The Sum of the Reciprocals of Pandigital Numbers

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Abstract

We show that the sum of the reciprocals of pandigital numbers is divergent.

It is well known that the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

diverges. In 1914, A. J. Kempner [3] proved that if the denominators include only those numbers which do not contain the digit 9, then the series converges (of course, the same can be done with any other digit, and using a base of numeration different of ten). This is somewhat surprising because Kempner's series seems to have "most of the" summands of the harmonic series, and thus, at first sight, Kempner's series should be also divergent.

However, the demonstration of its convergence is simple: there are 8 one digit numbers not containing the digit 9, the smallest of which is 1; $8 \cdot 9$ two digit numbers not containing the digit 9, the smallest of which is 10; $8 \cdot 9^2$ three digit numbers not containing the digit 9, the smallest of which is 100; and so on. Therefore the sum of the reciprocals of the positive integers not containing the digit 9 is less than

$$8 \cdot 1 + \frac{8 \cdot 9}{10} + \frac{8 \cdot 9^2}{10^2} + \cdots = \frac{8}{1 - \frac{9}{10}} = 80.$$

A more precise estimation for the sum of Kempner's series can be found in [1].

In a given base of numeration, a pandigital number is an integer in which any digit used in the base appears at least once. At first sight and without further reflection, our patterns with small numbers in the usual base ten says that "very few" numbers are pandigital, and thus that the series of the reciprocals of pandigital numbers

$$\sum_{p \text{ pandigital}} \frac{1}{p} \tag{1}$$

should also be convergent.

But, again, our first intuition fails (it is worth recalling the "strong law of small numbers", see [2], that can be formulated as "there aren't enough small numbers to meet the many demands made of them", and that focuses in the

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problem of ensuring if a mathematical pattern observed with small positive integers will persist). Actually, the divergence of (1) can be proved as a consequence of the convergence of Kempner's series.

Let us take W_j (for $j = 0, 1, \ldots, 9$, if we use base ten) the set of positive integers without digit j. If n is not a pandigital number, then $n \in W_j$ for some j. Then,

$$\sum_{n \text{ not-pandigital}} \frac{1}{n} \leq \sum_{n \in W_0} \frac{1}{n} + \sum_{n \in W_1} \frac{1}{n} + \cdots + \sum_{n \in W_9} \frac{1}{n} < \infty$$

due to the convergence of Kempner's series, and thus (1) diverges.

We can also give an easy proof of the divergence of (1) that is independent of Kempner's result. In base ten, take $a = 1\,234\,567\,890$. Then, all the numbers

$$n = 10^{10} r + a,$$

with $r \in \mathbb{N} \cup \{0\}$, are pandigital. Consequently, and because $a/10^{10} \leq 1$,

$$\begin{aligned} \sum_{n \text{ pandigital}} \frac{1}{n} &\geq \sum_{r=0}^{\infty} \frac{1}{10^{10} r + a} \\ &= \frac{1}{10^{10}} \sum_{r=0}^{\infty} \frac{1}{r + a/10^{10}} \\ &\geq \frac{1}{10^{10}} \sum_{r=0}^{\infty} \frac{1}{r+1} = \infty. \end{aligned}$$

The same proof applies in any base b taking the corresponding a and using 10^{10} with its meaning b^b in base b.

References

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