ELSEVIER

Journal of Pure and Applied Algebra 154 (2000) 143–158

JOURNAL OF PURE AND APPLIED ALGEBRA

www.elsevier.com/locate/jpaa

A tensor-hom adjunction in a topos related to vector topologies and bornologies ☆

Luis Español *, Laureano Lambán

Departamento de Matemáticas y Computación, Universidad de La Rioja, Logroño, Spain

Communicated by I. Moerdijk

Abstract

In this paper, $\mathbf{N} = \mathbb{N} \cup \{\infty\}$ is the one-point compactification of the discrete space of natural numbers, $\mathbf{M}$ is the monoid of continuous maps $f : \mathbf{N} \to \mathbf{N}$ such that $f(\infty) = \infty$, and $\mathscr{M}$ is the topos of $\mathbf{M}$-sets. We define two sheaf subtoposes $\mathscr{C}$ and $\mathscr{B}$ of $\mathscr{M}$ and construct a tensor-hom adjunction between certain categories of modules in $\mathscr{C}$ and $\mathscr{B}$. Finally, we prove that this construction induces an adjunction between adequate categories of topological and bornological real vector spaces.

MSC: 18B25; 18F20; 46A17; 54A20

Introduction

A basic fact in functional analysis is that the set of all continuous linear maps and the set of all bounded linear maps between two normed spaces are the same. In this fact, we can see an equivalence between a topological vector structure and a bornological vector structure in normed spaces. There are more general constructions associating a bornological vector space structure to a topological vector space and a topological vector space structure to a bornological vector space. Our goal is to obtain in a categorical way an adjunction (partly classical, see "internal duality" in [3])

$$^{\mathrm{c}}(-) \dashv {}^{\mathrm{b}}(-) : SVS_{\mathrm{sep}} \to BVS_{\mathrm{sep}}$$

between the category SVS_{sep} of separated subsequential vector spaces [5] and the category BVS_{sep} of separated bornological vector spaces. This adjunction (not a duality

☆ Partially supported by Universidad de La Rioja, projects 96PYB14LEG and API-98/B22.

* Corresponding author.

E-mail addresses: luis.espanol@dmc.unirioja.es (L. Español), laureano.lamban@dmc.unirioja.es (L. Lambán).

0022-4049/00/$ - see front matter
PII: S0022-4049(99)00188-7

but covariant functors) is given as a consequence of a tensor-hom adjunction between adequate categories of modules in the topos $\mathscr{M}$ of $\mathbf{M}$-sets, where $\mathbf{M}$ is the monoid of continuous maps $f:\mathbf{N}\to\mathbf{N}$ such that $f(\infty)=\infty$, with $\mathbf{N}=\mathbb{N}\cup\{\infty\}$ the one-point compactification of the discrete space of natural numbers.

This paper gives a partial answer to a question proposed by Lawvere (unpublished part [9] of a talk given together with [10]) who asked for a strong relation, induced by the ring of convergent sequences of real numbers, between two categories of modules in toposes defined by monoids of maps bigger than those we use. The present work subsumes the final part of Lambán [8] in such a way that this reference is not a prerequisite.

We use the topos $\mathscr{M}$ as a universe of presheaves, but our main interest concerns the relationship between certain categories of modules in two sheaf subtoposes, $\mathscr{C}$ and $\mathscr{B}$, of $\mathscr{M}$. Our topos $\mathscr{B}$, which will be called the *bounded topos*, is an adaptation of Lawvere's *bornological topos* [9] and our topos $\mathscr{C}$, the *continuous topos*, is adapted from Johnstone's *topological topos* [5]. Though Johnstone uses a bigger site, his topos is the topos of sheaves for a (Grothendieck) topology on the monoid of all continuous maps $f:\mathbf{N}\to\mathbf{N}$; on the other hand, the bornological topos can be presented by means of a topology on the monoid of all maps $f:\mathbf{N}\to\mathbf{N}$. We impose $f(\infty)=\infty$ because to consider only maps fixing the infinity point allows us to have (i) a unique topos $\mathscr{M}$ in which to construct both toposes as subtoposes of sheaves, and (ii) an inclusion functor $\mathscr{C}\hookrightarrow\mathscr{B}$. In this setting, $\mathscr{C}$ and $\mathscr{B}$ contain subsequential and bornological spaces respectively, however the embedding of spaces in either $\mathscr{C}$ or $\mathscr{B}$ is not full; but this decisive property can be recovered for vector spaces.

Now we give a brief description of the contents of this work. We devote Section 1 to introducing some basic properties of the topos of M-sets for an abstract monoid M. From Section 2 on, we only use the monoid of maps $\mathbf{M}$ defined above. This monoid has a unique constant map. To avoid the lack of constants in $\mathbf{M}$, we introduce an ideal $\mathbf{Z}$ of "pseudoconstant" maps which selects those ideals I (which will be called *extended* ideals) such that the images of the maps $f\in I$ cover $\mathbf{N}$. The ideals that we shall use in Sections 3 and 4, which are the core of this paper, are of this kind. In Section 2 we describe the basic properties of extended ideals and we present two sheaf subtoposes, $\mathscr{M}_{\neg\neg}$ and $\mathscr{E}$, of $\mathscr{M}$ induced by the unique constant map and the ideal $\mathbf{Z}$, respectively.

The most important toposes are $\mathscr{C}$ and $\mathscr{B}$, which are studied in Section 3. They are important because:

(i) for any subsequential space X, the $\mathbf{M}$-set $\Sigma_{\mathrm{c}}(X)$ of all convergent sequences $\sigma:\mathbf{N}\to X$ belongs to $\mathscr{C}$,

(ii) for any bornological space X, the $\mathbf{M}$-set $\Sigma_{\mathrm{b}}(X)$ of all bounded sequences $\sigma:\mathbf{N}\to X$ belongs to $\mathscr{B}$.

By taking Sections 2 and 3 jointly, we complete a chain of sheaf subtoposes of $\mathscr{M}$ and the corresponding chain of topologies on $\mathbf{M}$:

$$\mathscr{M}_{\neg\neg}\hookrightarrow\mathscr{E}\hookrightarrow\mathscr{C}\hookrightarrow\mathscr{B}\hookrightarrow\mathscr{M}\quad\text{respectively }\mathbf{J}_{\neg\neg}\supset\mathbf{J}_{\mathrm{e}}\supset\mathbf{J}_{\mathrm{c}}\supset\mathbf{J}_{\mathrm{b}}\supset\{\mathbf{M}\}.$$

Section 4 contains the main result. We consider the ring object $\mathbf{R}_c = \Sigma_c(\mathbb{R})$ of *continuous reals* in $\mathscr{M}$, and the ring object $\mathbf{R}_b = \Sigma_b(\mathbb{R})$ of *bounded reals* in $\mathscr{M}$. Hence we have two categories of modules, $\mathscr{M}\mathbf{od}_c$ and $\mathscr{M}\mathbf{od}_b$ corresponding to the rings $\mathbf{R}_c$ and $\mathbf{R}_b$, respectively. Since $\mathbf{R}_0 = \{\sigma \in \mathbf{R}_c;\ \sigma(\infty) = 0\}$ is an $\mathbf{R}_c - \mathbf{R}_b$-bimodule, there exist adjoint functors

$$\mathbf{R}_0 \otimes_b (-) \dashv Hom_c(\mathbf{R}_0, -) : \mathscr{M}\mathbf{od}_c \to \mathscr{M}\mathbf{od}_b,$$

where $\otimes_b$ means the tensor product over $\mathbf{R}_b$ in $\mathscr{M}\mathbf{od}_b$, and Hom_c means the object of $\mathbf{R}_c$-linear maps in $\mathscr{M}\mathbf{od}_c$. By using the sheafification functor relative to $\mathscr{C} \hookrightarrow \mathscr{M}$, and the fact that the three objects $\mathbf{R}_0 \hookrightarrow \mathbf{R}_c \hookrightarrow \mathbf{R}_b$ belong to $\mathscr{C}$ (and $\mathscr{B}$), we obtain a similar adjunction between smaller categories of modules $\mathbf{Mod}_c = \mathscr{C} \cap \mathscr{M}\mathbf{od}_c$ and $\mathbf{Mod}_b = \mathscr{B} \cap \mathscr{M}\mathbf{od}_b$. Finally, by means of two full and faithful functors

$$\Sigma_0 : SVS_{\mathrm{sep}} \to \mathbf{Mod}_c, \qquad \Sigma_b : BVS_{\mathrm{sep}} \to \mathbf{Mod}_b,$$

we get the adjunction ${}^c(-) \dashv {}^b(-)$ on the level of the usual functional analysis, as a corollary of the categorical tensor-hom adjunction.

1. Preliminaries on M-sets

We briefly review some of the basic properties of the topos $\mathscr{M}$ of M-sets for an abstract monoid M. The content of this section particularizes to M-sets some notions on Grothendieck toposes [5,11] in order to fix the notation that will be used. For instance, the site is now the one-object category associated to the monoid M and the sieves are the ideals of M.

Let $\mathscr{S}$ denote the category of sets and maps. Let M be a monoid with product $f \circ g$ and unit *id* (we use this notation because our typical example will be a monoid of maps, but in this section M is an abstract monoid). An *M-set* is a set X together with a (right) action $X \times M \to X$, denoted by xf, such that $(xf)g = x(f \circ g)$, $xid = x$. Given two M-sets X and Y, a map $h : X \to Y$ is said to be *equivariant* if $h(xf) = h(x)f$. M-sets and equivariant maps form a category $\mathscr{M}$. The monoid M is an M-set and there is a natural bijection $\mathscr{M}(M, X) \cong X$.

A *fixed point* of an M-set X is an element $x \in X$ such that $xf = f$ for each $f \in M$. We write $\Gamma(X)$ for the M-subset of all fixed points of X. This assignment defines the *global section* functor $\Gamma : \mathscr{M} \to \mathscr{S}$ which has a left adjoint Δ; given a set A, $\Delta(A)$ is the *trivial* M-set on A, i.e. the action $A \times M \to A$ is the projection. The category $\mathscr{M}$ has limits and colimits and they are calculated like in $\mathscr{S}$. Thus Δ preserves finite limits and we have a geometric morphism (Δ, Γ). If 1 is the final object then $\mathscr{M}(1, X) \cong \Gamma(X)$.

Given two M-sets X and Y, the set Y^X of all equivariant maps $\xi : M \times X \to Y$ with the action defined by $(\xi f)(g, x) = \xi(f \circ g, x)$ is an M-set such that $\Gamma(Y^X) \cong \mathscr{M}(X, Y)$. This construction can be easily extended to a functor $(-)^X : \mathscr{M} \to \mathscr{M}$ which is a right adjoint of the product functor $(-) \times X$, so that $\mathscr{M}$ has exponentials.

Let $Sub(X)$ denote the set of all subobjects of an M-set X. In particular, $\Omega = Sub(M)$ is the set of all (right) *ideals* I of M; that is, $f \in I$ and $g \in M$ implies $f \circ g \in I$. The set Ω is an M-set if we consider the action $\Omega \times M \to \Omega$ which associates to each ideal I and element $f \in M$ the ideal

$$(I : f) = \{g \in M;\ f \circ g \in I\}.$$

Thus, Ω results to be the subobject classifier of $\mathscr{M}$; for any M-subset $S \hookrightarrow X$, the classifying map $\varphi : X \to \Omega$, $\varphi(x) = \{g \in M;\ xg \in S\}$, is the only equivariant map such that $\varphi^{-1}(M) = S$. According to our notation, we write $(S : x)$ for the ideal $\varphi(x)$. Note that $\Omega^X \cong Sub(M \times X)$, $\Gamma(\Omega^X) \cong Sub(X)$, and $\Gamma(\Omega) = \{\emptyset, M\}$.

A special ideal is $C = \Gamma(M)$, the set of all *zeros* of the monoid M. Let us note that C is a two-sided ideal, $C \circ C = C$, each subset $I \subset C$ is an ideal, and $xC \subset \Gamma(X)$ for each M-set X and $x \in X$. The ideal C may be empty, for instance if M is a non-trivial group, but $C \neq \emptyset$ holds if M is the monoid of all endomaps of a non-empty set; then zeros are constant maps. Each $Sub(X)$ is a locale [11] with implication $S \to S' = \{x \in X;\ (S : x) \subset (S' : x)\}$. If $C \neq \emptyset$ then, by calculating with $\neg S = S \to \emptyset$, we obtain the double negation $\neg\neg S = \Gamma(X) \to S$; in particular we have $\neg\neg I = C \to I$ in Ω.

Recall [11] that a (*Grothendieck*) *topology* on M is an M-subset $\mathbf{J}$ of Ω such that $M \in \mathbf{J}$ and for each ideal I the following property holds: if $J \in \mathbf{J}$ and $(I : f) \in \mathbf{J}$ for each $f \in J$, then $I \in \mathbf{J}$. An M-subset $\mathbf{J}$ of Ω is a topology if and only if its classifying map j is a nucleus. For any ideal I the map $j_I = I \to (-) : \Omega \to \Omega$ is a nucleus as a map of locales, but it is not equivariant in general. Now we give some results which we shall use to obtain topologies in the next sections.

Lemma 1.1. *Given an ideal* I, *the family* $\mathbf{J}_I = \{J \in \Omega;\ I \subset J\}$ *is a topology* (*or the nucleus* $j_I = I \to (-) : \Omega \to \Omega$ *is equivariant*) *if and only if the ideal* I *is two-sided and satisfies the following property*:

$$(*) \quad \textit{for any } f \in I \textit{ and } J \in \Omega, \quad I \subset (J : f) \textit{ implies } f \in J.$$

Proof. Note that I is two-sided if and only if $I \subset (I : f)$ for all $f \in M$. Now, j_I equivariant means that $I \to (J : f) = ((I \to J) : f)$, and by taking $J = I$ it results $I \subset (I : f)$. On the other hand, $I \subset (J : f)$ implies $f \in I \to J$, so that if $f \in I$ then $f \in J$. Conversely, I two-sided implies that $\mathbf{J}_I$ is an M-subset of Ω, and $(*)$ gives us the third condition for a topology. □

Note that if I is a non-empty ideal and j_I is equivariant then I is idempotent and $C \subset I$. Moreover, the two-sided ideal C satisfies the condition $(*)$: if $c \in C$ and $I \in \Omega$ then either $(I : c) = M$ if $c \in I$ or $(I : c) = \emptyset$ if $c \notin I$. Hence the family $\mathbf{J}_C = \{I \in \Omega;\ C \subset I\}$ is a topology, which is the double negation topology $\mathbf{J}_{\neg\neg}$ (corresponding to the nucleus $j = \neg\neg$) when $C \neq \emptyset$.

Let I be an ideal. An M-set X is said to be an I-*sheaf* if for each equivariant map $H : I \to X$ there exists a unique $x \in X$ such that $H(f) = xf$ for all $f \in I$. In other words, there exists a unique equivariant extension of H from I to M. Given a topology $\mathbf{J}$, an M-set X is said to be a $\mathbf{J}$-*sheaf* if X is an I-sheaf for all $I \in \mathbf{J}$. Let $\mathscr{M}_j$ be the full subcategory of $\mathscr{M}$ which consists of all $\mathbf{J}$-sheaves, with inclusion functor $\mathbf{i} : \mathscr{M}_j \to \mathscr{M}$. Recall that $\mathscr{M}_j$ is a topos with the same exponentials that $\mathscr{M}$ and Ω_j (the image of the nucleus j) as subobject classifier. Moreover, there is a canonical geometric morphism $(\mathbf{a}, \mathbf{i})$ where the left adjoint $\mathbf{a} : \mathscr{M} \to \mathscr{M}_j$ is the *sheafification* functor. For the topologies $\mathbf{J}_I$ in Lemma 1.1 this geometric morphism is essential (see [8] for M-sets or [6] for a more general result). The proof of the following lemma is an easy exercise.

Lemma 1.2. *Let I be an ideal such that $\mathbf{J}_I$ is a topology. An $\mathbf{M}$-set X is a $\mathbf{J}_I$-sheaf if and only if X is a I-sheaf.*

The largest topology $\mathbf{J}$ for which the monoid M is a $\mathbf{J}$-sheaf, called the *canonical* topology on M, is denoted by $\mathbf{J}_{\mathrm{can}}$. If M is the monoid of all endomaps of a set, then $\mathbf{J}_{\mathrm{can}} = \mathbf{J}_C = \mathbf{J}_{\neg\neg}$. Only $\mathbf{J}_{\mathrm{can}} \subset \mathbf{J}_C$ holds in general, as we shall see in the next section. We say that an ideal I is *canonical* if $I \in \mathbf{J}_{\mathrm{can}}$, and a topology $\mathbf{J}$ is *subcanonical* if M is a $\mathbf{J}$-sheaf, that is, if $\mathbf{J} \subset \mathbf{J}_{\mathrm{can}}$. We give without proof the following result (see [8] for M-sets or [4] for general Grothendieck toposes).

Lemma 1.3. *An ideal I is canonical if and only if for every $f \in M$ the monoid M is an $(I : f)$-sheaf.*

2. Extended ideals

From now on in this paper, $\mathbf{M}$ is the monoid of all continuous maps $f : \mathbf{N} \to \mathbf{N}$ such that $f(\infty) = \infty$, and $\mathscr{M}$ the topos of $\mathbf{M}$-sets. Let us note that if $f(\infty) = \infty$, then f is continuous if and only if $f^{-1}(A)$ is finite for every finite subset $A \subset \mathbb{N}$. The constant onto ∞, denoted by $\mathbf{z}$, is the unique zero of the monoid $\mathbf{M}$, so that $C = \{\mathbf{z}\}$. Nevertheless, we have an ideal $\mathbf{Z} \supset C$ of "pseudoconstant" maps whose images cover $\mathbf{N}$. Now, we give some examples of maps and ideals in $\mathbf{M}$ and then we continue with two technical lemmas which will be used in Section 3.

Example 2.1. The following maps are elements of $\mathbf{M}$:
(i) Given any *infinite* subset $A \subset \mathbb{N}$, the maps f_A which is the unique 1–1 monotone map whose image is $A \cup \{\infty\}$.
(ii) Given any *finite* subset $A \subset \mathbb{N}$ and $k \in \mathbf{N}$, the map

$$\mathbf{z}_{A,k}(n) = k \quad \text{if } n \in A, \qquad \mathbf{z}_{A,k}(n) = \infty \quad \text{if } n \notin A.$$

We put $\mathbf{z}_A = \mathbf{z}_{A,0}$, $\mathbf{z}_k = \mathbf{z}_{\{0\},k}$, so that $\mathbf{z}_0 = \mathbf{z}_{\{0\}}$, $\mathbf{z}_{A,\infty} = \mathbf{z}_{\emptyset,k} = \mathbf{z}$, and $\mathbf{z}_{A,k} = \mathbf{z}_k \circ \mathbf{z}_A$. For any $f \in \mathbf{M}$ we calculate $\mathbf{z}_{A,k} \circ f = \mathbf{z}_{A',k}$ where $A' = f^{-1}(A)$, and $f \circ \mathbf{z}_{A,k} = \mathbf{z}_{A,f(k)}$.

Example 2.2. The following sets are ideals of $\mathbf{M}$:

(i) Let $\mathbf{Z}$ be the set of all maps $\mathbf{z}_{A,k}$ defined above. From Example 2.1(ii) we have that $\mathbf{Z}$ is a two-sided and idempotent ideal which is generated by the maps $\mathbf{z}_k$, $k \in \mathbb{N}$. Moreover, $\mathbf{Z}$ satisfies condition $(*)$ in Lemma 1.1, that is, if $\mathbf{Z} \subset (I : \mathbf{z}_{A,k})$ then $\mathbf{z}_{A,k} \in I$: if $A = \emptyset$, $\mathbf{z}_{A,k} = \mathbf{z}$ and the statement is obvious; if $A \neq \emptyset$, for any $a \in A$ the equality $\mathbf{z}_{A,k} \circ \mathbf{z}_{A,a} = \mathbf{z}_{A,k}$ holds so that $\mathbf{z}_{A,a} \in (I : \mathbf{z}_{A,k})$ implies $\mathbf{z}_{A,k} \in I$. Hence $\mathbf{J}_{\mathbf{Z}}$ is a topology on $\mathbf{M}$ and clearly $\mathbf{J}_{\mathbf{Z}} \subset \mathbf{J}_{\neg\neg}$. Note that $\mathbf{M}$ is $\mathbf{Z}$-*separated*, that is, given $f, g \in \mathbf{M}$, if $f \circ z = g \circ z$ for all $z \in \mathbf{Z}$ then $f = g$.

(ii) For each $A \subset \mathbb{N}$ we consider the ideal $I_A = \{f \in \mathbf{M};\ f^{-1}(A) = \emptyset\}$. We write I_a for $I_{\{a\}}$. In particular $I_\emptyset = \mathbf{M}$, $I_{\mathbb{N}} = C$, and if $A \subset B$ then $I_B \subset I_A$. If $a \in A$ and $f \in I_A$ then $\mathbf{z}_a \circ f = \mathbf{z} = \mathbf{z} \circ f$, so that $\mathbf{M}$ is not I_A-separated when $A \neq \emptyset$.

Lemma 2.3. (i) *Let (f) be the ideal generated by $f \in \mathbf{M}$ while $g \in \mathbf{M}$. Then $g \in (f)$ if and only if $Im\, g \subset Im\, f$.*

(ii) *Let $I \neq \emptyset$ be an ideal and $A \subset \mathbb{N}$ finite. If $\mathbf{z}_{A,k} \notin I$ then $(I : \mathbf{z}_{A,k}) = I_A$.*

Proof. (i) Let us suppose that $Im\, g \subset Im\, f$. We define the map $h : \mathbf{N} \to \mathbf{N}$ by $h(g^{-1}(\infty)) = \{\infty\}$ and $h(k) = \max f^{-1}(g(k))$ if $g(k) \in \mathbb{N}$. Then $f \circ h = g$ and we only need to prove that $h \in \mathbf{M}$. In fact, if $h(k) \neq \infty$ then $h^{-1}(h(k)) \subset g^{-1}(g(k))$ is finite. The converse is obvious.

(ii) By (i) clearly $(\mathbf{z}_{A,k}) = (\mathbf{z}_{B,k})$ holds for all $\emptyset \neq A$, $B \subset \mathbb{N}$. Then $\mathbf{z}_{B,k} \in I$, $B = f^{-1}(A)$, is impossible if $B \neq \emptyset$, hence $f \in (I : \mathbf{z}_{A,k})$ means $f \in I_A$. $\square$

The *extent* of a map $f \in \mathbf{M}$ is the subset $Ext(f) \subset \mathbb{N}$ defined by $Im(f) = Ext(f) \cup \{\infty\}$, and the *extent* of an ideal I is $Ext(I) = \bigcup\{Ext(f);\ f \in I\}$. For instance, $Ext(f_A) = A$, $Ext(\mathbf{z}_{A,k}) = \{k\}$, $Ext(C) = \emptyset$, $Ext(\mathbf{Z}) = \mathbb{N}$, $Ext(I_A)$ is the complement of A in $\mathbb{N}$. It is clear that $Ext(I) = \{k \in \mathbb{N};\ \mathbf{z}_k \in I\} = Ext(\mathbf{Z} \cap I)$ and then $\mathbf{J}_{\mathbf{Z}} = \{I \in \Omega;\ Ext(I) = \mathbb{N}\}$. From now on we shall call $\mathbf{J}_{\mathbf{Z}}$ the *extent* topology and we shall use $\mathbf{J}_{\mathrm{e}}$ to denote it. In the same way, we shall say that $I \in \mathbf{J}_{\mathrm{e}}$ is an *extended* ideal.

Lemma 2.4. *If I is an extended ideal then*:

(i) *Any equivariant map $H : I \to \mathbf{M}$ satisfies $H(\mathbf{Z}) \subset \mathbf{Z}$ and $H(\mathbf{z}) = \mathbf{z}$.*

(ii) *If $\mathbf{M}$ is an I-sheaf then there exists $A \subset \mathbb{N}$ infinite such that $f_A \in I$.*

Proof. (i) If we put $H(\mathbf{z}_k) = f_k$ then $H(\mathbf{z}_{A,k}) = H(\mathbf{z}_k \circ \mathbf{z}_A) = f_k \circ \mathbf{z}_A = \mathbf{z}_{A,b}$ with $b = f_k(0)$; in particular $f_k = \mathbf{z}_b$.

(ii) Suppose that there is no infinite subset $A \subset \mathbb{N}$ such that $f_A \in I$. From Lemma 2.3(i) we deduce that $Ext(f)$ is a finite subset for every $f \in I$. Thus, the map $\mathbf{z}_{\mathbb{N}} : \mathbf{N} \to \mathbf{N}$ induces an equivariant map $H : I \to \mathbf{M}$, $H(f) = \mathbf{z}_{\mathbb{N}} \circ f$, which has no extension to $\mathbf{M}$ because $\mathbf{z}_{\mathbb{N}}$ is not in $\mathbf{M}$ and I is extended. Then $\mathbf{M}$ is not an I-sheaf. $\square$

The ideals $C \subset \mathbf{Z}$ induce topologies $\mathbf{J}_{\neg\neg} \supset \mathbf{J}_{\mathrm{e}}$ and toposes $\mathscr{M}_{\neg\neg} \hookrightarrow \mathscr{E}$ respectively, where $\mathscr{E}$ denotes the topos of $\mathbf{J}_{\mathrm{e}}$-sheaves, which will be called the *extended* topos. We complete this section with two results about these toposes.

Proposition 2.5. *The double negation topology on the monoid* $\mathbf{M}$ *is* $\mathbf{J}_{\neg\neg} = \{I \in \Omega;\ I \neq \emptyset\}$. *Moreover*, $\Omega_{\neg\neg} = \{\emptyset, \mathbf{M}\}$ *and* $\mathscr{M}_{\neg\neg} = \mathscr{S}$.

Proof. We know that $C \neq \emptyset$ implies $\mathbf{J}_{\neg\neg} = \mathbf{J}_C$. Moreover, for any ideal I we have $\mathbf{z} \in I$ if and only if $I \neq \emptyset$. Now it is easy to verify that $\neg\neg I = \mathbf{M}$ if $I \neq \emptyset$. To give an equivariant map $C \to X$ is equivalent to give a fixed point in X, and for each $x \in X$ the fixed point $x\mathbf{z}$ satisfies $(x\mathbf{z})\mathbf{z} = x\mathbf{z}$; so that if X is a C-sheaf then $x\mathbf{z} = x$ for all $x \in X$. Hence, X is a C-sheaf if and only if X is a trivial $\mathbf{M}$-set. Now we apply Lemma 1.2. □

Note that the inclusion $\mathscr{M}_{\neg\neg} \hookrightarrow \mathscr{M}$ is the trivial $\mathbf{M}$-sets functor Δ and for this monoid Γ is both left and right adjoint to Δ. Thus, the (essential) geometric morphism $\mathscr{M}_{\neg\neg} \to \mathscr{M}$ is (Γ, Δ).

Given a set X, the set $X^{\mathbf{N}}$ of all *sequences* $\sigma : \mathbf{N} \to X$ is an $\mathbf{M}$-set with action the composition. Each element $\sigma \in X^{\mathbf{N}}$ can be seen as a pair $\sigma = \langle s, x \rangle$, formed by an ordinary sequence $s : \mathbb{N} \to X$ plus a distinguished point $x = \sigma(\infty) \in X$.

Proposition 2.6. *For any set X, the* $\mathbf{M}$*-set* $X^{\mathbf{N}}$ *belongs to* $\mathscr{E}$.

Proof. We must prove that $X^{\mathbf{N}}$ is a $\mathbf{Z}$-sheaf. Given an equivariant map $H : \mathbf{Z} \to X^{\mathbf{N}}$ we have the constant sequence $H(\mathbf{z}) = \{x, x, x, \ldots\}$; then, for each $k \in \mathbb{N}$, $H(\mathbf{z}_k) = \{x_k, x, x, \ldots\}$ because if $n \neq 0$ then

$$H(\mathbf{z}_k)(n) = H(\mathbf{z}_k)(\mathbf{z}_n(0)) = (H(\mathbf{z}_k) \circ \mathbf{z}_n)(0) = H(\mathbf{z}_k \circ \mathbf{z}_n)(0) = H(\mathbf{z})(0) = x.$$

Hence we can take the sequence $\sigma = \langle s, x \rangle : \mathbf{N} \to X$ such that $s(k) = x_k$, and verify that σ is the unique sequence such that $H(\mathbf{z}_{A,k}) = \sigma \circ \mathbf{z}_{A,k}$. □

Because of Proposition 2.6, the monoid $\mathbf{N}^{\mathbf{N}}$ belongs to $\mathscr{E}$, but $\mathbf{M}$ does not by Lemma 2.4(ii); $\mathbf{M}$ is only $\mathbf{Z}$-separated by Example 2.2(i).

3. The continuous topos and the bounded topos

Let $\mathbf{J}_{\mathrm{c}}$ be the canonical topology of $\mathbf{M}$, which we prefer to call *continuous* topology for this particular monoid. We also call *continuous* ideals the ideals $I \in \mathbf{J}_{\mathrm{c}}$, and the topos $\mathscr{C} \hookrightarrow \mathscr{M}$ of $\mathbf{J}_{\mathrm{c}}$-sheaves is called the *continuous* topos. The next theorem implies that $\mathbf{J}_{\mathrm{c}} \subset \mathbf{J}_{\mathrm{e}}$, so that $\mathscr{E} \hookrightarrow \mathscr{C}$. The topos $\mathscr{C}$ is closely related to Johnstone's topological topos; in particular, the following result is analogous to Proposition 3.4 in [5].

Theorem 3.1. *An ideal I is continuous if and only if I is extended and for any* $A \subset \mathbb{N}$ *infinite, there exists* $B \subset A$ *infinite such that* $f_B \in I$.

Proof. We shall use Lemma 1.3. Let I be a continuous ideal and $k \in \mathbb{N}$. If $\mathbf{z}_k \notin I$ then $(I : \mathbf{z}_k) = I_0$ by Lemma 2.3(ii). Since $\mathbf{M}$ is not I_0-separated (see Example 2.2(ii)),

$\mathbf{M}$ is not an $(I : \mathbf{z}_k)$-sheaf. Now let $A \subset \mathbb{N}$ be an infinite subset. Because $(I : f_A)$ is a continuous ideal we obtain $\mathbf{Z} \subset (I : f_A)$ and then, from Lemma 2.4(ii), there exists an infinite $U \subset \mathbb{N}$ such that $f = f_A \circ f_U \in I$. Thus, $B = Ext(f) \subset A$ is infinite and from Lemma 2.3(i) $f_B \in I$.

Conversely, if I is an ideal that satisfies the two conditions above, then $\mathbf{M}$ is an I-sheaf. In fact, given an equivariant map $H : I \to \mathbf{M}$ we know (see Lemma 2.4(i)) that $H(\mathbf{Z}) \subset \mathbf{Z}$, $H(\mathbf{z}) = \mathbf{z}$, and by defining $h : \mathbf{N} \to \mathbf{N}$, $h(k) = H(\mathbf{z}_k)(0)$, it results $H(g) = h \circ g$. If $h \in \mathbf{M}$ then it is unique because $\mathbf{M}$ is $\mathbf{Z}$-separated. So we must prove that h is continuous, that is, the sequence $h(k)$ converges to ∞ in $\mathbf{N}$. But the second condition in the statement implies that every subsequence of $h(k)$ contains a subsequence converging to ∞, hence h is continuous. Finally, we have to prove that $\mathbf{M}$ is also an $(I{:}f)$-sheaf for all $f \in \mathbf{M}$ but, as it has just been proved, this property follows if $(I{:}f)$ satisfies the two conditions in the statement of the theorem. It is clear that $\mathbf{Z} \subset I$ implies $\mathbf{Z} \subset (I{:}f)$. Now let $A \subset \mathbb{N}$ be infinite. If $f(A)$ is infinite, there exists $U \subset f(A) \cap \mathbb{N}$ infinite such that $f_U \in I$. So that the set $B = A \cap f^{-1}(U)$ is infinite and $Ext(f \circ f_B) = U = Ext(f_U)$ and hence, from Lemma 2.3(i), $f \circ f_B \in I$. If $f(A)$ is finite, there exists an infinite subset $B \subset A$ such that $B \subset f^{-1}(\infty)$ so that $f \circ f_B = \mathbf{z} \in I$. □

A *subsequential space* (see [5] or L^*-*spaces* in [1,7]) is a set X together with a subset $R \subset X^{\mathbb{N}} \times X$ satisfying the following properties, where we write $s \to x$ for $(s,x) \in R$ and say that x is a *limit* of s or that s *converges* to x:

(i) For every $x \in X$, $c_x \to x$ ($c_x : \mathbb{N} \to X$ is the constant map onto $x \in X$).

(ii) If $s \to x$ and f is 1–1 monotone then $s \circ f \to x$.

(iii) Given $s \in X^{\mathbb{N}}$ and $x \in X$, if for every 1–1 monotone map f there exists an 1–1 monotone g such that $s \circ f \circ g \to x$, then $s \to x$.

Note that a *subsequence* of $s : \mathbb{N} \to X$ is a sequence $s \circ f$ where the map $f : \mathbb{N} \to \mathbb{N}$ is 1–1 monotone. Recall that a subsequential space is said to be separated if $s \to x$ and $s \to y$ implies $x = y$ (unique limit). Given a subsequential space X and $x \in X$ we consider the $\mathbf{M}$-sets

$$\Sigma_{\mathrm{c}}(X) = \{\sigma = \langle s,x\rangle; s \to x\} \subset X^{\mathbf{N}}, \qquad \Sigma_{\mathrm{x}}(X) = \{\sigma \in \Sigma_{\mathrm{c}}(X); \sigma(\infty) = x\}$$

and identify the subsequential structure with $\Sigma_{\mathrm{c}}(X)$, which is the disjoint union of all $\Sigma_x(X)$, $x \in X$. If X is separated and $\sigma = \langle s,x\rangle \in \Sigma_x(X)$ then we can also identify σ with s. Constant elements in $\Sigma_{\mathrm{c}}(X)$ will be denoted by $\sigma_x = \langle c_x, x\rangle$. Given subsequential spaces X, Y, a map $h : X \to Y$ is said to be *continuous* if it preserves convergent sequences, that is, $s \to x$ implies $h \circ s \to h(x)$. Thus, a map $h : X \to Y$ is continuous if and only if $h \circ \sigma \in \Sigma_{\mathrm{c}}(Y)$ for all $\sigma \in \Sigma_{\mathrm{c}}(X)$. Note that $\Sigma_{\mathrm{c}}(X)$ is the set of continuous maps from $\mathbf{N}$ to X.

We shall show that the $\mathbf{M}$-sets $\Sigma_{\mathrm{c}}(X)$, $\Sigma_x(X)$, belong to $\mathscr{C}$. In particular, we are interested in the space $\mathbf{R}_{\mathrm{c}} = \Sigma_{\mathrm{c}}(\mathbb{R})$ of all convergent sequences of real numbers (called the object of *continuos reals*) and its subspace $\mathbf{R}_0 = \Sigma_0(\mathbb{R})$ of all sequences converging to 0. Moreover, we shall also prove that the space $\mathbf{R}_{\mathrm{b}} = \Sigma_{\mathrm{b}}(\mathbb{R})$ of all bounded sequences of real numbers (called the object of *bounded reals*) belongs to $\mathscr{C}$.

Theorem 3.2. *The following spaces are* $\mathbf{J}_c$-*sheaves*:
(i) $\Sigma_c(X)$ *and* $\Sigma_x(X)$ *for every subsequential space* X *and* $x \in X$ (*in particular*, $\mathbf{R}_0$ *and* $\mathbf{R}_c$).
(ii) *The space* $\mathbf{R}_b$ *of all bounded sequences of real numbers.*

Proof. (i) From the definition of subsequential space, we have that $\sigma = \langle s, x\rangle \in X^{\mathbf{N}}$ belongs to $\Sigma_x(X)$ if and only if there exists a continuous ideal I such that $\sigma \circ I \subset \Sigma_x(X)$. Let I be a continuous ideal and $H : I \to \Sigma_x(X)$ an equivariant map. We can take (Proposition 2.6) the unique sequence $\sigma = \langle s, x\rangle \in X^{\mathbf{N}}$ such that $H(\mathbf{z}_{A,k}) = \sigma \circ \mathbf{z}_{A,k}$. Then it is clear that $H(f) = \sigma \circ f$ for all $f \in I$. Since I is continuous, $\sigma \in \Sigma_x(X)$. Similary, it is easily verified that if $H : I \to \Sigma_c(X)$ is equivariant then there exists $x \in X$ such that $H(I) \subset \Sigma_x(X)$.

(ii) Once again, let I be continuous and $H : I \to \mathbf{R}_b$ equivariant. We must prove that the sequence $\sigma = \langle s, \lambda\rangle \in \mathbf{R}^{\mathrm{N}}$ defined by H is bounded. But I continuous implies that each subsequence of s has a bounded subsequence, so that s is bounded. □

In this way we have a faithful functor $\Sigma_c : SS \to \mathscr{C}$, where SS denotes the category of subsequential spaces and continuous maps. But Σ_c is not full; in fact, it is easy to see that the map $H : \Sigma_c(\mathbf{N}) \to \Sigma_c(\mathbf{N})$ given by $H(f) = f$ if $f \in \mathbf{M}$ and $H(f) = \sigma_{f(\infty)}$ if $f \notin \mathbf{M}$ is equivariant but there is no map $h : \mathbf{N} \to \mathbf{N}$ such that $H = h \circ (-)$. The last statement in Theorem 3.2 is not a general property of spaces of bounded sequences, that is, there exist bornological spaces X such that the $\mathbf{M}$-subsets $\Sigma_b(X) \subset X^{\mathbf{N}}$ (of all bounded sequences $\sigma : \mathbf{N} \to X$) are not a $\mathbf{J}_c$-sheaves (the example proposed by Frölicher and Kriegl [2, p. 8] to show a bornology which is not an ℓ^∞-structure can be used here).

Our next goal is to find in $\mathscr{M}$ a new subtopos of sheaves, bigger than $\mathscr{C}$, in which spaces of bounded sequences will be included. In the most general formulation, the canonical topology consists of the universally effective-epimorphic sieves, and Lemma 1.3 is the translation of this condition into the language of monoids. The next topology we introduce consists of the sieves containing a finite epimorphic family, and it is a subcanonical topology.

An ideal I is said to be *bounded* if there exist $f_1, \ldots, f_n \in I$ such that

$$\mathbb{N} = Ext(f_1) \cup \cdots \cup Ext(f_n).$$

Let $\mathbf{J}_b$ denote the set of all bounded ideals. It is clear that $\mathbf{M} \in \mathbf{J}_b \subset \mathbf{J}_e$.

Example 3.3. (i) The ideal (f_A) with $A \subset \mathbb{N}$ infinite is not bounded if $A \neq \mathbf{N}$, but if $B \subset \mathbb{N}$ is also infinite then $(f_A) \cup (f_B)$ is bounded if and only if $A \cup B = \mathbb{N}$.

(ii) If $Ext(f)$ is finite then $(I : f)$ is bounded for every extended ideal I. In fact, $f_K \in (I : f)$ holds where $K = (\max Ext(f), \infty)$. Moreover, since $(I : f)$ is extended, then the family $\{\mathbf{z}_1, \ldots, \mathbf{z}_n, f_K\} \subset (I : f)$ supplies a finite cover of $\mathbb{N}$, where $n = \max Ext(f) \in \mathbb{N}$.

Theorem 3.4. (i) $\mathbf{J}_{\mathrm{b}}$ *is a topology and* $\mathbf{J}_{\mathrm{b}} \subset \mathbf{J}_{\mathrm{c}}$.
(ii) *For every bornological space* $X, \Sigma_{\mathrm{b}}(X)$ *is a* $\mathbf{J}_{\mathrm{b}}$*-sheaf.*

Proof. (i) We prove that $(I : f)$ is bounded for each bounded ideal I and $f \in \mathbf{M}$. After Example 3.3(ii) we only consider the case $A = Ext(f)$ infinite. If $\mathbb{N} = A_1 \cup \cdots \cup A_n$, $A_i = Ext(f_i)$, $f_i \in I$, then $\mathbb{N} = B_1 \cup \cdots \cup B_{n+1}$ where $B_i = f^{-1}(A_i)$, $1 \le i \le n$, and $B_{n+1} = f^{-1}(\infty)$. We fix $i = 1, \ldots, n+1$. If B_i is finite then we have a finite cover of B_i by subsets $Ext(\mathbf{z}_{\mathrm{k}})$, $\mathbf{z}_{\mathrm{k}} \in (I : f)$. If not, $g_i = f_{B_i} \in (I : f)$ because $Ext(f \circ g_i) \subset Ext(f)$ implies $f \circ g_i \in I$ by Lemma 2.3(i). Finally, it is an easy exercise to verify that if I is bounded and $(J : f)$ is bounded for all $f \in I$ then J is bounded. Hence $\mathbf{J}_{\mathrm{b}}$ is a topology.

Now let I be a bounded ideal. If $U \subset \mathbb{N}$ is infinite then (using the same notation) $V = U \cap A_i$ is infinite for some $i = 1, \ldots, n$, and from Lemma 2.3(i) we conclude $f_V \in I$, so that I is continuous because it is extended too.

(ii)Let I be a bounded ideal and $H : I \to \Sigma_{\mathrm{b}}(X)$ an equivariant map. We have to prove that $\sigma \in \Sigma_{\mathrm{b}}(X)$, where $\sigma = \langle s, x \rangle \in X^{\mathbf{N}}$ is the sequence defined in Proposition 2.6, which satisfies $H(f) = \sigma \circ f$ for all $f \in I$. Because I is bounded, $s(\mathbb{N})$ is a finite union of subsets $B = s(Ext(f))$ where $f \in I$, so that if each B is bounded then s (or σ) is bounded. But $B \subset (\sigma \circ f)(\mathbf{N})$ is bounded because $f \in I$. $\square$

Hence we have the *bounded* topos $\mathscr{B}$ of $\mathbf{J}_{\mathrm{b}}$-sheaves, with an inclusion functor $\mathscr{C} \hookrightarrow \mathscr{B}$. If BS denotes the category of bornological spaces and bounded maps, there is a faithful functor $\Sigma_{\mathrm{b}} : BS \to \mathscr{B}$ which is not full (by using the same example as that for Σ_{c}). The functors Σ_{c} and Σ_{b} will be full if they are restricted to linear subcategories, as we shall see in the next section.

4. Modules in the toposes $\mathscr{C}$ and $\mathscr{B}$

We begin this section presenting some basic notions of linear algebra in the topos $\mathscr{M}$ of M-sets for an abstract monoid M (for more details see [8]). Then we shall apply these constructions to the monoid $\mathbf{M}$.

Let R be a ring object in $\mathscr{M}$ and let $\mathscr{M}\mathbf{od}_R$ denote the corresponding category of R-module objects in $\mathscr{M}$. Given two R-modules X and Y, the M-set $Hom_R(X, Y)$ of all R-linear maps is the M-subset of Y^X whose elements are those equivariant maps $\xi : M \times X \to Y$ such that $\xi(f, -)$ is linear for all $f \in M$. Moreover, the tensor product $X \otimes Y$ in $\mathscr{S}$ with the action given by $(x \otimes y)f = xf \otimes yf$, is the tensor product in $\mathscr{M}\mathbf{od}_R$. Given two rings R and S, each R–S-bimodule A gives us an internal adjunction

$$(\otimes) \qquad A \otimes_S (-) \dashv Hom_R(A, -) : \mathscr{M}\mathbf{od}_R \to \mathscr{M}\mathbf{od}_S$$

$$Hom_R(A \otimes_S X, Y) \cong Hom_S(X, Hom_R(A, Y)) \quad (\text{iso in } \mathscr{M}).$$

Given $\xi : M \times (A \otimes_S X) \to Y$ in the first set, the corresponding ζ in the second is $\zeta(f,x)(g,a) = \xi(f \circ g, a \otimes xg)$; conversely, $\xi(f, a \otimes x) = \zeta(f,x)(id,a)$. By taking global sections we obtain the external adjunction corresponding to ($\otimes$); that is, linear equivariant maps from $A \otimes_S X$ to Y and linear equivariant maps from X to $Hom_R(A,Y)$ are in 1–1 correspondence.

Now we consider the rings $\mathbf{R}_{\mathrm{c}}$, $\mathbf{R}_{\mathrm{b}}$ and the $\mathbf{R}_{\mathrm{c}} - \mathbf{R}_{\mathrm{b}}$-bimodule $\mathbf{R}_0$ in the topos $\mathscr{M}$ of $\mathbf{M}$-sets, so that we have adjoint functors

$$\mathbf{R}_0 \otimes_{\mathrm{b}} (-) \dashv Hom_{\mathrm{c}}(\mathbf{R}_0, -) : \mathscr{M}\mathbf{od}_{\mathrm{c}} \to \mathscr{M}\mathbf{od}_{\mathrm{b}},$$

where $\otimes_{\mathrm{b}}$ means the tensor product over $\mathbf{R}_{\mathrm{b}}$ in $\mathscr{M}\mathbf{od}_{\mathrm{b}}$, and Hom_{c} means the object of $\mathbf{R}_{\mathrm{c}}$-linear maps in $\mathscr{M}\mathbf{od}_{\mathrm{c}}$. But we are interested in modules which are sheaves, that is, we consider the categories $\mathbf{Mod}_{\mathrm{c}} = \mathscr{C} \cap \mathscr{M}\mathbf{od}_{\mathrm{c}}$ and $\mathbf{Mod}_{\mathrm{b}} = \mathscr{B} \cap \mathscr{M}\mathbf{od}_{\mathrm{b}}$.

Theorem 4.1. *There exists an internal adjunction in $\mathscr{M}$.*

$$\mathbf{a}(\mathbf{R}_0 \otimes_{\mathrm{b}} (-)) \dashv Hom_{\mathrm{c}}(\mathbf{R}_0, -) : \mathbf{Mod}_{\mathrm{c}} \to \mathbf{Mod}_{\mathrm{b}}.$$

Proof. Since $\mathbf{M} \times \mathbf{a}(X) \cong \mathbf{a}(\mathbf{M} \times X)$, the geometric morphism $(\mathbf{a}, \mathbf{i}) : \mathscr{C} \to \mathscr{M}$ gives us an internal adjunction

$$\mathbf{a} \dashv \mathbf{i} : \mathbf{Mod}_{\mathrm{c}} \hookrightarrow \mathscr{M}\mathbf{od}_{\mathrm{c}}, \; Hom_{\mathrm{c}}(\mathbf{a}(X), Y) \cong Hom_{\mathrm{c}}(X, Y) \quad (Y \; \mathbf{J}_{\mathrm{c}}\text{-sheaf}).$$

By composing with the internal adjunction ($\otimes$) induced by $\mathbf{R}_0$ we get an internal left adjoint $\mathbf{a}(\mathbf{R}_0 \otimes_{\mathrm{b}} (-))$ to $Hom_{\mathrm{c}}(\mathbf{R}_0, -) : \mathbf{Mod}_{\mathrm{c}} \to \mathscr{M}\mathbf{od}_{\mathrm{b}}$. But if X is a $\mathbf{J}_{\mathrm{c}}$-sheaf then $Hom_{\mathrm{c}}(\mathbf{R}_0, X)$ is a $\mathbf{J}_{\mathrm{c}}$-sheaf too, so that it is a $\mathbf{J}_{\mathrm{b}}$-sheaf; hence we can consider $Hom_{\mathrm{c}}(\mathbf{R}_0, -)$ valued in $\mathbf{Mod}_{\mathrm{b}}$ and restrict its left adjoint. □

Our last goal is to relate the external part of this adjunction to a well-known construction in functional analysis. The product of subsequential spaces allows us to form routinely the category *SVS* of subsequential (real) vector spaces and linear continuous maps.

Lemma 4.2. *$\Sigma_0 : SVS \to \mathbf{Mod}_{\mathrm{c}}$ is full and faithful.*

Proof. It suffices to prove that Σ_0 is full. Let E and F be subsequential vector spaces and $H : \Sigma_0(E) \to \Sigma_0(F)$ an equivariant $\mathbf{R}_{\mathrm{c}}$-linear map. We can define $h : E \to F$ by $h(x) = H(\varepsilon_x)(0)$ where $\varepsilon_x = \{x, 0, 0, \ldots\}$. The map h is linear because $\varepsilon_{x+y} = \varepsilon_x + \varepsilon_y$ and $\varepsilon_{\lambda x} = c_\lambda \varepsilon_x$ hold. Moreover, h satisfies $H = \Sigma_0(h)$: $h(\sigma(\infty)) = 0 = H(\sigma)(\infty)$ and if $k \in \mathbb{N}$ then $h(\sigma(k)) = H(\varepsilon_{\sigma(k)})(0) = H(\sigma)(k)$ by using $\varepsilon_{\sigma(k)} = \sigma \circ \mathbf{z}_{\mathrm{k}}$. Finally, the condition $H = h \circ (-)$ implies that h is continuos because we are dealing with subsequential spaces. □

Recall [3] that a *Kolmogorov* bornological (or *K-bornological*) space X is a bornological space such that if every sequence in $A \subset X$ is bounded then A is bounded, i.e. sequentially bounded subsets are bounded. We write *KS* for the category of K-bornological spaces and bounded maps which is a full (reflective) subcategory of BS. Let KVS be the category of K-bornological (real) vector spaces and bounded linear maps.

Lemma 4.3. $\Sigma_b : KVS \to \mathbf{Mod}_b$ *is full and faithful.*

Proof. It suffices to prove that Σ_b is full. Let E and F be K-bornological vector spaces and $H : \Sigma_b(E) \to \Sigma_b(F)$ an equivariant $\mathbf{R}_b$-linear map. Note that $E \cong \Gamma(\Sigma_b(E))$ and the same for F, so that we obtain $h \cong \Gamma(H)$ if we define $h : E \to F$ by $H(\sigma_x) = \sigma_{h(x)}$ or $h(x) = H(\sigma_x)(0)$. It is clear that h is linear and $H = \Sigma_b(h) = h \circ (-)$. In fact, by means of the equality $\tau \circ \mathbf{z} = \sigma_{\tau(\infty)}$ we see that $h(\tau(\infty)) = H(\tau)(\infty)$ for all $\tau \in \Sigma_b(E)$, and if $k \in \mathbb{N}$ then we multiply by the basic sequence $e_k = \{0, \dots, 0, 1_k, 0, \dots\} \in \mathbf{R}_b$,

$$e_k H(\tau) = H(e_k \tau) = H(e_k \sigma_{\tau(k)}) = e_k H(\sigma_{\tau(k)}) = e_k \sigma_{h(\tau(k))}$$

and obtain $H(\tau)(k) = h(\tau(k))$ by evaluating at k. Once again, $H = h \circ (-)$ implies that h is bounded because we are dealing with K-bornological spaces. $\square$

Let E be a subsequential vector space. A subset $A \subset E$ is said to be *bounded* if for every zero-converging real sequence λ and every sequence s in A, the product λs converges to zero in E, that is $\langle \lambda s, 0 \rangle \in \Sigma_0(E)$. It is easy to verify that the set E together with this family of bounded subsets is a K-bornological vector space which shall be denoted by ${}^{b}E$. This construction defines a functor

$${}^{b}(-) : SVS \to KVS.$$

If E is separated then ${}^{b}E$ is *separated*; i.e. $\{0\}$ is the only bounded vector subspace. Moreover, ${}^{b}E$ has the von Neumann bornology [3] if E is a topological vector space. The next result determines the bounded sequences in ${}^{b}E$ by means of the maps $\sigma : \mathbf{N} \to {}^{b}E$, because for any $x \in E$, the image of $\sigma = \langle s, x \rangle$ is bounded (σ is bounded) if and only if s is bounded.

Lemma 4.4. *Let E be a subsequential vector space. A map $\sigma : \mathbf{N} \to {}^{b}E$ is bounded if and only if $\lambda(\sigma \circ f) \in \Sigma_0(E)$ for all $\lambda \in \mathbf{R}_0$ and $f \in \mathbf{M}$.*

Proof. The direct part follows from the definition of ${}^{b}E$. Conversely, we have to show that if $\lambda(\sigma \circ f) \in \Sigma_0(E)$ for all $\lambda \in \mathbf{R}_0$ and $f \in \mathbf{M}$, then $Im\,\sigma$ is bounded. Actually, we shall prove that the subset $A = Im\,\sigma - \{\sigma(\infty)\}$ is bounded. If A is finite this is obvious, so we may suppose that A is infinite and we must prove that $\mu t \to 0$ in E holds for each $t : \mathbb{N} \to A$ and $\mu \in \mathbf{R}_0$. Because E is subsequential, it suffices to find for each 1–1 monotone $f : \mathbb{N} \to \mathbb{N}$ a new 1–1 monotone $g : \mathbb{N} \to \mathbb{N}$ such that $(\mu t) \circ f \circ g \to 0$ in E. So we fix t, μ and f as above and look for such a map g. If $Im(t \circ f)$ is finite, then there exist $a \in A$ such that $U = (t \circ f)^{-1}(a)$ is infinite and it is clear that we can take $g = f_U$. If $Im(t \circ f)$ is infinite, it is easy to verify that there exists a 1–1 monotone map $f' : \mathbb{N} \to \mathbb{N}$ such that $t \circ f \circ f'$ is 1–1, so we may suppose from now on that $Im(t \circ f)$ is infinite and $t \circ f$ is 1–1. Thus we have two maps $s, t \circ f : \mathbb{N} \to E$ (the second injective) such that $Im(t \circ f) \subset Im(s)$. By using set theory we can construct two 1–1 monotone maps $g,h:\mathbb{N}\to\mathbb{N}$ such that $t\circ f\circ g=s\circ h$. (If the set $R=\{n\in\mathbb{N}; s^{-1}((t\circ f)(n))$ infinite$\}$ is infinite we can take $g=f_R$, and if R is finite then we must use that the complement of $R, L=\{n\in\mathbb{N}; s^{-1}((t\circ f)(n))$ finite$\}$, is infinite). Note that each $f:\mathbb{N}\to\mathbb{N}$ 1–1 monotone can be uniquely extended to $f\in\mathbf{M}$ by $f(\infty)=\infty$, and if we take $\tau=\langle t,x\rangle:\mathbf{N}\to E$ with $\sigma=\langle s,x\rangle$ then $\tau\circ f\circ g=\sigma\circ h$ holds, so that

$$(\mu\tau)\circ f\circ g=\langle(\mu t)\circ f\circ g,x\rangle=\lambda(\sigma\circ f)\in\Sigma_0(E),\quad \lambda=\mu\circ f\circ g\in\mathbf{R}_0.$$

and the proof is complete. □

Theorem 4.5. (i) *Let E be a subsequencial vector space. There exists an isomorphism $Hom_{\mathrm{c}}(\mathbf{R}_0,\Sigma_0(E))\cong\Sigma_{\mathrm{b}}({}^{\mathrm{b}}E)$ in* $\mathbf{Mod}_{\mathrm{b}}$.

(ii) *The ring objects $Hom_{\mathrm{c}}(\mathbf{R}_0,\mathbf{R}_0)$ and $\mathbf{R}_{\mathrm{b}}$ are isomorphic.*

Proof. (i) Let $\xi:M\times\Sigma_0(E)\to\Sigma_0(E)$ be an equivariant $\mathbf{R}_{\mathrm{c}}$-linear map. We are going to prove that ξ is determined by a bounded sequence $\sigma:\mathbf{N}\to{}^{\mathrm{b}}E$, which we define by $\sigma(n)=\xi(\mathbf{z}_n,e_0)(0)$. Then, for each $n\in\mathbf{N}$, we have $\xi(f,\lambda)=\lambda(\sigma\circ f)$:

$$\xi(f,\lambda)(n)=\xi(f,\lambda)(\mathbf{z}_n)(0)=\xi(\mathbf{z}_{f(n)},\varepsilon_{\lambda(n)})(0)=(\varepsilon_{\lambda(n)}\xi(\mathbf{z}_{f(n)},e_0))(0)=\lambda(n)\sigma_{f(n)}.$$

Hence we conclude $\sigma\in\Sigma_{\mathrm{b}}({}^{\mathrm{b}}E)$ by Lemma 4.4. Conversely, given $\sigma\in\Sigma_{\mathrm{b}}({}^{\mathrm{b}}E)$, the map defined by $\xi(f,\lambda)=\lambda(\sigma\circ f)$ is an equivariant $\mathbf{R}_{\mathrm{c}}$-linear map. In this way, we have constructed a bijection from $Hom_{\mathrm{c}}(\Sigma_0(E),\Sigma_0(E))$ to $\Sigma_{\mathrm{b}}({}^{\mathrm{b}}E)$ which is clearly equivariant. Finally, it is easy to verify that this bijection is $\mathbf{R}_{\mathrm{b}}$-linear.

(ii) In particular, we have the $\mathbf{R}_{\mathrm{b}}$-linear isomorphism (note that this case can be proved directly, without using Lemma 4.4). Moreover, it is easy to verify that this bijection is a ring morphism (the product in $Hom_{\mathrm{c}}(\mathbf{R}_0,\mathbf{R}_0)$ is given by composition $(\xi\circ\xi')(f,\lambda)=\xi'(f,\xi(f,\lambda))$). □

Let F be a bornological vector space. Now we are going to consider a classical notion of sequential convergence in F. Following [3] we say that a sequence $s=\{x_n\}:\mathbb{N}\to F$ *converges bornologically* to a point $x\in F$ (denoted $s\to x$) if there exist a circled bounded subset $B\subset F$ and a sequence $\lambda=\{\lambda_n\}\in\mathbf{R}_0$ such that $x_n-x\in\lambda_n B$ for all $n\in\mathbb{N}$. This condition is equivalent to say that there exists a sequence $\lambda\in\mathbf{R}_0$, $\lambda_n>0$, such that $\{(x_n-x)/\lambda_n\}$ is bounded in F; moreover, we note that it is possible to choose such a sequence λ monotone decreasing. Bornological convergence is also called *Mackey convergence*. Clearly, the relations $\lambda\to a$, $s\to x$ and $t\to y$ imply $\lambda s\to ax$ and $s+t\to x+y$. Thus, the set $\mathbf{c}_0(F)$ of all sequences bornologically converging to 0 determines the family of bornologically convergent sequences in F.

If F is separated then every Mackey convergent sequence in F has a unique limit, so that there is a 1–1 correspondence between the set $\mathbf{c}(F)$ of all bornologically convergent sequences in F and the relation $R\subset F^{\mathbb{N}}\times F$ of all pairs (s,x) such that $s\to x$. It is easy to verify that R has the properties (i) and (ii) in the definition of subsequential space given in Section 3. The subsequential structure $R^*\subset F^{\mathbb{N}}\times F$ generated by R can be defined (see L-convergence and L^*-spaces in [1,7]) as follows: $(s,x)\in R^*$

if every subsequence s' of s contains a subsequence s'' such that $\langle s'',x\rangle$ belongs to R. We denote by ${}^{\mathrm{c}}F=(F,R^*)$ the corresponding subsequential space. It is easy to verify that ${}^{\mathrm{c}}F$ is a separated subsequential vector space. Moreover, the image of any Mackey convergent sequence under a bounded linear map is Mackey convergent, so this construction defines a functor

$$^{\mathrm{c}}(-): BVS_{\mathrm{sep}} \to SVS_{\mathrm{sep}}.$$

Each element $\sigma\in\Sigma_{\mathrm{c}}({}^{\mathrm{c}}F)$ has the form $\sigma=\langle s,x\rangle$ with $(s,x)\in R^*$, so we can consider $\mathbf{c}_0(F)\subset\Sigma_0({}^{\mathrm{c}}F)$ if we identify $\sigma=\langle s,0\rangle$ with s (recall that F is separated). Note that $s\in\Sigma_0({}^{\mathrm{c}}F)$ means "$s\to 0$" in the subsequential sense of $(s,0)\in R^*$, and $s\in\mathbf{c}_0(F)$ means $s\to 0$ in the bornological sense of the Mackey convergence. Now we see $\mathbf{c}_0(F)$ as an $\mathbf{M}$-set with the action given by $sf=\langle s,0\rangle\circ f$. Hence we finally obtain a mono $\mathbf{c}_0(F)\hookrightarrow\Sigma_0({}^{\mathrm{c}}F)$ in $\mathscr{M}\mathbf{od}_{\mathrm{c}}$.

Theorem 4.6. *Let F be a separated bornological vector space. There exists an isomorphism* $\mathbf{a}(\mathbf{R}_0\bigotimes_{\mathrm{b}}\Sigma_{\mathrm{b}}(F))\cong\Sigma_0({}^{\mathrm{c}}F)$ *in* $\mathbf{Mod}_{\mathrm{c}}$.

Proof. We shall prove two isomorphisms (i) $\mathbf{R}_0\bigotimes_{\mathrm{b}}\Sigma_{\mathrm{b}}(F)\cong\mathbf{c}_0(F)$ and (ii) $\mathbf{a}(\mathbf{c}_0(F))\cong\Sigma_0({}^{\mathrm{c}}F)$. Then the theorem follows by transforming (i) through $\mathbf{a}$ and using (ii).

Proof of (i): It is clear that if $\sigma\in\Sigma_{\mathrm{b}}(F)$ and $\lambda\in\mathbf{R}_0$ then $\lambda\sigma=\langle s,0\rangle$ and the sequence s converges bornologically to 0; moreover, the natural map $H:\mathbf{R}_0\times\Sigma_{\mathrm{b}}(F)\to\mathbf{c}_0(F)$, $H(\lambda,\sigma)=s$, is $\mathbf{R}_{\mathrm{b}}$-bilinear. Note that H is onto in such a way that we can choose a monotone decreasing sequence $\lambda\in\mathbf{R}_0$ of positive real numbers, and $\sigma\in\Sigma_{\mathrm{b}}(F)$ with $\sigma(\infty)=0$, such that $s=H(\lambda,\sigma)$. In these conditions we say that (λ,σ) is a canonical pair of s. Now we prove that each $\mathbf{R}_{\mathrm{b}}$-bilinear map $G:\mathbf{R}_0\times\Sigma_{\mathrm{b}}(F)\to T$ is constant over the fibers of H. First, if $(\lambda,\sigma),(\lambda',\sigma')$ are two canonical pairs of s then we take $\mu=\max\{\lambda,\lambda'\}$, so that λ/μ and λ'/μ are bounded and we calculate

$$G(\mu(\lambda/\mu),\sigma)=G(\mu,(\lambda/\mu)\sigma)=G(\mu,(\lambda'/\mu)\sigma')=G(\mu(\lambda'/\mu),\sigma'),$$

hence $G(\lambda,\sigma)=G(\lambda',\sigma')$. This argument requires that $\mu_n\neq 0$ for all $n\in\mathbb{N}$. If not, we shall have $\lambda=\{\lambda_1,\dots,\lambda_k,0,0,\dots\},\lambda_1\geq\cdots\geq\lambda_k\neq 0$, and the same for λ' with $k'\leq k$ for instance; then we take $\alpha=\{1,\dots,1_k,0,0,\dots\}$ and we can calculate

$$G(\lambda,\sigma)=G(\alpha\lambda,\sigma)=G(\alpha,\lambda\sigma)=G(\alpha,\lambda'\sigma')=G(\lambda',\sigma').$$

Second, if $\lambda\sigma=\langle s,0\rangle$ but (λ,σ) is not a canonical pair, then we can construct by induction a new sequence μ such that $(\mu,(\lambda/\mu)\sigma)$ is a canonical pair of s. Since H is onto and G is constant over the fibers of H, there exists a unique map K such that $K\circ H=G$, that is, $K(s)=G(\lambda,\sigma)$ where (λ,σ) is a canonical pair of s. Now we can verify that K is $\mathbf{R}_{\mathrm{b}}$-linear, in particular $\mathbf{R}_{\mathrm{c}}$-linear. For instance, if $(\lambda,\sigma),(\lambda',\sigma')$ are canonical pairs of s and s', respectively, and $\mu=\max\{\lambda,\lambda'\}$, we can find canonical pairs of $s,s',s+s'$ with the same first component μ, then we apply that G is bilinear.

Proof of (ii): We know that $\mathbf{c}_0(F)$ is an $\mathbf{M}$-subset of $\Sigma_0({}^{\mathrm{c}}F)$, which is a sheaf, so that $\mathbf{a}(\mathbf{c}_0(F))$ is its closure, that is, the $\mathbf{M}$-set of those elements $s\in\Sigma_0({}^{\mathrm{c}}F)$ such that the ideal $I_s = \{f \in \mathbf{M}; sf \in \mathbf{c}_0(F)\}$ belongs to $\mathbf{J}_c$. But from the definition of ${}^{c}F$ it follows that this property holds for all $s \in \Sigma_0({}^{c}F)$. □

Corollary 4.7. *There exists an adjunction* ${}^{c}(-) \dashv {}^{b}(-) : SVS_{\mathrm{sep}} \to BVS_{\mathrm{sep}}$.

Proof. Let E be a separated subsequential vector space and F a separated K-bornological vector space. The natural bijection between linear maps ${}^{c}F \to E$ and linear maps $F \to {}^{b}E$ is given by the following chain:

$$\begin{array}{ll}
{}^{c}F \to E & \\
\hline & \text{Lemma 4.2 and Theorem 4.6} \\
\mathbf{a}(\mathbf{R}_0 \otimes_{\mathrm{b}} \Sigma_{\mathrm{b}}(F)) \to \Sigma_0(E) & \\
\hline & \text{Theorem 4.1} \\
\mathbf{R}_0 \otimes_{\mathrm{b}} \Sigma_{\mathrm{b}}(F)) \to Hom_{\mathrm{c}}(\mathbf{R}_0, \Sigma_0(E)) & \\
\hline & \text{Theorem 4.5(i) and Lemma 4.3} \\
F \to {}^{b}E &
\end{array}$$

where each step is a natural bijection. □

For instance, if E and F are normed spaces, then the elementary equality $Lin_{\mathrm{c}}(E,F) = Lin_{\mathrm{b}}(E,F)$ between continuous and bounded linear maps appears as a very particular case of an adjunction tensor-hom which is a conceptual construction in toposes. Naturally, the adjunction in Corollary 4.7 can also be obtained by working into functional analysis directly.

We finish the paper with some remarks. On one hand, after the isomorphism (i) in the proof of Theorem 4.6, it seems that the internal adjunction $\mathbf{R}_0 \otimes_{\mathrm{b}} (-) \dashv Hom_{\mathrm{c}}(\mathbf{R}_0, -)$ in $\mathscr{M}$ will lead us to an external adjunction ${}^{c}(-) \dashv {}^{b}(-)$ between two categories of vector structures weaker than subsequential and bornological spaces, respectively. On the other hand, actually the functor $Hom_{\mathrm{c}}(\mathbf{R}_0, -)$ can be valued into $\mathscr{C} \cap \mathscr{M}\mathbf{od}_{\mathrm{b}}$, so that the object $\Sigma_{\mathrm{b}}({}^{b}E)$ belongs to $\mathscr{C}$; this means that the bornonologies of the subsequential spaces are ℓ^{∞}-structures in the sense of Frölicher and Kriegl [2].

References

[1] R.M. Dudley, On sequencial convergence, Trans. Amer. Math. Soc. 112 (1964) 483–507.
[2] A. Frölicher, A. Kriegl, Linear Spaces and Differentiation Theory, Wiley-Interscience, New York, 1988.
[3] H. Hogbe-Nlend, Bornologies and Functional Analysis, North-Holland, Amsterdam, 1977.
[4] P.T. Johnstone, Topos Theory, Academic Press, London, 1977.
[5] P.T. Johnstone, On a topological topos, Proc. London Math. Soc. 3 (38) (1979) 237–271.
[6] G.M. Kelly, F.W. Lawvere, On the complete lattice of essential localizations, Bull. Soc. Math. Belgique 41 (1989) 289–319.
[7] J. Kisynski, Convergence du type L, Colloq. Math. 7 (1960) 205–211.
[8] L. Lambán, Construcciones en topos que extienden relaciones entre categorías de espacios topológicos y bornológicos, Zaragoza, Universidad de Zaragoza, Publ. del Sem. Mat. García de Galdeano, Ser. II, Sec. 2, no 27, 1990.

[9] F.W. Lawvere, Taking categories seriously. Talks given at Bogotá, Seminario-Taller sobre la Teoría de Categorías y sus aplicaciones, August 1983, unpublished part.
[10] F.W. Lawvere, Taking categories seriously, Rev. Colombiana Mat. 20 (1986) 147–178.
[11] S. Mac Lane, I. Moerdijk, Sheaves in Geometry and Logic. A First Introduction to Topos Theory, Springer, New York, 1992.