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On bornologies, locales and toposes of M-sets ☆

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Abstract

Let M be the monoid of all endomaps of a non-empty set N, Ω the locale of all ideals of M, and let $\mathscr{M}$ be the topos of all M-sets. The core of this paper is formed by a locale $\mathbf{B}$, a subtopos $\mathscr{B} \hookrightarrow \mathscr{M}$ and two theorems, where $\mathbf{B}$ is the locale of all bornologies defined on subsets of N and $\mathscr{B}$ is the topos of j-sheaves for a topology $j : \Omega \to \Omega$. The first theorem shows a morphism of locales $\mathbf{B} \to \Omega$ with nucleus j which induces an isomorphism of locales between $\mathbf{B}$ and the sublocale $\Omega_j \hookrightarrow \Omega$. The second theorem, which generalizes the first one, gives an equivalence between the category of Kolmogorov bornological spaces and bounded maps, and the full subcategory $\mathscr{B}' \hookrightarrow \mathscr{B}$ formed by all j-sheaves which are separated for the double negation topology of $\mathscr{B}$.

MSC: 18B25; 18F20; 46A17

0. Introduction

We consider the set $\mathbf{B}$ of all bornologies into a non-empty set N and the topos $\mathscr{M}$ of all M-sets and equivariant maps between them, where M is the monoid of all endomaps $N \to N$. In abstract functional analysis one considers bornologies related to sequences $s : N \to X$, where N is the set of natural numbers, but the constructions in this paper work for every non-empty set N and they have a very particular sense when N is finite. The core of this paper is formed by a theorem about bornologies into

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N in the context of locales and a second theorem about bornological spaces, which generalizes the first one in a context of big categories.

This paper subsumes some results in [6] which are different of those improved in [2]. The basic ideas for the relation between bornologies and toposes were communicated by Lawvere in several talks [7] during the Bogotá 1983 workshop on category theory, but they were not included in the later paper [8]. For locales and toposes we refer to [1,4,11] and [3] for bornologies.

Now we give a more detailed description of the contents of this work. In Section 1 we deal with the locale $\mathbf{B}$, the locale Ω of all ideals of M (the subobject classifier of $\mathscr{M}$) and the boolean locale $\mathscr{P}(N)$ of all subsets of N. We define two open morphisms of locales $\mathscr{P}(N) \to \mathbf{B}$ and $\mathscr{P}(N) \to \Omega$ with similar properties. Then we complete in Section 2 a commutative triangle with a morphism of locales $\mathbf{B} \to \Omega$ which gives us, as usual, a nucleus $j : \Omega \to \Omega$ and a sublocale $\Omega_j = j(\Omega) \hookrightarrow \Omega$. Our first theorem says that there exists an isomorphism of locales $\mathbf{B} \cong \Omega_j$.

Section 3 is devoted to the topos $\mathscr{M}$, in particular to study the double negation topology $\mathbf{J}_{\neg\neg}$ and the associated subtopos $\mathscr{M}_{\neg\neg}$. We calculate $\mathbf{J}_{\neg\neg} = \{I \in \Omega; C \subset I\}$ and $\mathscr{M}_{\neg\neg} \cong \mathscr{S}$, where $\mathscr{S}$ is the topos of sets. We also consider the full subcategory $\mathscr{M}'$ of all $\neg\neg$-separated M-sets. In Section 4 we note that the nucleus j above is equivariant, so that it defines a Grothendieck topology $\mathbf{J} = j^{-1}(M)$ and the subtopos $\mathscr{B} \hookrightarrow \mathscr{M}$ of $\mathbf{J}$-sheaves. Since Ω_j is the subobject classifier of $\mathscr{B}$, the locale $\mathbf{B}$, with a natural structure of M-set, is also an object of true values in the topos $\mathscr{B}$. We describe the sheafification functor $\mathscr{M} \to \mathscr{B}$ over the subcategory $\mathscr{B} \cap \mathscr{M}'$. Then, we consider the double negation topology $k : \Omega_j \to \Omega_j$ in the topos $\mathscr{B}$ and we prove that the subcategory $\mathscr{B}'$ of all j-sheaves which are k-separated is $\mathscr{B} \cap \mathscr{M}'$ and $\mathscr{B}_k \cong \mathscr{S}$.

Finally, we obtain in Section 5 a commutative triangle formed with the functors $\mathscr{S} \hookrightarrow$ K-BOR, $\mathscr{S} \hookrightarrow \mathscr{M}'$, where K-BOR is the category of all Kolmogorov bornological spaces (and bounded maps between them), and the functor K-BOR $\hookrightarrow \mathscr{M}'$ defined by means of bounded sequences. This diagram is an extension of the diagram of locales in Section 2. Then we give the second theorem: there exists an equivalence between the categories K-BOR and $\mathscr{B}'$. Let us note that Johnstone [5, Proposition 3.6] has proved a similar result involving the category of all subsequential spaces and continuous maps between them, and the category $\mathscr{T}'$ corresponding to the topos $\mathscr{T}$ of all T-sets, where T is the monoid of all continuous endomaps of $\mathbb{N}^+$ (the one point compactification of the discrete space of natural numbers).

1. Locales of bornologies and locales of ideals

Let N be a non-empty set. A *bornology into* N is a non-empty family of subsets of N (called *bounded* subsets) which is hereditary under inclusion and stable under finite union. Let $\mathbf{B}$ denote the set of all bornologies into N, which is ordered in a natural way. The intersection of bornologies into N is a bornology into N, so that $\mathbf{B}$ is a locale with maximum $\mathscr{P}(N)$, the set of all subsets of N, and minimum $\{\emptyset\}$. Let us note that the locale $\mathbf{B}$ also depends on N, but we omit this fact in the notation. The supremum of a family $\{\beta_i\}_i$ in $\mathbf{B}$ is the bornology β whose bounded sets are all the subsets of finite unions of bounded sets of the different β_i's. If β is a bornology into

N, we denote by $E(\beta)$ the union of all the subsets of N belonging to β. If $A = E(\beta)$ we say that β is a *bornology on A* or a bornology with *extent A*; we also say that (A, β) is a *bornological space.*

Given a locale L, we denote by $\neg x$ the negation of an element $x \in L$ and by $L_{\neg\neg}$ the image of the double negation nucleus $\neg\neg : L \to L$ (in general, L_j will denote the image of a nucleus $j : L \to L$). It is easy to verify that the negation of the locale $\mathbf{B}$ is given by $\neg\beta = \mathscr{P}(N - E(\beta))$, hence $\neg\neg\beta = \mathscr{P}(E(\beta))$. Let us note that there are monotone maps

$$E : \mathbf{B} \to \mathscr{P}(N) \quad \text{and} \quad \mathscr{K},\ \mathscr{P} : \mathscr{P}(N) \to \mathbf{B},$$

where $\mathscr{K}(A)$ is the bornology that consists of all finite subsets of A and $\mathscr{P}(A)$ is the discrete bornology on A (all subsets of A), so that the double negation map of $\mathbf{B}$ is $\neg\neg = \mathscr{P} \circ E : \mathbf{B} \to \mathbf{B}$. We shall use the open sublocale $(\mathscr{K}(N)]$ of $\mathbf{B}$ formed by all bornologies contained in $\mathscr{K}(N)$, with the corresponding nucleus, $j = \mathscr{K}(N) \to (-) : \mathbf{B} \to \mathbf{B}$, given by the implication in the locale $\mathbf{B}$. The following properties are easy to prove.

Lemma 1.1. (i) *There exist Galois connections* $\mathscr{K} \dashv E \dashv \mathscr{P}$.
(ii) *The equalities* $E \circ \mathscr{K} = \mathrm{id} = E \circ \mathscr{P}$ *hold.*
(iii) $\mathscr{K}(A \cap E(\beta)) = \mathscr{K}(A) \cap \beta$ (*Frobenius formula*).

Proposition 1.2. *The locale* $\mathbf{B}$ *satisfies*:
(i) *The double negation nucleus is* $\mathscr{K}(N) \to (-)$.
(ii) *There exist isomorphisms of locales* $\mathbf{B}_{\neg\neg} \cong \mathscr{P}(N) \cong (\mathscr{K}(N)]$.

Proof. (i) By Lemma 1.1, we know that the double negation map of $\mathbf{B}$ is the nucleus $\mathscr{P} \circ E$ associated to the morphism of locales $(E, \mathscr{P}) : \mathscr{P}(N) \to \mathbf{B}$. We shall prove that $\mathscr{P}(E(\beta)) = j(\beta)$ for each bornology β, where $j = \mathscr{K}(N) \to (-)$. The inclusion $\mathscr{P}(E(\beta)) \subset j(\beta)$ is equivalent to the counit $\mathscr{K}(E(\beta)) \subset \beta$ because the left-hand side is equal to $\mathscr{K}(N) \cap \mathscr{P}(E(\beta))$ by the Frobenius equality. On the other hand, by two adjunctions, the inclusion $j(\beta) \subset \mathscr{P}(E(\beta))$ is equivalent to $\mathscr{K}(E(j(\beta))) \subset \beta$, where, by the Frobenius equality, the left-hand side is equal to $\mathscr{K}(N) \cap (\mathscr{K}(N) \to \beta)$, that is $\mathscr{K}(N) \cap \beta$.

(ii) By Lemma 1.1 the locale $\mathscr{P}(N)$ is isomorphic to $\mathbf{B}_{\neg\neg}$. In fact we have $\mathbf{B}_{\neg\neg} = \mathrm{Im}\,\mathscr{P}$, which is the set of all discrete bornologies into N. Moreover, each bornology $\beta \subset \mathscr{K}(N)$ is of the form $\beta = \mathscr{K}(E(\beta))$ (Frobenius equality) and the locale $(\mathscr{K}(N)]$ is isomorphic to $\mathscr{P}(N)$ through $\mathscr{K}$ and E. $\square$

Our next step is to consider (right) ideals in the monoid M of all endomaps of N (with composition), that is, subsets I of M such that $f \circ g \in I$ for any $f \in I$ and $g \in M$. Let Ω be the set of all ideals I of M. Given an ideal I and $f \in M$ the set $(I : f) = \{g \in M; f \circ g \in I\}$ is also an ideal. It is well known that Ω is a locale with the usual union, intersection, and

$$I \to J = \{f \in M; (I{:}f) \subset (J{:}f)\}, \quad \neg I = I \to \emptyset = \{f \in M; \forall g \in M, f \circ g \notin I\},$$
$$\neg\neg I = \{f \in M; \forall g \in M, \exists h \in M, f \circ g \circ h \in I\}.$$

Let C be the set of all *constant* maps of M, so that $C \cong N$. The subset C is an ideal and each subset of C is also an ideal, that is, $\mathscr{P}(C) = (C]$ is the open sublocale of Ω defined by C. Since C is non-empty we have $\neg C = \emptyset$. It is easy to see that $C \cap \neg\neg I = C \cap I$, so that $\neg\neg I \subset C \to I$; conversely, $C \to I \subset \neg\neg I$ is equivalent to $\neg I \cap (C \to I) = \emptyset$, but this is clear in a Heyting algebra since the left-hand side is contained in $\neg C$. Hence, we have $\neg\neg I = C \to I$, and $\neg\neg I = M$ if and only if $C \subset I$.

By means of the formulas

$$\mathrm{Ext}(I) = \bigcup\{\mathrm{Im}\,(f); f \in I\},\ \mathrm{Cont}(A) = \{f \in M; \mathrm{Im}\, f \subset A\},\ c(A) = C \cap \mathrm{Cont}(A),$$

we define three monotone maps $\mathrm{Ext} : \Omega \to \mathscr{P}(N)$ and c, $\mathrm{Cont} : \mathscr{P}(N) \to \Omega$, and we say that $\mathrm{Ext}(I)$ is the *extent* of the ideal I and $\mathrm{Cont}(A)$ is the *content* of the subset A. We shall see later the reason why we call "extent" both $E(\beta)$ and $\mathrm{Ext}(I)$. If c_n is the constant map valued onto $n \in N$, then it is clear that $\mathrm{Ext}(I) = \{n \in N; c_n \in I\} \cong C \cap I$ and $c(A) = \{c_n; n \in A\} \cong A$. It is useful to note that $c_n \in I$ if $f \in I$ and $n \in \mathrm{Im}\,(f)$. We omit the proof of the next lemma because it is straightforward. Let us note that $C = c(N)$, so that this lemma is similar to Lemma 1.1.

Lemma 1.3. (i) *There exist Galois connections* $c \dashv \mathrm{Ext} \dashv \mathrm{Cont}$.
(ii) *The equalities* $\mathrm{Ext} \circ c = \mathrm{id} = \mathrm{Ext} \circ \mathrm{Cont}$ *hold.*
(iii) $c(A \cap \mathrm{Ext}(I)) = c(A) \cap I$ (*Frobenius formula*).

Proposition 1.4. *The locale* Ω *satisfies*:
(i) *The double negation nucleus is* $\mathrm{Cont} \circ \mathrm{Ext} = C \to (-)$.
(ii) *There exist isomorphisms of locales* $\Omega_{\neg\neg} \cong \mathscr{P}(N) \cong (C]$.

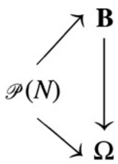
Proof. We know that $\neg\neg = C \to (-)$. By Lemma 1.3 $(\mathrm{Ext}, \mathrm{Cont}) : \mathscr{P}(N) \to \Omega$ is a morphism of locales with nucleus $\mathrm{Cont} \circ \mathrm{Ext}$. Since Lemma 1.3 is similar to Lemma 1.1, the proof of this proposition is similar to that given in Proposition 1.2. □

We have found two isomorphic copies of the boolean locale $\mathscr{P}(N)$ into the locales $\mathbf{B}$ and Ω. These diagrams are examples of "unity and identity of adjoint opposites" (UIAO) using the terminology of Lawvere [10]. For instance, the map E *unifies* the *opposites* $\mathscr{K}$ and $\mathscr{P}$, with the idempotent maps $\mathscr{K} \circ E$ and $\neg\neg$, respectively. In these cases, the Frobenius formula gives an isomorphism between the opposites.

Example 1.5. If N is finite then $\mathscr{K} = \mathscr{P}$ and $\mathbf{B} = \{\mathscr{P}(A); A \subset N\} \cong \mathscr{P}(N)$. Hence, in this case we only consider the isomorphism on the right-hand side in the above formula. When $N = \{1\}$ we have $M = \{\mathrm{id}\}$, $\mathscr{P}(N) = \{\emptyset, N\} \cong \{\emptyset, M\} = \Omega$. When $N = \{1, 2\}$ then $\mathscr{P}(N) = \{\emptyset, \{1\}, \{2\}, N\}$, $M = \{\mathrm{id}, \tau, c_1, c_2\}$ with $\tau^2 = \mathrm{id}$, $C = \{c_1, c_2\}$ and $\Omega = \{\emptyset, C_1, C_2, C, M\}$ where $C_i = \{c_i\}$, $i = 1, 2$; now we can calculate $\mathbf{J}_{\neg\neg} = \{C, M\}$ and $\Omega_{\neg\neg} = \{\emptyset, C_1, C_2, M\}$.

2. The isomorphism of locales

Our aim in this section is to compare both the locales **B** and Ω by means of a morphism of locales which induces a nucleus j of Ω such that $\mathbf{B} \cong \Omega_j$. This morphism of locales and the two morphisms studied in the last section complete a commutative diagram.



First, we extend to bornologies the notion of content given for subsets. Let β be a bornology into N. We shall say that $\mathrm{Cont}(\beta) = \{f \in M; \mathrm{Im}\, f \in \beta\}$ is the *content* of β. Since the sets $\mathrm{Cont}(A)$ and $\mathrm{Cont}(\mathscr{P}(A))$ are equals, this terminology is coherent and we have a monotone map $\mathrm{Cont} : \mathbf{B} \to \Omega$ extending the content $\mathscr{P}(N) \to \Omega$ through $\mathscr{P}$. On the other hand, to any $f \in M$ and $I \in \Omega$ we associate the bornology

$$\mathrm{Bor}(I) = \sup\{\mathrm{Bor}(f); f \in I\} \quad \text{where } \mathrm{Bor}(f) = \mathscr{P}(\mathrm{Im}\, f),$$

where the sup is taken in the locale **B**, so that bounded sets in $\mathrm{Bor}(I)$ are subsets of finite unions of some images $\mathrm{Im}\, f$, with $f \in I$. Let us note that the sets $E(\beta)$ and $\mathrm{Ext}(I)$ are equal if $\beta = \mathrm{Bor}(I)$. The map $\mathrm{Bor} : \Omega \to \mathbf{B}$ is monotone and also is monotone the composition $j = \mathrm{Cont} \circ \mathrm{Bor} : \Omega \to \Omega$ (in fact, we shall see that j is a nucleus). For the forthcoming first theorem, we need some particular non-constructive properties of our monoid M that we state without proof.

Lemma 2.1. *The following properties of the monoid M hold*:

(i) *For every non-empty $A \subset N$, there exists $f \in M$ such that $A = \mathrm{Im}\, f$.*
(ii) *Given $f, g \in M$, if $\mathrm{Im}\, f \subset \mathrm{Im}\, g$ there exists $h \in M$ such that $f = g \circ h$.*
(iii) *Given $f, g \in M$ and $I \in \Omega$, if $\mathrm{Im}\, f \subset \mathrm{Im}\, g$ and $g \in I$ then $f \in I$.*

Theorem 2.2. *There exists an isomorphism of locales $\mathbf{B} \cong \Omega_j$ induced by the nucleus $j = \mathrm{Cont} \circ \mathrm{Bor} : \Omega \to \Omega$.*

Proof. We shall prove that

$$\mathrm{Bor} \dashv \mathrm{Cont}, \ \mathrm{Bor}(I \cap J) = \mathrm{Bor}(I) \cap \mathrm{Bor}(J) \text{ and } \mathrm{Bor} \circ \mathrm{Cont} = \mathrm{id}.$$

Hence the pair (Bor, Cont) is a regular monomorphism of locales and the theorem follows from the Proposition 1.5.4 in [1].

It is easy to verify the adjunction and $\mathrm{Bor}(I \cap J) \subset \mathrm{Bor}(I) \cap \mathrm{Bor}(J)$. Now, if we suppose that B is a bounded subset in $\mathrm{Bor}(I) \cap \mathrm{Bor}(J)$ then there exist $f_1, \ldots, f_m \in I$

and $g_1,\dots,g_n \in J$ such that

$$B \subset \operatorname{Im} f_1 \cup \cdots \cup \operatorname{Im} f_m \quad \text{and} \quad B \subset \operatorname{Im} g_1 \cup \cdots \cup \operatorname{Im} g_n,$$

so that $B \subset \bigcup B_{ij}$, where $B_{ij} = \operatorname{Im} f_i \cap \operatorname{Im} g_j$. For the moment we prove that if $A = \operatorname{Im} f \cap \operatorname{Im} g$, $f \in I$, $g \in J$, then $A \subset \operatorname{Bor}(I \cap J)$; in fact, by Lemma 2.1(i) we have $A = \operatorname{Im} h$ with $h \in M$, but then $\operatorname{Im} h \subset \operatorname{Im} f$ so that $h \in I$ by Lemma 2.1(iii), and the same argument shows that $h \in J$; hence $h \in I \cap J$ and this means that A is a bounded in $\operatorname{Bor}(I \cap J)$. In this way, we have proved that $B_{ij} \in \operatorname{Bor}(I \cap J)$ for all B_{ij} and this implies that $B \in \operatorname{Bor}(I \cap J)$.

Finally, the counit of the adjunction means that $\operatorname{Bor}(\operatorname{Cont}(\beta)) \subset \beta$; conversely, if B is a non-empty bounded set in β, then $B = \operatorname{Im} f$ by Lemma 2.1(i) and this means that $f \in \operatorname{Cont}(\beta)$, that is $B \in \operatorname{Bor}(\operatorname{Cont}(\beta))$. □

We complete this section by giving an explicit description of the M-subsets Ω_j and $\mathbf{J} = j^{-1}(M)$ of Ω. The proof of the next proposition uses Lemma 2.1 and it is straightforward.

Proposition 2.3. *For each ideal $I \in \Omega$, the following characterizations hold*:
(i) *$I \in \Omega_j$ if and only if* $\operatorname{Bor}(I) = \{B \subset N; \exists f \in I, B \subset \operatorname{Im} f\}$.
(ii) *$I \in \mathbf{J}$ if and only if* $\operatorname{Bor}(I) = \mathscr{P}(N)$.

The condition $\operatorname{Bor}(I) = \mathscr{P}(N)$ means that $N \in \operatorname{Bor}(I)$, that is, $N = \bigcup_{1 \leqslant i \leqslant m} \operatorname{Im} f_i$, where each f_j belongs to I. Let us note that every ideal of the form $\operatorname{Cont}(\beta)$ belongs to Ω_j, and that Proposition 2.3(ii) implies $C \subset I$ ($N = \operatorname{Ext}(I)$). Moreover, if N is finite then $j = \neg\neg$.

3. The double negation in the topos of M-sets

A set X with an action of a monoid M is called an M-set. An action of M on X is a map $X \times M \to X$, usually denoted simply by xf, such that $x(\mathrm{id}) = x$ and $(xf)g = x(f \circ g)$ for every $x \in X$, $f, g \in M$. Let $\mathscr{M}$ be the topos of M-sets and *equivariant* maps $\phi : X \to Y$, that is, maps preserving the actions: $\phi(xf) = (\phi(x))f$, for every $x \in X$ and $f \in M$. The subobject classifier of $\mathscr{M}$ is Ω, which is an M-set with the action defined by $(I : f) = \{g \in M; f \circ g \in I\}$, $f \in M$. For each M-subset $U \subset X$, the characteristic morphism $\varphi : X \to \Omega$ is given by $\varphi(x) = (U : x)$, where $(U : x) = \{f \in M; xf \in U\}$. In particular, the characteristic morphism $\varphi : M \to \Omega$ of an ideal $I \subset M$ is the unique equivariant map defined by $\varphi(\mathrm{id}) = I$. Let $\operatorname{Sub}(X)$ be the set of all subobjects of X in $\mathscr{M}$, so that $\Omega \cong \operatorname{Sub}(M)$. The logical operations in the locale $\operatorname{Sub}(X)$ are defined like in the case $X = M$ (see Section 1) and for every $x \in X$ the equalities

$$(U \cup V : x) = (U : x) \cup (V : x), \quad (U \to V : x) = (U : x) \to (V : x), \text{ etc.}$$

hold. In particular, the logical operations of Ω are equivariant and hence the nucleus $\neg\neg = C \to (-)$ of Ω is an equivariant map. Recall that $I \to (-)$ is not equivariant for every ideal I (see Lemma 1.1 in [2] for a characterization of this kind of ideals).

We give a example: if we suppose $m \neq n \in N$ and take the ideals $I = \{c_m\}$, $J = \{c_n\}$, then we have $I \to (J : c_m) = \neg\neg I$ and $((I \to J) : c_m) = \emptyset$, but $c_n \in \neg\neg I$; hence $I \to (-)$ is not equivariant.

Any M-set X has a subset of fixed points ($x \in X$ is a fixed point if $xf = x$ for every $f \in M$) denoted $\Gamma(X)$. In particular $C = \Gamma(M)$. We have $\neg\neg U = \Gamma(X) \to U$, for every M-subset $U \subset X$, and $C \subset (\Gamma(X) : x)$ for every $x \in X$. It is clear that from the evaluation map $M \times N \to N$ and the action $X \times M \to X$ we can obtain the set $\Gamma(X)$ as a coequalizer of two maps of the form $N \times X \times M \to N \times X$. Then the canonical map $N \times X \to \Gamma(X)$ corresponds to

$$\mu : X \to \Gamma(X)^N, \quad \mu(x)(n) = xc_n,$$

where $\Gamma(X)^N$ is the set of all maps $N \to \Gamma(X)$ in $\mathscr{S}$. Given a set S, the set S^N of all maps $N \to S$ in $\mathscr{S}$, which we shall call *sequences* of S, is an M-set with action the composition of sequences $N \to S$ and endomaps in M. We have $\Gamma(S^N) \cong S$ and $\mu \cong \mathrm{id}$ when $X = S^N$. For the trivial M-set S (action given by the projection $S \times M \to S$) the map $\mu : S \to S^N$ is the natural inclusion given by the constant sequences. In $\mathscr{S}$, maps $g : X \to S^N$ are in one-to-one correspondence with maps $G : N \times X \to S$, and g is equivariant if and only if G factorizes by the coequalizer $\Gamma(X)$, so that we have an adjunction $\Gamma \dashv (-)^N$ with unit the natural transformation μ defined above. The one-to-one correspondence between the set of all maps $h : \Gamma(X) \to S$ and the set of all equivariant maps $H : X \to S^N$ is given by $H(x) = h \circ \mu(x)$, and if x is a fixed point then $h(x)$ is the constant value of $H(x)$. Hence, the topos $\mathscr{S}$ of sets is equivalent to the full subcategory of $\mathscr{M}$ formed by all M-sets X such that μ is an isomorphism in the level X.

It is well known that there exists an geometric morphism $(\Delta, \Gamma) : \mathscr{M} \to \mathscr{S}$, where, for every set S, $\Delta(S)$ is the set S with the trivial action given by the projection $S \times M \to S$. For instance, for each M-set X, $\Gamma(X)$ is a trivial M-subset of X. Actually, there exists an essential geometric morphism

$$\Delta \dashv \Gamma \dashv (-)^N : \mathscr{S} \to \mathscr{M}, \quad \Gamma \circ (-)^N \cong \mathrm{id}, \quad \Gamma \circ \Delta = \mathrm{id}.$$

Let us note that this is another example of UIAO (in this case it is also called essential localization): Γ unifies the opposites Δ and $(-)^N$ which, by composing with Γ, give the idempotent functors $\Delta \circ \Gamma$ and $\Sigma = (-)^N \circ \Gamma : \mathscr{M} \to \mathscr{M}$, respectively. This diagram of functors is an extension of that given by the Galois connections $c \dashv \mathrm{Ext} \dashv \mathrm{Cont}$ in Lemma 1.3, since we have natural inclusion functors $\mathscr{P}(N) \hookrightarrow \mathscr{S}$ and $\Omega \hookrightarrow \mathscr{M}$.

Now we shall see this construction in terms of sheaves. We recall that an equivariant map $j : \Omega \to \Omega$ is the characteristic morphism of the M-subset $\mathbf{J} = j^{-1}(M)$, and j is a nucleus if and only if $\mathbf{J}$ is a topology in the sense of Grothendieck. An M-set X is an I-sheaf if for every equivariant map $H : I \to X$, there exists a unique equivariant extension $H' : M \to X$ of H, that is, there exists a unique $x \in X$ such that $H(f) = xf$ for every $f \in I$; and X is a $\mathbf{J}$-sheaf (j-sheaf) if it is an I-sheaf for every ideal $I \in \mathbf{J}$. If we only suppose that the element $x \in X$ is unique when it exists then we say that X is *I-separated* or $\mathbf{J}$-separated (j-separated), respectively. The full subcategory $\mathscr{M}_j \hookrightarrow \mathscr{M}$ of all $\mathbf{J}$-sheaves is a topos with subobject classifier Ω_j. The inclusion has a left exact left adjoint functor called the sheafification functor. Limits and exponentials in $\mathscr{M}_j$ are like in $\mathscr{M}$, and colimits in $\mathscr{M}_j$ are constructed by sheafification.

Now we consider the subtopos $\mathscr{M}_{\neg\neg} \hookrightarrow \mathscr{M}$ of sheaves for the nucleus $\neg\neg : \Omega \to \Omega$ (Proposition 1.4), so that the associated topology is $\mathbf{J}_{\neg\neg} = \{I \in \Omega; C \subset I\}$. The following elementary results are given without proof.

Lemma 3.1. *Every M-set X satisfies*:

(i) *X is a $\mathbf{J}_{\neg\neg}$-sheaf if and only if it is a C-sheaf.*
(ii) *X is C-separated if and only if μ is a monomorphism.*
(iii) *X is C-sheaf if and only if μ is an isomorphism.*

Proposition 3.2. *The topos $\mathscr{M}_{\neg\neg}$ verifies*:

(i) *$\Sigma : \mathscr{M} \to \mathscr{M}_{\neg\neg}$ is the sheafification functor.*
(ii) *The functor $(-)^N$ induces an equivalence of categories $\mathscr{S} \cong \mathscr{M}_{\neg\neg}$.*

We shall add two simple comments. The condition C-separated for an M-set X means that $xc_n = yc_n$ for all $n \in N$ implies $x = y$, and if X is a set of sequences, this is the principle of extensionality for sequences. The monoid M is clearly a C-sheaf, and actually $\mathbf{J}_{\neg\neg}$ is the canonical topology of the topos $\mathscr{M}$, since if M is $\mathbf{J}$-sheaf then $\mathbf{J}_{\neg\neg} \subset \mathbf{J}$. We prove it: if $I \in \mathbf{J}$ then $(I : c_n) \in \mathbf{J}$ for all $n \in N$, but M is $(I : c_n)$-sheaf so that $(I : c_n) \neq \emptyset$, that is $(I : c_n) = M$ and $c_n \in I$; hence $C \subset I$.

By using Proposition 3.2 we have a new look for the above UIAO. The functor $\Sigma : \mathscr{M} \to \mathscr{M}_{\neg\neg}$ unifies the opposites $\Delta \circ \Gamma : \mathscr{M}_{\neg\neg} \hookrightarrow \mathscr{M}$ and the inclusion functor with the corresponding idempotents $\Delta \circ \Gamma$, $\Sigma : \mathscr{M} \to \mathscr{M}$, respectively. Let us note that $2 = \{0, 1\}$ is the subobject classifier of $\mathscr{S}$ and we can use $\mathscr{P}(N) \cong 2^N$ as subobject classifier of $\mathscr{M}_{\neg\neg}$ (Propositions 1.4 and 3.2), in this case the characteristic morphism corresponding to an M-subset $U \subset X$ in $\mathscr{M}_{\neg\neg}$ is $\varphi : X \to \mathscr{P}(N)$, $\varphi(x) = \{n \in N; xc_n \in U\}$. If X is a set of sequences then φ selects the points of each sequence belonging to U.

Let $\mathscr{M}'$ be the full subcategory of $\mathscr{M}$ formed by all C-separated M-sets, so that we have a chain $\mathscr{M}_{\neg\neg} \hookrightarrow \mathscr{M}' \hookrightarrow \mathscr{M}$ of full subcategories which are reflexive. The reflector functor $(-)' : \mathscr{M} \to \mathscr{M}'$ is given by $X' = \mu(X)$ and the reflector $\mathscr{M}' \to \mathscr{M}_{\neg\neg}$ is Γ, with $\Gamma(\mu(X)) \cong \Gamma(X)$. Let us note that the UIAO's considered above can be reformulated taking $\mathscr{M}'$ instead of $\mathscr{M}$. It is well known that $\mathscr{M}'$ is a quasitopos (Theorem 10.1 in [5], or Theorem 43.6 in [12]) but we do not use this structure in the present paper.

Example 3.3. The case $N = \{1\}$ is trivial since $\mathscr{M} = \mathscr{M}_{\neg\neg} = \mathscr{S}$, but $N = \{1, 2\}$ is an interesting case. By using the representation given by Lawvere [9], $\mathscr{M}$ is the category of all reflexive directed graphs with an involutive operation $x^* = x\tau$ (corresponding to the transposition $\tau \in M$, $\tau^2 = \mathrm{id}$, see Example 2.4), $\Gamma(X)$ represents the set of all vertices of X and $\Sigma(X) = \Gamma(X) \times \Gamma(X)$, so that $\mu(x) = (a, b)$ means that x is an arrow from the vertex a to the vertex b (then $\mu(x^*) = (b, a)$). A C-separated graph X (object in $\mathscr{M}'$) is an equivalence relation on the set $\Gamma(X)$. Finally, a C-sheaf is the equality relation on $\Gamma(X)$.

4. A topos of M-sets for bornologies

We shall consider $\mathbf{B}$ as an M-set with the action given by

$$\beta f = \{B \subset N; f(B) \in \beta\}.$$

Then the logical operations in $\mathbf{B}$, the maps $\mathrm{Cont}: \mathbf{B} \to \Omega$ and $\mathrm{Bor}: \Omega \to \mathbf{B}$, the nucleus $j = \mathrm{Cont} \circ \mathrm{Bor}$ and the bijection $\Omega_j \cong \mathbf{B}$ (see Section 2) all are equivariant. Since the nucleus j is equivariant, the M-subset $\mathbf{J} = j^{-1}(M)$ of Ω is a (Grothendieck) topology and we can consider the subtopos $\mathscr{B} \hookrightarrow \mathscr{M}$ of $\mathbf{J}$-sheaves. The subobject classifier of $\mathscr{B}$ is $\Omega_j \cong \mathbf{B}$, so that we can consider $\mathbf{B}$ as an M-set of true values of the topos $\mathscr{B}$. In this way, given an M-subset $U \subset X$ in $\mathscr{B}$, we take the characteristic morphism in the form $\varphi: X \to B$,

$$\varphi(x) = \{B \subset N; \exists f \in M, xf \in U,\ B \subset \mathrm{Im}\, f\}.$$

If $U = \Gamma(X)$, then $E(\varphi(x)) = N$ by using the constant maps. We shall call $\mathscr{B}$ the *bornological topos* because there exists the subobject classifier $\mathbf{B}$ and also because it contains a full reflexive subcategory of bornological spaces as we shall see in the next section. (In [2] we have used a topos, called bounded topos, defined in a similar way by using another monoid of maps.)

Before exploring the relation of the topos $\mathscr{B}$ to bornological spaces, we shall consider the full subcategories $\mathscr{B}_k \hookrightarrow \mathscr{B}' \hookrightarrow \mathscr{B}$ as defined in Section 3, but now from the double negation $k: \Omega_j \to \Omega_j$ in $\mathscr{B}$, which is the nucleus given by $k(I) = j(\neg\neg I)$, where $\neg\neg$ is the double negation in Ω. We shall denote by $\mathbf{J}_k$ the corresponding topology $k^{-1}(M) \subset \Omega_j$.

Proposition 4.1. (i) $\mathbf{J}_k = \mathbf{J}_{\neg\neg} \cap \Omega_j$. (ii) $\mathscr{B}' = \mathscr{M}' \cap \mathscr{B}$. (iii) $\mathscr{B}_k \cong \mathscr{S}$.

Proof. (i) Given $I \in \Omega_j$ we have $k(I) = j(C \to I) = j(C) \to I$ (with the implication in Ω). Hence $I \in \mathbf{J}_k$ if and only if $j(C) \subset I$, that is $C \subset I$ since $I \in \Omega_j$.

(ii) Like in Lemma 3.1, it is clear that an object X in $\mathscr{B}$ is $\mathbf{J}_k$-separated if and only if X is $j(C)$-separated, but $C \subset j(C)$ so that X is $j(C)$-separated if and only if X is C-separated as an object in $\mathscr{M}$.

(iii) Like in (ii), $\mathscr{B}_k = \mathscr{M}_{\neg\neg} \cap \mathscr{B} = \mathscr{M}_{\neg\neg}$, and then we use Proposition 3.2. □

By the isomorphism $\Omega_j \cong \mathbf{B}$ (Theorem 2.2) the topology $\mathbf{J}_k \subset \Omega_j$ is transformed in the topology $\{\beta \in \mathbf{B}; E(\beta) = N\} \subset \mathbf{B}$ corresponding to the double negation in $\mathbf{B}$. Moreover, $j(C) = \{f \in M; \mathrm{Im}\, f \text{ finite}\}$ is the ideal of Ω_j associated to the bornology $\mathscr{K}(N)$.

Topos theory says that there exists a sheafification functor, left exact and left adjoint to the inclusion $\mathscr{B} \hookrightarrow \mathscr{M}$, which we shall denote by $\mathbf{b}: \mathscr{M} \to \mathscr{B}$. In the next proposition we give an explicit description of $\mathbf{b}$ over the subcategory $\mathscr{M}'$.

Proposition 4.2. *If X is C-separated then*

$$\mathbf{b}(X) = \{s: N \to \Gamma(X); s(N) \in \beta\} \subset \Sigma(X),$$

where β is the bornology on $\Gamma(X)$ generated by the family of discrete bornologies $\{\mathscr{P}(\operatorname{Im}\mu(x));\ x \in X\}$.

Proof. By Proposition 2.3(ii) the topology $\mathbf{J}$ is subcanonical, hence $\mathscr{M}_{\neg\neg} \hookrightarrow \mathscr{B}$ and each M-set of the form S^N is a $\mathbf{J}$-sheaf, in particular $\Sigma(X)$ for all M-set X. If X is C separated then $\mu : X \to \Sigma(X)$ is mono and hence $\mathbf{b}(X)$ is the closure of $\mu(X)$ in $\Sigma(X)$, that is, a sequence $s : N \to \Gamma(X)$ belongs to $\mathbf{b}(X)$ if and only if $(\mu(X) : s) \in \mathbf{J}$, in other words, there exist maps $f_i \in M$ and elements $x_i \in X$, $1 \leqslant i \leqslant n$, such that $N = \operatorname{Im} f_1 \cup \cdots \cup \operatorname{Im} f_n$ and $s \circ f_i = \mu(x_i)$ for every index. Now it is clear that $s \in \mathbf{b}(X)$ implies $s(N) \in \beta$, where β is the bornology in the statement (it is clear that for every $x \in \Gamma(X)$ the condition $\{x\} \in \beta$ holds). Conversely, if $s(N) \in \beta$ then we have $s(N) \cap \operatorname{Im}\mu(x_1) \cup \cdots \cup \operatorname{Im}\mu(x_n)$ for some $x_i \in X$, $1 \leqslant i \leqslant n$, and for each index $s(N) \cap \operatorname{Im}\mu(x_i) \neq \emptyset$, so that we can find maps $f_i \in M$ such that $\operatorname{Im}(s \circ f_i) \subset \operatorname{Im}\mu(x_i)$; but then there exists $y_i \in X$ such that $s \circ f_i = \mu(y_i)$, $1 \leqslant i \leqslant n$. In fact, if we define a map $g \in M$ by choosing $g(n)$ in the non-empty fibre $\mu(x_i)^{-1}((s \circ f_i)(n))$ then it is easy to check that $(s \circ f_i) = \mu(x_i g)$. Hence we conclude that $s \in \mathbf{b}(X)$. $\square$

If we take in Proposition 4.2 a trivial M-set S we obtain the finite bornology $\mathscr{K}(S)$, so that in this case $\mathbf{b}(S) = \{s : N \to \Gamma(S); s(N) \text{finite}\}$.

Moreover, we obtain $\mathscr{P}(S)$ from the M-set S^N of sequences. Let us note that there exists a UIAO (or an essential localization)

$$\Lambda \dashv \Gamma \dashv (-)^N : \mathscr{S} \to \mathscr{B}, \quad \Gamma \circ (-)^N = \mathrm{id} = \Gamma \circ \Lambda$$

where $\Lambda = \mathbf{b} \circ \Delta$. If we take $\mathbf{B}$ in $\mathscr{B}$ then we have $\Lambda(\Gamma(\mathbf{B})) = \Gamma(\mathbf{B})^N \cong \mathscr{P}(N)$ with the action $Af = f^{-1}(A)$. Let us note that $\Gamma(\mathbf{B}) \cong \Gamma(\mathscr{P}(N)) \cong 2$. The equivariant maps $\mathscr{P}$, E are, respectively, the unit and the counit of the adjunction $\Lambda \dashv \Gamma$. Since the subcategory $\mathscr{M}' \hookrightarrow \mathscr{M}$ is reflexive (Section 3) the subcategory $\mathscr{B}' \hookrightarrow \mathscr{M}$ is reflexive too.

5. Kolmogorov bornological spaces and M-sets

We have a chain of categories $\mathscr{S} \hookrightarrow \mathscr{B}' \hookrightarrow \mathscr{B} \hookrightarrow \mathscr{M}$, where $\mathscr{B}'$ is a quasitopos (like $\mathscr{M}'$, see Section 3) and all others are toposes. In this section we shall identify $\mathscr{B}'$ with a category of bornological spaces.

Let BOR be the category of all bornological spaces (S, β) and bounded maps between them. (Recall that, given bornological spaces (S, β) and (S', β'), a map $f : S \to S'$ is bounded if for every $B \in \beta$ we have $f(B) \in \beta'$.) We have a forgetful functor $E : \mathrm{BOR} \to \mathscr{S}$ with left adjoint $\mathscr{K}$ (finite bornology) and right adjoint $\mathscr{P}$ (discrete bornology) such that the equalities $E \circ \mathscr{K} = \mathrm{id} = E \circ \mathscr{P}$ hold, like in Lemma 1.1. Actually, there are natural inclusions $\mathscr{P}(N) \hookrightarrow \mathscr{S}$ and $\mathbf{B} \hookrightarrow \mathrm{BOR}$, the last one sending every bornology $\beta \in \mathbf{B}$ to the bornological space $(E(\beta), \beta)$. Our aim in this section is to extend the diagram of locales $\mathscr{P}(N) \to \mathbf{B} \to \Omega$ in Section 2 to a diagram of categories $\mathscr{S} \to \mathrm{BOR} \to \mathscr{M}$, and then to induce an equivalence between categories from the second functor.

We define the following functor by associating to each bornological space the M-set of its bounded sequences,

$$\Sigma_b : \mathrm{BOR} \to \mathscr{M}, \quad \Sigma_b(S,\beta) = \{s : N \to S; s(N) \in \beta\} \subset S^N,$$

with the obvious action $\Sigma(h)(s) = h \circ s$ on the bounded maps. If we consider N with the discrete bornology, then $\Sigma_b = \mathrm{BOR}(N,-)$. It is clear that for each bornology $\beta \in \mathbf{B}$ we have $\Sigma_b(E(\beta),\beta) = \mathrm{Cont}(\beta)$, so that given an ideal I we have in particular $\Sigma_b(\mathrm{Ext}(I),\mathrm{Bor}(I)) = j(I)$.

Lemma 5.1. *Σ_b is faithful and factorizes through $\mathscr{M}'$.*

Proof. Given a bounded map $h : S \to S'$ we can recover h from $\Sigma_b(h)$ since the constant sequences c_x are bounded and $\Sigma_b(h)(c_x) = h \circ c_x = c_h(x)$ for every $x \in S$; hence Σ_b is faithful. Moreover, we have seen (Proposition 3.2) that S^N is a C-sheaf, hence $\Sigma_b(S,\beta) \subset S^N$ is C-separated (note that the inclusion is the mono μ in this case). □

By Lemma 5.1 we can reduce to $\mathscr{M}'$ the codomain of Σ_b and the new codomain suffices to contain the images of both functors Δ and $(-)^N$, so that it is a good extension of $\mathscr{S}$ yet.

Now we analyse the domain. We say that (S,β) is a *Kolmogorov bornological* (or *K-bornological*) space [3] if every subset $B \subset S$ such that $s(N) \in \beta$ for all $s : N \to B$ satisfies $B \in \beta$. Let K-BOR be the full subcategory of BOR given by all K-bornological spaces. There exists the universal K-bornological space over (S,β), which is the same set S but with the bornology β^-, $\beta \subset \beta^-$, given by

$$\beta^- = \{B \subset S; s(N) \in \beta \text{ for all } s : N \to B\}.$$

Let us note that a K-bornological space is determined by its bounded sequences, and the bornological spaces (S,β) and (S,β^-) have the same bounded sequences. All spaces $(E(\beta),\beta)$, $\beta \in \mathbf{B}$, are K-bornological and the spaces obtained by using $\mathscr{K}$ or $\mathscr{P}$ also; hence K-BOR is a good extension of $\mathbf{B}$ for the domain of the new functor of Σ_b.

From now on, we shall consider the commutative diagram in the form

$$\begin{array}{ccc} & & \text{K-BOR} \\ & \nearrow & \\ \mathscr{S} & & \downarrow \Sigma_b \\ & \searrow & \\ & & \mathscr{M}' \end{array}$$

Given an M-set X, by taking the set $\Gamma(X)$ with the K-bornology generated by the bornology defined in Proposition 4.1, we produce from Γ the functor $\Gamma_b : \mathscr{M}' \to$ K-BOR. In fact, for every equivariant map $H : X \to Y$, the restriction $H : \Gamma_b(X) \to \Gamma_b(Y)$ is bounded because, given $x \in X$ and the generating bounded sequence $\mu(x) : N \to \Gamma_b(X)$, the map $H \circ \mu(x) : N \to \Gamma_b(Y)$ is bounded since $H \circ \mu(x) = \mu(H(x))$ by the naturality of μ. In particular, $\Gamma_b(M)$ is (isomorphic to) the set N with the discrete bornology, and every point $x : M \to X$ is transformed by Γ_b in the sequence

$\mu(x) = \{xc_n\} : N \to \Gamma(X)$ of all fixed points in its orbit. Now, we give the second theorem of this paper.

Theorem 5.2. *There exists and adjunction $\Gamma_b \dashv \Sigma_b : K\text{-}BOR \to \mathscr{M}'$ which induces an equivalence of categories $K\text{-}BOR \cong \mathscr{B}'$.*

Proof. If we consider the one-to-one correspondence defined by the adjunction $\Gamma \dashv (-)^N$ (see Section 3) then it is easy to verify that $h : \Gamma_b(X) \to S$ is bounded if and only if $H : X \to S^N$, $H(x) = h \circ \mu(x)$, satisfies $H(X) \subset \Sigma_b(S)$. Hence, we have the adjunction $\Gamma_b \dashv \Sigma_b$ and we shall describe the induced equivalence.

For the counit we have $\Gamma_b \circ \Sigma_b \cong \text{id}$. In fact, given a K-bornological space (S, β), since $\Gamma_b(\Sigma_b(S)) \cong S$ as sets, we observe that the new K-bornology β' on S is generated by the inclusion $\mu : \Sigma_b(S) \hookrightarrow S^N$, that is, by the sets $s(N)$ where s is a bounded sequence for β. We must prove that $\beta = \beta'$. Every $B \subset S$ such that $B \subset s_1(N) \cup \cdots \cup s_r(N)$ with $s_i \in \Sigma_b(S)$, $1 \leqslant i \leqslant r$, belongs to β, so that $\beta' \subset \beta$ since β is K-bornology. Conversely, given $B \in \beta$, we must prove that $B \in \beta'$, which is a consequence, because β' is K-bornology, of the condition $s(N) \in \beta'$ for every $s : N \to B$; but $s \in \Sigma_b(S)$ since $B \in \beta$, so that the condition follows.

For the unit, by Proposition 4.2, we must prove that given a C-separated M-set X, the equivariant map $\mu_b : X \to \Sigma_b(\Gamma_b(X))$ (μ with restricted codomain) is an isomorphism if and only if X is a **J**-sheaf. But by Proposition 4.2 and the property $\Sigma_b(S, \beta) = \Sigma_b(S, \beta^-)$ we have $\mathbf{b}(X) = \Sigma_b(\Gamma_b(X))$, so that μ_b is an iso if and only if $X \cong \mathbf{b}(X)$ is a **J**-sheaf. □

The equivalence $\mathscr{S} \cong \mathscr{M}_{\neg\neg} = \mathscr{B}_k$ (Propositions 3.2 and 4.1) is the restriction to the discrete bornologies of the equivalence in Theorem 5.2. Note that if N is finite all the Kolmogorov bornologies are discrete. As a corollary of Theorem 5.2 we conclude that the category K-BOR is a quasitopos. Actually, we can obtain also this result from the fact that BOR is a quasitopos [12, p. 99]. Given two object S, T in K-BOR, we shall describe the exponential T^S in terms of exponential in $\mathscr{B}'$. If $X = \Sigma_b(S)$ and $Y = \Sigma_b(T)$, then $\Gamma_b(Y^X) \cong \mathscr{B}'(X, Y) \cong \text{K-BOR}(S, T)$ with the K-bornology determined by the bounded sequences $\mu(\xi) : N \to \Gamma_b(Y^X)$, $\xi \in Y^X$. Recall that $\xi : M \times X \to Y$ is an equivariant map and $\mu(\xi)(n) = \xi c_n$, $n \in N$, is the equivariant map given by $(\xi c_n)(f, s) = \xi(c_{f(n)}, s)$, $f \in M$, $s \in X$. By the above bijection, $\mu(\xi)$ corresponds to $w : N \to \text{K-BOR}(S, T)$, where for every $n \in N$ $w(n)$ define an equivariant map $\omega_n \in \Gamma_b(Y^X)$ by $\omega_n(f, s)(m) = w(f(n))(s(m))$, but this means that the map $w^\wedge : N \times S \to T$, $w^\wedge(n, x) = w(n)(x)$ is bounded with N discrete. Hence, the bornology in T^S is in fact the equibounded bornology.

Finally, we conclude that if N is the set $\mathbb{N}$ of all natural numbers then the bornological topos $\mathscr{B}$ is a convenient topos to study the boundedness properties of spaces in functional analysis. A natural number object in $\mathscr{B}$ is $\mathbb{N}_b = \mathbf{b}(\mathbb{N}) = \{s : \mathbb{N} \to \Gamma(X); s(\mathbb{N}) \text{ finite}\}$, and then the objects of integers and rationals are $\mathbb{Z}_b = \mathbf{b}(\mathbb{Z})$ and $\mathbb{Q}_b = \mathbf{b}(\mathbb{Q})$, respectively; but $\mathbf{b}(\mathbb{R})$ is not the object of Dedekind reals. It can be proved that this object is $\mathbb{R}_b = \Sigma_b(\mathbb{R})$ when we consider the set $\mathbb{R}$ of all real numbers with the usual bornology, that is, $\mathbb{R}_b$ is the classical space ℓ^∞ of real bounded sequences. Hence, the category of

K-bornological real vector spaces is equivalent to the category MOD_b of $\mathbb{R}_b$-modules in $\mathscr{B}'$.

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