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On global convergence for an efficient third-order iterative process[☆]

J.A. Ezquerro, M.A. Hernández-Verón, Á.A. Magreñán [*]

University of La Rioja, Department of Mathematics and Computation, Calle Madre de Dios, 53, 26006 Logroño, La Rioja, Spain

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ABSTRACT

We establish a global convergence result for an efficient third-order iterative process which is constructed from Chebyshev's method by approximating the second derivative of the operator involved by combinations of the operator. In particular, from the use of auxiliary points, we provide domains of restricted global convergence that allow obtaining balls of convergence and locate solutions. Finally, we use different numerical examples, including a Chandrashekar's integral equation problem, to illustrate the study.

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1. Introduction

When solving a nonlinear equation, we feel generally obliged to apply iterative processes to approximate a solution of the nonlinear equation because we cannot calculate the solution exactly. Thus, if we consider an equation of the form $F(x) = 0$, where F is a nonlinear operator, $F : \Omega \subseteq X \to Y$, defined on a non-empty open convex domain Ω of a Banach space X with values in a Banach space Y, we often apply a one-point iterative process of the form

$$x_{n+1} = \mathcal{P}(x_n), \quad n \geq 0, \quad \text{for given } x_0, \tag{1}$$

where $\mathcal{P} : X \to X$, to approximate a solution of $F(x) = 0$. For this, we need to see that the solution exists and the sequence (1) converges to this solution.

In the general approach we just mentioned, the nonlinear equation $F(x) = 0$ can represent a system of nonlinear equations, a differential equation, a boundary value problem, an integral equation, etc. If we pay a little attention to the equation $F(x) = 0$, we see that it can be represented in the form of a fixed point equation. For this, just write $F(x) = 0$ as $x = G(x)$ with $G(x) = x - F(x)$, so that if we approximate a fixed point of the equation $x = G(x)$, we then obtain a solution of $F(x) = 0$. A common procedure to approximate a fixed point of an equation is to apply a fixed point result like this [1,2]:

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* Corresponding author.
E-mail addresses: mahernan@unirioja.es (M.A. Hernández-Verón), angel-alberto.magrenan@unirioja.es, angel-alberto.magrenan@unirioja.es (Á.A. Magreñán).

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If the operator $H : S \to S$, where S is a convex and compact set of a Banach space X, is a contraction, then the operator H has a unique fixed point in S, that is approximated by the method of successive approximations, $u_{n+1} = H(u_n)$, $n \geq 0$, from any $u_0 \in S$.

This fixed point result has two remarkable characteristics: it proves the existence of the fixed point and provides an iterative process to approximate it. The iterative process has the advantage that it is globally convergent in the set S and the disadvantage of its slow convergence (linear convergence only).

If we now think of any iterative process of the form (1), we see that we cannot obtain a result with features similar to those given by the previous fixed point result, since we cannot obtain them simultaneously. Observe that we usually analyze the convergence of sequence (1) in two different ways [3–6]. The first consists in giving conditions on the starting point x_0 and on the operator F, so that it proves the existence of a solution x^* of the equation $F(x) = 0$, provides a ball of existence of solution of $F(x) = 0$, $\overline{B(x_0, \rho_1)}$, called existence ball, and the convergence of (1) starting at x_0. This is called semilocal convergence. The second assumes previously that a solution x^* of $F(x) = 0$ exists and, under certain conditions on the operator F, we obtain a ball $\overline{B(x^*, \rho_2)}$, called convergence ball, where the convergence of (1) is guaranteed by taking any point on the ball as the starting point x_0, so that we obtain global convergence for (1) in the ball $\overline{B(x^*, \rho_2)}$. This is called local convergence. We then see that none of these results has the features of the above fixed point result at the same time. In this work, we look for a convergence result for an iterative process of the form (1) that combines the thesis of local and semilocal convergence.

Our main aim in this work is to obtain a result of convergence for an efficient third-order iterative process of the form (1) such that the existence of a solution x^* of $F(x) = 0$ is proved and it provides a domain in which the sequence given by the iterative process converges globally, so that it converges starting at any point of the domain.

The iterative process considered,

$$\begin{cases} x_0 \text{ given in } \Omega, \\ y_n = x_n - [F'(x_n)]^{-1}F(x_n), \\ z_n = x_n + p\,(y_n - x_n), \quad p \in (0, 1], \\ x_{n+1} = y_n - \dfrac{1}{p^2}\,[F'(x_n)]^{-1}\left((p-1)F(x_n) + F(z_n)\right), \quad n \geq 0, \end{cases} \tag{2}$$

is constructed in [7] from Chebyshev's method and by using a slight modification of a technique developed in [8] to obtain iterative processes of the form (1). The idea used in [7] is to approximate the second Fréchet derivative of the operator F in Chebyshev's method by means of only combinations of F in different points, so that F'' is not used and F' is only evaluated in x_n. With this modification of Chebyshev's method, the number of evaluations of functions and the computational cost are considerably reduced, the efficiency is improved and the order of convergence is kept, see [7]. Notice that, for $p = 1$, the iterative process (2) corresponds to the two-step frozen Newton method [9].

To achieve our aim, we use a technique based on auxiliary points [2,10], and, from certain conditions on the operator F and on an auxiliary point, we prove the existence of a solution x^* of $F(x) = 0$ and guarantee the convergence of (2) to x^* starting at any point of a ball centered at the auxiliary point. Moreover, we obtain results of semilocal and local convergence for iterative process (2) from two particular choices of the auxiliary point.

Throughout the paper, we denote $\overline{B(x, \varrho)} = \{y \in X : \|y - x\| \leq \varrho\}$ and $B(x, \varrho) = \{y \in X : \|y - x\| < \varrho\}$ and the set of bounded linear operators from Y to X by $\mathcal{L}(Y, X)$.

2. Convergence conditions

Now, we introduce the conditions under which we study the global convergence of the iterative process (2), which are:

(C1) For some $\widetilde{x} \in \Omega$, there exists $\widetilde{\Gamma} = [F'(\widetilde{x})]^{-1} \in \mathcal{L}(Y, X)$ with $\|\widetilde{\Gamma}\| \leq \beta$ and $\|\widetilde{\Gamma}F(\widetilde{x})\| \leq \eta$.
(C2) There exists a constant $K \geq 0$ such that $\|F'(x) - F'(y)\| \leq K\|x - y\|$, for all $x, y \in \Omega$.

If we observe the condition (C2), it is easy to follow that

$$\|F'(x) - F'(\widetilde{x})\| \leq \widetilde{K}\|x - \widetilde{x}\|, \text{ for all } x \in \Omega,$$

with $\widetilde{K} \leq K$, once $\widetilde{x} \in \Omega$ is fixed.

Next, we introduce some approximations that are used later to prove the convergence of the iterative process (2).

Lemma 1. *Let $F : \Omega \subseteq X \to Y$ be a once continuously differentiable operator defined on a non-empty open convex domain Ω of a Banach space X with values in a Banach space Y. Then,*

(a) $F(x_0) = F(\widetilde{x}) + F'(\widetilde{x})(x_0 - \widetilde{x}) + \displaystyle\int_{\widetilde{x}}^{x_0} \left(F'(x) - F'(\widetilde{x})\right)(x_0 - x)\,dx$, *with* $x_0, x \in \Omega$.

(b) For $x_n, z_n \in \Omega$, we have (see [7])

$$F(z_n) = (1-p)F(x_n) + \int_{x_n}^{z_n} \left(F'(x) - F'(x_n)\right)(z_n - x)\,dx,$$

(c) For $x_n, x_{n+1} \in \Omega$, we have (see [7])

$$
\begin{aligned}
F(x_{n+1}) &= F(x_n) + F'(x_n)(x_{n+1} - x_n) + \int_{x_n}^{x_{n+1}} \left(F'(x) - F'(x_n)\right)\, dx \\
&= \frac{1}{p}\int_0^1 \left[F'(x_n) - F'(x_n + pt(y_n - x_n))\right](y_n - x_n)\, dt \\
&\quad + \int_0^1 \left[F'(x_n + t(x_{n+1} - x_n)) - F'(x_n)\right](x_{n+1} - x_n)\, dt.
\end{aligned}
$$

(d) For $x_n \in \Omega$, we have

$$
F(x_n) + F'(x_n)(\widetilde{x} - x_n) = F(\widetilde{x}) - \int_{x_n}^{\widetilde{x}} \left(F'(x) - F'(x_n)\right)(\widetilde{x} - x)\, dx.
$$

In addition, from $\|I - \widetilde{\Gamma}F'(x)\| \leq \|\widetilde{\Gamma}\left(F'(\widetilde{x}) - F'(x)\right)\|$ and the Banach lemma on invertible operators, we have

$$
\|\Gamma\| = \|[F'(x)]^{-1}\| \leq \frac{\beta}{1 - \widetilde{K}\beta R} = d \qquad \text{and} \qquad \|\Gamma F'(\widetilde{x})\| \leq \frac{1}{1 - \widetilde{K}\beta R},
$$

provided that there exists $R > 0$ such that $x \in B(\widetilde{x}, R) \subset \Omega$ and

$$
\widetilde{K}\beta R < 1. \tag{3}
$$

3. Analysis of global convergence

In this section, we analyze the global convergence of the iterative process (2). For this, we first construct a system of recurrence relations, where a scalar sequence is involved, from the real parameters given in conditions (C1) and (C2), and the global convergence of the iterative process is then guaranteed.

3.1. Recurrence relations

From (3), it follows that $\|\Gamma_0\| = \|[F'(x_0)]^{-1}\| \leq d$ and $\|\Gamma_0 F'(\widetilde{x})\| \leq \frac{1}{1 - \widetilde{K}\beta R}$, provided that $x_0 \in B(\widetilde{x}, R)$. Next, from item (a) of Lemma 1, we have

$$
\|y_0 - x_0\| \leq \|\Gamma_0 F'(\widetilde{x})\|\|\widetilde{\Gamma}F(x_0)\| \leq \frac{2\eta + 2R + \widetilde{K}\beta R^2}{2(1 - \widetilde{K}\beta R)} = e.
$$

In addition, from item (d) of Lemma 1, it follows

$$
\begin{aligned}
\|y_0 - \widetilde{x}\| &= \left\|-\Gamma_0\left(F(x_0) + F'(x_0)(\widetilde{x} - x_0)\right)\right\| \\
&\leq \|\Gamma_0 F'(\widetilde{x})\|\|\widetilde{\Gamma}F(\widetilde{x})\| + \frac{K\beta}{2(1 - \widetilde{K}\beta R)}\|x_0 - \widetilde{x}\|^2 \\
&\leq \frac{2\eta + K\beta R^2}{2(1 - \widetilde{K}\beta R)}.
\end{aligned}
$$

so that $y_0 \in B(\widetilde{x}, R)$, provided that

$$
\frac{2\eta + K\beta R^2}{2(1 - \widetilde{K}\beta R)} \leq R. \tag{4}
$$

Moreover, as

$$
\begin{aligned}
&\|z_0 - x_0\| = p\|y_0 - x_0\| \leq p\, e, \\
&\|z_0 - \widetilde{x}\| \leq (1 - p)\|x_0 - \widetilde{x}\| + p\|y_0 - \widetilde{x}\| \leq R,
\end{aligned}
$$

it follows that $z_0 \in B(\widetilde{x}, R)$, provided that (4) is satisfied. Furthermore,

$$
\begin{aligned}
\|x_1 - x_0\| &= \left\|-\frac{1}{p^2}\Gamma_0\left((p^2 + p - 1)F(x_0) + F(z_0)\right)\right\| \leq (1 + a_0/2)\,\|y_0 - x_0\|, \\
\|x_1 - \widetilde{x}\| &= \left\|y_0 - \widetilde{x} - \frac{1}{p^2}\Gamma_0\left((p - 1)F(x_0) + F(z_0)\right)\right\| \leq \frac{2\eta + K\beta R^2}{2(1 - \widetilde{K}\beta R)} + \frac{e}{2}a_0.
\end{aligned}
$$

where $a_0 = Kde$, and $x_1 \in B(\widetilde{x}, R)$, provided that

$$
\frac{2\eta + K\beta R^2}{2(1 - \widetilde{K}\beta R)} + \frac{e}{2}a_0 \leq R. \tag{5}
$$

Observe that (4) holds if (5) is satisfied.

Next, from $\|y_1 - x_1\| \leq \|\Gamma_1\| \|F(x_1)\|$ and item (c) of Lemma 1, we have

$$\|y_1 - x_1\| \leq f(a_0)\|y_0 - x_0\|,$$

where

$$f(t) = \frac{t}{8}(t^2 + 4t + 8). \tag{6}$$

As f is increasing for $t > 0$, it follows that

$$\begin{aligned}
Kd\|y_1 - x_1\| &\leq a_1,\\
\|y_1 - \widetilde{x}\| &\leq \frac{2\eta + K\beta R^2}{2(1 - \widetilde{K}\beta R)},\\
\|z_1 - x_1\| &\leq pf(a_0)\|y_0 - x_0\|,\\
\|z_1 - \widetilde{x}\| &\leq R,\\
\|x_2 - x_1\| &\leq (1 + a_1/2)\,\|y_1 - x_1\|,\\
\|x_2 - \widetilde{x}\| &\leq \frac{2\eta + K\beta R^2}{2(1 - \widetilde{K}\beta R)} + \frac{e}{2}a_0 f(a_0)^2,
\end{aligned}$$

and $y_1, z_1, x_2 \in B(\widetilde{x}, R)$, provided that

$$\frac{2\eta + K\beta R^2}{2(1 - \widetilde{K}\beta R)} + \frac{e}{2}a_0 f(a_0)^2 \leq R.$$

Notice that the last condition holds if (5) and $f(a_0) < 1$ are satisfied.

Now, we consider $a_1 = a_0 f(a_0)$ and define

$$a_n = a_{n-1} f(a_{n-1}), \quad n \geq 1.$$

Notice that $f(a_0) < 1$ if $a_0 < 0.7064\ldots$ and, as a consequence, the sequence $\{a_n\}$ is decreasing.

The following aim is to guarantee that sequence (2) is well-defined. For this, we introduce the following lemma.

Lemma 2. *Let f be the real function given in (6). If there exists $R > 0$ such that condition (5) and $a_0 < 0.7064\ldots$ are satisfied, then the following items are true for all $n \geq 1$:*

$$\begin{aligned}
\|y_n - x_n\| &\leq f(a_{n-1})\|y_{n-1} - x_{n-1}\|,\\
Kd\|y_n - x_n\| &\leq a_n,\\
\|y_n - \widetilde{x}\| &\leq \frac{2\eta + K\beta R^2}{2(1 - \widetilde{K}\beta R)} \leq R,\\
\|z_n - x_n\| &\leq pf(a_{n-1})\|y_{n-1} - x_{n-1}\|,\\
\|x_{n+1} - x_n\| &\leq (1 + a_n/2)\|y_n - x_n\|,\\
\|x_{n+1} - \widetilde{x}\| &\leq \frac{2\eta + K\beta R^2}{2(1 - \widetilde{K}\beta R)} + \frac{e}{2}a_0\left(\prod_{i=0}^{n-1} f(a_i)^2\right) \leq R,
\end{aligned} \tag{7}$$

To prove the last lemma, we simply apply mathematical induction and use the same arguments that we have previously used to prove the first step of induction.

From the use of auxiliary points, we notice that the global convergence of the iterative process (2) follows steps similar to those used to obtain results of semilocal convergence [7]. Since condition (5) is usually satisfied for different values of R, we then choose the most favorable value. Note that the greater value of R gives us the best global convergence domain and the smallest gives us the best location of the solution.

3.2. Global convergence

The restricted global convergence of the iterative process (2) follows now from the next result.

Theorem 3. *Let F be a once continuously differentiable operator defined on a non-empty open convex domain Ω of a Banach space X with values in a Banach space Y. Assume that the conditions (C1) and (C2) hold and the existence of $R > 0$ satisfying (5) and such that $B(\widetilde{x}, R) \subset \Omega$. If conditions (3) and $a_0 < 0.7064\ldots$ are satisfied, then the iterative process (2) is well-defined and converges to a solution x^* of the equation $F(x) = 0$ in the domain $\overline{B(\widetilde{x}, R)}$ from every point x_0 belonging to $B(\widetilde{x}, R)$.*

Proof. Observe that the sequence given by the iterative process (2) is of Cauchy without taking into account (7) and

$$\begin{aligned}\|x_{n+m}-x_n\| &\leq \sum_{i=n}^{n+m-1}(1+a_i/2)\|y_i-x_i\|\\ &\leq (1+a_0/2)\sum_{i=n}^{n+m-1} f(a_{i-1})\|y_{i-1}-x_{i-1}\|\\ &\leq (1+a_0/2)f(a_0)^n\|y_0-x_0\|\sum_{i=0}^{m-1} f(a_0)^i\\ &\leq (1+a_0/2)f(a_0)^n\|y_0-x_0\|\frac{1-f(a_0)^m}{1-f(a_0)}.\end{aligned}$$

Therefore, the sequence given by the iterative process (2) is convergent.

In addition, by the continuity of F in $\overline{B(\widetilde{x},R)}$, we obtain $F(x^*)=0$, since $\|F(x_n)\|\to 0$ as a consequence of $\|\Gamma_n F(x_n)\|\to 0$, $\|F(x_n)\|\leq \|F'(x_n)\|\|\Gamma_n F(x_n)\|$ and $\{\|F'(x_n)\|\}$ is bounded, due to the fact that $\|F'(x_n)\|\leq \|F'(\widetilde{x})\|+\widetilde{K}R$, for all $n\geq 0$. ■

Note that this way of analyzing the global convergence of the iterative process (2) allows locating the solution x^* in the ball $\overline{B(\widetilde{x},R)}$ and defining a ball of global convergence $\overline{B(\widetilde{x},R)}$.

4. Uniqueness of solution

In the following, we establish the uniqueness of solution of the equation $F(x)=0$.

Theorem 4. *If the conditions (C1) and (C2) are satisfied, then the solution x^* of the equation $F(x)=0$ is unique in $B(\widetilde{x},\varepsilon)\cap\Omega$, where ε is a positive root of*

$$\widetilde{K}\beta(R+\varepsilon)-2=0. \tag{8}$$

Proof. Assume that v^* is another solution of equation $F(x)=0$ in $B(\widetilde{x},\varepsilon)\cap\Omega$. Then,

$$0=\widetilde{\Gamma}\left(F(v^*)-F(x^*)\right)=\left(\int_0^1\widetilde{\Gamma}F'(x^*+t(v^*-x^*))\,dt\right)(v^*-x^*)=Q(v^*-x^*),$$

so that $v^*=x^*$ if the operator $Q=\int_0^1\Gamma_0 F'(x^*+t(v^*-x^*))\,dt$ is invertible. For this, we see that

$$\begin{aligned}\|I-Q\| &\leq \|\widetilde{\Gamma}\|\left\|\int_0^1\left(F'(\widetilde{x})-F'(x^*+t(v^*-x^*))\right)\,dt\right\|\\ &\leq \beta\int_0^1\widetilde{K}\|\widetilde{x}-x^*-t(v^*-x^*)\|\,dt\\ &< \frac{1}{2}\widetilde{K}\beta(R+\varepsilon)\\ &=1\end{aligned}$$

and, by the Banach lemma on invertible operators, we conclude that Q is invertible. ■

Example 5. In this example we present an application of the previous analysis to the Chandrasekhar equation:

$$x(s)=1+\frac{\varpi_0}{2}\,s\,x(s)\int_0^1\frac{x(t)}{s+t}\,dt,\quad s\in[0,1], \tag{9}$$

which arises in the theory of radiative transfer [11]; where ϖ_0 is the albedo for single scattering and $x(s)$ is the unknown function which is sought in $C[0,1]$. The physical background of this equation is fairly elaborate. It was developed by Chandrasekhar [11] to solve the problem of determination of the angular distribution of the radiant flux emerging from a plane radiation field. This radiation field must be isotropic at a point, that is the distribution in independent of direction at that point. Explicit definitions of these terms may be found in the literature [11]. It is considered to be the prototype of the equation,

$$x(s)=1+s\,x(s)\int_0^1\frac{\varphi(s)}{s+t}x(t)\,dt,\quad s\in[0,1],$$

for more general laws of scattering, where $\varphi(s)$ is an even polynomial in s with

$$\int_0^1 \varphi(s)\,ds \le \frac{1}{2}. \tag{10}$$

Integral equations of the above form also arise in the other studies [12,13].

We consider (9) for $\varpi_0 = \frac{1}{4}$ which satisfies (10). To solve this equation is equivalent to solve $F(x) = 0$, where $F : C[0,1] \to C[0,1]$ and

$$[F(x)](s) = x(s) - 1 - \frac{s}{8}\, x(s) \int_0^1 \frac{x(t)}{s+t}\,dt, \quad s \in [0,1], \tag{11}$$

therefore $\Omega = C[0,1]$.

First of all, from Eq. (9) we obtain that $x(0) = 1$ and as a consequence we can select as initial guess $\widetilde{x}(s) = 1$, in the domain of s. Moreover, we obtain that

$$[F(x_0)](s) = -\frac{s}{8}\int_0^1 \frac{dt}{s+t} = -\frac{s}{8}\ln\left[\frac{1+2}{s}\right], \quad s \in [0,1].$$

Consequently,

$$\|[F(x_0)](s)\| = \frac{\ln 2}{8}.$$

Furthermore, we get

$$[F'(x)y](s) = y(s) - \frac{s}{8}x(s)\int_0^1 \frac{y(t)}{s+t}dt - \frac{s}{8}y(s)\int_0^1 \frac{x(t)}{s+t}dt, \quad s \in [0,1]$$

then

$$\|I - F'(x_0)\| \le \frac{\ln 2}{4},$$

and, from the Banach lemma, we obtain

$$\beta = \frac{1}{1 - \frac{\ln 2}{4}} \approx 1.2096 \qquad \text{and} \qquad \eta = \frac{\frac{\ln 2}{8}}{1 - \frac{\ln 2}{4}} \approx 0.1048.$$

Moreover, it is easy to check that $K = \widetilde{K} = \dfrac{\ln 2}{4} \approx 0.173287$. Now, notice that the condition

$$\frac{2\eta + K\beta R^2}{2(1 - \widetilde{K}\beta R)} + \frac{e}{2}a_0 \le R$$

it is verified for $R \in [0.114254, 1.45189]$. For all these values $K\beta R < 1$ is verified. But the condition $a_0 =< 0.7064\ldots$ it is verified for $R \in [0.114254, 1.38785]$. So, both conditions are satisfied and as a consequence the iterative process (2) is well-defined and converges to a solution x^* of the equation $F(x) = 0$ in the domain $\overline{B(1,R)}$ for $R \in [0.114254, 1.38785]$. Therefore, the best ball of existence for the location of the solution is $\overline{B(1, 0.114254)}$ and the best ball for the global convergence for the iterative process (2) is $\overline{B(1, 1.38785)}$.

On the other hand, from Theorem 4, we obtain as the ball of uniqueness of solution $B(1, 8.15371)$

To obtain a numerical solution of equation, we first discretize the problem and approach the integral by a Gauss–Legendre numerical quadrature with eight nodes

$$\int_0^1 f(t)\,dt \approx \sum_{j=1}^{8} w_j f(t_j),$$

where the nodes and weights are given in Table 1.

If we denote $x_i = x(t_i)$, $i = 1, 2, \ldots, 8$, equation is transformed into the following nonlinear system:

$$x_i = 1 + \frac{x_i}{8}\sum_{j=1}^{8} a_{ij}x_j, \quad i = 1, 2, \ldots, 8, \tag{12}$$

where, $a_{ij} = \dfrac{t_i w_j}{t_i + t_j}$. Take into account the previous continuous study, we can now use the initial choice

$$x_0 = (1, 1, 1, 1, 1, 1, 1, 1)^T.$$

Table 1
Nodes and weights of Gauss–Legendre.

j	t_j	w_j
1	0.01985507175123188...	0.050614268145188129...
2	0.10166676129318663...	0.111190517226687235...
3	0.23723379504183550...	0.156853322938943643...
4	0.40828267875217509...	0.181341891689180991...
5	0.59171732124782490...	0.181341891689180991...
6	0.76276620495816449...	0.156853322938943643...
7	0.89833323870681336...	0.111190517226687235...
8	0.98014492824876811...	0.050614268145188129...

Table 2
Solution of system of Eqs. (12).

j	x_j^*
1	1.0101781...
2	1.0329569...
3	1.0547234...
4	1.0719797...
5	1.0844979...
6	1.0930361...
7	1.0984086...
8	1.1012071...

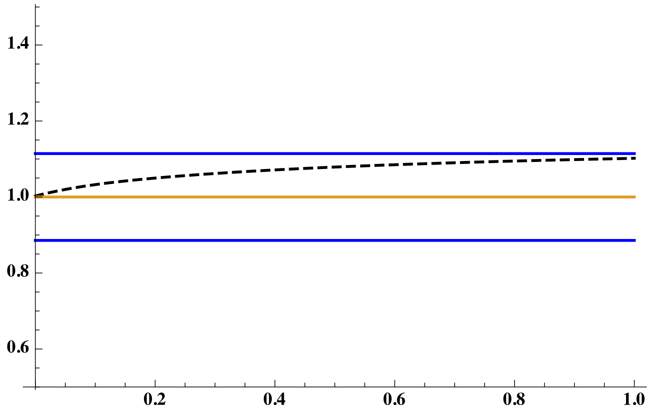

Fig. 1. Interpolatory polynomial x_{int} (dotted) and the ball of existence (blue). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

If we take $p = 1$, the two-step frozen Newton's method, after three iterates and a tolerance of 10^{-30}, iterative process (2) converges to the numerical solution x^* of system of Eqs. (12) given in Table 2.

Once the solution x^* has been obtained, we construct a function x_{int} (see Figs. 1–2) by an interpolating procedure to use the values obtained from the numerical solution of the arithmetic problem $\{(t_j, x_j^*)\}_{j=1}^{8}$. Observe that the interpolated approximation x_{int} lies within the ball of existence and the ball of global convergence of solutions obtained above in the continuous study.

5. Particular cases

As we have indicated in Section 1, we can obtain results of local and semilocal convergence for iterative process (2) from two particular choices of the auxiliary point $\widetilde{x}$.

First, if $\widetilde{x} = x^*$, we can establish the next local convergence theorem for iterative process (2).

Theorem 6. *Let F be a once continuously differentiable operator defined on a non-empty open convex domain Ω of a Banach space X with values in a Banach space Y. Assume that the conditions (C1) and (C2) hold for $\widetilde{x} = x^*$ and the existence of $R > 0$ satisfying (5) and such that $B(x^*, R) \subset \Omega$. If conditions (3) and $a_0 < 0.7064\ldots$ are satisfied, then the iterative process (2) is well-defined and converges to a solution x^* of the equation $F(x) = 0$ in the domain $\overline{B(x^*, R)}$ from every point x_0 belonging to $B(x^*, R)$.*

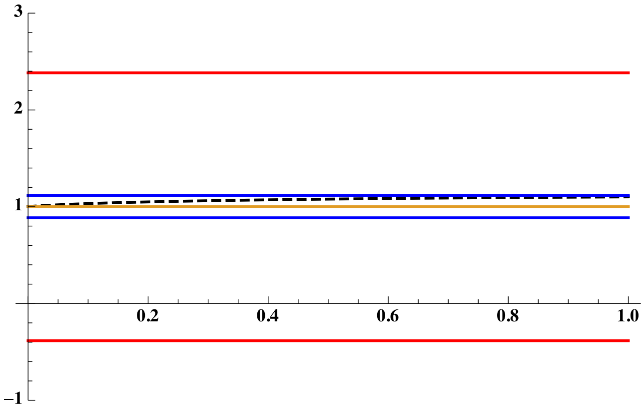

Fig. 2. Interpolatory polynomial x_{int} (dotted) and the ball of existence (blue) and the ball of global convergence (red). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Example 7. We consider the simple system of equations given by

$$\begin{cases} \dfrac{\ln(3-x)^2}{8} + y - 3 = 0 \\ xy - 2y = 0, \end{cases}$$

it is easy to check that $(2, 3)$ is a solution of this system of equations.

So, let $F(x, y) = 0$ be given by $F : \Omega = (2-\delta, 2+\delta) \times (3-\delta, 3+\delta) \to \mathbb{R}^2$, with $\delta \in (0, 1)$, and

$$F(x, y) = \left(\frac{\ln(3-x)^2}{8} + y - 3, xy - 2y \right). \tag{13}$$

Then, we have

$$F'(x, y) = \begin{pmatrix} \dfrac{-1}{4(x-3)} & 1 \\ y & x-2 \end{pmatrix}$$

We take the sup-norm in $\mathbb{R}^2$ and the norm $\|A\| = \max\{|a_{11}| + |a_{12}|, |a_{21}| + |a_{22}|\}$ for a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Then, for $\delta = 0.5$, we obtain

$$\|F'(x, y) - F'(u, v)\| \leq \|(x, y) - (u, v)\|$$

and then $K = 1$.

We now choose $\widetilde{x} = x^* = (2, 3)$, then $\widetilde{K} = 1$. Moreover, we obtain $\beta = 13/12$ and $\eta = 0$. So, the condition

$$\frac{2\eta + K\beta R^2}{2(1 - \widetilde{K}\beta R)} + \frac{e}{2} a_0 \leq R$$

is verified for $R \in [0, 0.27738]$. Therefore, $B((2, 3), (0.27738)) \subset \Omega$, $\widetilde{K}\beta R = 0.300495\ldots < 1$ and $a_0 = 0.70639\ldots < 0.7064\ldots$. As a consequence, the iterative process (2) is well-defined and converges to the solution $x^* = (2, 3)$ of the equation $F(x) = 0$ in the domain $\overline{B((2, 3), 0.27738)}$ from every point x_0 belonging to $B((2, 3), 0.27738)$.

We can see in Fig. 3 that our result gives a local convergence ball which is contained in the immediate basin of attraction of solution $(2, 3)$, painted in dark cyan after 100 iterations and a tolerance of 10^{-3}, the black color correspond to the points of non-convergence to that root. The circle in white is the local convergence ball.

Second, if $\widetilde{x} = x_0$, we can establish a semilocal convergence result for iterative process (2), whose proof is followed similarly to that of Theorem 3 without more than taking into account that items

$$\begin{aligned} &\|y_n - x_n\| \leq f(b_{n-1})\|y_{n-1} - x_{n-1}\|, \\ &Kd\|y_n - x_n\| \leq b_n, \\ &\|y_n - x_0\| \leq \left(f(b_0)^n + (1 + b_0/2)\frac{1 - f(b_0)^{n+1}}{1 - f(b_0)} \right)\eta \leq R, \end{aligned}$$

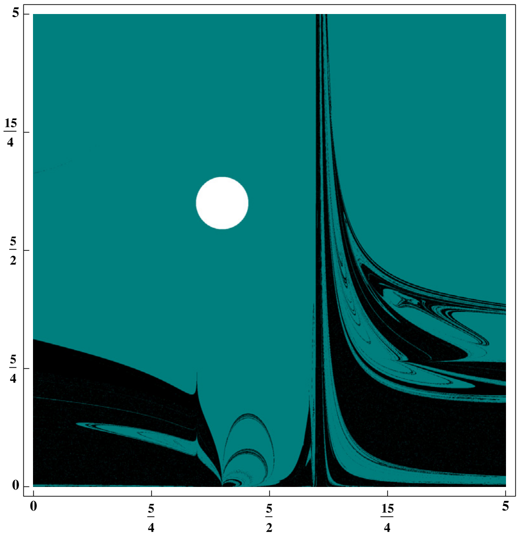

Fig. 3. Basins of attraction associated to the method (2), for $p = 1$, applied to function (13) and ball of local convergence.

$$\begin{aligned}
\|z_n - x_n\| &\leq pf(b_{n-1})\|y_{n-1} - x_{n-1}\|,\\
\|x_{n+1} - x_n\| &\leq (1 + b_n/2)\|y_n - x_n\|,\\
\|x_{n+1} - x_0\| &\leq (1 + b_0/2)\frac{1 - f(b_0)^{n+1}}{1 - f(b_0)}\,\eta \leq R,
\end{aligned}$$

are true, for all $n \geq 1$, if $\frac{(1+b_0/2)\eta}{1-f(b_0)} \leq R$ and $b_n = b_{n-1}f(b_{n-1})$ with $b_0 = Kd\eta < 0.7064\ldots$ So, we have the following theorem.

Theorem 8. *Let F be a once continuously differentiable operator defined on a non-empty open convex domain Ω of a Banach space X with values in a Banach space Y. Assume that the conditions (C1) and (C2) hold for $\widetilde{x} = x_0$. If there exists $R > 0$ such that*

$$\frac{(1 + b_0/2)\eta}{1 - f(b_0)} \leq R,$$

where $b_0 = Kd\eta$, and satisfies

$$\widetilde{K}\beta R < 1, \qquad b_0 < 0.7064\ldots$$

and $B(x_0, R) \subset \Omega$, then the iterative process (2) is well-defined and converges to a solution x^* of $F(x) = 0$ in $\overline{B(x_0, R)}$ starting at x_0.

Example 9. Now we consider the following Planck's radiation law problem found in [14]:

$$\varphi(\lambda) = \frac{8\pi cP\lambda^{-5}}{e^{\frac{cP}{\lambda BT}} - 1} \tag{14}$$

which calculates the energy density within an isothermal blackbody, where

- λ is the wavelength of the radiation,
- T is the absolute temperature of the blackbody,
- B is Boltzmann's constant,
- P is the Planck's constant,
- c is the speed of light.

Our main purpose it to determine wavelength λ which corresponds to maximum energy density $\varphi(\lambda)$. From (14), we get

$$\varphi'(\lambda) = \left(\frac{8\pi cP\lambda^{-6}}{e^{\frac{cP}{\lambda BT}} - 1}\right)\left(\frac{(\frac{cP}{\lambda BT})e^{\frac{cP}{\lambda BT}}}{e^{\frac{cP}{\lambda kT}} - 1} - 5\right) \tag{15}$$

Table 3
Tolerance of iterations for method (2) with different values of p.

Tolerance	$p = 0.25$	$p = 0.5$	$p = 0.75$	1
$\|x_1 - x_0\|$	1.5288	1.5303	1.5321	1.5344
$\|x_2 - x_1\|$	0.0059	0.0045	0.0026	0.0004
$\|x_3 - x_2\|$	1.107×10^{-9}	3.346×10^{-10}	4.176×10^{-11}	5.389×10^{-14}
$\|x_4 - x_3\|$	7.033×10^{-30}	1.373×10^{-31}	1.573×10^{-34}	1.022×10^{-43}
$\|x_5 - x_4\|$	1.799×10^{-90}	9.504×10^{-96}	8.412×10^{-105}	6.986×10^{-133}

The maxima for φ occurs, using $x = \frac{cP}{\lambda BT}$, when

$$1 - \frac{x}{5} = e^{-x} \tag{16}$$

So, now defining

$$F(x) = e^{-x} - 1 + \frac{x}{5}, \tag{17}$$

then

$$\lambda = \frac{cP}{x^* BT}. \tag{18}$$

being x^* a solution of Eq. (17).

This function F is continuous and from $F(4) = -0.181684\ldots$, $F(12) = 1.400006144\ldots$ and for the Rolle Theorem, F has at least one root in that interval $(4, 12)$. Now, we consider $\Omega = (4, 12)$ and $x_0 = 6.5$ as initial guess. So, we obtain $\beta = 5.03787$ and $\eta = 1.51894$. Moreover, it is easy to check that $K = \widetilde{K} = 0.0183156$. Then,

$$\frac{(1 + b_0/2)\eta}{1 - f(b_0)} \le R,$$

for all $R \in [2.03199, 8.18845]$.

If we consider $R = 2.03199$, then

$$B(6.5, 2.03199) = (4.46801, 8.53199) \subseteq \Omega,$$

$\widetilde{K}\beta R = 0.187495 < 1$ and $b_0 = 0.172497 < 0.7064$. So, the iterative process (2) is well-defined and converges to a solution x^* of $F(x) = 0$ in $\overline{B(6.5, 2.03199)}$ starting at $x_0 = 6.5$. Moreover, the solution x^* of the equation $F(x) = 0$ is unique in $B(6.5, \varepsilon) \cap \Omega$, where $\varepsilon = 19.6431$.

As it can be seen in Table 3, the convergence seems higher as the values of p are greater, and the best value is obtained when $p = 1$, two-step Newton's method. Moreover, we obtain the numerical solution $x^* = 4.965114231\ldots$, so we get $\lambda = \dfrac{cP}{4.965114231\ldots BT}$.

6. Conclusions

In this work, we have considered the family of iterative methods (2) which depends on a parameter $p \in (0, 1]$. When $p = 1$ we obtain the frozen two-step Newton's method. The family (2), which has cubic convergence, if more efficient than Newton's and Chebyshev's methods. For this family, we have obtained a global convergence result in a domain, which corresponds to a ball in our case, and allows us to locate solutions as well as find, in an easier way, favorable starting points to obtain the convergence of the iterative method to a solution of the problem considered. Moreover, we have obtained a uniqueness result which allows us to distinguish different solution of the problem considered. On the other hand, these results allow to obtain, in particular cases, new local and semilocal convergence results for the iterative family studied. In our numerical development, we have applied the results to a Chandrasekhar integral equation problem, locating the solution and obtaining a global convergence domain for the method in a way that, once we have discretized the equation, we can find a numerical solution that approximates the solution of that integral equation. Finally, in order to prove the theoretical results, we have applied the local and semilocal findings to methods of the family (2) to different examples.

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