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Solving nonlinear integral equations with non-separable kernel via a high-order iterative process☆

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ABSTRACT

In this work we focus on location and approximation of a solution of nonlinear integral equations of Hammerstein-type when the kernel is non-separable through a high order iterative process. For this purpose, we approximate the non-separable kernel by means of a separable kernel and then, we perform a complete study about the convergence criteria for the approximated solution obtained to the solution of our first problem. Different examples have been tested in order to apply our theoretical results.

1. Introduction

It well known, that different problems of engineering give place to nonlinear integral equations, between these we find the nonlinear integral equations of Hammerstein type ([11,13]), that appears in neural networks, theory of structures, etc. It is clear that these equations present a special difficulty when the kernel is non-separable. The purpose of this work deals with equations of this type that are given by the following expression:

$$h(x) = \ell(x) + \mu \int_p^q N(x,t)\, \mathcal{P}_k(h)(t)\,dt, \quad x \in [p,q], \quad \mu \in \mathbb{R}, \tag{1}$$

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where ℓ is a continuous function in $[p,q]$, the kernel N is continuous and nonnegative in $[p,q]\times[p,q]$ and $\mathcal{P}_k$ is a Nemystkii operator [17], $\mathcal{P}_k:\mathscr{C}[p,q]\longrightarrow\mathscr{C}[p,q]$, with $\mathcal{P}_k(h)(t)=\sum_{i=0}^{k}a_ih(t)^i$, $a_i\in\mathbb{R}$ for $i=0,1,\dots,k$.

To approximate a solution of (1), we consider the operator $\mathcal{F}:\mathcal{D}\subseteq\mathscr{C}[p,q]\longrightarrow\mathscr{C}[p,q]$ where $\mathcal{D}$ is a nonempty convex domain in $\mathscr{C}[p,q]$ given by

$$\mathcal{F}(h)(x)=h(x)-\ell(x)-\mu\int_p^q N(x,t)\,\mathcal{P}_k(h)(t)dt,\quad x\in[p,q] \text{ and } \mu\in\mathbb{R}. \tag{2}$$

So, in this situation, a solution h^* of $\mathcal{F}(h)=0$ is a solution of equation (1) and vice versa. For approximating a solution of $\mathcal{F}(h)=0$, we will apply the fifth order iterative process:

$$\begin{cases} h_0\in\mathcal{D},\\ g_n=h_n-\Lambda_n\mathcal{F}(h_n),\\ f_n=g_n-5\,\Lambda_n\mathcal{F}(g_n),\\ h_{n+1}=f_n-\frac{1}{5}\Lambda_n(-16\,\mathcal{F}(g_n)+\mathcal{F}(f_n)), n\geq 0. \end{cases} \tag{3}$$

where $\Lambda_n=\left[\mathcal{F}'(h_n)\right]^{-1}$ for $n\in\mathbb{N}$. It is known [15], that if the kernel of nonlinear integral equation given in (1) is separable we can calculate explicitly $\left[\mathcal{F}'(h_n)\right]^{-1}$. However, if the kernel N is non-separable then it is not possible to obtain directly the operator $\left[\mathcal{F}'(h_n)\right]^{-1}$ for $h\in\mathscr{C}[p,q]$, being

$$[\mathcal{F}'(h)g](x)=g(x)-\mu\int_p^q N(x,t)[\mathcal{P}_k'(h)]g(t)dt.$$

Remember that, the kernel N is separable if it can be expressed

$$N(x,t)=\sum_{j=1}^{m}\alpha_j(x)\beta_j(t), \tag{4}$$

being α_j and β_j two real functions in $[p,q]$.

In this paper, we consider that kernel N is non-separable. So, to apply the iterative process (3) we can consider two alternatives. In first place, a procedure is to modify the iterative process by approximating the operator $[\mathcal{F}'(h_n)]^{-1}$ ([14,15]). This procedure causes the iterative process to lose its convergence order. For example in [14], we saw that when we approximate the operator $[\mathcal{F}'(h_n)]^{-1}$ in a third-order iterative process linear convergence is obtained. In second place, another procedure, is to approximate integral equation (1) by another integral equation of the form

$$h(x)=\ell(x)+\mu\int_p^q \widetilde{N}(x,t)\,\mathcal{P}_k(h)(t)dt,\quad x\in[p,q],\quad \mu\in\mathbb{R}, \tag{5}$$

where $\widetilde{N}$ is a separable kernel which is an approximation of N given by

$$N(x,t)=\widetilde{N}(x,t)+\mathcal{E}(\epsilon,x,t),\ \text{ with } \widetilde{N}(x,t)=\sum_{i=1}^{m}\alpha_i(x)\beta_i(t), \tag{6}$$

for the real functions α_j and β_j in $[p,q]$ and $\epsilon,x,t\in[p,q]$. There are different procedures to achieve the expression (6). For example, if $N(x,t)$ is sufficiently derivable in any of its arguments, we can apply Taylor's development in that argument. Other possible procedures may be their approximation using Bernstein polynomials or other interpolation approximations [16,18]. All these approximations verify that their error $\mathcal{E}$ tends to zero when m tends to infinity.

Notice that the proximity between the solutions of both integral Eq. (1) and (5), will mark the accuracy of the solution obtained by applying the iterative process (3) to solve $\widetilde{\mathcal{F}}(h)=0$ with

$$\widetilde{\mathcal{F}}(h)(s)=h(x)-\ell(x)-\mu\int_p^q \widetilde{N}(x,t)\,\mathcal{P}_k(h)(t)dt,\quad x\in[p,q],\quad \mu\in\mathbb{R}, \tag{7}$$

regarding the solution $\mathcal{F}(h)=0$.

In the situation of (7), we can explicitly calculate $[\widetilde{\mathcal{F}}'(h_n)]^{-1}$ and by means of the iterative process (3) to approximate a solution $\tilde{h}$ of $\widetilde{\mathcal{F}}(h)=0$. This procedure, as we saw in [12], may have the disadvantage, compared to the first procedure, of increasing the operational cost.

Two are the main goals in this work. The first is to justify that when considering a high-order iterative process, in our case a fifth order iterative process, the operational cost relative to the application of the iterative process (3) is not significantly increased due to its convergence rate, making the proximity between the solutions of the integral Eq. (1) and (5) improve, as we will see in Section 2. The second is a theoretical goal. It is known that there are semilocal convergence studies about iterative process (3) involving Lipschitz and Holder operators (see [19]). Thus, an important goal in this work is to reach a higher theoretical level, so we generalize previous semilocal convergence results obtained by other authors (see [2] and [19]). For this, as our focus is to approximate a solution of integral equation (1) applying iterative process (3) by resolution of the equation (5), we will need to set convergence conditions for operator (7), which will allow us to generalize the usually convergence conditions considered by other authors.

Notice that a very commonly measure of speed of convergence in Banach spaces is the order of convergence [20]. There are several adaptive methods with which the accuracy of a process can be checked (see [3,4,8–10]).

The development of this work begins in the Section 2, where we establish a way to obtain sufficient tolerance to measure the distance between the solutions of Eq. (1) and (5). In Section 3, our novel analysis of the semilocal convergence of the iterative process (3) is developed. While in Section 4, we apply the results obtained at work to various numerical problems. Finally, in Section 5 we draw some conclusions.

2. Motivation

As we indicate previously, we cannot directly apply the iterative process (3), for approximating a solution h^* of (1), because the kernel N is non-separable. Thus, we will consider the approximate integral equation (5) and apply the iterative process (3) to the operator $\widetilde{\mathcal{F}}$ obtaining a sequence of approximations $\{\widetilde{h}_n\}$ that will approximate a solution $\widetilde{h}$ of (5). This procedure will be effective when the distance between the solutions of the integral equations (1) and (5) is small enough. Therefore, we will start by analyzing the distance between the two solutions. Obviously,

$$\|h^* - \widetilde{h}_n\| \leq \|h^* - \widetilde{h}\| + \|\widetilde{h} - \widetilde{h}_n\|, \tag{8}$$

then, when $\lim_n \widetilde{h}_n = \widetilde{h}$, it is the amount $\|h^* - \widetilde{h}\|$ that will set us the accuracy we get in the approximation of h^*. If operator N supports a separable decomposition as (6), we have

$$\begin{aligned} h^*(x) - \widetilde{h}(x) &= \mu\left[\int_p^q N(x,t)\,\mathcal{P}_k(h^*)(t)dt - \int_p^q \widetilde{N}(x,t)\,\mathcal{P}_k(\tilde{h})(t)\,dt\right] \\ &= \mu\left[\int_p^q N(x,t)\,[\mathcal{P}_k(h^*) - \mathcal{P}_k(\widetilde{h})](t)\,dt. - \int_p^q \mathcal{E}(\epsilon,x,t)\,\mathcal{P}_k(\widetilde{h})(t)\,dt\right], \end{aligned}$$

On the other hand,

$$\|\mathcal{P}_k(h) - \mathcal{P}_k(g)\| \leq \|a_1(h-g) + a_2(h^2-g^2) + \cdots + a_{k-1}(h^{k-1}-g^{k-1}) + a_k(h^k-g^k)\|$$

$$\leq \Big[|a_1| + |a_2|(\|h\|+\|g\|) + \cdots + |a_k|(\|h\|^{k-1} + \|h\|^{k-2}\|g\| + \cdots + \|h\|\,\|g\|^{k-2} + \|g\|^{k-1})\Big]\|h-g\|$$

So, the Nemystkii operator $\mathcal{P}_k$ verifies the following condition

$$\|\mathcal{P}_k(h) - \mathcal{P}_k(g)\| \leq \kappa(\|h\|,\|g\|)\,\|h-g\|, \text{ for } h,g \in \mathcal{D}, \tag{9}$$

being $\kappa : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a nondecreasing continuous function in both arguments with $\kappa(0,0) \geq 0$. Then, from (9), it follows

$$\|h^* - \widetilde{h}\| \leq \frac{|\mu|\,\|\int_p^q \mathcal{E}(\epsilon,x,t)\,dt\|\,\|\mathcal{P}_k(\widetilde{h})\|}{1 - |\mu|\,\|\int_p^q N(x,t)\,dt\|\;\kappa(\|h^*\|,\|\widetilde{h}\|)}, \tag{10}$$

provide that $|\mu|\,\|\int_p^q N(x,t)\,dt\|\;\kappa(\|h^*\|,\|\widetilde{h}\|) < 1$.

So, given a tolerance Tol, if we set out to calculate an approximation $\widetilde{h}_{N_0}$ with $\|h^* - \widetilde{h}_{N_0}\| < Tol$, as $\lim_n \widetilde{h}_n = \widetilde{h}$, from (8) it is sufficient that $\|h^* - \widetilde{h}\| < Tol$. Then, by taking into account (10), it will be sufficient to determine the number of terms m that we have to consider in the separable development of operator N, see (6).

On the other hand, taking into account our theoretical goal, to apply the iterative process (3) to approximate a solution $\widetilde{h}$ of the equation $\widetilde{\mathcal{F}}(h) = 0$, the first derivative of the operator $\widetilde{\mathcal{F}}$, given by (7), is

$$[\widetilde{\mathcal{F}}'(h)g](s) = g(s) - \mu\int_p^q \widetilde{N}(x,t)[\mathcal{P}_k'(h)]g(t)dt, \quad g \in \mathcal{D}.$$

So, as

$$\|\mathcal{P}_k'(h) - \mathcal{P}_k'(g)\| \leq \|2a_2(h-g) + 3a_3(h^2-g^2) + \cdots + (k-1)a_{k-1}(h^{k-2}-g^{k-2}) + ka_k(h^{k-1}-g^{k-1})\|$$

$$\leq \Big[2|a_2| + 3|a_3|(\|h\|+\|g\|) + \cdots + k|a_k|(\|h\|^{k-2} + \cdots + \|h\|\,\|g\|^{k-3} + \|g\|^{k-2})\Big]\|h-g\|,$$

then

$$\|\widetilde{\mathcal{F}}'(h) - \widetilde{\mathcal{F}}'(g)\| \leq \sigma(\|h\|,\|g\|)\|h-g\|, \qquad h,g \in \mathcal{D}, \tag{11}$$

being

$$\sigma(\|h\|,\|g\|) = |\mu|L\Big[2|a_2| + 3|a_3|(\|h\|+\|g\|) + \cdots + k|a_k|(\|h\|^{k-2} + \cdots + \|h\|\,\|g\|^{k-3} + \|g\|^{k-2})\Big], \tag{12}$$

being $L = \max_{x\in[p,q]}\left|\int_p^q \widetilde{N}(x,t)dt\right|$. Therefore, we will have to take into account the condition (11) to obtain a semilocal convergence result so that the iterative process (3) can be applied to approximate a solution of the equation $\widetilde{\mathcal{F}}(h) = 0$.

3. Convergence analysis

In the Introduction, we have mentioned that to obtain a semilocal convergence result for the iterative process (3) is a main theoretical goal of this work. Specifically, our goal is to generalize the semilocal convergence results obtained by other authors ([2,19]) for the iterative process (3). For this purpose we will consider the condition (11) that generalizes the conditions considered so far. Since space $\mathscr{C}[p,q]$ with the infinity norm is a Banach space, to be able to apply the result that we obtain to this space, we will get a semilocal convergence result over Banach spaces. So, we will consider $\mathcal{G}:\mathcal{D}\subseteq\mathscr{X}\to\mathscr{Y}$ a nonlinear twice Fréchet differentiable operator in an open convex domain $\mathcal{D}$, being $\mathscr{X}$, $\mathscr{Y}$ two Banach spaces.

Notice that, in [2], semilocal convergence of the iterative process (3) is tested when the first derivative of the operator $\mathcal{G}$ is a K-Lipschitz continuous operator:

$$\|\mathcal{G}'(u)-\mathcal{G}'(v)\|\le K\|u-v\|,\quad \text{with } u,v\in\mathcal{D}. \tag{13}$$

Whereas, in [19], the semilocal convergence of the iterative process (3) is tested when the first derivative of the operator $\mathcal{G}$ is a (K,ρ)-Hölder continuous operator:

$$\|\mathcal{G}'(u)-\mathcal{G}'(v)\|\le K\|u-v\|^{\rho},\quad \text{with } u,v\in\mathcal{D} \text{ and } \rho\in[0,1]. \tag{14}$$

However, as we have seen in Section 2, we wil consider the condition (11) which obviously generalizes the conditions (13) and (14).

3.1. Convergence conditions

Next, we consider the iterative process (3) when is applied to a general equation $\mathcal{G}(u)=0$ given by

$$\begin{cases} u_0\in\mathcal{D},\\ v_n=u_n-\Lambda_n\mathcal{F}(u_n),\\ w_n=v_n-5\,\Lambda_n\mathcal{F}(v_n),\\ u_{n+1}=f_n-\frac{1}{5}\Lambda_n(-16\,\mathcal{F}(v_n)+\mathcal{F}(w_n)),\, n\ge 0. \end{cases} \tag{15}$$

where $\Lambda_n=\left[\mathcal{G}'(u_n)\right]^{-1}$ for $n\in\mathbb{N}$.

From the previous ideas, we want to prove the semilocal convergence of iterative process (15) under the following conditions:

(CI) Let $u_0\in\mathcal{D}$ such that there exists $\Lambda_0=[\mathcal{G}'(u_0)]^{-1}\in\mathcal{L}(\mathscr{Y},\mathscr{X})$, being $\mathcal{L}(\mathscr{Y},\mathscr{X})$ the set of bounded linear operators from $\mathscr{Y}$ to $\mathscr{X}$, with $\|\Lambda_0\|\le\beta$ and $\|\Lambda_0\mathcal{G}(u_0)\|\le\eta$,

(CII) There exists a nondecreasing function $\kappa:\mathbb{R}_+\times\mathbb{R}_+\longrightarrow\mathbb{R}_+$ continuous in both arguments with $\kappa(0,0)\ge 0$, such that $\|\mathcal{G}'(u)-\mathcal{G}'(v)\|\le\kappa(\|u\|,\|v\|)\,\|u-v\|^{\rho}$, with $u,v\in\mathcal{D}$ and $\rho\in(0,1]$.

Notice that, condition (*CII*) generalizes the conditions (13) and (14) taking $\kappa(\|u\|,\|v\|)$ a constant function. Moreover, obviously, (*CII*) generalizes (11) too.

3.2. Auxiliary scalar sequences

Under conditions (*CI*) and (*CII*) we will start by assuming that there exists $R>0$ such that $B(u_0,R\eta)\subseteq\mathcal{D}$, being $B(u_0,R\eta)=\{u\in\mathscr{X}:\|u-u_0\|<R\eta\}$. Now, we establish relationships between the real parameters that are introduced under conditions given previously for the pair $(\mathcal{G},u_0)$.

Let $c_0=Q(R\eta)\beta\eta^{\rho}$, where $Q(s)=\kappa(\|u_0\|+s,\|u_0\|+s)$, and, for $n\geqslant 0$, we consider the real sequence given by

$$c_{n+1}=c_n\psi(c_n)^{1+\rho}\xi(c_n)^{\rho}, \tag{16}$$

where

$$\psi(s)=\frac{1}{1-s(1+\phi(s))^{\rho}}, \tag{17}$$

$$\xi(s)=(s+1)\phi(s)+\frac{s}{1+\rho}(1+\phi(s)^{1+\rho}), \tag{18}$$

being

$$\phi(s)=\frac{s+s^2}{1+\rho}+\frac{5^{\rho}s^{2+\rho}}{(1+\rho)^{2+\rho}}.. \tag{19}$$

Below, we study the properties that the sequence $\{c_n\}$ verifies from the following results.

Lemma 1. *Given the function $\zeta(s)=s(1+\phi(s))^{\rho}-1$ in (0,1), we denote by α its smallest positive real root. Then,*

(i) $\psi(s)$ *is an increasing function with* $\psi(s) > 1$ *for* $s \in (0, \alpha)$,
(ii) $\xi(s)$ *and* $\phi(s)$ *are both increasing functions for* $s \in (0, \alpha)$.

Now, we define the following auxiliary function

$$\delta(s) = \xi(s)^{\rho} - (\ 1 - s(1 + \phi(s))^{\rho})^{1+\rho}.$$

Obviously, $\delta(s)$ is an increasing function with $\delta(0) < 0$, $\delta(\alpha) > 0$ and $\delta'(s) > 0$. Therefore, this function has a root α_1 in $(0, \alpha)$.

Lemma 2. *If* $c_0 \in (0, \alpha_1)$*, then*

(i) $\psi(c_0)^{1+\rho}\xi(c_0)^{\rho} < 1$.
(ii) $\{c_n\}$ *is a decreasing sequence with* $c_n < \alpha_1$ *for all* $n \geq 0$.
(iii) $c_n(1 + \phi(c_n))^{\rho} < 1$.

Proof. The first strictly inequality follows by the definition of $\delta(s)$. Replacing s for c_0 in $\delta(s)$, we get $\psi(c_0)^{1+\rho}\xi(c_0)^{\rho} < 1$ for $c_0 \in (0, \alpha_1)$. What it proves (i).

We will prove (ii) by mathematical induction. So, for $n = 0$, from (16) we have $c_1 = c_0\psi(c_0)^{1+\rho}\xi(c_0)^{\rho} < c_0$. Now, assume that $c_k < c_{k-1}$ for $k \leq n$. Therefore, $c_{n+1} = c_n\psi(c_n)^{1+\rho}\xi(c_n)^{\rho} < c_{n-1}\psi(c_n)^{1+\rho}\xi(c_n)^{\rho} < c_{n-1}\psi(c_{n-1})^{1+\rho}\xi(c_{n-1})^{\rho} = c_n$. Hence, $\{c_n\}$ is a decreasing sequence and $c_n < \alpha_1$ for $n \geq 0$.

Next, to prove (iii), we consider $c_n(1 + \phi(c_n))^{\rho} < c_{n-1}(1 + \phi(c_{n-1}))^{\rho} < c_0(1 + \phi(c_0))^{\rho} < 1$, for all $c_0 \in (0, \alpha_1)$. Hence (iii) is proved. □

3.3. Recurrence relations

Now, we will set the recurrence relations for (15) under the assumptions considered above. As Λ_0 exists, therefore, v_0 and w_0 are well defined and suppose that $u_0, v_0, w_0 \in \mathcal{D}$. So,

$$\begin{aligned} w_0 - v_0 &= -5\Lambda_0\mathcal{G}(v_0) \\ &= -5\Lambda_0\int_0^1 \left(\mathcal{G}'(u_0 + \tau(v_0 - u_0)) - \mathcal{G}'(u_0)\right)(v_0 - u_0)d\tau. \end{aligned}$$

Therefore,

$$\begin{aligned} \|w_0 - v_0\| &\leq 5\beta\left(\int_0^1 \kappa(\|u_0 + \tau(v_0 - u_0)\|, \|u_0\|)\tau^{\rho}d\tau\|v_0 - u_0\|^{\rho}\right)\|v_0 - u_0\| \\ &\leq \frac{5}{1+\rho}\beta Q(R\eta)\|v_0 - u_0\|^{1+\rho} \\ &= \frac{5}{1+\rho}c_0\|v_0 - u_0\|. \end{aligned}$$

On the other hand,

$$\begin{aligned} u_1 - u_0 &= v_0 - u_0 - \frac{9}{5}\Lambda_0\mathcal{G}(v_0) - \frac{1}{5}\Lambda_0\left(\mathcal{G}(v_0) + \int_0^1 \mathcal{G}'(v_0 + \tau(w_0 - v_0))(w_0 - v_0)d\tau\right) \\ &= v_0 - u_0 + \frac{1}{5}(w_0 - v_0) - \frac{1}{5}\Lambda_0\int_0^1 \left(\mathcal{G}'(v_0 + \tau(w_0 - v_0)) - \mathcal{G}'(u_0)\right)d\tau(w_0 - v_0), \end{aligned}$$

then, taking norms in the previous equality, we have

$$\begin{aligned} \|u_1 - u_0\| &\leq \|v_0 - u_0\| + \frac{1}{5}\|w_0 - v_0\| + \frac{\beta}{5}\int_0^1 \Big(\kappa(\|v_0 + \tau(w_0 - v_0)\|, \|v_0\|)\tau^{\rho}\|w_0 - v_0\|^{\rho} \\ &\quad + \kappa(\|v_0\|, \|u_0\|)\|v_0 - u_0\|^{\rho}\Big)d\tau\|w_0 - v_0\| \\ &\leq \left(1 + \frac{c_0}{1+\rho} + \frac{c_0^2}{1+\rho} + \frac{5^{\rho}c_0^{2+\rho}}{(1+\rho)^{2+\rho}}\right)\|v_0 - u_0\| \\ &= \left(1 + \phi(c_0)\right)\|v_0 - u_0\|. \end{aligned} \tag{20}$$

Then, using mathematical induction, we will test the following recurrence relations $n \geq 1$.

(I) $\|\Lambda_n\| \leq \psi(c_{n-1})\|\Lambda_{n-1}\|$,
(II) $\|v_n - u_n\| \leq \psi(c_{n-1})\xi(c_{n-1})\|v_{n-1} - u_{n-1}\|$,
(III) $\|w_n - v_n\| \leq \frac{5}{1+\rho}c_n\|v_{n-1} - u_{n-1}\|$,

(IV) $Q(R\eta)\|\Lambda_n\|\|v_n - u_n\|^\rho \le c_n$,
(V) $\|u_n - u_{n-1}\| \le (1+\phi(c_{n-1}))^\rho \|v_{n-1} - u_{n-1}\|$.

If we suppose $u_1, v_1, w_1 \in \mathcal{D}$ and $c_0 < \alpha_1$, as $\|I - \Lambda_0\mathcal{G}'(u_1)\| = \|\Lambda_0\|\|\mathcal{G}'(u_1) - \mathcal{G}'(u_0)\| \le \beta\kappa(\|u_1\|, \|u_0\|)\|u_1 - u_0\|^\rho \le \beta Q(R\eta)\|v_0 - u_0\|^\rho (1+\phi(c_0))^\rho = c_0(1+\phi(c_0))^\rho < 1$, by Banach Lemma, there exists Λ_1 with

$$\|\Lambda_1\| \le \frac{1}{1 - c_0(1+\phi(c_0))^\rho}\|\Lambda_0\| = \psi(c_0)\|\Lambda_0\|. \tag{21}$$

Now, by the Taylor's expansion of $\mathcal{G}(u_1)$, we have

$$\begin{aligned}\mathcal{G}(u_1) = &\int_0^1 \left(\mathcal{G}'(u_0 + \tau(v_0 - u_0)) - \mathcal{G}'(u_0)\right)(v_0 - u_0)d\tau + \left(\mathcal{G}'(v_0) - \mathcal{G}'(u_0)\right)(u_1 - v_0)\\ &+ \mathcal{G}'(u_0)(u_1 - v_0) + \int_0^1 \left(\mathcal{G}'(v_0 + \tau(u_1 - v_0)) - \mathcal{G}'(v_0)\right)(u_1 - v_0)d\tau\end{aligned}$$

and, taking norms, we get

$$\begin{aligned}\|\mathcal{G}(u_1)\| \le &\frac{1}{1+\rho}Q(R\eta)\|v_0 - u_0\|^{1+\rho} + Q(R\eta)\left(\frac{c_0}{1+\rho} + \frac{c_0^2}{1+\rho} + \frac{5^\rho c_0^{2+\rho}}{(1+\rho)^{2+\rho}}\right)\|v_0 - u_0\|^{1+\rho}\\ &+ \frac{1}{\beta}\left(\frac{c_0}{1+\rho} + \frac{c_0^2}{1+\rho} + \frac{5^\rho c_0^{2+\rho}}{(1+\rho)^{2+\rho}}\right)\\ &+ \frac{1}{1+\rho}Q(R\eta)\left(\frac{c_0}{1+\rho} + \frac{c_0^2}{1+\rho} + \frac{5^\rho c_0^{2+\rho}}{(1+\rho)^{2+\rho}}\right)^{1+\rho}\|v_0 - u_0\|^{1+\rho}\end{aligned}$$

Therefore,

$$\begin{aligned}\|\Lambda_1\mathcal{G}(u_1)\| \le &\psi(c_0)\left(\frac{c_0}{1+\rho} + (c_0+1)\left(\frac{c_0}{1+\rho} + \frac{c_0^2}{1+\rho} + \frac{5^\rho c_0^{2+\rho}}{(1+\rho)^{2+\rho}}\right)\right.\\ &\left.+ \frac{c_0}{1+\rho}\left(\frac{c_0}{1+\rho} + \frac{c_0^2}{1+\rho} + \frac{5^\rho c_0^{2+\rho}}{(1+\rho)^{2+\rho}}\right)^{1+\rho}\right)\|v_0 - u_0\|\\ \le &\psi(c_0)\xi(c_0)\|v_0 - u_0\|,\end{aligned} \tag{22}$$

then

$$\|v_1 - u_1\| \le \|\Lambda_1\|\|\mathcal{G}(u_1)\| \le \psi(c_0)\xi(c_0)\|v_0 - u_0\|. \tag{23}$$

Now

$$\begin{aligned}\|w_1 - v_1\| &\le 5\beta\psi(c_0)\left(\int_0^1 \kappa(\|u_1 + \tau(v_1 - u_1)\|, \|u_1\|)\tau^\rho d\tau\|v_1 - u_1\|^\rho\right)\|v_1 - u_1\|\\ &\le \frac{5}{1+\rho}\beta Q(R\eta)\psi(c_0)\|v_1 - u_1\|^{1+\rho}\\ &\le \frac{5}{1+\rho}c_0\psi(c_0)^{1+\rho}\xi(c_0)^\rho\|v_1 - u_1\|\\ &= \frac{5}{1+\rho}c_1\|v_1 - u_1\|\end{aligned} \tag{24}$$

Using (21) and (23), we get

$$\begin{aligned}Q(R\eta)\|\Lambda_1\|\|v_1 - u_1\|^\rho &\le \beta Q(R\eta)\psi(c_0)(\psi(c_0))^\rho(\xi(c_0))^\rho\|v_0 - u_0\|^\rho,\\ &\le c_0(\psi(c_0))^{1+\rho}(\xi(c_0))^\rho = c_1.\end{aligned} \tag{25}$$

From (21), (23), (24) and (25), the recurrence relations (I)-(IV) hold for $n = 1$. The fifth one is proved in (20). Now, by using mathematical induction, it can easily be proved that (I)-(V) holds for $n \ge 1$.

3.4. A semilocal convergence result

From previous development, we are already able to test our new semilocal convergence result.

Theorem 1. *Let us assume that the conditions* $(CI) - (CII)$ *hold and* $u_0 \in \mathcal{D}$. *Suppose that there exists at least one positive real root of equation*

$$R = \frac{5}{1+\rho}c_0 + \frac{(1+\phi(c_0))^\rho}{1 - \psi(c_0)\xi(c_0)} \tag{26}$$

and denote by R, the smallest positive real root. If $c_0 < \alpha_1$ and $B(u_0, R\eta) \subseteq \mathcal{D}$, then the sequence $\{u_n\}$, defined in (15), converges to a solution u^* of the equation $\mathcal{G}(u) = 0$. Moreover, the solution u^* and the iterates u_n, v_n and w_n belong to $\overline{B(u_0, R\eta)} = \{u \in \mathscr{X} : \|u - u_0\| \leq R\eta\}$. Moreover,

$$\|u^* - u_n\| \leq \frac{(1+\phi(c_0))}{1-\psi(c_0)\xi(c_0)}(\psi(c_0)\xi(c_0))^n\eta, \quad \text{for } n \geqslant 0. \tag{27}$$

Proof. From the recurrence relation (V), we have

$$\|u_n - u_0\| \leq (1+\phi(c_0))\sum_{k=0}^{n-1}(\psi(c_0)\xi(c_0))^k\|v_0 - u_0\|. \tag{28}$$

So,

$$\begin{aligned}\|v_n - u_0\| &\leq \|v_n - u_n\| + \|u_n - u_0\|\\ &\leq (1+\phi(c_0))\sum_{k=0}^{n}(\psi(c_0)\xi(c_0))^k\|v_0 - u_0\|,\\ &\leq (1+\phi(c_0))\frac{1-(\psi(c_0)\xi(c_0))^{n+1}}{1-\psi(c_0)\xi(c_0)}\|v_0 - u_0\| < R\eta. \end{aligned} \tag{29}$$

Now, from (III) and (29), we get

$$\begin{aligned}\|w_n - u_0\| &\leq \|w_n - v_n\| + \|v_n - u_0\|\\ &\leq \frac{5}{1+\rho}c_0(\psi(c_0)\xi(c_0))^n\|v_0 - u_0\| + (1+\phi(c_0))\frac{1-(\psi(c_0)\xi(c_0))^{n+1}}{1-\psi(c_0)\xi(c_0)}\|v_0 - u_0\|\\ &\leq \left(\frac{5}{1+\rho}c_0 + (1+\phi(c_0))\frac{1-(\psi(c_0)\xi(c_0))^{n+1}}{1-\psi(c_0)\xi(c_0)}\right)\eta < R\eta.\end{aligned}$$

Hence v_n and $w_n \in \mathcal{D}$.

On the other hand,

$$\|u_{n+1} - u_n\| \leq (1+\phi(c_n))\left(\prod_{j=0}^{n-1}\psi(c_j)\xi(c_j)\right)\|v_0 - u_0\|$$

and therefore

$$\begin{aligned}\|u_{n+m} - u_n\| &\leq \|u_{n+m} - u_{n+m-1}\| + \|u_{n+m-1} - u_{n+m-2}\| + \|u_{n+m-2} - u_{n+m-3}\| + \ldots + \|u_{n+1} - u_n\|\\ &\leq (1+\phi(c_0))\sum_{l=0}^{m-1}\left(\prod_{j=0}^{n+l-1}\psi(c_j)\xi(c_j)\right)\eta\\ &\leq (1+\phi(c_0))\sum_{l=0}^{m-1}(\psi(c_0)\xi(c_0))^{l+n}\eta.\end{aligned}$$

So,

$$\|u_{n+m} - u_n\| \leq (1+\phi(c_0))\frac{1-(\psi(c_0)\xi(c_0))^m}{1-\psi(c_0)\xi(c_0)}(\psi(c_0)\xi(c_0))^n\eta. \tag{30}$$

Hence $\{u_n\}$ is a Cauchy sequence because $\psi(c_0)\xi(c_0) < 1$, then the sequence $\{u_n\}$ converges.

Moreover, for $n = 0$ and $m \geq 1$, we get

$$\|u_m - u_0\| \leq (1+\phi(c_0))\frac{1-(\psi(c_0)\xi(c_0))^m}{1-\psi(c_0)\xi(c_0)} < R\eta, \tag{31}$$

therefore $u_m \in \overline{B}(u_0, R\eta)$.

Now, taking $m \to \infty$ in (31) we get $u^* \in \overline{B}(u_0, R\eta)$ and, from (30), it follows (27).

To prove that u^* is a solution of $\mathcal{G}(u) = 0$, we consider

$$\|\mathcal{G}(u_n)\| \leq \|\mathcal{G}'(u_n)\|\|v_n - u_n\| \leq \|\mathcal{G}'(u_n)\|(\psi(c_0)\xi(c_0))^n\|v_0 - u_0\|,$$

as $\|\mathcal{G}'(u_n)\|$ is bounded since that $\|\mathcal{G}'(u_n)\| \leq \|\mathcal{G}'(u_n) - \mathcal{G}'(u_0)\| + \|\mathcal{G}'(u_0)\| \leq \|\mathcal{G}'(u_0)\| + \kappa(\|u_0\| + R, \|u_0\|)R^\rho\eta^\rho$, then $\|\mathcal{G}(u_n)\| \to 0$ as $n \to \infty$. Now, by the continuity of $\mathcal{G}$ in $\mathcal{D}$, we obtain that $\mathcal{G}(u^*) = 0$. □

Theorem 2. *Under conditions of Theorem (1), if there exists at least one positive real root of the equation*

$$\beta\kappa(\|u_0\|, \|u_0\| + s)\int_0^1((1-\theta)R\eta + \theta r)^\rho d\theta = 1$$

and denote by r the smallest positive root, then u^* *is unique in* $B(u_0, r) \cap \mathcal{D}$.

Proof. If we suppose that there exists $y^* \in B(u_0, r)$ such that $\mathcal{G}(y^*) = 0$ and $y^* \neq u^*$. Then

$$0 = \mathcal{G}(y^*) - \mathcal{G}(u^*) = \int_0^1 \mathcal{G}'(u^* + \theta(y^* - u^*))d\theta(y^* - u^*) = S(y^* - u^*)$$

Now, as

$$\begin{aligned}\|I - \Lambda_0 S\| &\leq \|\Lambda_0\| \left\| \int_0^1 \left(\mathcal{G}'(u^* + \theta(y^* - u^*)) - \mathcal{G}'(u_0)\right)d\theta \right\| \\ &< \beta\kappa(\|u_0\| + r, \|u_0\|) \int_0^1 ((1-\theta)\|u^* - u_0\| + \theta\|y^* - u_0\|)^\rho d\theta \\ &< \beta\kappa(\|u_0\| + r, \|u_0\|) \int_0^1 ((1-\theta)R\eta + \theta r)^\rho d\theta \\ &= 1,\end{aligned}$$

so, there exists S^{-1} exists and hence $y^* = u^*$.Then, by reducing to absurdity the result is proven. □

3.5. Particular cases

Now, three particular cases are studied. In first place, we obtain a semilocal convergence result for operators with first derivative K-Lipschitz continuous from the Theorem 1. In second place, we realize a similar study to the previous one but considering operators with first derivative (K, ρ)-Hölder continuous. In the third place, we consider the application of the Theorem 1 to ensure that iterative process (15) converges to a solution of the equation $\widetilde{\mathcal{F}}(u) = 0$.

3.5.1. $\mathcal{G}'$ *iIs Llipschitz continuous*

In this first case, if we take $\kappa(\|u\|, \|v\|) = K$ and $\rho = 1$ in condition (CII), we can apply Theorem 1 for operators with first derivative K-Lipschitz continuous under the following conditions:

(LI) Let $u_0 \in \mathcal{D}$ such that there exists $\Lambda_0 = [\mathcal{G}'(u_0)]^{-1} \in \mathcal{L}(\mathscr{Y}, \mathscr{X})$, with $\|\Lambda_0\| \leq \beta$ and $\|\Lambda_0 \mathcal{G}(u_0)\| \leq \eta$,
(LII) $\|\mathcal{G}'(u) - \mathcal{G}'(v)\| \leq K\|u - v\|$, with $u, v \in \mathcal{D}$.

Then, we obtain the following semilocal convergence result.

Theorem 3. *Suppose that conditions (LI) and (LII) are satisfied. Let us denote* $c_0 = K\beta\eta$ *and* $R = \frac{5}{2}c_0 + \frac{1 + \phi(c_0)}{1 - \psi(c_0)\xi(c_0)}$. *If* $c_0 <$ 0.2931 ... *and* $B(u_0, R\eta) \subset \mathcal{D}$, *then the sequence* $\{u_n\}$ *given by (15) and starting at* u_0, *converges to a solution* u^* *of the equation* $\mathcal{G}(u) = 0$ *and* $u^*, u_n, v_n, w_n \in \overline{B(u_0, R\eta)}$, *for all* $n \in \mathbb{N}$. *Moreover, the solution* u^* *is unique in* $B\left(u_0, \frac{2}{K\beta} - R\eta\right) \cap \mathcal{D}$.

Notice that, as $\kappa(\|u\|, \|y\|) = K$ then c_0 is a constant value and R is directly obtained. So, the previous result is the same that the result obtained in [2].

3.5.2. $\mathcal{G}'$ *Is hölder continuous*

In this second case, if we take $\kappa(\|u\|, \|v\|) = K$ and $\rho \in [0, 1]$ in condition (CII), we can to apply the Theorem 1 for operators with first derivative (K, ρ)-Hölder continuous under the following conditions:

(H I) Let $u_0 \in \mathcal{D}$ such that there exists $\Lambda_0 = [\mathcal{G}'(u_0)]^{-1} \in \mathcal{L}(\mathscr{Y}, \mathscr{X})$, with $\|\Lambda_0\| \leq \beta$ and $\|\Lambda_0 \mathcal{G}(u_0)\| \leq \eta$,
(H II) $\|\mathcal{G}'(u) - \mathcal{G}'(v)\| \leq K\|u - v\|^\rho$, with $u, v \in \mathcal{D}$ and $\rho \in [0, 1]$..

Now, from the Theorem 1, we obtain the following result

Theorem 4. *Suppose that conditions (H I) and (H II) are satisfied. Let us denote* $c_0 = K\beta\eta^\rho$ *and* $R = \frac{5}{2}c_0 + \frac{1 + \phi(c_0)}{1 - \psi(c_0)\xi(c_0)}$. *If* $c_0 < \alpha_1$, *where* α_1 *is a root in* $(0, \alpha)$ *of auxiliary function*

$$\delta(s) = \xi(s)^\rho - (\ \ 1 - s(1 + \phi(s))^\rho)^{1+\rho},$$

being α *she smallest positive real root of* $\zeta(s) = s(1 + \phi(s))^\rho - 1$ *in (0,1) and* $B(u_0, R\eta) \subset \mathcal{D}$, *then the sequence* $\{u_n\}$ *given by (15) and starting at* u_0, *converges to a solution* u^* *of the equation* $\mathcal{G}(u) = 0$ *and* $u^*, u_n, v_n, w_n \in \overline{B(u_0, R\eta)}$, *for all* $n \in \mathbb{N}$. *Moreover, the solution* u^* *is unique in* $B\left(u_0, \frac{2}{K\beta} - R\eta\right) \cap \mathcal{D}$.

As in the case K-Lipschitz, $\kappa(\|u\|, \|v\|) = K$ then c_0 is a constant value and R is directly obtained. Thus, we have obtained a result of semilocal convergence, under (K, ρ)-Hölder conditions, similar to that obtained in [19], although the auxiliary function ϕ here obtained differs from that considered in [19].

3.5.3. How to approximate a solution of (1)

Now, we consider $\widetilde{\mathcal{F}} : \mathscr{C}[p,q] \to \mathscr{C}[p,q]$ a nonlinear integral operator given by

$$\widetilde{\mathcal{F}}(h)(s) = h(x) - \ell(x) - \mu \int_p^q \widetilde{N}(x,t)\, \mathcal{P}_k(x)(t)dt, \quad x \in [p,q], \quad \mu \in \mathbb{R},$$

where $\widetilde{N}$ is a separable kernel obtained from (6). Then, we have

$$\|\widetilde{\mathcal{F}}'(h) - \widetilde{\mathcal{F}}'(g)\| \le \sigma(\|h\|, \|g\|)\|h-g\|, \qquad h, g \in \mathcal{D},$$

where $\sigma(\|h\|, \|g\|)$ is given in (12).

In this case, it is easy to check that $\alpha = 0.6114\ldots$ and $\alpha_1 = 0.2493\ldots$. Moreover, we get

$$[\widetilde{\mathcal{F}}'(h)g](x) = g(x) - \mu \int_p^q \widetilde{N}(x,t)[\mathcal{P}_k'(h)]g(t)dt,$$

then, for $\widetilde{h}_0 \in \mathscr{C}[p,q]$ we have $\|I - \widetilde{\mathcal{F}}'(\widetilde{h}_0)\| \le |\mu| L \|\mathcal{P}_k'(\widetilde{h}_0)\|$, with $L = \max_{x\in[p,q]}\left|\int_p^q \widetilde{N}(x,t)dt\right|$. So, if $|\mu|L\|\mathcal{P}_k'(\widetilde{h}_0)\| < 1$ there exists $[\widetilde{\mathcal{F}}'(\widetilde{h}_0)]^{-1}$ with

$$\|[\widetilde{\mathcal{F}}'(\widetilde{h}_0)]^{-1} \le \frac{1}{1 - |\mu|L\|\mathcal{P}_k'(\widetilde{h}_0)\|} = \beta$$

Therefore, under following conditions:

(IE I) Let $\widetilde{h}_0 \in \mathscr{C}[p,q]$ such that there exists $\Lambda_0 = [\widetilde{\mathcal{F}}'(\widetilde{h}_0)]^{-1} \in \mathcal{L}(\mathscr{C}[p,q], \mathscr{C}[p,q])$, with $\|\Lambda_0\| \le \beta$ and $\|\Lambda_0 \mathcal{G}(\widetilde{h}_0)\| \le \eta$,

(IE II) There exists a nondecreasing function $\sigma : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ continuous in both arguments with $\sigma(0,0) \ge 0$, such that $\|\widetilde{\mathcal{F}}'(h) - \widetilde{\mathcal{F}}'(g)\| \le \sigma(\|h\|, \|g\|)\, \|h-g\|$, with $h, g \in \mathscr{C}[p,q]$,

and by Theorem 1, one has the following semilocal convergence result.

Theorem 5. *Suppose that conditions (IE I) and (IE II) are satisfied. Let us denote* $c_0 = \sigma(\|\widetilde{h}_0\| + R\beta\delta, \|\widetilde{h}_0\| + R\beta\delta)\, \beta^2\, \delta$, *with* $\|\widetilde{\mathcal{F}}(\widetilde{h}_0)\| = \delta$, *and suppose that there exists R, the smallest positive real solution of equation*

$$R = \frac{5}{2}c_0 + \frac{1 + \phi(c_0)}{1 - \psi(c_0)\xi(c_0)}. \tag{32}$$

If $c_0 < 0.2493\ldots$, *then the sequence* $\{\widetilde{h}_n\}$ *given by (3) and starting at* $\widetilde{h}_0$, *converges to a solution* $\widetilde{h}$ *of the equation* $\widetilde{\mathcal{F}}(h) = 0$ *and* $\widetilde{h}, \widetilde{h}_n, \widetilde{g}_n, \widetilde{f}_n \in \overline{B(\widetilde{h}_0, R\beta\delta)}$, *for all* $n \in \mathbb{N}$. *Moreover, the solution* $\widetilde{h}$ *is unique in* $B(h_0, r)$, *where r is the smallest positive real root of the equation*

$$r = \frac{2}{\beta\sigma(\|\widetilde{h}_0\| + r, \|\widetilde{h}_0\|)} - R\beta\delta.$$

There are works, like [5–7] and [10], where convergence or error analysis can be viewed to predict error or convergence rate.

4. Numerical experiments

Here we study two numerical examples in favor of our theoretical findings. In the first numerical test, we will see that if we consider a good approximation of the solutions of both integral Eq. (1) and (5), by means of a small number of iterations we will approximate the solution of the original equation (1) with the prefixed precision, which produces a reduction in the operational cost. In the second numerical experiment, we will apply the Theorem 5 to an integral equation of the type (1) for obtaining the convergence of the iterative process (3). Noting that the theoretical results obtained for the iterative process (3) given in [2] and [19] can not be used in this case.

Example 1. We focus on the nonlinear integral equation of Hammerstein-type given by

$$h(x) = \frac{e^x(50 - e^2) + 1}{50} + \frac{1}{50}\int_0^1 \mathrm{e}^{xt}(x+2)h(t)^2 dt. \tag{33}$$

It can be checked that, $h^*(x) = e^x$ is a solution of (33), in this case $\mathcal{D} = \mathscr{C}[0,1]$ and

$$\mathcal{F}(h)(s) = h(x) - \frac{e^x(50 - e^2) + 1}{50} - \frac{1}{50}\int_0^1 \mathrm{e}^{xt}(x+2)h(t)^2 dt, \tag{34}$$

where the kernel $N(x,t) = \mathrm{e}^{xt}(x+2)$ is non-separable. In this case, we can obtain $N(x,t) = \widetilde{N}(x,t) + \mathcal{E}(\theta, x, t)$ with

$$\widetilde{N}(x,t) = (x+2)\left(\sum_{i=0}^{m} \frac{x^i t^i}{i!}\right) \text{ and } \mathcal{E}(\theta, x, t) = \frac{(x+2)\mathrm{e}^{x\theta}}{(m+1)!} x^{m+1} t^{m+1}. \tag{35}$$

Table 1
Tolerance obtained for different m values.

m	$\widetilde{\sigma}_1$	*Tol* given by (37)
2	3.721606828848387	4.0291e-02
6	3.924044382792517	2.6903e-05
10	3.924179224736788	2.2648e-09
14	3.924179235679935	5.1849e-14
18	3.924179235680181	4.4590e-19
22	3.924179235680181	1.7484e-24

Table 2
Numerical results by taking starting function $h_0(x) = 3/2\ e^x$.

iterative process (3)			
m	2	6	14
Iterations	3	3	3
$\|\widetilde{h}_n(x) - \widetilde{h}_{n-1}(x)\|$	1.5685e-16	5.9635e-16	5.9650e-16
$\|\widetilde{h}_n(x) - h^*(x)\|$	1.9246e-02	1.1281e-05	2.1704e-14

Then,

$$\widetilde{\mathcal{F}}(x)(s) = h(x) - \frac{e^x(50-e^2)+1}{50} - \frac{1}{50}\int_0^1 \left[(x+2)\sum_{i=0}^{m}\frac{x^i t^i}{i!}\right] h(t)^2 dt. \tag{36}$$

If we denote by $h^*(x)$ ad $\widetilde{h}(x)$ the solutions of (34) and (36) respectively, taking into account (10), we obtain

$$\|h^* - \widetilde{h}\| \le \frac{\frac{1}{50}\|\int_0^1 \frac{e^{x\theta}(x+2)}{(m+1)!}x^{m+1}\,t^{m+1}\,dt\|\,\|\widetilde{h})\|^2}{1-\frac{1}{50}\|\int_0^1 e^{xt}(x+2)\,dt\|\ (\|h^*\|+\|\widetilde{h}\|)} \le Tol, \tag{37}$$

provide that $\|\int_0^1 e^{xt}(x+2)\,dt\|\ (\|h^*\|+\|\widetilde{h}\|) < 50$. Now, to locate previously $\widetilde{h}(x)$ in $\mathscr{C}([0,1])$, we see from (33) that

$$\|\widetilde{h}\| - \frac{1}{50}\|e^x(50-e^2)+1\| - \frac{1}{50}\widetilde{M}_1\|\widetilde{h}\|^2 \le 0,$$

where $\widetilde{M}_1 = \max_{x\in[0,1]}\left|\int_0^1 (x+2)\left(\sum_{i=0}^{m}\frac{x^i t^i}{i!}\right)dt\right|$.

In addition, $h^*(s)$ satisfies $\|h^*\| \le \sigma_1^* = e$, when the value of m is fixed to obtain $\widetilde{N}(x,t)$, $\widetilde{h}(x)$ does $\|\widetilde{h}\| \le \widetilde{\sigma}_1$, where $\widetilde{\sigma}_1$ is the smallest positive root of the scalar equation $t - \frac{1}{50}\|e^x(50-e^2)+1\| - \frac{1}{50}\widetilde{M}_1 t^2 = 0$.

In Table 1 we show different values for the tolerance *Tol* given for (37).

Now, we consider the iterative process (3) and we apply the proposed iterative process for different value of m for testing the results of Table 1.

Note that when there is poor proximity between the solutions of both integral Eq. (1) and (5), as in the case $m = 2$, the iterative process (3) converges quickly but to a distant solution that we want to approximate. However, as the proximity between the solutions of both integral Eq. (1) and (5) is improved, cases $m = 6$ and $m = 14$, we see that the approximation to the solution of equation (1) is fast and more accurate. So, by using iterative process (3), we can obtain the approximated solution to main problem (1) with the precision prefixed by performing a few iterations (see Table 2). We run the algorithms by using Matlab 2019 with 50 digits a stopping criteria 10^{-15} since we use values of $m = 2, 6, 14$, see in Table 1 the tolerance that we can reach.

Example 2. We consider the nonlinear integral equation of Hammerstein-type given by

$$h(x) = \frac{x}{2} + \frac{1}{3}\int_0^1 e^{xt}\mathcal{P}_3(h)(t)\,dt, \tag{38}$$

with $\mathcal{P}_3(h)(t) = \sum_{i=0}^{3}\frac{1}{(i+1)^4}h(t)^i$ and $x \in [0,1]$.

As the kernel $N(x,t) = e^{xt}$ is non-separable and operator $\mathcal{F}$ defined in (2) is such that $\mathcal{F}: \mathscr{C}([0,1]) \to \mathscr{C}([0,1])$ with

$$\mathcal{F}(h)(x) = h(x) - \frac{x}{2} - \frac{1}{3}\int_0^1 e^{xt}\mathcal{P}_3(h)(t)\,dt.$$

Table 3
Semilocal convergence radius.

$\|\widetilde{h}_0(x)\|$	β	δ	R	$R\eta$	r
0.5	1.046341	1.589275	1.263230	2.100656	14.999999
0.25	1.041148	1.328816	1.299999	1.798543	14.999998

As the kernel $N(x,t)=e^{xt}$ is non-separable, the use of iterative process (3) for solving (38) is not possible. For this, we approximate, by using Taylor's series, the kernel $N(x,t)=e^{xt}$. So,

$$e^{xt}=\widetilde{N}(x,t)+\mathcal{E}(\epsilon,x,t) \quad \text{with} \quad \widetilde{N}(x,t)=\sum_{i=0}^{m}\frac{x^i t^i}{i!} \quad \text{and} \quad \mathcal{E}(\epsilon,x,t)=\frac{e^{x\epsilon}}{(m+1)!}x^{m+1}t^{m+1},$$

where $\epsilon\in(\min\{0,t\},\max\{0,t\})$. Then, we consider the integral operator

$$\widetilde{\mathcal{F}}(h)(x)-\frac{x}{2}-\frac{1}{3}\int_0^1\widetilde{N}(x,t)\mathcal{P}_3(h)(t)dt, \quad x\in[0,1]. \tag{39}$$

Now, by denoting the solutions of (38) and (39) by $h^*(s)$ and $\widetilde{h}(s)$, respectively, we try to locate them in $\mathscr{C}([0,1])$, we obtain that

$$\|h^*(s)\|-\frac{1}{2}-\frac{M_2}{3}\sum_{i=0}^{3}\frac{\|h^*(s)\|^i}{(i+1)^4}\le 0 \qquad \text{and} \qquad \|\widetilde{h}(s)\|-\frac{1}{2}-\frac{\widetilde{M_2}}{3}\sum_{i=0}^{3}\frac{\|\widetilde{h}(s)\|^i}{(i+1)^4}\le 0,$$

where $M_2=\max_{x\in[0,1]}\left|\int_0^1 e^{xt}dt\right|=e-1=1.7182\ldots$ and $\widetilde{M_2}=\max_{x\in[0,1]}\left|\int_0^1\widetilde{N}(x,t)dt\right|$. In addition, $h^*(s)$ satisfies $\|h^*\|\le 1.1252\ldots=\sigma_2^*$, where σ_2^* is the smallest positive solution of scalar equation $t-\frac{1}{2}-\frac{M_2}{3}\sum_{i=0}^{3}\frac{t^i}{(i+1)^4}=0$, and $\widetilde{h}(s)$ does $\|\widetilde{h}\|\le\widetilde{\sigma}_2$, where $\widetilde{\sigma}_2$ is the smallest positive solution of scalar equation: $t-\frac{1}{2}-\frac{\widetilde{M_2}}{3}\sum_{i=0}^{3}\frac{t^i}{(i+1)^4}=0$ and once the value of m is fixed to obtain $\widetilde{N}(x,t)$ for obtaining a tolerance prefixed. For instance, if $m=3$ we have $\widetilde{\sigma}_2=1.1213$, then by using theoretical results obtained in Theorem 7 and (10) we obtain $Tol=8.7660\cdot10^{-3}$ and with $m=7$, one gets $Tol=2.9001\cdot10^{-6}$.

Finally, we consider $m=3$ and construct auxiliary functions defined in subsection 3.2 and by using Theorem 7 in order to get the radius of the semilocal convergence convergence ball, we have, for $\widetilde{h}_0(x)=\frac{1}{2}$, that the $B(\frac{1}{2},2.100656)$ contains all the iterates and the solution, being the uniqueness radius $r=14.999999$. For any suitable starting guess which norm let be 0.25 the corresponding semilocal convergence radius can be seen in Table 3.

Finally, we apply the iterative process (3) for obtaining the approximation to the solution of equation (38) by taking $m=3$ and using Matlab 2019 working with 50 digits and stopping criteria 10^{-4}, we get in 2 iterations the solution $\widetilde{h}_2(x)=0.01491\ldots x^3+0.05943\ldots x^2+0.6773\ldots x+0.3511\ldots$, with residual error $1.0422e-07$ so by applying (8) and (10) we have that the distance to the exact solution will be

$$\|h^*-\widetilde{h}_2\|\le 8.7660\cdot10^{-3}+1.0422\cdot10^{-7}+\le 9\cdot10^{-3}. \tag{40}$$

But with $m=6$ and stopping criteria 10^{-7}, after 3 iterations the approximation is $\widetilde{h}_3(x)=8.931\cdot10^{-6}x^7+7.138\cdot10^{-5}x^6+0.000499x^5+0.002989x^4+0.01491x^3+0.05944x^2+0.6774x+0.3511$, and it is obtained with the following bounds:

$$\|h^*-\widetilde{h}_3\|\le 2.9001\cdot10^{-6}\le 4.0625\cdot10^{-39}\le 3.0\cdot10^{-6}. \tag{41}$$

Remark

It is very important to point out that, in case we have a nonlinear integral equation of Hammerstein-type where the Nemystki operator is a continuous function which is not a polynomial, we can use a polynomial interpolation approximation of this operator and then we can apply the theoretical results obtained in this paper for solving the problem.

5. Conclusions

In this work, we have approximated the solution of a nonlinear integral equation of Hammerstein-type with polynomial Nemystkii operator and nonseparable kernel using a high-order iterative process. For this, we have developed two objectives. First, we have approximated the original integral equation by a nonlinear integral equation of Hammerstein-type with polynomial Nemystkii operator and separable kernel to subsequently apply the high-order iterative process to approximate the solution of this integral equation. Seeing that by making a good approximation of the original integral equation, we can get a solution of the original integral equation quickly and with a prefixed precision. Second, we have obtained a new semilocal convergence result for the iterative process (3) by adapting the hypotheses to be able to apply it to the integral nonlinear equation of Hammerstein-type considered. This has allowed us to generalize semi-local convergence results for the iterative process (3) (see [1,2] and [19]). So, we have obtained the semilocal convergence ball for the nonlinear problem corresponding to nonlinear integral equations Hammerstein-type, this fact allows us to assure the existence of solution for this problem, and also the location of the solution and, of course, the approximation to the solution with a given prefixed tolerance.

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