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Closed model categories for uniquely S-divisible spaces

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Abstract

For each integer $n > 1$ and a multiplicative system S of non-zero integers, we give a distinct closed model category structure to the category of pointed spaces $\mathrm{Top}_\star$ and we prove that the corresponding localized category $\mathrm{Ho}(\mathrm{Top}_\star^{(S,n)})$, obtained by inverting the weak equivalences, is equivalent to the standard homotopy category of uniquely (S,n)-divisible, $(n-1)$-connected spaces. A space X is said to be uniquely (S,n)-divisible if for $k \geqslant n$ the homotopy group $\pi_k X$ is uniquely S-divisible. This equivalence of categories is given by an (S,n)-colocalization functor that carries a pointed space X to a space $X^{(S,n)}$. There is also a natural map $X^{(S,n)} \to X$ which is (finally) universal among all the maps $Z \to X$ with Z a uniquely (S,n)-divisible, $(n-1)$-connected space. The structure of closed model category given by Quillen to $\mathrm{Top}_\star$ is based on maps which induce isomorphisms on all homotopy group functors π_k and for any choice of base point. For each pair (S,n), the closed model category structure given here take as weak equivalences those maps that for the given base point induce isomorphisms on the homotopy groups functors $\pi_k(\mathbb{Z}[S^{-1}];-)$ with coefficients in $\mathbb{Z}[S^{-1}]$for $k \geqslant n$. We note that the category $\mathrm{Ho}(\mathrm{Top}_\star^{(\mathbb{Z}-\{0\},2)})$ is the homotopy category of rational 1-connected spaces.

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1. Introduction

Quillen [13] introduced the notion of closed model category and proved that the categories of spaces and of simplicial sets have the structure of a closed model category.

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Moreover, Quillen [14] used this structure to construct localization functors for 1-connected spaces and to find algebraic models for rational homotopy theory. In this paper, we use this categorical structure to construct colocalization functors and to study the homotopy category of uniquely S-divisible spaces. These colocalization functors are just given by the cofibrant approximations of an object in the closed model structures developed in this work.

For each $n > 1$, we take as weak (S, n)-equivalences those maps of $\mathrm{Top}_\star$ which induce isomorphisms on the homotopy group functors $\pi_k(\mathbb{Z}[S^{-1}]; -)$ for $k \geqslant n$, where S is a multiplicative system of non-zero integers and $\pi_k(\mathbb{Z}[S^{-1}]; -)$ denotes the kth homotopy group functor with coefficients in $\mathbb{Z}[S^{-1}]$. The abelian group $\mathbb{Z}[S^{-1}]$ is the subgroup (subring) of the rationals $\mathbb{Q}$ of all fractions of the form a/b with $b \in S$. The class of weak (S, n)-equivalences is completed with classes of (S, n)-fibrations and (S, n)-cofibrations and using these classes we are able to prove the following important result:

Theorem 3.1. *For each $n > 1$, the category $\mathrm{Top}_\star$ together with the families of (S, n)-fibrations, (S, n)-cofibrations and weak (S, n)-equivalences, has the structure of a closed model category.*

In this paper we give some algebraic characterizations of the spaces which up to weak equivalence are (S, n)-cofibrant spaces. Recall that an abelian group H is said to be a uniquely S-divisible group if for any $h \in H$ and $s \in S$ there is a unique $x \in H$ such that $sx = h$. A space is said to be uniquely (S, n)-divisible if the homotopy groups $\pi_k X$ are uniquely S-divisible for $k \geqslant n$. In the last section of this paper, we have proved the following result:

Theorem 4.1. *Let X be a pointed space, then the following statements are equivalent*:

(i) *X is weakly equivalent to an (S, n)-cofibrant space*,
(ii) *for every abelian group B right-orthogonal to $\mathbb{Z}[S^{-1}]$ the reduced singular cohomology groups $\tilde{H}^q(X; B)$ are trivial and X is an $(n-1)$-connected space*
(iii) *for every $s \in S$ the reduced singular homology groups $\tilde{H}_q(X; \mathbb{Z}/s)$ are trivial and X is an $(n-1)$-connected space.*
(iv) *the reduced singular homology groups of X are uniquely S-divisible groups and X is an $(n-1)$-connected space*,
(v) *X is a uniquely (S, n)-divisible, $(n-1)$-connected space.*

As a consequence of this characterization of (S, n)-cofibrant spaces we also have obtained the following equivalence of categories:

Theorem 4.2. *The localized category $\mathrm{Ho}(\mathrm{Top}_\star^{(S,n)})$ is equivalent to the homotopy category of uniquely (S, n)-divisible, $(n-1)$-connected spaces.*

In particular for $S = \mathbb{Z} - \{0\}$ and $n = 2$, one has the rational category of 1-connected spaces.

An interesting functor induced by this closed model category is the cofibrant approximation. Given a pointed space Y if we take the (S,n)-cofibrant approximation, then the corresponding canonical map $Y^{(S,n)} \to Y$ is finally universal among all the maps $Z \to Y$ with Z a uniquely (S,n)-divisible, $(n-1)$-connected space. This property is dual to the universal property of the Quillen–Sullivan localization $X \to X \otimes \mathbb{Z}[S^{-1}]$. Moreover, in the localized category of $(n-1)$-connected spaces the Quillen–Sullivan localization functor is left adjoint to the (S,n)-colocalization functor introduced in this paper.

2. Preliminaries on closed model categories

We begin by recalling the definition of a closed model category (CMC) given by Quillen [14]. For more properties of closed model categories we refer the reader to [4,5,8,10].

Definition 2.1. A closed model category $\mathscr{C}$ is a category endowed with three distinguished families of maps called cofibrations, fibrations and weak equivalences satisfying the axioms CM1–CM5 below:

CM1. $\mathscr{C}$ is closed under finite projective and inductive limits.

CM2. If f and g are maps such that gf is defined then if two of these f, g and gf are weak equivalences then so is the third.

Recall that the maps in $\mathscr{C}$ form the objects of a category Maps($\mathscr{C}$) having commutative squares for morphisms. We say that a map f in $\mathscr{C}$ is a retract of g if there are morphisms $\varphi : f \to g$ and $\psi : g \to f$ in Maps($\mathscr{C}$) such that $\psi\varphi = id_f$.

A map which is a weak equivalence and a fibration is said to be a trivial fibration and, similarly, a map which is a weak equivalence and a cofibration is said to be a trivial cofibration.

CM3. If f is a retract of g and g is a fibration, cofibration or weak equivalence then so is f.

CM4. (Lifting.) Given a solid arrow diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

the dotted arrow exists in either of the following situations:

(i) i is a cofibration and p is a trivial fibration,
(ii) i is a trivial cofibration and p is a fibration.

CM5. (Factorization.) Any map f may be factored in two ways:

(i) $f = pi$ where i is a cofibration and p is a trivial fibration,
(ii) $f = qj$ where j is a trivial cofibration and q is a fibration.

We say that a map $i : A \to B$ in a category has the left lifting property (LLP) with respect to another map $p : X \to Y$ and p is said to have the right lifting property (RLP) with respect to i if the dotted arrow exists in any diagram of the form above.

The initial object of $\mathscr{C}$ is denoted by $\emptyset$ and the final object by $\bigstar$. An object X of $\mathscr{C}$ is said to be fibrant if the morphism $X \to \bigstar$ is a fibration and it is said cofibrant if $\emptyset \to X$ is a cofibration.

In the category Top of spaces, a map is said to be a Serre fibration if it has the right lifting property with respect to the maps

$$I^k \to I^{k+1}, (t_1, \dots, t_k) \to (t_1, \dots, t_k, 0)$$

($\{0\} = I^0 \to I^1 = I$, maps 0 into 0) for $k \geqslant 0$, where I denotes the closed unit interval.

In this paper we will consider the closed model category $\mathrm{Top}_\bigstar$ of pointed topological spaces with the following structure: Given a map $f : X \to Y$ in $\mathrm{Top}_\bigstar$, f is said to be a fibration if it is a Serre fibration in the non-pointed category Top; f is a weak equivalence if f induces isomorphisms $\pi_q(f)$ for $q \geqslant 0$ and for any choice of base point and f is a cofibration if it has the LLP with respect to all trivial fibrations. For this structure we refer the reader to Quillen [13]. We also recall that $\mathrm{Top}_\bigstar$ has also a compatible simplicial structure. If K is a finite simplicial object and X is a pointed space then $X \otimes K$ is defined to be

$$X \otimes K = X \times |K|^+ / (X \times \bigstar \cup \bigstar \times |K|^+),$$

where $|K|^+$ is the disjoint union of $|K|$ and the one point space $\bigstar$.

In particular we have the standard pointed cylinder

$$X \otimes I = X \otimes \Delta[1].$$

Let $\mathrm{Ho}(\mathrm{Top}_\bigstar)$ denote localized categories obtained by formal inversion of weak equivalences defined above.

In the category of pointed topological spaces and continuous maps, $\mathrm{Top}_\bigstar$, we consider a family $\mathscr{F} = \{M_\lambda \mid \lambda \in \Lambda\}$ of spaces which are suspensions of CW-complexes ($M_\lambda = \Sigma N_\lambda$ where N_λ is a CW-complex). In this section we give a CMC structure in the category of pointed spaces that will be used in the next sections to prove the main theorems of this paper. This structure is inspired by the CMC structure given in [6], which for the case of one space has been developed in [3]. We have included the significant facts that allow us to prove that the category of pointed spaces admits this CMC structure. In order to see the difference with the CMC structures given in [10] we have included a characterization of the family of fibrations in Theorem 2.1. Notice that the family of fibrations of our CMC structure is larger than the class of Serre fibrations.

We consider the following classes of maps:

Definition 2.2. Let $f : X \to Y$ be a map in $\mathrm{Top}_\bigstar$,

(i) f is a weak $\mathscr{F}$-equivalence if the induced map

$$[\Sigma^k M_\lambda, f] \colon [\Sigma^k M_\lambda, X] \to [\Sigma^k M_\lambda, Y]$$

is an isomorphism for each $k \geqslant 0$ and $\lambda \in \Lambda$, where $[-,-]$ denotes the standard set of pointed homotopy classes.

(ii) f is an $\mathscr{F}$-fibration if it has the RLP in the category of pointed spaces with respect to the family $\mathscr{T}(\mathscr{F})$ of inclusions

$$(C\Sigma^k N_\lambda \times 0) \cup (\Sigma^k N_\lambda \otimes I) \to C\Sigma^k N_\lambda \otimes I$$

for every $k \geqslant 0$ and $\lambda \in \Lambda$.

A map which is both an $\mathscr{F}$-fibration and a weak $\mathscr{F}$-equivalence is said to be a trivial $\mathscr{F}$-fibration.

(iii) f is an $\mathscr{F}$-cofibration if it has the LLP with respect to any trivial $\mathscr{F}$-fibration.

A map which is both an $\mathscr{F}$-cofibration and a weak $\mathscr{F}$-equivalence is said to be a trivial $\mathscr{F}$-cofibration.

A pointed space X is said to be $\mathscr{F}$-fibrant if the map $X \to \bigstar$ is an $\mathscr{F}$-fibration, and X is said to be $\mathscr{F}$-cofibrant if the map $\bigstar \to X$ is an $\mathscr{F}$-cofibration.

Remark 2.1. Let C be the path-component of the given base point of X. Note that the inclusion $C \to X$ is always a weak $\mathscr{F}$-equivalence. It as also clear that all objects in $\mathrm{Top}_\bigstar$ are $\mathscr{F}$-fibrant.

Theorem 2.1. *Suppose that $\mathscr{F}$ has at least a non-trivial* CW-*complex*, *and for a map $f: X \to Y$ in* $\mathrm{Top}_\bigstar$, *denote by $f_0: X_0 \to Y_0$ the induced map on the path-components of the given base points. Then f is an $\mathscr{F}$-fibration if and only if f_0 is a Serre fibration.*

Proof. Since the maps of the family $\mathscr{T}(\mathscr{F})$ are between 0-connected spaces, one has that in the category of pointed spaces a map f has the RLP with respect to $\mathscr{T}(\mathscr{F})$ if and only if f_0 has the RLP with respect to $\mathscr{T}(\mathscr{F})$. Suppose that f_0 is a Serre fibration, because the maps of $\mathscr{T}(\mathscr{F})$ are trivial cofibrations, it follows that f_0 has the right lifting property with respect to $\mathscr{T}(\mathscr{F})$, and therefore f is an $\mathscr{F}$-fibration. Conversely, for simplicity write by $i_k: U_k \to V_k$ a map of the family $\mathscr{T}(\mathscr{F})$ which is a cofibration between contractible CW-complexes. Take $0 \in I^k$ as a base point and write $I \vee I^k$ for the pointed sum. Since the family has at least a non-trivial CW-complex, for a large integer K it is possible to embed $I \vee I^k$ in U_K and to extend to an embedding $I \vee I^{k+1} \to V_K$ in such a way that the embeddings are both cofibrations and the point 1 of I has been identified to the base point of V_K. Lifting in the diagram

$$\begin{array}{ccc} I \vee I^k & \longrightarrow & I \vee I^k \\ \downarrow & & \downarrow \\ U_K & \longrightarrow & * \end{array}$$

gives a retraction $r{:}U_K \to I \vee I^k$. If we consider the induced map $r + id{:}U_K \bigcup_{(I\vee I^k)} (I \vee I^{k+1}) \to I \vee I^{k+1}$ we can lift in the diagram

$$\begin{array}{ccc} U_K \bigcup\limits_{(I\vee I^k)} (I \vee I^{k+1}) & \longrightarrow & I \vee I^{k+1} \\ \Big\downarrow & & \Big\downarrow \\ V_K & \longrightarrow & * \end{array}$$

to obtain that $(I \vee I^k, 1) \to (I \vee I^{k+1}, 1)$ is a retract of the map $(U_K, *) \to (V_K, *)$.

Now assume that f has the RLP with respect to $U_K \to V_K$ in the pointed setting, then f_0 has the same lifting property. Therefore f_0 has the RLP with respect to $I \vee I^k \to I \vee I^{k+1}$ in the pointed setting. Since the domain of f_0 is 0-connected, it follows that f_0 ha the RLP with respect $I^k \to I^{k+1}$ in the non-pointed setting. This implies that f_0 is a Serre fibration. □

We have the following characterization of the class of trivial $\mathscr{F}$-fibrations:

Proposition 2.1. *For a map* $f: X \to Y$ *in* $\mathrm{Top}_\star$, *the following statements are equivalent*:

(i) f *is a trivial* $\mathscr{F}$*-fibration*,
(ii) f *has the RLP with respect to the family* $\mathscr{C}(\mathscr{F})$ *of inclusions*

$$\star \to M_\lambda, \quad \lambda \in \Lambda,$$

$$\Sigma^k M_\lambda \to C\Sigma^k M_\lambda, \qquad k \geqslant 0, \ \lambda \in \Lambda.$$

Proof. Let F be the homotopy fibre of f in $\mathrm{Top}_\star$. Suppose that f has the RLP with respect to the maps of $\mathscr{C}(\mathscr{F})$. This fact implies that $[M_\lambda, X] \to [M_\lambda, Y]$ is surjective and $[\Sigma^k M_\lambda, F] \cong 0$ for $k \geqslant 0$ and $\lambda \in \Lambda$. Therefore we have that f is a weak $\mathscr{F}$-equivalence.

In order to prove that f is an $\mathscr{F}$-fibration, for $k \geqslant 0$ and $\lambda \in \Lambda$, consider a commutative diagram of the form

$$\begin{array}{ccc} (C\Sigma^k N_\lambda \otimes 0) \cup (\Sigma^k N_\lambda \otimes I) & \xrightarrow{u} & X \\ \Big\downarrow & & \Big\downarrow f \\ C\Sigma^k N_\lambda \otimes I & \xrightarrow[v]{} & Y \end{array}$$

Define $\bar{u}$: $(\Sigma^k N_\lambda \otimes I) \cup (C\Sigma^k N_\lambda \otimes 1) \to X$, $\bar{v}{:}C\Sigma^k N_\lambda \otimes I \to Y$ by the formulas $\bar{u}[x,t] = u[x, 1-t]$, $\bar{v}[x,t] = v[x, 1-t]$. Since $\bar{u}[x,0] = u[x,1]$ and $\bar{v}[x,0] = v[x,1]$, the following diagram is commutative:

$$\begin{array}{ccc}(C\Sigma^k N_\lambda \otimes 0) \cup (\Sigma^k N_\lambda \otimes I) \cup (\Sigma^k N_\lambda \otimes I) \cup (C\Sigma^k N_\lambda \otimes 1) & \xrightarrow{u+\bar{u}} & X \\ \downarrow & & \downarrow f \\ (C\Sigma^k N_\lambda \otimes I) \cup (C\Sigma^k N_\lambda \otimes I) & \xrightarrow[v+\bar{v}]{} & Y \end{array}$$

However, one has that

$$\Sigma^k M_\lambda \cong (C\Sigma^k N_\lambda \otimes 0) \cup (\Sigma^k N_\lambda \otimes I) \cup (\Sigma^k N_\lambda \otimes I) \cup (C\Sigma^k N_\lambda \otimes 1)$$

$$C\Sigma^k M_\lambda \cong (C\Sigma^k N_\lambda \otimes I) \cup (C\Sigma^k N_\lambda \otimes I).$$

Therefore we can apply that f has the RLP with respect to $\mathscr{C}(\mathscr{F})$ to obtain a lifting $h : (C\Sigma^k N_\lambda \otimes I) \cup (C\Sigma^k N_\lambda \otimes I) \to X$ for the diagram above. Now the restriction of h to the first copy $h/(C\Sigma^k N_\lambda \otimes I)$ is the desired lifting. Hence one concludes that f is an $\mathscr{F}$-fibration.

That part (i) implies (ii) is straightforward. □

Using the characterization of trivial (S, n)-fibrations by the RLP with respect to a family of maps, one can prove following result.

Theorem 2.2. *The category* Top$_\star$ *together with the classes of $\mathscr{F}$-fibrations, $\mathscr{F}$-cofibrations and weak $\mathscr{F}$-equivalences, has the structure of a closed model category.*

Proof. At this point we assume that all the axioms have been verified (the proofs are easy) with the exception of axioms CM5 and CM4(ii).

To prove the factorization axiom CM5, the following generalization of the small object argument is very useful, see [1,7,10]. This generalization had also been considered by Joyal [11] to give a closed model structure to the category of simplicial objects in a Grothendieck Topos.

Let $f : X \to Y$ be a map in Top$_\star$, we have to prove that f can be factored in two ways:

(i) $f = pi$, where i is an $\mathscr{F}$-cofibration and p is a trivial $\mathscr{F}$-fibration,
(ii) $f = qj$, where j is a weak $\mathscr{F}$-equivalence having the LLP with respect to all $\mathscr{F}$-fibrations and q is an $\mathscr{F}$-fibration.

For instance, in order to obtain the first factorization, we choose a limit ordinal γ whose cardinality is greater than the cardinal of the set of cells of M_λ for every $\lambda \in \Lambda$.

First we can consider all maps of the form $v : M_\lambda \to Y$, $\lambda \in \Lambda$ to construct the space $X^0 = X \bigvee (\bigvee_v M_{\lambda(v)})$ and the map $p^0 : X^0 \to Y$ defined by the sum of f and all the maps v. This map $p^0 : X^0 \to Y$ has the RLP with respect to the maps $\star \to M_\lambda$. Now we construct the following γ-sequence, for any ordinal $\beta \leqslant \gamma$

$$X^0 \to X^1 \to X^2 \to \cdots \to X^\beta \to \cdots$$

and compatible maps $p^{\beta}: X^{\beta} \to Y$. For $\beta = 0$, we have the map $p^0: X^0 \to Y$. Given an ordinal β, suppose that we have $p^{\alpha}: X^{\alpha} \to X$ for any $\alpha < \beta$. Now we consider two cases:

Case 1: β is the successor ordinal of α, then we take all commutative diagrams D of the form

$$\begin{array}{ccc} \Sigma^k M_\lambda & \xrightarrow{u^D} & X^{\alpha} \\ \downarrow & & \downarrow p^{\alpha} \\ C\Sigma^k M_\lambda & \xrightarrow[v^D]{} & Y \end{array}$$

where $k \geqslant 0$ and $\lambda \in \Lambda$. Define $j^{\beta}: X^{\alpha} \to X^{\beta}$, by the pushout

$$\begin{array}{ccc} \bigvee_D \Sigma^k M_\lambda & \longrightarrow & X^{\alpha} \\ \downarrow & & \downarrow j^{\beta} \\ \bigvee_D C\Sigma^k M_\lambda & \longrightarrow & X^{\beta} \end{array}$$

and define $p^{\beta}: X^{\beta} \to Y$ by the sum of p^{α} and all the v^D.

Case 2: β is a limit ordinal. In this case we take

$$X^{\beta} = \operatorname*{Colim}_{\alpha<\beta} X^{\alpha},$$

$$p^{\beta} = \operatorname*{Colim}_{\alpha<\beta} p^{\alpha}.$$

By transfinite induction we obtain an $\mathscr{F}$-cofibration $i: X \to X^{\gamma}$ and a trivial $\mathscr{F}$-fibration $p: X^{\gamma} \to Y$.

The other factorization $f = qj$ is similarly obtained. In this case, we also have that j has the LLP with respect to all $\mathscr{F}$-fibrations.

Next we verify Axiom CM4(ii). Suppose that i is a trivial $\mathscr{F}$-fibration, by CM5, i can be factored as $i = qj$ where $j: A \to W$ is a weak $\mathscr{F}$-equivalence having the LLP with respect to all $\mathscr{F}$-fibrations and $q: W \to B$ is an $\mathscr{F}$-fibration. Since CM2 is verified, q is a trivial $\mathscr{F}$-fibration. Then, there is a lifting $r: B \to W$ for the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{j} & W \\ i \downarrow & & \downarrow q \\ B & \xrightarrow{id} & B \end{array}$$

So, the map i is a retract of j. Therefore i also has the LLP with respect to all $\mathscr{F}$-fibrations. $\square$

Remark 2.2. We note that the factorizations above are functorial. This will be interesting when we consider left-derived and right-derived functors. This also implies that we have functorial cylinders and cocylinders.

Remark 2.3. Hirschhorn [10] and Dror-Farjoun [7] have been working with cellularization functors associated to a set A of objects in a closed model category. Hirschhorn proves that there is a closed model structure on $\mathrm{Top}_\star$ taking as fibrations the usual Serre fibrations of $\mathrm{Top}_\star$, as weak equivalences they consider A-cellular equivalences and the A-cellular cofibrations are defined by the LLP with respect to all the maps which are both fibrations and A-cellular equivalences. Taking as set of objects $A = \{\bigvee_{\lambda\in\Lambda} M_\lambda\}$ if we consider the closed model structure given by Hirschhorn, we have that the class of weak $\mathscr{F}$-equivalences is exactly the class of A-cellular equivalences. To see this fact it is necessary to take into account that $\bigvee_{\lambda\in\Lambda} M_\lambda$ is a suspension space that induces nice properties in the corresponding function space. However, one has that the class of $\mathscr{F}$-fibrations is larger than the class of fibrations. For example, because $0 \to I$ is not a Serre fibration we have that $\star + 0 \to \star + I$ is an $\mathscr{F}$-fibration which is not a Serre fibration. Therefore the CMC structure given in this work is different to the CMC structure given in [10]. However, it is interesting to note that a space is $\mathscr{F}$-cofibrant if and only if it is connected and cofibrant in the closed model category given by Hirschhorn.

3. Some closed model categories associated to $\mathbb{Z}[S^{-1}]$

In this paper we consider a multiplicative system S of non-zero integers and a fixed $n > 1$. In order to introduce a model structure associated with (S,n) we recall briefly the definition of homotopy groups with coefficients. For a more complete description and properties we refer the reader to Hilton [9]. For $k \geqslant 1$, we have the canonical space $M(\mathbb{Z}[S^{-1}];k)$ which is usually called the Moore space with coefficient group $\mathbb{Z}[S^{-1}]$ and degree k. For a pointed space X, consider the set of pointed homotopy classes $\pi_k(\mathbb{Z}[S^{-1}];X) = [M(\mathbb{Z}[S^{-1}],k),X]$. This hom-set admits the structure of a group for $k \geqslant 2$ which abelian for $k \geqslant 3$. It is said that $\pi_k(\mathbb{Z}[S^{-1}];X)$ is the kth homotopy group of X with coefficients in $\mathbb{Z}[S^{-1}]$. In this paper, for $q \geqslant 2$ we shall frequently use the following exact sequence:

$$0 \to \mathrm{Ext}(\mathbb{Z}[S^{-1}],\pi_{q+1}X) \to \pi_q(\mathbb{Z}[S^{-1}];X) \to \mathrm{Hom}(\mathbb{Z}[S^{-1}],\pi_q X) \to 0.$$

In the category of pointed topological spaces and continuous maps, $\mathrm{Top}_\star$, for a set S of non-zero integers and $n > 1$, we consider the following families of maps:

Definition 3.1. Let $f : X \to Y$ be a map in $\mathrm{Top}_\star$,

(i) f is a weak (S,n)-equivalence if the induced map

$$\pi_l(\mathbb{Z}[S^{-1}];f)\colon \pi_l(\mathbb{Z}[S^{-1}];X) \to \pi_l(\mathbb{Z}[S^{-1}];Y)$$

is an isomorphism for each $l \geqslant n$.

(ii) f is an (S,n)-fibration if it has the RLP with respect to the family $\mathscr{T}(S,n)$ of inclusions

$$(C\Sigma^k M(\mathbb{Z}[S^{-1}];n-1)\times 0)\cup(\Sigma^k M(\mathbb{Z}[S^{-1}];n-1)\otimes I)$$
$$\to C\Sigma^k M(\mathbb{Z}[S^{-1}];n-1)\otimes I$$

for every $k \geqslant 0$.

A map which is both an (S,n)-fibration and a weak (S,n)-equivalence is said to be a trivial (S,n)-fibration.

(iii) f is an (S,n)-cofibration if it has the LLP with respect to any trivial (S,n)-fibration.

A map which is both an (S,n)-cofibration and a weak (S,n)-equivalence is said to be a trivial (S,n)-cofibration.

A pointed space X is said to be (S,n)-fibrant if the map $X \to \bigstar$ is an (S,n)-fibration, and X is said to be (S,n)-cofibrant if the map $\bigstar \to X$ is an (S,n)-cofibration.

We note that the homotopy group $\pi_q(\mathbb{Z}[S^{-1}];X)$ only depends on the path component C of the given base point of X. Therefore the inclusion $C \to X$ is always a weak (S,n)-equivalence. It as also clear that all objects in $\mathrm{Top}_\bigstar$ are (S,n)-fibrant.

If we take the family $\mathscr{F}=\{M(\mathbb{Z}[S^{-1}];n)\}$, which only has the Moore space obtained by the suspension of $M(\mathbb{Z}[S^{-1}];n-1)$, the classes of $\mathscr{F}$-fibrations, $\mathscr{F}$-cofibrations and weak $\mathscr{F}$-equivalences given in Definition 2.2 are exactly the classes of Definition 3.1. Then the following result is a consequence of Theorem 2.2.

Theorem 3.1. *For each $n>1$, the category $\mathrm{Top}_\bigstar$ together with the families of (S,n)-fibrations, (S,n)-cofibrations and weak (S,n)-equivalences, has the structure of a closed model category.*

We denote by $\mathrm{Top}_\bigstar^{(S,n)}$ the closed model category $\mathrm{Top}_\bigstar$ with the distinguished families of (S,n)-fibrations, (S,n)-cofibrations and weak (S,n)-equivalences and by $\mathrm{Ho}(\mathrm{Top}_\bigstar^{(S,n)})$ the category of fractions obtained from $\mathrm{Top}_\bigstar^{(S,n)}$ by formal inversion of the family of weak (S,n)-equivalences.

In a closed model category a map between objects which are cofibrant and fibrant is a homotopy equivalence if and only if it is a weak equivalence. Then one has:

Theorem 3.2 (Whitehead theorem). *Let $f: X \to Y$ be a map in $\mathrm{Top}_\bigstar$ and suppose that X and Y are (S,n)-cofibrant, then f is a pointed homotopy equivalence if and only if $\pi_k(\mathbb{Z}[S^{-1}];f)$ is an isomorphism for every $k \geqslant n$.*

Definition 3.2. The (S,n)-cofibrant space obtained through the factorization of $\bigstar \to Y$ as the composite of an (S,n)-cofibration and a trivial (S,n)-fibration, will be called the (S,n)-colocalization of Y and it will be denoted by $Y^{(S,n)}$. The trivial (S,n)-fibration $Y^{(S,n)} \to Y$ will be called the (S,n)-colocalization map of Y.

Definition 3.3. An abelian group A is said to be left-orthogonal to B and B is said to be right-orthogonal to A if $\mathrm{Hom}(A,B)\cong 0$ and $\mathrm{Ext}(A,B)\cong 0$. Given classes $\mathscr{A}$

and $\mathscr{B}$, if for every A of $\mathscr{A}$ and every B of $\mathscr{B}$, A is left-orthogonal to B, the class $\mathscr{A}$ is said to be left-orthogonal to $\mathscr{B}$ and $\mathscr{B}$ is said to be right-orthogonal to $\mathscr{A}$. We denote by $\mathscr{A}^{\mathrm{ort}}$ the class of abelian groups which are right-orthogonal to $\mathscr{A}$ and by ${}^{\mathrm{ort}}\mathscr{B}$ the class abelian groups which are left-orthogonal to $\mathscr{B}$. An abelian group which is right-orthogonal to $\mathbb{Z}[S^{-1}]$ will be also called an Ext-S-complete abelian group. A space X is said to be Ext-(S,n)-complete if for $k \geqslant n$, $\pi_k X$ is Ext-S-complete.

Theorem 3.3. *Let X be an (S,n)-cofibrant space, and $Y^{(S,n)} \to Y$ the (S,n)-colocalization of a space Y, then*

$$\mathrm{Ho}(\mathrm{Top}_\star)(X, Y^{(S,n)}) \to \mathrm{Ho}(\mathrm{Top}_\star)(X,Y)$$

is an isomorphism. In particular, if Y is an Ext-(S,n)-*complete space, then*

$$\mathrm{Ho}(\mathrm{Top}_\star)(X,Y) \cong \star.$$

Remark 3.1. If X is an (S,n)-cofibrant space and B is an abelian group which is right-orthogonal to $\mathbb{Z}[S^{-1}]$, then the reduced cohomology of X with coefficients in B is trivial.

Remark 3.2. If B is an abelian group which is right-orthogonal to $\mathbb{Z}[S^{-1}]$, if we denote $C[S^{-1}] = \mathbb{Z}[S^{-1}]/\mathbb{Z}$, one has that $\mathrm{Hom}(C[S^{-1}],B) \cong 0$ and $\mathrm{Ext}(C[S^{-1}],B) \cong B$. If S is the multiplicative system generated by a prime p, for a given abelian group D the group $\mathrm{Ext}(C[S^{-1}],D)$ is usually called the Ext-p-completion of D. We refer the reader to [12] for questions related with Ext-p-completions.

Remark 3.3. In order to give the factorizations of axiom CM5, we have chosen a determined limit ordinal. Since the standard Moore space $M(\mathbb{Z}[S^{-1}],n)$ has a countable number of cells, then for this case we can choose the continuum limit ordinal $\aleph_1$.

4. (S, n)-cofibrant spaces and uniquely S-divisible spaces

In this section, we observe that an (S,n)-cofibrant space is $(n-1)$-connected and its reduced singular homology groups are uniquely S-divisible. This fact implies that the homotopy groups of an (S,n)-cofibrant space are also uniquely S-divisible abelian groups. On the other hand, we prove that if X is a uniquely (S,n)-divisible, $(n-1)$-connected space, then X is weakly equivalent to an (S,n)-cofibrant space. This gives an algebraic characterization of (S,n)-cofibrant spaces.

Proposition 4.1. *If X is an (S,n)-cofibrant space, then X is $(n-1)$-connected.*

Proof. For any ordinal $\beta \leqslant \aleph_1$, consider the $\aleph_1$-sequence given in Theorem 2.2:

$$X^0 \to X^1 \to X^2 \to \cdots \to X^\beta \to \cdots$$

where $X^0 = \bigvee_f M(\mathbb{Z}[S^{-1}];n)_f$ for all maps $f: M(\mathbb{Z}[S^{-1}];n) \to X$. For X^β we have two cases:

If β is the successor ordinal of α, then X^β is the homotopy cofibre of a map of the form $\bigvee_D M(\mathbb{Z}[S^{-1}];m_D) \to X^\alpha$, $m_D \geqslant n$.

If β is a limit ordinal. We have that

$$X^\beta = \operatorname*{Colim}_{\alpha<\beta} X^\alpha.$$

By transfinite induction we obtain an (S,n)-cofibrant space $X^{\aleph_1}$ and a trivial (S,n)-fibration $p: X^{\aleph_1} \to X$. Now we can apply Theorem 3.2 to obtain that p is also a weak equivalence.

It is easy to see that X^0 is an $(n-1)$-connected space and taking into account that a Moore space of the form $M(\mathbb{Z}[S^{-1}];m_D)$ with $m_D \geqslant n$ is $(n-1)$-connected, one has that the pushouts and colimits of the construction above give again $(n-1)$-connected spaces. Therefore $X^{\aleph_1}$ is an $(n-1)$-connected space. Because p is a weak equivalence one has that X is $(n-1)$-connected. □

Definition 4.1. A pointed space X is said to be uniquely (S,n)-divisible if for $k \geqslant n$ the homotopy groups $\pi_k(X)$ are uniquely S-divisible. Similarly we have the notion of (S,n)-divisible and of (S,n)-torsion.

Proposition 4.2. *Suppose that X is a uniquely $(S,2)$-divisible, 1-connected space. Then for $q \geqslant 1$ one has*:

$$\pi_q X \cong \operatorname{Hom}(\mathbb{Z}[S^{-1}], \pi_q X) \cong \pi_q(\mathbb{Z}[S^{-1}];X).$$

Proof. We note that if B is a uniquely S-divisible abelian group, then $\operatorname{Hom}(\mathbb{Z}[S^{-1}],B) \cong B$ and $\operatorname{Ext}(\mathbb{Z}[S^{-1}],B) \cong 0$. For the last isomorphism you can see that any epimorphism $B \to \mathbb{Z}[S^{-1}]$ has a section. Now from the exact sequence

$$0 \to \operatorname{Ext}(\mathbb{Z}[S^{-1}], \pi_{q+1}X) \to \pi_q(\mathbb{Z}[S^{-1}];X) \to \operatorname{Hom}(\mathbb{Z}[S^{-1}], \pi_q X) \to 0$$

the desired result follows. □

The following result gives up to weak equivalence some algebraic characterizations of (S,n)-cofibrant spaces.

Theorem 4.1. *Let X be a pointed space, then the following statements are equivalent*:

(i) *X is weakly equivalent to an (S,n)-cofibrant space*,
(ii) *for every abelian group B right-orthogonal to $\mathbb{Z}[S^{-1}]$ the reduced singular cohomology groups $\tilde{H}^q(X;B)$ are trivial and X is an $(n-1)$-connected space*,
(iii) *for every $s \in S$ the reduced singular homology groups $\tilde{H}_q(X;\mathbb{Z}/s)$ are trivial and X is an $(n-1)$-connected space*,
(iv) *the reduced singular homology groups of X are uniquely S-divisible groups and X is an $(n-1)$-connected space*,
(v) *X is a uniquely (S,n)-divisible, $(n-1)$-connected space.*

Proof. (i) = > (ii). Proposition 4.1 and Remark 3.1.

(ii) = > (iii). Note that if $s \in S$ then any $\mathbb{Z}/s$-module M is right-orthogonal to $\mathbb{Z}[S^{-1}]$. Therefore the reduced cohomology of X with coefficients in a $\mathbb{Z}/s$-module M vanishes if $s \in S$. By the universal coefficient theorem for $\mathbb{Z}/s$-module chain complexes we have that $\mathrm{Hom}(\tilde{H}_q(X;\mathbb{Z}/s), M) \cong 0$ and $\mathrm{Ext}(\tilde{H}_q(X;\mathbb{Z}/s), M) \cong 0$. In particular one has that $\mathrm{Hom}(\tilde{H}_q(X;\mathbb{Z}/s), \tilde{H}_q(X;\mathbb{Z}/s)) \cong 0$. This implies that $\tilde{H}_q(X;\mathbb{Z}/s) \cong 0$. Then we have that the reduced homology of X with coefficients in $\mathbb{Z}/s$ is trivial.

(iii) < = > (iv). This is obvious from the universal coefficient theorem. If A is an abelian group, then $A \otimes \mathbb{Z}/s\mathbb{Z}$ and $\mathrm{Tor}(A, \mathbb{Z}/s\mathbb{Z})$ are respectively the cokernel and kernel of multiplication by$s$ on A, so these group vanish if and only if A is uniquely s-divisible.

(iv) < = > (v). It follows from Serre mod $\mathscr{C}$ theory, see [15]. A 1-connected space has uniquely S-divisible homology groups if and only if it has uniquely S-divisible homotopy groups.

(v) = > (i). Let $p: X^{(S,n)} \to X$ be the S-cofibrant approximation of X. By Proposition 4.2, if (v) holds, then for every $q \geqslant n$, $\pi_q(\mathbb{Z}[S^{-1}]; X)$ is isomorphic to $\pi_q(X)$ and similarly for $X^{(S,n)}$. Therefore, p is a weak equivalence and X is weakly equivalent to an (S,n)-cofibrant space. □

Theorem 4.2. *The localized category* $\mathrm{Ho}(\mathrm{Top}_\star^{(S,n)})$ *is equivalent to the homotopy category of uniquely* (S,n)*-divisible,* $(n-1)$*-connected spaces.*

Proof. By Theorem 3.1 $\mathrm{Top}_\star^{(S,n)}$ has the structure of a closed model category. Therefore the localized category $\mathrm{Ho}(\mathrm{Top}_\star^{(S,n)})$ is equivalent to homotopy category of (S,n)-cofibrant spaces (in this case all spaces are (S,n)-fibrant). Now by Theorem 4.1 one has that a space X is (S,n)-cofibrant if and only if X is a uniquely (S,n)-divisible, $(n-1)$-connected space. Then $\mathrm{Ho}(\mathrm{Top}_\star^{(S,n)})$ is equivalent to homotopy category of uniquely (S,n)-divisible, $(n-1)$-connected spaces. □

Now we study the relationship between the homotopy groups of a space X and the homotopy groups of its (S,n)-colocalization $X^{(S,n)}$.

Proposition 4.3. *Let* $X^{(S,n)}$ *be the* (S,n)*-colocalization of a pointed space* X*. Then for* $q \geqslant n$ *the following sequence is exact*

$$0 \to \mathrm{Ext}(\mathbb{Z}[S^{-1}], \pi_{q+1}X) \to \pi_q(X^{(S,n)}) \to \mathrm{Hom}(\mathbb{Z}[S^{-1}], \pi_q X) \to 0.$$

Corollary 4.1. *Suppose that* B *is an abelian group and* $K(B,q)$ *the Eilenberg–Mac Lane space at degree* $q \geqslant 2$*. Then* $K(B,n)^{(S,n)}$ *is an Eilenberg–Mac Lane space such that* $\pi_n(K(B,n)^{(S,n)}) \cong \mathrm{Hom}(\mathbb{Z}[S^{-1}], B)$*. For* $m > n$*,* $K(B,m)^{(S,n)}$ *has two possible non trivial homotopy groups*

$$\pi_{m-1}(K(B,m)^{(S,n)}) \cong \mathrm{Ext}(\mathbb{Z}[S^{-1}], B),$$

$$\pi_m(K(B,m)^{(S,n)}) \cong \mathrm{Hom}(\mathbb{Z}[S^{-1}], B).$$

Remark 4.1. The multiplicative system S can be seen as a cofinite directed set. We can assume that all the integers of S are positive. Given s', $s \in S$ we say that $s' \geqslant s$ if there is an integer t such that $s' = ts$. Given an abelian group B we can construct the pro-group ${}^S B$ directed by S as follows: Define ${}^S B(s) = B$ for all $s \in S$. If $s' = ts$ the bounding map ${}^S B_s^{s'}: {}^S B(s') \to {}^S B(s)$ is defined by ${}^S B_s^{s'}(x) = tx$ for every $x \in {}^S B(s')$. It is easy to check that

$$\mathrm{Hom}(\mathbb{Z}[S^{-1}], B) \cong \lim({}^S B),$$

$$\mathrm{Ext}(\mathbb{Z}[S^{-1}], B) \cong \lim{}^1({}^S B).$$

Remark 4.2. Given a pointed space X, we can associate to the loop space ΩX the pro-space ${}^S\Omega X$ directed by S as follows: Define ${}^S\Omega X(s) = \Omega X$ for all $s \in S$. If $s' = ts$ the bounding map ${}^S\Omega X(s') \to {}^S\Omega X(s)$ is defined by sending a loop ω to the composite ω^t. The homotopy limit $\mathrm{holim}^S \Omega X$ is weakly equivalent to the space $\Omega(X^{(S,n)})$.

Remark 4.3. For $n > 1$, and a multiplicative system S of non-zero integers, we have the Sullivan–Quillen localization, that for an $(n-1)$-connected space X gives a localization map $X \to X \otimes \mathbb{Z}[S^{-1}]$, such that $X \otimes \mathbb{Z}[S^{-1}]$ is a uniquely (S,n)-divisible, $(n-1)$-connected space. This map is initially universal among every map $f: X \to Z$ with Z a uniquely (S,n)-divisible, $(n-1)$-connected space. Using the (homotopy) universal property of the map $Y^{(S,n)} \to Y$, for every $(n-1)$-connected space X , and a pointed space Y, one has

$$\mathrm{Ho}(\mathrm{Top}_\star)(X \otimes \mathbb{Z}[S^{-1}], Y) \cong \mathrm{Ho}(\mathrm{Top}_\star)(X, Y^{(S,n)}).$$

In particular for $q \geqslant n$, the homotopy groups of $Y^{(S,n)}$ can be obtained as

$$\pi_q(Y^{(S,n)}) \cong \mathrm{Ho}(\mathrm{Top}_\star)(S^q, Y^{(S,n)}) \cong \mathrm{Ho}(\mathrm{Top}_\star)(S^q \otimes \mathbb{Z}[S^{-1}], Y).$$

Remark 4.4. For $n = 2$, and the multiplicative system $S = \mathbb{Z} - \{0\}$ we have a "corationalization" functor and a canonical map $Y^{(S,2)} \to Y$ that gives an equivalence of the localized category and homotopy category of rational 1-connected spaces. We remark that we can also define a functor and a colocalization map $Y^{(S,1)} \to Y$. The functor $Y^{(S,1)}$ becomes more complicated because 1-connected spaces do not have nice properties.

Remark 4.5. Suppose that X is an 1-connected space and for the ring $R = \mathbb{Z}/p$, we consider the Bousfield–Kan R-completion $X \to R_\infty X$, see [2,12]. If we take the multiplicative system S generated by p and $n=2$, we have the (S,n)-colocalization $X^{(S,n)} \to X$ and the homotopy fibre F of the map $X \to R_\infty X$. Since the homotopy groups of $R_\infty X$ are Ext-p-complete, one has that the hom-set $\mathrm{Ho}(\mathrm{Top}_\star)(X^{(S,2)}, R_\infty X)$ is trivial. Therefore there is a canonical map $X^{(S,n)} \to F$. On the other hand, if we assume that in each $\pi_k X$ the p-torsion elements are of bounded order, one has the exact sequence

$$\cdots \to \pi_{k+1}X \to \mathrm{Ext}(C[\tfrac{1}{p}], \pi_{k+1}X) \to \pi_k F \to \pi_k X \to \mathrm{Ext}(C[\tfrac{1}{p}], \pi_k X \to \cdots$$

(recall that $C[1/p]$ denotes the quotient $\mathbb{Z}[1/p]/\mathbb{Z}$) and consequently the exact sequence

$$0 \to \mathrm{Ext}(\mathbb{Z}[\tfrac{1}{p}], \pi_{k+1}X) \to \pi_k F \to \mathrm{Hom}(\mathbb{Z}[\tfrac{1}{p}], \pi_k X) \to 0.$$

Therefore we have that $\pi_k F$ are uniquely p-divisible. With the additional condition $\mathrm{Ext}(\mathbb{Z}[\frac{1}{p}], \pi_2 X) \cong 0$, we have that F is also 1-connected. In this case, there is also a canonical map $F \to X^{(S,2)}$ and we have that F is weakly equivalent to $X^{(S,2)}$.

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