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A TRANSFERENCE THEOREM FOR HERMITE EXPANSIONS

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A transference theorem for multipliers of Hermite expansions is proved. The result allows to transfer weighted $L^2(\mathbb{R}^n)$ estimates from lower to higher dimensions.

1. Introduction and the transference theorem.

Let $H = -\Delta + |\cdot|^2$ be the Hermite operator on $\mathbb{R}^n$. The eigenfunctions of H, for a multiindex μ, are given by

$$\Phi_\mu(x) = \prod_{j=1}^{n} h_{\mu_j}(x_j),$$

where $h_k(t) = (2^k k!\sqrt{\pi})^{-1/2} H_k(t) e^{-\frac{t^2}{2}}$, with H_k the Hermite polynomials. So, taking $|\mu| = \mu_1 + \ldots + \mu_n$, we have $H\Phi_\mu = (2|\mu| + n)\Phi_\mu$. The sequence $\{\Phi_\mu\}_{\mu\in\mathbb{N}^n}$ forms a complete orthonormal system for $L^2(\mathbb{R}^n, dx)$. Given a function f on $\mathbb{R}^n$ we define its Hermite expansion by

$$f \sim \sum_{\mu} \hat{f}(\mu)\Phi_\mu, \quad \hat{f}(\mu) = \int_{\mathbb{R}^n} f(y)\Phi_\mu(y)\, dy.$$

Defining P_k to be the projection onto the kth eigenspace

$$P_k f = \sum_{|\mu|=k} \hat{f}(\mu)\Phi_\mu,$$

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we obtain that $f \sim \sum_{k=0}^{\infty} P_k f$.

Hermite expansions have been widely studied as we can see in [2] and the literature cited there. The aim of this note is to prove a transference theorem for multipliers of Hermite expansions. Our result allows to transfer some weighted L^2 estimates from lower to higher dimensions. In [1], a similar theorem for radial multipliers of the Fourier transform is proved. For references about transference theorems, see [1].

Taking $m \in L^\infty(\mathbb{R}_+)$, we consider the operators given by

$$T_m f = \sum_{k=0}^{\infty} m(2k+n) P_k f. \tag{1}$$

The following result holds for this kind of multipliers:

THEOREM. *Let $w(s)$ be a nonnegative measurable function in $\mathbb{R}_+$, and let us suppose that the operator T_m, defined by (1), is bounded from $L^2(\mathbb{R}^n, w(|x|)\,dx)$ into itself, with $n \geq 2$, i. e.,*

$$\|T_m f\|_{L^2(\mathbb{R}^n, w(|x|)\,dx)} \leq C \|f\|_{L^2(\mathbb{R}^n, w(|x|)\,dx)}$$

$(f \in L^2(\mathbb{R}^n))$. *Then the same inequality is true in all the spaces $\mathbb{R}^{n+2j}$, $j \geq 1$, and with the same constant C as above.*

2. Series of Laguerre functions and the proof of the Theorem.

In the proof of our result we will use multipliers for series of Laguerre functions. We define, for $\alpha \geq -1/2$,

$$\mathcal{L}_n^\alpha(r) = \left(2 \frac{n!}{\Gamma(\alpha+n+1)}\right)^{1/2} e^{-r^2/2} L_n^\alpha(r^2), \; n = 0, 1, \ldots,$$

where L_n^α denotes the Laguerre polynomial of order α. These functions form a complete orthonormal system for $L^2((0,\infty), r^{2\alpha+1}\,dr)$. Given f defined on $(0,\infty)$ we have

$$f \sim \sum_{k=0}^{\infty} a_k(f) \mathcal{L}_k^\alpha, \quad a_k(f) = \int_0^\infty f(t) \mathcal{L}_k^\alpha(t) t^{2\alpha+1}\,dt.$$

So, taking again $m \in L^\infty(\mathbb{R}_+)$, we define the operators

$$\mathcal{T}_{m,\alpha} f = \sum_{k=0}^{\infty} m(2(2k+\alpha+1)) a_k(f) \mathcal{L}_k^\alpha.$$

Proof of the Theorem. The main step is the following equivalence: The operator T_m is bounded from $L^2(\mathbb{R}^n, w(|x|)\,dx)$ into itself if and only if the inequalities

$$\|\mathcal{T}_{m,i+n/2-1}f\|_{L^2((0,\infty),r^{2i+n-1}w(r)\,dr)} \le C\|f\|_{L^2((0,\infty),r^{2i+n-1}w(r)\,dr)}, \tag{2}$$

hold for all $i \ge 0$ and all f continuous with compact support.

So, the proof is trivial because, if the sequence of inequalities (2) is satisfied for an integer $n \ge 2$, then it is also satisfied, with an equal or smaller C, for $n + 2j$ (there are just j inequalities less to verify).

Now, we are going to show the equivalence between the boundedness of T_m and (2). It can be proved using the following result (see [2, Theorem 3.4.1. in p. 82]): Let us assume that $f(x) = f_0(|x|)\mathcal{P}_i(x)$ where $\mathcal{P}_i$ is a solid harmonic of degree i. Then we get that $P_{2l+i}(f) = F_l(|x|)\mathcal{P}_i(x)$ where

$$F_l(r) = R_l^\delta(f_0)L_l^\delta(r^2)e^{-r^2/2} \tag{3}$$

with $\delta = i + n/2 - 1$ and

$$R_l^\delta(f_0) = 2\frac{l!}{\Gamma(l+\delta+1)}\int_0^\infty f_0(r)L_l^\delta(r^2)e^{-r^2/2}r^{2\delta+1}\,dr. \tag{4}$$

For other values of k, $P_k f = 0$.

In this way, given f continuous with compact support in $\mathbb{R}^n$, we consider its expansion into spherical harmonics $f(x) = \sum_{i=0}^\infty f_i(|x|)\mathcal{P}_i(x)$, where like above $\mathcal{P}_i$ is a solid harmonic of degree i and such that $\int_{\mathbb{S}^{n-1}} |\mathcal{P}_i(\theta)|^2\,d\sigma(\theta) = 1$. By the definition of T_m and using (3) and (4) we have

$$\begin{aligned}
T_m(f,x) &= \sum_{k=0}^\infty m(2k+n)P_k\left(\sum_{i=0}^\infty f_i(|x|)\mathcal{P}_i(x)\right)\\
&= \sum_{i=0}^\infty \mathcal{P}_i(x)\sum_{l=0}^\infty m(4l+2i+n)R_l^{i+n/2-1}(f_i)L_l^{i+n/2-1}(|x|^2)e^{-|x|^2/2}\\
&= \sum_{i=0}^\infty \mathcal{P}_i(x)\sum_{l=0}^\infty m(2(2l+i+n/2))a_l(f_i)\mathcal{L}_l^{i+n/2-1}(|x|)\\
&= \sum_{i=0}^\infty \mathcal{P}_i(x)\mathcal{T}_{m,i+n/2-1}(f_i,|x|).
\end{aligned}$$

By integrating in polar coordinates, first with respect to the angular variable, and from the orthogonality of $\{\mathcal{P}_i\}_{i\geq 0}$, we obtain

$$\int_{\mathbb{R}^n} |f(x)|^2 w(|x|)\,dx = \sum_{i=0}^{\infty} \int_0^{\infty} |f_i(r)|^2 r^{n+2i-1} w(r)\,dr$$

$$\int_{\mathbb{R}^n} |T_m(f,x)|^2 w(|x|)\,dx = \sum_{i=0}^{\infty} \int_0^{\infty} |\mathcal{T}_{m,i+n/2-1}(f_i,r)|^2 r^{n+2i-1} w(r)\,dr.$$

Comparing the right-hand sides, and taking into account that the functions f_i may be arbitrary, the necessity and sufficiency of (2) become obvious.

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