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Letter to the Editor

A modification of the classical Kantorovich conditions for Newton's method ☆

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Abstract

In the classical Kantorovich theorem on Newton's method it is assumed that the second Fréchet derivative of the involved operator satisfies the condition $||F''(x)|| \leqslant K$ in an appropiate domain. In this paper we study a modification of this condition, assuming that $||F''(x)|| \leqslant \omega(||x||)$, where ω is a continuous and non-decreasing real function.

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1. Introduction

The classical Kantorovich conditions are the most famous ones in the study of Newton's method in Banach spaces. These conditions are the following (we denote them by $(\mathscr{K})$ throughout this paper):

(i) F is a Fréchet differentiable operator defined on a non-empty open convex set Ω included in a Banach space X and with values in another Banach space Y.
(ii) There exists a point $x_0 \in \Omega$ where the operator $\Gamma_0 = F'(x_0)^{-1}$ is defined, $||\Gamma_0|| \leqslant b$ and $||\Gamma_0 F(x_0)|| \leqslant a$.
(iii) $||F''(x)|| \leqslant K,\ x \in \Omega$.
(iv) $h = abK \leqslant 1/2$.

☆ This work is dedicated to the memory of Prof. J. Javier Guadalupe (Chicho).

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(v) $\overline{B(x_0,t^*)} \subseteq \Omega$, where

$$t^* = \frac{1-\sqrt{1-2h}}{bK} \quad \text{and} \quad \overline{B(x_0,t^*)} = \{x; ||x-x_0|| \leqslant t^*\}.$$

Under conditions ($\mathscr{K}$), it is proved that Newton's method,

$$x_{n+1} = x_n - \Gamma_n F(x_n), \quad \Gamma_n = F'(x_n)^{-1}, \quad n \geqslant 0, \tag{1}$$

converges to a solution of the equation $F(x)=0$. Further, Kantorovich-type results provide the regions where the solution is located and unique, along with some error estimates.

There is a wide bibliography concerning Newton–Kantorovich results, such as the classical text of Kantorovich and Akilov [5]. Some authors have studied different modifications of ($\mathscr{K}$). These changes mainly affect condition (iii). Instead of assuming that the derivative F'' satisfies condition (iii), results have been obtained when F' satisfies a Lipschitz condition [5] or when F' is a (k,p)-Hölder function [2], or more generally when F' satisfies

$$||F'(x) - F'(y)|| \leqslant \omega(||x-y||),$$

where w is a given real function [1].

Results have also been obtained by assuming a Lipschitz condition on the second Fréchet derivative (see [3]). In [4], Newton's method is studied under the condition

$$||F''(x) - F''(x_0)|| \leqslant k||x-x_0||.$$

In this case, the Lipschitz condition is weakened because one of the points is fixed.

In this paper, we investigate whether it is possible to weaken the conditions on the second Fréchet derivative assuming only that

$$\text{(iii}'\text{)} \quad ||F''(x)|| \leqslant \omega(||x||), \quad x \in \Omega,$$

where the function $\omega(z)$ is a continuous and non-decreasing real function for $z > 0$ and such that $\omega(0) \geqslant 0$. As we will see, this change also influences conditions (iv) and (v).

From now on, we assume that the operator F satisfies the following conditions (we denote them by ($\mathscr{K}'$)):

(i′) F is a Fréchet differentiable operator defined on a non-empty open convex set Ω included in a Banach space X and with values in another Banach space Y.

(ii′) There exists a point $x_0 \in \Omega$ where the operator $\Gamma_0 = F'(x_0)^{-1}$ is defined, $||\Gamma_0|| \leqslant b$ and also $||\Gamma_0 F(x_0)|| \leqslant a$.

(iii′) $||F''(x)|| \leqslant \omega(||x||)$, $x \in \Omega$, where the function $\omega(z)$ is a continuous and non-decreasing real function for $z > 0$, and such that $\omega(0) \geqslant 0$.

(iv′) If we denote the real function $M(t) = \omega(||x_0|| + t)$, there exists at least one positive root of the equation

$$2[1 - bM(t)t]a - [2 - 3bM(t)t]t = 0.$$

We denote the small positive root of this equation by R.

(v′) $\overline{B(x_0,R)} \subseteq \Omega$.

Our goal, in this paper, is to prove that, under conditions ($\mathscr{K}'$), Newton's method converges to a solution of $F(x)=0$. In addition, we give the domain where the solution is located. To do that we follow a new technique based on the use of recurrence relations instead of the classical majorizing sequences.

2. The main result

First of all, we give some technical lemmas.

Lemma 1. *Under conditions* $(\mathscr{K}')$, *we define* $f(x)=(1-x)^{-1}$ *and denote* $\delta=bM(R)R$. *Then, the following hold*:
(i) $\frac{1}{2}\delta f(\delta)<1$.
(ii) $\left(\sum_{j=0}^{n}\left[\frac{1}{2}\delta f(\delta)\right]^j\right)a=\left(\frac{1-[(1/2)\delta f(\delta)]^{n+1}}{1-(1/2)\delta f(\delta)}\right)a<\left(\frac{1}{1-(1/2)\delta f(\delta)}\right)a=R.$

Proof. Notice that R verifies

$$2[1-bM(R)R]a-[2-3bM(R)R]R=0,$$

and therefore

$$M(R)=\frac{2(R-a)}{Rb(3R-2a)}.$$

So, $\delta<2/3$ and (i) is then proved. The proof of (ii) follows easily from (i). □

Lemma 2. *With the notations of the previous lemma, we have*:

1. *There exists* Γ_n *and* $||\Gamma_n F'(x_0)||\leqslant f(\delta)$, $n\geqslant 1$.
2. $||F(x_n)||\leqslant\frac{1}{2}M(R)R||x_n-x_{n-1}||$, $n\geqslant 2$.
3. $||x_{n+1}-x_n||\leqslant\frac{1}{2}\delta f(\delta)||x_n-x_{n-1}||<R$, $n\geqslant 2$.
4. $||x_{n+1}-x_0||\leqslant(1+\frac{1}{2}\delta f(\delta)+(\frac{1}{2}\delta f(\delta))^2+\cdots+(\frac{1}{2}\delta f(\delta))^n)a<R$, $n\geqslant 1$.

Proof. Firstly, notice that, for all $x\in\overline{B(x_0,R)}$, we have

$$||I-\Gamma_0F'(x)||\leqslant||\Gamma_0||\,||F'(x)-F'(x_0)||\leqslant\delta<1.$$

Then, $F'(x)^{-1}$ exists and

$$||F'(x)^{-1}F'(x_0)||\leqslant\frac{1}{1-\delta}=f(\delta).$$

In addition, we have that $||x_1-x_0||\leqslant a<R$, and then $x_1\in B(x_0,R)$. Therefore $\Gamma_1=F'(x_1)^{-1}$ exists with $||F'(x_1)^{-1}F'(x_0)||\leqslant f(\delta)$. Further, by Taylor's formula, we obtain

$$\begin{aligned}F(x_1)&=F(x_0)+F'(x_0)(x_1-x_0)+\int_{x_0}^{x_1}F''(x)(x_1-x)\,\mathrm{d}x\\&=\int_0^1F''(x_0+t(x_1-x_0))(1-t)(x_1-x_0)^2\,\mathrm{d}t.\end{aligned}$$

Consequently

$$||F(x_1)||\leqslant\tfrac{1}{2}M(R)R||x_1-x_0||,$$

and

$$||x_2 - x_1|| \leqslant ||\Gamma_1 F'(x_0)||\,||\Gamma_0||\,||F(x_1)|| \leqslant \tfrac{1}{2}\delta f(\delta)||x_1 - x_0|| < a < R. \tag{2}$$

So, by Lemma 1,

$$\begin{aligned} ||x_2 - x_0|| &\leqslant ||x_2 - x_1|| + ||x_1 - x_0|| \leqslant \left(1 + \tfrac{1}{2}\delta f(\delta)\right)||x_1 - x_0|| \\ &\leqslant \left(1 + \tfrac{1}{2}\delta f(\delta)\right) a < R. \end{aligned} \tag{3}$$

Secondly, it is possible to go on with the process because $x_2 \in B(x_0, R)$. Then $\Gamma_2 = F'(x_2)^{-1}$ exists and moreover

$$||\Gamma_2 F'(x_0)|| \leqslant f(\delta).$$

In addition, as in the previous step, we have

$$\begin{aligned} ||F(x_2)|| &= \left|\left|\int_0^1 F'(x_1 + t(x_2 - x_1))(1 - t)(x_2 - x_1)^2\,\mathrm{d}t\right|\right| \\ &\leqslant \tfrac{1}{2}M(R)R||x_2 - x_1||. \end{aligned}$$

Then

$$\begin{aligned} ||x_3 - x_2|| &\leqslant ||\Gamma_2 F'(x_0)||\,||\Gamma_0||\,||F(x_2)|| \leqslant \tfrac{1}{2}\delta f(\delta)||x_2 - x_1|| \\ &< ||x_2 - x_1|| < ||x_1 - x_0|| < a < R \end{aligned}$$

and finally, by Lemma 1, (2) and (3),

$$\begin{aligned} ||x_3 - x_0|| \leqslant ||x_3 - x_2|| + ||x_2 - x_0|| &\leqslant \tfrac{1}{2}\delta f(\delta)||x_2 - x_1|| + \left(1 + \tfrac{1}{2}\delta f(\delta)\right) a \\ &\leqslant \left(1 + \tfrac{1}{2}\delta f(\delta) + \left(\tfrac{1}{2}\delta f(\delta)\right)^2\right) a < R. \end{aligned}$$

The rest of the proof follows inductively. Assume that the conditions of the Lemma hold for $2, 3, \ldots, n-1$. As $x_n \in \overline{B(x_0, R)} \subseteq \Omega$, then Γ_n exists and

$$||\Gamma_n F'(x_0)|| \leqslant f(\delta).$$

Next, as $||x_n - x_{n-1}|| < R$, we have

$$||F(x_n)|| \leqslant \frac{1}{2}M(R)R||x_n - x_{n-1}||. \tag{4}$$

So, it follows that

$$||x_{n+1} - x_n|| \leqslant ||\Gamma_n F'(x_0)||\,||\,||\Gamma_0||F(x_n)|| \leqslant \tfrac{1}{2}\delta f(\delta)||x_n - x_{n-1}|| < R.$$

Now, take into account that

$$||x_n - x_{n-1}|| \leqslant ||\Gamma_{n-1}F'(x_0)||\,||\Gamma_0||\,||F(x_{n-1})|| \leqslant \tfrac{1}{2}\delta f(\delta)||x_{n-1} - x_{n-2}|| \leqslant \cdots \leqslant \left(\tfrac{1}{2}\delta f(\delta)\right)^{n-2}||x_2 - x_1|| \leqslant \left(\tfrac{1}{2}\delta f(\delta)\right)^{n-1} a < R,$$

by Lemma 1, we have

$$||x_{n+1} - x_n|| \leqslant \left(\tfrac{1}{2}\delta f(\delta)\right)^{n} a < R. \tag{5}$$

Finally, from (5) and the previous lemma:

$$\begin{aligned} ||x_{n+1} - x_0|| &\leqslant ||x_{n+1} - x_n|| + ||x_n - x_0|| \\ &\leqslant \left(1 + \tfrac{1}{2}\delta f(\delta) + \left(\tfrac{1}{2}\delta f(\delta)\right)^2 + \cdots + \left(\tfrac{1}{2}\delta f(\delta)\right)^{n-1}\right) a + \tfrac{1}{2}\delta f(\delta)||x_n - x_{n-1}|| \\ &\leqslant \left(1 + \tfrac{1}{2}\delta f(\delta) + \left(\tfrac{1}{2}\delta f(\delta)\right)^2 + \cdots + \left(\tfrac{1}{2}\delta f(\delta)\right)^{n}\right) a < R. \qquad \square \end{aligned}$$

Theorem 3. *Under conditions* $(\mathscr{K}')$ *we have that Newton's method* (1) *converges to a solution* x^* *of the equation* $F(x) = 0$.

Proof. From Lemma 2, it follows that

$$\begin{aligned} ||x_{n+m} - x_n|| &\leqslant ||x_{n+m} - x_{n+m-1}|| + \cdots + ||x_{n+1} - x_n|| \\ &\leqslant ||\Gamma_{n+m-1}F'(x_0)||\,||\Gamma_0||\,||F(x_{n+m-1})|| + \cdots + ||\Gamma_nF'(x_0)||\,||\Gamma_0||\,||F(x_n)|| \\ &\leqslant f(\delta)b\sum_{j=n}^{n+m-1}||F(x_j)|| \leqslant \tfrac{1}{2}\delta f(\delta)a\sum_{j=n}^{n+m-1}\left[\tfrac{1}{2}\delta f(\delta)\right]^{j-1}. \end{aligned}$$

Then $\{x_n\}$ is a Cauchy sequence and therefore converges. Let x^* be the limit. By letting $n \to \infty$ in (4), we obtain $F(x^*) = 0$. $\square$

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