\alpha+1,\beta)}(x), \quad n \geq 1, \tag{46}$$

where

$$c_n = \frac{1}{\xi + \sqrt{\xi^2 - 1}} + O\left(\frac{1}{n}\right),$$

with $\sqrt{\xi^2-1} > 0$. Moreover,

$$t_n^2 := \|T_n\|^2_{L^p(\mu_2)} \approx \frac{2^{\alpha+\beta}}{n} c_n. \tag{47}$$

The sequence $\{Q_n\}_n$ satisfies a relation similar to (12) in Section 3:

$$\frac{n+\alpha+\beta-1}{2n+\alpha+\beta-1}P_n^{(\alpha,\beta-1)}(x)-\frac{n+\beta-1}{2n+\alpha+\beta-1}c_{n-1}P_{n-1}^{(\alpha,\beta-1)}(x)$$
$$=Q_n(x)-a_{n-1}Q_{n-1}(x),\quad n\geq 3,\quad a_n=O(1/n^2). \quad (48)$$

Notice that when we derive (48) we obtain

$$\frac{1}{2}(n+\alpha+\beta-1)T_{n-1}(x)=Q_n'(x)-a_{n-1}Q_{n-1}'(x). \quad (49)$$

Let $\{S_n\}_n$ be the sequence of orthonormal polynomials with respect to μ_2. Let $s_n(f)$ denote the Fourier coefficients with respect to S_n, i.e.

$$s_n(f)=\int_{-1}^{1}f(x)S_n(x)\,d\mu_2(x).$$

Lemma 4.2. *Let $\alpha>-1$, $\beta>0$ and $\tau=\max\{\alpha+1,\beta\}$. Then*

$$\|S_n\|_{L^p(\mu_2)}\leq C\begin{cases}1, & 2\tau>p\tau-2+p/2,\\ (\log n)^{1/p}, & 2\tau=p\tau-2+p/2,\\ n^{\tau+1/2-2(\tau+1)/p}, & 2\tau<p\tau-2+p/2.\end{cases} \quad (50)$$

Proof. From (46) and (47) we have that

$$\|S_n\|_{L^p(\mu_2)}\leq C\|B_n^{(\alpha+1,\beta)}\|_{L^p(\mu_{\alpha+1,\beta})}.$$

□

Using (48) we may prove the following lemma.

Lemma 4.3. *For $n\geq 3$ the Fourier coefficients $e_n(f)=(R_n,f)_S$ can be expressed as*

$$e_n(f)=\frac{n+\alpha+\beta-1}{2n+\alpha+\beta-1}\frac{d_n^{(\alpha,\beta-1)}}{q_n}b_n^{(\alpha,\beta-1)}(f)$$
$$-\frac{n+\beta-1}{2n+\alpha+\beta-1}c_{n-1}\frac{d_{n-1}^{(\alpha,\beta-1)}}{q_n}b_{n-1}^{(\alpha,\beta-1)}(f)$$
$$+\frac{\lambda}{2}\frac{(n+\alpha+\beta-1)t_{n-1}}{q_n}s_{n-1}(f')+\frac{a_{n-1}q_{n-1}}{q_n}e_{n-1}(f). \quad (51)$$

In this case the asymptotic for $q_n^2=(Q_n,Q_n)_S$, following the same arguments of Theorem 2 of [9], remains in this way.

Lemma 4.4. *Let $\alpha>-1$ and $\beta>0$ then*

$$g_1(n)+\frac{\lambda}{4}(n+\alpha+\beta-1)^2t_{n-1}^2\;\leq\;q_n^2\;\leq\;g_2(n)+\frac{\lambda}{4}(n+\alpha+\beta-1)^2t_{n-1}^2, \quad (52)$$

where $g_1(n) \approx C/n$ *and* $g_2(n) \approx C/n$.

Lemma 4.5. *Let* $\alpha > -1$, $\beta > 0$ *and* $\tau = \max\{\alpha+1, \beta\}$. *Then*

$$\|R_n\|_{W^p_{1,2}} \leq C \begin{cases} 1, & 2\tau > p\tau - 2 + p/2, \\ (\log n)^{1/p}, & 2\tau = p\tau - 2 + p/2, \\ n^{\tau+1/2-2(\tau+1)/p}, & 2\tau < p\tau - 2 + p/2. \end{cases} \tag{53}$$

Proof. From (48) and (49)

$$\|Q_n\|_{L^p(\mu_1)} \leq C\|P_n^{(\alpha,\beta-1)}\|_{L^p(\mu_1)}, \quad \|Q_n'\|_{L^p(\mu_2)} \leq Cn\|T_{n-1}\|_{L^p(\mu_2)}.$$

Then from (46) and (47)

$$\|Q_n\|_{W^p_{1,2}} \leq C(\|P_n^{(\alpha,\beta-1)}\|_{L^p(\mu_{\alpha,\beta-1})} + \lambda n\|P_{n-1}^{(\alpha+1,\beta)}\|_{L^p(\mu_{\alpha+1,\beta})}),$$

From (10) and (6) the following inequalities hold

$$\|P_j^{(\alpha+1,\beta)}\|_{L^p(\mu_{\alpha+1,\beta})} \leq C\|P_n^{(\alpha+1,\beta)}\|_{L^p(\mu_{\alpha+1,\beta})}, \quad j \leq n$$

and

$$\|P_n^{(\alpha,\beta-1)}\|_{L^p(\mu_{\alpha,\beta-1})} \leq Cn\|P_n^{(\alpha+1,\beta)}\|_{L^p(\mu_{\alpha+1,\beta})}.$$

Then, for $\alpha > -1$, we can write

$$\begin{aligned} \|Q_n\|_{W^p_{1,2}} \leq C(n\|P_n^{(\alpha+1,\beta)}\|_{L^p(\mu_{\alpha+1,\beta})} + \lambda n\|P_n^{(\alpha+1,\beta)}\|_{L^p(\mu_{\alpha+1,\beta})}) \\ \leq Cn\|P_n^{(\alpha+1,\beta)}\|_{L^p(\mu_{\alpha+1,\beta})}. \end{aligned}$$

Taking into account (52) and (47)

$$q_n \geq Cn^{1/2},$$

then

$$\|R_n\|_{W^p_{1,2}} = \frac{\|Q_n\|_{W^p_{1,2}}}{q_n^{1/2}} \leq C\frac{n\|P_n^{(\alpha+1,\beta)}\|_{L^p(\mu_{\alpha+1,\beta})}}{n^{1/2}} \leq C\|B_n^{(\alpha+1,\beta)}\|_{L^p(\mu_{\alpha+1,\beta})}.$$

Thus, from (10), we have proved the result. □

4.2. Proof of Theorem 1.1, ii)

First note, that applying Hölder inequality and (50) we have that for $g \in L^p(\mu_2)$

$$|s_k(g)| \leq C\|g\|_{L^p(\mu_2)}\|S_k\|_{L^q(\mu_2)} \leq C\|g\|_{L^p(\mu_2)}, \tag{54}$$

The last inequality is true if (4) holds. With analogous arguments (21) remains true in type II. Taking

$$\begin{aligned}
M_0(x) :=& G_2 f(x),\\
M_1(x) :=& \sum_{k=3}^{n} \frac{(k+\alpha+\beta-1)d_k^{(\alpha,\beta-1)}}{q_k^2(2k+\alpha+\beta-1)} b_k^{(\alpha,\beta-1)}(f)\\
& \left(\frac{k+\alpha+\beta-1}{2k+\alpha+\beta-1}P_k^{(\alpha,\beta-1)}(x) - \frac{k+\beta-1}{2k+\alpha+\beta-1}c_{k-1}P_{k-1}^{(\alpha,\beta-1)}(x)\right),\\
M_2(x) :=& -\sum_{k=3}^{n} \frac{(k+\beta-1)c_{k-1}d_{k-1}^{(\alpha,\beta-1)}}{q_k^2(2k+\alpha+\beta-1)} b_{k-1}^{(\alpha,\beta-1)}(f)\\
& \left(\frac{k+\alpha+\beta-1}{2k+\alpha+\beta-1}P_k^{(\alpha,\beta-1)}(x) - \frac{k+\beta-1}{2k+\alpha+\beta-1}c_{k-1}P_{k-1}^{(\alpha,\beta-1)}(x)\right),\\
M_3(x) :=& \frac{\lambda}{2}\sum_{k=3}^{n} \frac{(k+\alpha+\beta-1)t_{k-1}}{q_k^2} s_{k-1}(f')\\
& \left(\frac{k+\alpha+\beta-1}{2k+\alpha+\beta-1}P_k^{(\alpha,\beta-1)}(x) - \frac{k+\beta-1}{2k+\alpha+\beta-1}c_{k-1}P_{k-1}^{(\alpha,\beta-1)}(x)\right),\\
M_4(x) :=& \sum_{k=3}^{n} \frac{a_{k-1}q_{k-1}}{q_k^2} e_{k-1}(f)\\
& \left(\frac{k+\alpha+\beta-1}{2k+\alpha+\beta-1}P_k^{(\alpha,\beta-1)}(x) - \frac{k+\beta-1}{2k+\alpha+\beta-1}c_{k-1}P_{k-1}^{(\alpha,\beta-1)}(x)\right),\\
M_5(x) :=& \sum_{k=3}^{n} \frac{(k+\alpha+\beta-1)d_k^{(\alpha,\beta-1)}a_{k-1}q_{k-1}}{q_k^2(2k+\alpha+\beta-1)} b_k^{(\alpha,\beta-1)}(f)R_{k-1}(x),\\
M_6(x) :=& -\sum_{k=3}^{n} \frac{(k+\beta-1)c_{k-1}d_{k-1}^{(\alpha,\beta-1)}a_{k-1}q_{k-1}}{q_k^2(2k+\alpha+\beta-1)} b_{k-1}^{(\alpha,\beta-1)}(f)R_{k-1}(x),\\
M_7(x) :=& \frac{\lambda}{2}\sum_{k=3}^{n} \frac{(k+\alpha+\beta-1)a_{k-1}t_{k-1}q_{k-1}}{q_k^2} s_{k-1}(f')R_{k-1}(x),\\
M_8(x) :=& \sum_{k=3}^{n} \frac{a_{k-1}^2 q_{k-1}^2}{q_k^2} e_{k-1}(f)R_{k-1}(x).
\end{aligned}$$

Using (48) and (51) we can write

$$G_n f(x) = M_0(x) + M_1(x) + \cdots + M_8(x).$$

From (20), (21), (54), Proposition 2.2 and Theorem 2.1, we prove the sufficient conditions. Necessary conditions are proved with the same arguments of Theorem 1.1.

Remark 1. If we took $M > 0$ in the second measure, it would appear in $M_3(x)$ a term of this kind

$$\sum_{k=3}^{n} \frac{C}{k} \frac{1}{t_{k-1}} |T_{k-1}(\xi)|,$$

and as

$$\frac{|T_{k-1}(\xi)|}{t_{k-1}} \leq C,$$

the series could be divergent. Thus, we cannot obtain the boundedness of the operator.

5. Proof of Theorem 1.2

We start proving i). Let (μ_1, μ_2) be a Jacobi coherent pair of measures with $M = 0$ for type II and $\alpha \neq 0$ for type I. Then, we may prove the result using directly Theorem 4.1 and Corollary 4.1 of [14]. For type I with $\alpha = 0$, we take

$$d\tilde{\mu}_1(x) = (1+x)^{\beta-1}dx, \quad d\mu_2(x) = (1+x)^{\beta}dx,$$

and $\overline{W}_{1,2}^p$ will be the space of measurable functions f on $[-1,1]$ such that there exists f' almost everywhere and

$$\|f\|_{\overline{W}_{1,2}^p}^p = \|f\|_{W_{1,2}^p}^p - M|f(1)|^p < \infty.$$

From Theorem 4.1 of [14], the set $\mathcal{C}_c^\infty(\mathbb{R})$ is dense in the Sobolev space $\overline{W}_{1,2}^p$. That is, given $f \in \overline{W}_{1,2}^p$ and $\tilde{\varepsilon} > 0$, there exists $g \in \mathcal{C}_c^\infty(\mathbb{R})$ such that

$$\|f - g\|_{\overline{W}_{1,2}^p} < \frac{\tilde{\varepsilon}}{2}.$$

Let $h \in \mathcal{C}_c^\infty(\mathbb{R})$ be such that

$$\|h\|_{\overline{W}_{1,2}^p} < \frac{\tilde{\varepsilon}}{2} \quad \text{and} \quad h(1) = f(1) - g(1).$$

Then, using Minkowski inequality

$$\begin{aligned}\|f-(g+h)\|_{W_{1,2}^p} &= \left(\|f-(g+h)\|_{\overline{W}_{1,2}^p}^p + M|f(1)-(g+h)(1)|^p\right)^{1/p} \\ &\leq \|f-g\|_{\overline{W}_{1,2}^p} + \|h\|_{\overline{W}_{1,2}^p} < \tilde{\varepsilon}.\end{aligned}$$

On the other hand, given $\tilde{\varepsilon} > 0$ there exists a polynomial p_n of degree n such that

$$\|g + h - p_n\|_\infty < \tilde{\varepsilon}, \quad \|(g+h)' - p_n'\|_\infty < \tilde{\varepsilon}.$$

Therefore, given $\varepsilon = \tilde{\varepsilon}(1 + 1/\beta^p + M + 1/(\beta+1)^p)^{1/p} > 0$ and $f \in W_{1,2}^p$ there exists

p_n polynomial of degree n such that

$$\|f - p_n\|_{W^p_{1,2}} \leq \|f - (g+h)\|_{W^p_{1,2}} + \|(g+h) - p_n\|_{W^p_{1,2}} < \varepsilon.$$

So we have proved i).

Again for type II measures with $M = 0$ and type I measures with $\alpha \neq 0$, we deduce ii) from [14]. When $\alpha = 0$, let $\{f_n\}_n$ be a Cauchy sequence in $W^p_{1,2}$. Our target is to show that $\{f_n\}_n$ is convergent in $W^p_{1,2}$. $\{f_n\}_n$ will also be a Cauchy sequence in $\overline{W}^p_{1,2}$ that is a complete space. Thus, there exists $f \in \overline{W}^p_{1,2}$ such that

$$\|f_n - f\|^p_{L^p(\tilde{\mu}_1)} + \lambda\|f'_n - f'\|^p_{L^p(\mu_2)} \longrightarrow 0.$$

Therefore,

$$\|f_n - f\|^p_{L^p(\tilde{\mu}_1)} \longrightarrow 0. \tag{55}$$

On the other hand, f_n will also be a Cauchy sequence in $L^p(\mu_1)$. And as $L^p(\mu_1)$ is a complete space, we obtain that there exists $g \in L^p(\mu_1)$ such that

$$\|f_n - g\|^p_{L^p(\tilde{\mu}_1)} + M|f_n(1) - g(1)|^p \longrightarrow 0,$$

and then

$$\|f_n - g\|^p_{L^p(\tilde{\mu}_1)} \longrightarrow 0. \tag{56}$$

From (55) and (56), we deduce $f(x) = g(x)$ almost everywhere.

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