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Lie–Yamaguti algebras related to $\mathfrak{g}_2$

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Abstract

Lie–Yamaguti algebras (or generalized Lie triple systems) are intimately related to reductive homogeneous spaces. Simple Lie–Yamaguti algebras whose standard enveloping Lie algebra is the simple Lie algebra of type G_2 are described, making use of the octonions. These examples reveal the much greater complexity of these systems, compared to Lie triple systems.

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1. Introduction

Let G be a Lie group acting smoothly and transitively on a manifold M and let H be the isotropy subgroup at a fixed point; the homogeneous space $M \simeq G/H$ is said to be *reductive* (see [19]) in case there is a subspace $\mathfrak{m}$ of the Lie algebra $\mathfrak{g}$ of G such that

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \tag{1.1}$$

(where $\mathfrak{h}$ is the Lie subalgebra of the closed group H) and that $(\mathrm{Ad}\, H)(\mathfrak{m}) \subseteq \mathfrak{m}$; so that $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$ (and the converse is true if H is connected).

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A pair $(\mathfrak{g}, \mathfrak{h})$ formed by a Lie algebra $\mathfrak{g}$ over a field k and a subalgebra $\mathfrak{h}$ such that there is a complementary subspace $\mathfrak{m}$ of $\mathfrak{h}$ in $\mathfrak{g}$ with $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$ is called a *reductive pair* (see [21]) and decomposition (1.1) a *reductive decomposition*. In this situation, consider the binary and ternary multiplications on $\mathfrak{m}$ given by

$$\begin{aligned} &x \cdot y = \pi_{\mathfrak{m}}([x, y]),\\ &[x, y, z] = [\pi_{\mathfrak{h}}([x, y]), z] \end{aligned} \tag{1.2}$$

for any $x, y, z \in \mathfrak{m}$, where $\pi_{\mathfrak{h}}$ and $\pi_{\mathfrak{m}}$ denote the projections on $\mathfrak{h}$ and $\mathfrak{m}$ relative to the reductive decomposition and where $[\ ,\ ]$ denotes the Lie bracket in $\mathfrak{g}$. It is clear that $(\mathfrak{m}, \cdot, [\ ,\ ,\ ])$ satisfies the conditions in the following definition [17, Definition 5.1]:

Definition 1.1. A *Lie–Yamaguti algebra* $(\mathfrak{m}, \cdot, [\ ,\ ,\ ])$ is a vector space $\mathfrak{m}$ equipped with a bilinear operation $\cdot : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ and a trilinear operation $[\ ,\ ,\ ] : \mathfrak{m} \times \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ such that, for all $x, y, z, u, v, w \in \mathfrak{m}$:

(LY1) $x \cdot x = 0$,
(LY2) $[x, x, y] = 0$,
(LY3) $\sum_{(x,y,z)}([x, y, z] + (x \cdot y) \cdot z) = 0$,
(LY4) $\sum_{(x,y,z)}[x \cdot y, z, t] = 0$,
(LY5) $[x, y, u \cdot v] = [x, y, u] \cdot v + u \cdot [x, y, v]$,
(LY6) $[x, y, [u, v, w]] = [[x, y, u], v, w] + [u, [x, y, v], w] + [u, v, [x, y, w]]$.

Here $\sum_{(x,y,z)}$ means the cyclic sum on x, y, z.

The Lie–Yamaguti algebras with $x \cdot y = 0$ for any x, y are exactly the Lie triple systems, which appear in the study of the symmetric spaces, while the Lie–Yamaguti algebras with $[\ ,\ ,\ ] = 0$ are the Lie algebras.

Nomizu [19, Theorem 8.1] proved that the set of invariant affine connections on a reductive homogeneous space G/H is in bijection with $\mathrm{Hom}_H(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$, where $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is the corresponding reductive decomposition and $\mathfrak{m}$ is a module for H under the adjoint action. For connected H, $\mathrm{Hom}_H(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m}) = \mathrm{Hom}_{\mathfrak{h}}(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$. The *canonical connection* corresponds to the zero map, while the *natural connection* (which has trivial torsion) to the map $\alpha : \mathfrak{m} \otimes \mathfrak{m} \to \mathfrak{m}$, $x \otimes y \mapsto \frac{1}{2}x \cdot y = \frac{1}{2}\pi_{\mathfrak{m}}([x, y])$. This bijection makes several classes of nonassociative algebras (defined on $\mathfrak{m}$) play a role in differential geometry (see for instance [21,18]).

Nomizu [19, Section 19] also showed that, given any affinely connected and connected manifold M with parallel torsion T and curvature R, the tangent space at any point in M satisfies the above definition with $x \cdot y = -T(x, y)$ and $[x, y, z] = -R(x, y)z$.

The notion of a Lie–Yamaguti algebra is a natural abstraction made by Yamaguti [24] of Nomizu's considerations. Yamaguti called these systems *general Lie triple systems*, while Kikkawa [13] termed them *Lie triple algebras*. The term Lie–Yamaguti algebra, adopted here, appeared for the first time in [17]. These algebras have been studied by several authors [14,15,20,21,23], although there is not a general structure theory. In particular, a classification of the simple Lie–Yamaguti algebras seems to be a very difficult task.

Given a Lie–Yamaguti algebra $(\mathfrak{m}, \cdot, [\ ,\ ,\ ])$ and any two elements $x, y \in \mathfrak{m}$, the linear map $D(x, y) : \mathfrak{m} \to \mathfrak{m}, z \mapsto D(x, y)(z) = [x, y, z]$ is, due to (LY5) and (LY6), a derivation of both the binary and ternary products. Moreover, if $D(\mathfrak{m}, \mathfrak{m})$ denotes the linear span of these maps, it is closed under commutation thanks to (LY6). Let $\mathfrak{g}(\mathfrak{m}) = D(\mathfrak{m}, \mathfrak{m}) \oplus \mathfrak{m}$ with anticommutative multiplication given, for any $x, y, z, t \in \mathfrak{m}$, by

$$\begin{aligned}
&[D(x, y), D(z, t)] = D([x, y, z], t) + D(z, [x, y, t]),\\
&[D(x, y), z] = D(x, y)(z) = [x, y, z],\\
&[z, t] = D(z, t) + z \cdot t. \qquad (1.3)
\end{aligned}$$

Then it is straightforward [24] to check that $\mathfrak{g}(\mathfrak{m})$ is a Lie algebra, called the *standard enveloping Lie algebra* of the Lie–Yamaguti algebra $\mathfrak{m}$. The pair $(\mathfrak{g}(\mathfrak{m}), D(\mathfrak{m}, \mathfrak{m}))$ is a reductive pair and the operations in $\mathfrak{m}$ coincide with those given by (1.2), where $\mathfrak{h} = D(\mathfrak{m}, \mathfrak{m})$.

Proposition 1.2. (i) *Let* $(\mathfrak{m}, \cdot, [\ ,\ ,\ ])$ *be a Lie–Yamaguti algebra. If its standard enveloping Lie algebra is simple, so is* $(\mathfrak{m}, \cdot, [\ ,\ ,\ ])$.

(ii) *Let* $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ *be a reductive decomposition of a simple Lie algebra* $\mathfrak{g}$, *with* $\mathfrak{m} \neq 0$. *Then* $\mathfrak{g}$ *is isomorphic to the standard enveloping Lie algebra of the Lie–Yamaguti algebra* $(\mathfrak{m}, \cdot, [\ ,\ ,\ ])$ *given by* (1.2).

Proof. For (i) note that if $\mathfrak{n}$ is an ideal of $\mathfrak{m}$ (that is, $\mathfrak{m} \cdot \mathfrak{n} \subseteq \mathfrak{n}$ and $[\mathfrak{m}, \mathfrak{n}, \mathfrak{m}] \subseteq \mathfrak{n}$) then $D(\mathfrak{m}, \mathfrak{n}) \oplus \mathfrak{n}$ is easily checked to be an ideal of $\mathfrak{g}(\mathfrak{m})$.

For (ii) it is enough to note that $\pi_{\mathfrak{h}}([\mathfrak{m}, \mathfrak{m}]) \oplus \mathfrak{m}$ $(=[\mathfrak{m}, \mathfrak{m}] + \mathfrak{m})$ is an ideal of $\mathfrak{g}$ and that $\{x \in \mathfrak{h} : [x, \mathfrak{m}] = 0\}$ is an ideal too. Hence if $\mathfrak{g}$ is simple, $\pi_{\mathfrak{h}}([\mathfrak{m}, \mathfrak{m}]) = \mathfrak{h}$ which embeds naturally in $D(\mathfrak{m}, \mathfrak{m}) \subseteq \mathrm{End}_k(\mathfrak{m})$ (k being the ground field). From here, a natural isomorphism from $\mathfrak{g}$ onto $\mathfrak{g}(\mathfrak{m})$ is constructed. □

Our purpose in this paper is to provide examples of simple Lie–Yamaguti algebras $\mathfrak{m}$. We will restrict ourselves to the algebras whose standard enveloping Lie algebra $\mathfrak{g}(\mathfrak{m}) = \mathfrak{h} \oplus \mathfrak{m}$ is a central simple Lie algebra of type G_2 with a nonabelian reductive subalgebra $\mathfrak{h}$ in $\mathfrak{g}(\mathfrak{m})$ (that is, $\mathfrak{g}(\mathfrak{m})$ is a completely reducible module for $\mathfrak{h}$ under the adjoint action). It will be shown that even such restrictive conditions give a large variety of very different possibilities. This setting is motivated by the existence of several well-known reductive homogeneous spaces which are quotients of the compact Lie group $G = G_2$: the six-dimensional sphere $S^6 \simeq G/SU(3)$ (see [8] and references therein), the Stiefel manifold $V_{7,2} \simeq G/SU(2)$, the Grassmann manifold $\mathrm{Gr}_{7,2} = G/U(2)$, as well as the symmetric space $G/SO(4)$ or the isotropy irreducible space $G/SO(3)$.

While for a simple Lie triple system $\mathfrak{m}$ over an algebraically closed field of characteristic zero with standard enveloping Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, either $\mathfrak{h}$ is semisimple or reductive with a one-dimensional center and $\mathfrak{m}$ is either an irreducible module over $\mathfrak{h}$ (this is always the case if $\mathfrak{h}$ is semisimple) or the direct sum of two irreducible contragredient modules, the examples of Lie–Yamaguti algebras given here will show that no results of this type should be expected for Lie–Yamaguti algebras. The possibilities for the structure of $\mathfrak{m}$ as a module for $\mathfrak{h} = D(\mathfrak{m}, \mathfrak{m})$ do not seem to follow any pattern.

Sagle [20] defined a Lie–Yamaguti algebra $(\mathfrak{m}, \cdot, [\ ,\ ,\ ])$ to be *homogeneous* if there are scalars α, β, γ such that, for any $x, y, z \in \mathfrak{m}$

$$[x, y, z] = \alpha(x \cdot y) \cdot z + \beta(y \cdot z) \cdot x + \gamma(z \cdot x) \cdot y. \tag{1.4}$$

Sagle proved that, given any simple finite-dimensional homogeneous Lie–Yamaguti algebra $(\mathfrak{m}, \cdot, [\ ,\ ,\ ])$ over a field of characteristic zero, either $(\mathfrak{m}, \cdot)$ is a simple Lie or Malcev algebra, or it is in another variety defined by the following identity of degree four:

$$J(x, y, z) \cdot w = J(w, x, y \cdot z) + J(w, y, z \cdot x) + J(w, z, x \cdot y),$$

where $J(x, y, z) = (x \cdot y) \cdot z + (y \cdot z) \cdot x + (z \cdot x) \cdot y$ for any x, y, z.

It will be shown (Corollary 5.8) that none of the examples considered in this paper are homogeneous, so that homogeneity seems to be a very restrictive condition.

The paper is organized as follows. Section 2 will review the Cayley–Dickson process, used to construct the Cayley algebras by means of two copies of a quaternion algebra. This will be used to obtain a family of reductive subalgebras of the exceptional simple Lie algebras of type G_2. In Section 3 it will be shown that, over fields of characteristic 0, these subalgebras are essentially all the possible nonabelian reductive subalgebras, up to conjugation. Section 4 will be devoted to the detailed description, over algebraically closed fields of characteristic 0, of the associated reductive pairs and Lie–Yamaguti algebras. Most of them will be described in terms of the classical *transvections*, inspired by the work of Dixmier [4]. A misprint in [4, 6.2] will be corrected along the way. Finally, Section 5 will deal mainly with properties of the binary anticommutative algebras $(\mathfrak{m}, \cdot)$ that will have appeared so far. They will be proven to be simple, and their Lie algebras of derivations and Lie multiplication algebras will be computed. In particular, this will show that none of these Lie–Yamaguti algebras are homogeneous in Sagle's sense. The holonomy algebras will be computed too in Section 5.

2. The Cayley–Dickson process and related reductive pairs

Throughout this section, k will denote a ground field of characteristic $\neq 2, 3$. The Cayley algebras over k are the eight-dimensional unital composition algebras. Let us recall briefly some well-known features of these algebras, which can be found in [22,11].

The Cayley algebras can be obtained from k by three consecutive applications of the Cayley–Dickson process, which works as follows. Let A be any unital algebra over k with a scalar involution $x \mapsto \bar{x}$ (so that $x + \bar{x}$ and $x\bar{x} = \bar{x}x$ belong to $k = k1$ for any $x \in A$) and let $0 \neq \alpha \in k$. Then the Cayley–Dickson process gives a new algebra $B = (A, \alpha) = A \oplus Au$ (direct sum of two copies of A, here u is just a symbol) with multiplication given by

$$(a + bu)(c + du) = (ac + \alpha\bar{d}b) + (da + b\bar{c})u \tag{2.1}$$

and scalar involution

$$\overline{a + bu} = \bar{a} - bu. \tag{2.2}$$

In case A is a composition algebra with norm $n(x) = x\bar{x} = \bar{x}x \in k$, then B is again a composition algebra if and only if A is associative.

Then, given nonzero scalars $\alpha, \beta, \gamma \in k$, $K = (k, \alpha)$ is a quadratic étale algebra (that is, K is either a quadratic separable field extension of k or isomorphic to $k \times k$), $Q = (K, \beta)$ is a quaternion algebra and $C = (Q, \gamma)$ is a Cayley algebra.

Conversely, let C be any Cayley algebra over k, with norm n and standard involution $x \mapsto \bar{x}$. For any $u_1, u_2, u_3 \in C$ such that $n(u_i) \neq 0$, $i = 1, 2, 3$, and u_1 is orthogonal to 1, u_2 orthogonal to 1 and u_1, and u_3 orthogonal to $1, u_1, u_2, u_1u_2$, then $K = k1 + ku_1$ is a quadratic étale algebra, $Q = K \oplus Ku_2$ is a quaternion algebra and $C = Q \oplus Qu_3$ and, in the three cases, formulae (2.1), (2.2) are satisfied.

Any element $x \in C$ satisfies the quadratic relation

$$x^2 - t(x)x + n(x) = 0, \tag{2.3}$$

where $t(x) = x + \bar{x} = n(x, 1)$ (here $n(x, y)$ is the symmetric bilinear form associated to the norm: $n(x, y) = n(x + y) - n(x) - n(y)$ for any $x, y \in C$). Moreover, let K be the quadratic étale subalgebra of C above, then by Artin's Theorem, $(ab)x = a(bx)$ for any $a, b \in K$ and $x \in C$, and C becomes in this way a rank 4 free left K-module. Take $0 \neq l \in K$ with $t(l) = 0$ (for instance, $l = u_1$), then the map:

$$\begin{aligned} \sigma : C \times C &\longrightarrow K, \\ (x, y) &\mapsto n(l)n(x, y) - n(lx, y)l \end{aligned} \tag{2.4}$$

is a nondegenerate hermitian form which, up to a scalar, does not depend on l.

The Lie algebra of derivations $\mathfrak{g}_2 = \operatorname{Der} C$ is a central simple Lie algebra of type G_2, $C_0 = \{x \in C : t(x) = 0\}$ is the nontrivial irreducible module for $\mathfrak{g}_2$ of minimal dimension ($C = k \oplus C_0$ and k is a trivial module for $\mathfrak{g}_2$) and there is a surjective $\mathfrak{g}_2$-invariant map:

$$\begin{aligned} C \otimes C &\longrightarrow \mathfrak{g}_2, \\ x \otimes y &\mapsto D_{x,y} = L_{[x,y]} - R_{[x,y]} - 3[L_x, R_y], \end{aligned} \tag{2.5}$$

where $L_x : y \mapsto xy$, $R_x : y \mapsto yx$ denote the left and right multiplications.

If $u = u_3$, then the quaternion subalgebra $Q = K \oplus Ku_2$ above satisfies

$$C = Q \oplus Qu \tag{2.6}$$

and this is a $\mathbb{Z}_2$-grading of C because of (2.1). Moreover, $Q^\perp = Qu$ (for any subspace S of C, $S^\perp$ denotes the orthogonal subspace relative to n). The restriction of σ to any of K, Q, $K^\perp$ and $Q^\perp$ is nondegenerate.

The $\mathbb{Z}_2$-grading in (2.6) induces the corresponding $\mathbb{Z}_2$-grading $\mathfrak{g}_2 = (\mathfrak{g}_2)_{\bar{0}} \oplus (\mathfrak{g}_2)_{\bar{1}}$, by considering even and odd derivations. Note that $(\mathfrak{g}_2)_{\bar{0}} = D_{Q,Q} + D_{Q^\perp,Q^\perp}$, while $(\mathfrak{g}_2)_{\bar{1}} = D_{Q,Q^\perp}$.

To state our main result of this section we need one extra ingredient. Given any $w \in K \setminus k$ with $n(w) = 1$ (take for instance $w = x/\bar{x}$ for $x \in K \setminus k$ with $n(x) \neq 0 \neq t(x)$), the map $\tau_w : C \to C$ such that $\tau_w(x) = x$ and $\tau_w(xu) = (wx)u = x(wu)$ for any $x \in Q$, is an automorphism of C.

In the next result, several natural subalgebras of $\mathfrak{g}_2$ related to the chain of subalgebras $k \subseteq K \subseteq Q \subseteq C$ in the Cayley–Dickson process are considered.

Theorem 2.1. *Let* k, K, Q, C *and* u, w *be as above. Then there are the following isomorphisms of Lie algebras*:

(i) $\mathfrak{h}^1 := \{d \in \mathfrak{g}_2 : d(Q) \subseteq Q\} \cong \mathfrak{so}(Q, n)$ (*the orthogonal Lie algebra*).
(ii) $\mathfrak{h}^2 := \{d \in \mathfrak{g}_2 : d(Q) \subseteq Q,\ d(K) = 0\} \cong \mathfrak{u}(Q^\perp, \sigma)$ (*the unitary Lie algebra*).
(iii) $\mathfrak{h}^3 := \{d \in \mathfrak{g}_2 : d(Q) = 0\} \cong \mathfrak{su}(Q^\perp, \sigma)$ (*the special unitary Lie algebra*).
(iv) $\mathfrak{h}^4 := \{d \in \mathfrak{g}_2 : d\tau_w = \tau_w d\} \cong \mathfrak{u}(Q, \sigma)$.
(v) $\mathfrak{h}^5 := D_{Q,Q} \cong \mathfrak{su}(Q, \sigma)$.
(vi) $\mathfrak{h}^6 := \{d \in \mathfrak{g}_2 : d(K) = 0\} \cong \mathfrak{su}(K^\perp, \sigma)$.
(vii) $\mathfrak{h}^7 := \{d \in \mathfrak{g}_2 : d(Q) \subseteq Q,\ d(u) = 0\} \cong \mathfrak{so}(Q_0, n)$ ($Q_0 = \{x \in Q : t(x) = 0\}$).

Moreover, all the Lie subalgebras $\mathfrak{h}^i$, $i = 1, \ldots, 7$ *are reductive Lie subalgebras of* $\mathfrak{g}_2$ (*that is*, $\mathfrak{g}_2$ *is a completely reducible module for* $\mathfrak{h}^i$). *In particular, all the pairs* $(\mathfrak{g}_2, \mathfrak{h}^i)$ *are reductive*.

Proof. (i) Since $\mathfrak{g}_2 \subseteq \mathfrak{so}(C, n)$, $Q^\perp = Qu$ is invariant under $\mathfrak{h}^1$. Hence $\mathfrak{h}^1 = (\mathfrak{g}_2)_{\bar{0}} = \{d \in \mathfrak{g}_2 : d(Q) \subseteq Q,\ d(Qu) \subseteq Qu\}$ is the even part of the $\mathbb{Z}_2$-grading of $\mathfrak{g}_2$ induced by the $\mathbb{Z}_2$-grading in (2.6). In particular, $(\mathfrak{g}_2, \mathfrak{h}^1)$ is not only a reductive pair, but a symmetric pair. Now, since $Q^\perp = Qu$ generates C as an algebra, the linear map:

$$\begin{aligned} \Phi : \mathfrak{h}^1 &\longrightarrow \mathfrak{so}(Q, n), \\ d &\mapsto \quad \phi_d, \end{aligned} \tag{2.7}$$

where $d(xu) = \phi_d(x)u$ for any $x \in Q$, is well defined and one-to-one. Note that ϕ_d is the restriction to Q of $R_u^{-1} d R_u$. For any $a, b \in Q_0$, the maps L_a, R_b belong to $\mathfrak{so}(Q, n)$ and the map

$$\begin{aligned} Q_0^- \oplus Q_0^- &\longrightarrow \mathfrak{so}(Q, n), \\ (x, y) &\mapsto L_x - R_y, \end{aligned} \tag{2.8}$$

is an isomorphism of Lie algebras. Here Q_0^- denotes the Lie algebra Q_0 with the multiplication given by $[x, y] = xy - yx$, hence a central simple three-dimensional Lie algebra (and any such algebra arises in this way). The map in (2.8) is clearly one-to-one so, by dimension count, it is a bijection. Alternatively, $\mathfrak{so}(Q, n)$ is spanned by the maps $z \mapsto n(x, z)y - n(y, z)x = (z\bar{x} + x\bar{z})y - x(\bar{z}y + \bar{y}z) = -(x\bar{y})z + z(\bar{x}y) = -\frac{1}{2}\left(L_{x\bar{y} - y\bar{x}} - R_{\bar{x}y - \bar{y}x}\right)(z)$.

On the other hand, using (2.1), it is checked that, for any $x \in Q_0$, the linear maps d_x, D_x defined by

$$\begin{aligned} &d_x(Q) = 0,\ \ d_x(qu) = (xq)u, \\ &D_x(q) = [x, q],\ \ D_x(qu) = (-qx)u \end{aligned} \tag{2.9}$$

for any $q \in Q$, are derivations of C and $\Phi(d_x) = L_x$, $\Phi(D_x) = -R_x$, so that Φ in (2.7) is an isomorphism, as required.

This argument also shows that $\mathfrak{h}^1 = \mathfrak{s}^L \oplus \mathfrak{s}^R$ (direct sum of ideals), where

$$\mathfrak{s}^L = d_{Q_0} \quad \text{and} \quad \mathfrak{s}^R = D_{Q_0}, \tag{2.10}$$

both ideals being isomorphic to the simple Lie algebra Q_0^-.

(ii) The linear map $\Psi : \mathfrak{h}^1 \to \mathfrak{so}(Q^\perp, n)$, $d \mapsto d|_{Q^\perp}$ (the restriction to $Q^\perp$) is one-to-one too, and $\Psi(\mathfrak{h}^2) \subseteq \mathfrak{so}(Q^\perp, n) \cap \mathrm{End}_K(Q^\perp) = \mathfrak{u}(Q^\perp, \sigma)$ (σ as in (2.4)), which is four-dimensional. Moreover, $\mathfrak{s}^L \subseteq \mathfrak{h}^2$ and for $x \in Q_0$, $D_x \in \mathfrak{h}^2$ if and only if $[x, K] = 0$, if and only if $x \in K$. Hence $\mathfrak{s}^L \oplus D_{K_0} = \mathfrak{h}^2$, where $K_0 = K \cap C_0$. By dimension count $\mathfrak{h}^2 = \mathfrak{s}^L \oplus D_{K_0} \cong \mathfrak{u}(Q^\perp, \sigma)$.

(iii) Since $\mathfrak{s}^L \subseteq \mathfrak{h}^3 \subseteq \mathfrak{h}^2 = \mathfrak{s}^L \oplus D_{K_0}$ and $D_{K_0} \not\subseteq \mathfrak{h}^3$, it follows that $\mathfrak{h}^3 = \mathfrak{s}^L = [\mathfrak{h}^2, \mathfrak{h}^2] \cong \mathfrak{su}(Q^\perp, \sigma)$.

(iv) Since $\tau_w|_Q = I_Q$ (the identity map on Q), $\mathfrak{h}^4 \subseteq \mathfrak{h}^1$. Consider now the map Φ in (2.7) and take any $d \in \mathfrak{h}^1$ and $x \in Q$. Then

$$\begin{aligned} &d\tau_w(xu) = d((wx)u) = \phi_d(wx)u, \\ &\tau_w d(xu) = (w\phi_d(x))u, \end{aligned} \tag{2.11}$$

so that $d \in \mathfrak{h}^4$ if and only if $\phi_d \in \mathfrak{so}(Q, n) \cap \mathrm{End}_K(Q) = \mathfrak{u}(Q, \sigma)$, and $\Phi(\mathfrak{h}^4) = \mathfrak{u}(Q, \sigma)$.

Also, for any $x \in Q_0$, $\Phi(D_x) = -R_x \in \mathrm{End}_K(Q)$, but $\Phi(d_x) = L_x \in \mathrm{End}_K(Q)$ if and only if $x \in Q_0 \cap K = K_0$. Hence $\mathfrak{h}^4 = \Phi^{-1}(\mathfrak{u}(Q, \sigma)) = d_{K_0} \oplus \mathfrak{s}^R$.

(v) $\mathfrak{h}^5 = D_{Q,Q} = D_{Q_0,Q_0}$ is an ideal of $\mathfrak{h}^1 = \mathfrak{s}^L \oplus \mathfrak{s}^R$ because $[d, D_{x,y}] = D_{dx,y} + D_{x,dy}$ for any $x, y \in C$ and $d \in \mathfrak{g}_2$. Its dimension is at most three (as it is the image of the exterior power $\Lambda^2 Q_0$) and does not annihilate Q. Hence $\mathfrak{h}^5 = D_{Q,Q} = \mathfrak{s}^R = [\mathfrak{h}^4, \mathfrak{h}^4] \cong \mathfrak{su}(Q, \sigma)$.

(vi) This is proven in [8, Proposition 4.7].

(vii) $C = Q \oplus Qu = k1 \oplus ku \oplus Q_0 \oplus Q_0 u$ and any element in $\mathfrak{h}^7$ is determined by its action on Q_0. Hence the map $\varphi : \mathfrak{h}^7 \to \mathfrak{so}(Q_0, n)$, $d \mapsto d|_{Q_0}$ is one-to-one. Also, for any $x \in Q_0$, $d_x + D_x \in \mathfrak{h}^7$ and, by dimension count $\mathfrak{h}^7 = \{d_x + D_x : x \in Q_0\} \cong \mathfrak{so}(Q_0, n)$. Note that $\mathfrak{h}^7$ is a "diagonal subalgebra" in $\mathfrak{h}^1 = \mathfrak{s}^L \oplus \mathfrak{s}^R = d_{Q_0} \oplus D_{Q_0}$ and, with $\delta_x = d_x + D_x$,

$$\delta_x(q) = [x, q], \quad \delta_x(qu) = [x, q]u \tag{2.12}$$

for any $q \in Q$.

Finally, it is clear that we have the reductive decompositions $\mathfrak{g}_2 = \mathfrak{h}^i \oplus \mathfrak{m}^i$ with $\mathfrak{m}^1 = (\mathfrak{g}_2)_{\bar{1}} = \{d \in \mathfrak{g}_2 : d(Q) \subseteq Q^\perp,\ d(Q^\perp) \subseteq Q\}$, $\mathfrak{m}^2 = (\mathfrak{g}_2)_{\bar{1}} \oplus D_{K^\perp \cap Q}$, $\mathfrak{m}^3 = (\mathfrak{g}_2)_{\bar{1}} \oplus D_{Q_0}$, $\mathfrak{m}^4 = (\mathfrak{g}_2)_{\bar{1}} \oplus d_{K^\perp \cap Q}$, $\mathfrak{m}^5 = (\mathfrak{g}_2)_{\bar{1}} \oplus d_{Q_0}$, $\mathfrak{m}^6 = D_{K,K^\perp}$ (see [8, Section 4]) and $\mathfrak{m}^7 = (\mathfrak{g}_2)_{\bar{1}} \oplus \{d_x - D_x : x \in Q_0\}$.

It remains to be shown that all the $\mathfrak{h}^i$'s are reductive in $\mathfrak{g}_2$ and for this, it is enough to check it under the additional assumption of k being algebraically closed.

Then for $\mathfrak{h}^1 = \mathfrak{s}^L \oplus \mathfrak{s}^R$, $\mathfrak{g}_2 = \mathfrak{h}^1 \oplus \mathfrak{m}^1$ with $\mathfrak{m}^1 = (\mathfrak{g}_2)_{\bar{1}}$ being the tensor product of the two-dimensional natural module for $\mathfrak{s}^L \cong \mathfrak{sl}_2(k)$ and the four-dimensional irreducible module for $\mathfrak{s}^R \cong \mathfrak{sl}_2(k)$ [1, Theorem 3.2], and hence $\mathfrak{m}^1$ is irreducible for $\mathfrak{h}^1$. Now $\mathfrak{h}^2 = \mathfrak{s}^L \oplus D_{K_0}$ and $K_0 = ka$ with $t(a) = 0 \neq n(a)$. Then D_a is a semisimple element of $\mathfrak{s}^R$ and $\mathfrak{g}_2 = \mathfrak{h}^2 \oplus \mathfrak{m}^2$ with $\mathfrak{m}^2 = \mathfrak{m}^1 \oplus D_{K^\perp \cap Q}$, $\mathfrak{m}^1$ being the direct sum of four copies of the natural two-dimensional module for $\mathfrak{s}^L$ and $D_{K^\perp \cap Q}$ the sum of two one-dimensional trivial modules for $\mathfrak{s}^L$. On each of these summands, D_a acts as a scalar. Hence $\mathfrak{h}^2$ is reductive on $\mathfrak{g}_2$. For $i = 3$, $\mathfrak{g}_2 = \mathfrak{h}^3 \oplus \mathfrak{m}^3$ with $\mathfrak{h}^3 = \mathfrak{s}^L$ and $\mathfrak{m}^3 = \mathfrak{s}^R \oplus \mathfrak{m}^1$, $\mathfrak{m}^1$ being the sum of four copies of the natural module for $\mathfrak{h}^3 = \mathfrak{s}^L$ and $\mathfrak{s}^R$ being a trivial module for $\mathfrak{h}^3$. Also, $\mathfrak{h}^4 = d_{K_0} \oplus \mathfrak{s}^R$ so $\mathfrak{m}^4 = d_{K^\perp \cap Q} \oplus \mathfrak{m}^1$, $\mathfrak{m}^1$ being the sum of two copies of the four-dimensional irreducible module for $\mathfrak{s}^R \cong \mathfrak{sl}_2(k)$ and $d_{K^\perp \cap Q}$ the sum of two one-dimensional trivial modules for $\mathfrak{s}^R$,

d_a acting as a scalar on each of these irreducible modules. Similarly, $\mathfrak{h}^5 = \mathfrak{s}^R$ so $\mathfrak{m}^5 = \mathfrak{s}^L \oplus \mathfrak{m}^1$, $\mathfrak{m}^1$ being the sum of two copies of the four-dimensional irreducible module for $\mathfrak{s}^R$ and $\mathfrak{s}^L$ being a trivial $\mathfrak{s}^R$-module. For $\mathfrak{h}^6$ it is known that $\mathfrak{m}^6$ is the sum of the two contragredient three-dimensional irreducible modules for $\mathfrak{h}^6 \simeq \mathfrak{sl}_3(k)$ (see [8, Section 4] or [1, Section 5]). Finally, for $\mathfrak{h}^7 = \{d_x + D_x : x \in Q_0\}$, let $\delta_x = d_x + D_x$ as above. Then (2.12) shows that C_0 is the sum of two adjoint modules for $\mathfrak{h}^7 \cong Q_0^-$ and the trivial module ku. Then the vector subspace $\mathfrak{m}^7 = \{d_x - \frac{1}{3}D_x : x \in Q_0\} \oplus \mathfrak{m}^1$ can be checked to be the orthogonal complement to $\mathfrak{h}^7$ relative to the Killing form in $\mathfrak{g}_2$ (see also Theorem 4.5). The first summand in $\mathfrak{m}^7$ is an adjoint module for $\mathfrak{h}^7$, while $\mathfrak{m}^1 = D_{Q_0, Qu} = D_{Q_0,u} + \mathrm{span}\{D_{x,xu} : x \in Q_0\}$ (which follows from the identity $D_{xy,z} + D_{yz,x} + D_{zx,y} = 0$ [22, (3.73)]). But the span of $\{D_{x,xu} : x \in Q_0\}$ is, up to isomorphism, a quotient of the symmetric power $S^2(Q_0)$, which is the direct sum of the five-dimensional irreducible module for $\mathfrak{h}^7$ and a trivial one-dimensional module. Besides, $D_{Q_0,u}$ is an adjoint module for $\mathfrak{h}^7$. Since the dimension of $\mathfrak{m}^1$ is 8, it follows that $\mathfrak{m}^1$ is the sum of the adjoint module $D_{Q_0,u}$ and the irreducible five-dimensional module $\mathrm{span}\{D_{x,xu} : x \in Q_0\}$ for $\mathfrak{h}^7$. This finishes the proof. $\square$

3. Nonabelian reductive subalgebras of $\mathfrak{g}_2$

The purpose in this section is to show that the subalgebras in Theorem 2.1 essentially cover all the nonabelian reductive subalgebras $\mathfrak{h}$ of $\mathfrak{g}_2$ (that is, $\mathfrak{g}_2$ is a completely reducible $\mathfrak{h}$-module).

Throughout this section, the characteristic of the ground field k will be assumed to be 0.

Since $\mathfrak{g}_2 = \mathrm{Der}\, C$ is a completely reducible Lie algebra of linear transformations on the Cayley algebra C, so is any reductive subalgebra $\mathfrak{h}$ [12, Chapter III, exercise 20]. This implies [12, Chapter III, Theorem 10] that $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}] \oplus Z(\mathfrak{h})$, with $[\mathfrak{h}, \mathfrak{h}]$ semisimple and $Z(\mathfrak{h})$ the center of $\mathfrak{h}$, whose elements are semisimple transformations on C.

First recall [22] that a Cayley algebra C is termed *split* in case it contains an idempotent $e \neq 0, 1$, and this happens if and only if its norm n is isotropic. If this is the case, there is a Peirce decomposition:

$$C = ke_1 \oplus ke_2 \oplus U \oplus V, \tag{3.1}$$

with $e_1 = e$, $e_2 = 1 - e$, $U = e_1U = Ue_2$, $V = e_2V = Ve_1$ and $e_2U = Ue_1 = e_1V = Ve_2 = 0$. Then dual bases $\{x_1, x_2, x_3\}$ of U and $\{y_1, y_2, y_3\}$ of V can be chosen so that $(1 \leqslant i, j \leqslant 3)$:

$$\begin{aligned}
&e_1x_i = x_i = x_ie_2, \quad e_2x_i = 0 = x_ie_1,\\
&e_2y_i = y_i = y_ie_1, \quad e_1y_i = 0 = y_ie_2,\\
&x_i^2 = 0 = y_i^2,\\
&x_ix_j = \varepsilon_{ijk}x_k, \quad y_iy_j = \varepsilon_{ijk}y_k,\\
&x_iy_j = -\delta_{ij}e_1, \quad y_ix_j = -\delta_{ij}e_2,
\end{aligned} \tag{3.2}$$

where ε_{ijk} is the totally skewsymmetric tensor with $\varepsilon_{123} = 1$ and δ_{ij} is the usual Kronecker symbol, and

$$\begin{aligned} &n(e_1) = 0 = n(e_2), \quad n(e_1, e_2) = 1, \\ &n(e_i, x_j) = 0 = n(e_i, y_j), \\ &n(x_i) = 0 = n(y_i), \quad n(x_i, y_j) = \delta_{ij}. \end{aligned} \tag{3.3}$$

If $\mathfrak{h} = \{d \in \operatorname{Der} C \mid d(e_1) = 0 = d(e_2)\}$, then we are in the situation $\mathfrak{h} = \mathfrak{h}^6$ of Theorem 2.1, where $K = ke_1 \oplus ke_2$ and $\mathfrak{h} \cong \mathfrak{sl}_3(k)$, U and V being contragredient modules for $\mathfrak{h}$. Moreover, $Q = k1 \oplus (\oplus_{i=1}^3 k(x_i + y_i))$ is a quaternion subalgebra, and $C = Q \oplus Qu$ with $u = e_1 - e_2$, so that $\{d \in \operatorname{Der} C : d(Q) \subseteq Q,\ d(u) = 0\}$ is a type $\mathfrak{h}^7$ subalgebra inside $\mathfrak{h} = \mathfrak{h}^6$.

Lemma 3.1. *Let C be a Cayley algebra over an algebraically closed field k of characteristic* 0 *and let* $\mathfrak{s}$ *be a three-dimensional simple subalgebra of* Der C. *Then either*:

(i) *there exists a quaternion subalgebra Q of C invariant under* $\mathfrak{s}$, *or*
(ii) $\mathfrak{s}$ *acts irreducibly on* C_0 $(=\{x \in C : t(x) = 0\})$.

Proof. Notice that since k is algebraically closed, $\mathfrak{s} \cong \mathfrak{sl}_2(k)$.

The set $H = \{x \in C : \mathfrak{s}x = 0\}$ is a composition subalgebra of C and $H \neq C$ since $\mathfrak{s} \neq 0$. Therefore the dimension of H is either 1, 2 or 4. If $\dim_k H = 4$, H is a quaternion subalgebra and (i) is satisfied.

Assume now that $\dim_k H = 2$. Because of the hypotheses on k, $H = ke_1 \oplus ke_2$ for orthogonal idempotents e_1 and e_2 and hence, with the above notations, $\mathfrak{s} \subseteq \mathfrak{h} = \{d \in \operatorname{Der} C : d(e_1) = 0 = d(e_2)\}$ and U and V are three-dimensional $\mathfrak{s}$-modules. Since $U \cap H = 0 = V \cap H$, U and V are adjoint modules for $\mathfrak{s}$. A basis $\{d_1, d_2, d_3\}$ of $\mathfrak{s}$ can be taken so that $[d_i, d_j] = \varepsilon_{ijk} d_k$ for any $i, j = 1, 2, 3$ (recall that ε_{ijk} is the totally antisymmetric tensor with $\varepsilon_{123} = 1$). Then a basis $\{x_1, x_2, x_3\}$ of U can be taken with $d_i x_j = \varepsilon_{ijk} x_k$ and the dual basis $\{y_1, y_2, y_3\}$ in V relative to the pairing given by the norm $n : U \times V \to k$. Since $0 = n(d_i x_j, y_k) + n(x_j, d_i y_k)$ for any i, j, k, $d_i y_j = \varepsilon_{ijk} y_k$ too. Besides, $n(x_i x_j, x_k) = n(x_i, x_k \bar{x}_j) = -n(x_i, x_k x_j) = n(x_i, x_j x_k)$ for any i, j, k and it follows that $x_i x_j = \alpha \varepsilon_{ijk} y_k$ and, similarly, $y_i y_j = \beta \varepsilon_{ijk} x_k$ for suitable $\alpha, \beta \in k$. But $1 = n((x_1 + y_1)(x_2 + y_2)) = n(\beta x_3 + \alpha y_3) = \alpha\beta$.

Let $\mu \in k$ with $\mu^3 = \beta = \alpha^{-1}$. Note that $\{e_1, e_2, \mu x_i, \mu^{-1} y_i : i = 1, 2, 3\}$ is a basis of C with multiplication table as in (3.2). Then $Q = k1 \oplus \left(\oplus_{i=1}^3 k(\mu x_i + \mu^{-1} y_i)\right)$ is a quaternion subalgebra of C invariant under $\mathfrak{s}$.

Finally, assume $\dim_k H = 1$, so $H = k1$, and let $V(m)$ be the irreducible module for $\mathfrak{sl}_2(k)$ of dimension $m+1$. Then, by complete reducibility, $C_0 \cong \oplus_{i=1}^r V(m_i)$ with $m_1 + \cdots + m_r + r = 7$ and $m_1, \ldots, m_r \geqslant 1$. If $r = 1$ we are done (and we are in case (ii)). By invariance of the norm n, the submodules of C_0 corresponding to $V(m_i)$ and $V(m_j)$ with $m_i \neq m_j$ are orthogonal and it is well-known that $V(m)$ possess a nonzero $\mathfrak{sl}_2(k)$-invariant quadratic form if and only if m is even. The only possibility left with $r > 1$ is $C_0 \cong V(2) \oplus V(1) \oplus V(1)$. But, by invariance of the multiplication and since $V(2) \otimes V(2) \cong V(4) \oplus V(2) \oplus V(0)$, the submodule of C corresponding to $k1 \oplus V(2)$ is a quaternion subalgebra invariant under $\mathfrak{s}$. $\square$

Remark 3.2. With the same notations as in the above proof, if $\mathfrak{s}$ acts irreducibly on C_0, then C_0 becomes the unique seven-dimensional irreducible module $V(6)$ for $\mathfrak{s}$ and the multiplication $C_0 \otimes C_0 \to C_0$, $x \otimes y \mapsto xy + \frac{1}{2}n(x, y)1 = \frac{1}{2}[x, y]$ and the restriction of the norm to C_0 give, up to scalars, the unique $\mathfrak{s}$-invariant maps $V(6) \otimes V(6) \to V(6)$ and $V(6) \otimes V(6) \to V(0)$. Since $[[x, y], y] = xy^2 + y^2x - 2yxy = 4xy^2 - 2(xy + yx)y = 2n(x, y)y - 4n(y)x$ for any $x, y \in C_0$, the norm is determined by the multiplication. Then, up to conjugation by an automorphism of C, there is a unique possibility for such an $\mathfrak{s}$. In this case $\mathfrak{g}_2$ is isomorphic to $V(2) \oplus V(10)$ as a module for $\mathfrak{s}$. (See [4] for a model of C and $\mathfrak{g}_2$ based on such a subalgebra $\mathfrak{s} \cong \mathfrak{sl}_2(k)$.)

Theorem 3.3. *Let C be a Cayley algebra over an algebraically closed field k of characteristic* 0, *let* $\mathfrak{g}_2 = \operatorname{Der} C$ *and let* $\mathfrak{h}$ *be a reductive nonabelian subalgebra in* $\mathfrak{g}_2$. *Then either*:

(i) *there exists* $i = 1, \ldots 7$ *such that* $\mathfrak{h} = \mathfrak{h}^i$ *as in Theorem* 2.1 (*for suitable* K, Q, u, w), *or*
(ii) $\mathfrak{h} = \mathfrak{h}^8$ *is three-dimensional simple and* C_0 *is irreducible for* $\mathfrak{h}$.

Proof. Let $\mathfrak{h}$ be a reductive nonabelian subalgebra in $\mathfrak{g}_2$, then the rank of $\mathfrak{h}$ is either 1 or 2. If it is 1, then $\mathfrak{h} \cong \mathfrak{sl}_2(k)$ and because of Lemma 3.1 either item (ii) is satisfied or there exists a quaternion subalgebra Q of C invariant under $\mathfrak{h}$. Hence $\mathfrak{h} \subseteq \{d \in \operatorname{Der} C : d(Q) \subseteq Q\} = \mathfrak{h}^1 = \mathfrak{s}^L \oplus \mathfrak{s}^R$ (notation as in Theorem 2.1 and its proof). Let π_L and π_R be the projections of $\mathfrak{h}^1$ onto $\mathfrak{s}^L$ and $\mathfrak{s}^R$, respectively. If $\pi_R(\mathfrak{h}) = 0$, $\mathfrak{h} = \mathfrak{s}^L = \mathfrak{h}^3$, while if $\pi_L(\mathfrak{h}) = 0$, $\mathfrak{h} = \mathfrak{s}^R = \mathfrak{h}^5$. Otherwise $\pi_L|_{\mathfrak{h}}$ and $\pi_R|_{\mathfrak{h}}$ are isomorphisms by simplicity, and so is $(\pi_L|_{\mathfrak{h}})(\pi_R|_{\mathfrak{h}})^{-1} : \mathfrak{s}^R \to \mathfrak{s}^L$. Therefore, there is a Lie algebra automorphism $\varphi : Q_0^- \to Q_0^-$ such that $\mathfrak{h} = \{d_{\varphi(x)} + D_x : x \in Q_0\}$. But φ extends to an automorphism ψ of Q ($\psi(1) = 1$ and $\psi(x) = \varphi(x)$ for any $x \in Q_0$), because, as in Remark 3.2, the norm is determined by the Lie bracket in Q_0 and hence it is invariant under φ. By the Skolem–Noether theorem (see, for instance, [10, Theorem 4.3.1]), there is an invertible element $c \in Q$ such that $\psi(x) = cxc^{-1}$ for any $x \in Q$. But $C = Q \oplus Qu = Q \oplus Qv$ with $v = cu$ and because of (2.9) $(d_{\varphi(x)} + D_x)(v) = (\varphi(x)c)u - (cx)u = (cxc^{-1}c - cx)u = 0$ and hence, changing u by v, $\mathfrak{h} \subseteq \mathfrak{h}^7$ in Theorem 2.1 and, by dimension count, $\mathfrak{h} = \mathfrak{h}^7$.

Assume now that the rank of $\mathfrak{h}$ is 2; then either $\mathfrak{h}$ is a sum of a three-dimensional simple ideal and a one-dimensional center, or a sum of two simple three-dimensional ideals, or $\mathfrak{h}$ is simple of type either A_2 or C_2. In the latter case (type C_2), $\mathfrak{h}$ has no irreducible modules of dimension 2, 3 or 7. Then $\{x \in C_0 : \mathfrak{h}x = 0\} \neq 0$, so the composition subalgebra $H = \{x \in C : \mathfrak{h}x = 0\}$ has dimension 2 or 4, and hence $\mathfrak{h} \subseteq \mathfrak{h}^3$ or $\mathfrak{h} \subseteq \mathfrak{h}^6$, a contradiction.

If $\mathfrak{h}$ is simple of type A_2, its irreducible modules have dimensions 1, 3, 6 or $\geqslant 8$ and hence again $H = \{x \in C : \mathfrak{h}x = 0\} \neq k$ is a composition subalgebra of dimension 2 or 4, and the only possibility here is $H = K$ (two-dimensional) and $\mathfrak{h} = \mathfrak{h}^6$.

Now, if $\mathfrak{a}$ is a three-dimensional simple ideal of $\mathfrak{h}$, $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{b}$ with $\mathfrak{b} = \{d \in \mathfrak{h} : [d, \mathfrak{a}] = 0\} \neq 0$. The $\mathfrak{a}$-module C_0 cannot be irreducible, since then the elements of $\mathfrak{b}$ should act on C_0 as scalars (Schur's lemma) and nonzero scalars cannot be derivations. Hence, the arguments at the beginning of the proof show that $\mathfrak{a}$ is either $\mathfrak{h}^3 = \mathfrak{s}^L$, $\mathfrak{h}^5 = \mathfrak{s}^R$, or $\mathfrak{h}^7$. In the latter case, the last part of the proof of Theorem 2.1 shows that $\mathfrak{g}_2$ is the direct sum of three copies of the adjoint module for $\mathfrak{h}^7$ plus a five-dimensional irreducible module, and hence $\mathfrak{b} = 0$, a contradiction. If $\mathfrak{a} = \mathfrak{s}^L$ (respectively, $\mathfrak{s}^R$), then $\mathfrak{b}$ is contained in $\{d \in \mathfrak{g}_2 : [d, \mathfrak{a}] = 0\}$ which equals $\mathfrak{s}^R$ (respectively, $\mathfrak{s}^L$), therefore either $\mathfrak{h} = \mathfrak{s}^L \oplus \mathfrak{s}^R = \mathfrak{h}^1$, or $\mathfrak{h} = \mathfrak{s}^L \oplus kD_x$ or $\mathfrak{h} = kd_x \oplus \mathfrak{s}^R$ for some $x \in Q_0$. In the last two cases, since $\mathfrak{h}$ is reductive in $\mathfrak{g}_2$, D_x or d_x is a semisimple element of $\mathfrak{g}_2$ and hence $n(x) \neq 0$. This shows that either $\mathfrak{h} = \mathfrak{h}^2$ or $\mathfrak{h} = \mathfrak{h}^4$ with $K = k1 + kx$. $\square$

Remark. The semisimple subalgebras of the simple Lie algebras over $\mathbb{C}$ have been described in [6]. Up to conjugation there are four possibilities for $\mathfrak{sl}_2(k)$ to be a subalgebra of $\mathfrak{g}_2$ and exactly one possibility for $\mathfrak{sl}_2(k) \oplus \mathfrak{sl}_2(k)$ and for $\mathfrak{sl}_3(k)$.

Because of Remark 3.2 and since any two quadratic étale subalgebras (respectively, any two quaternion subalgebras) of a Cayley algebra over an algebraically closed field are conjugate under an automorphism of the algebra, the next result follows:

Corollary 3.4. *Over an algebraically closed field of characteristic* 0, *there are exactly eight conjugacy classes of reductive nonabelian subalgebras in the Lie algebra* $\mathfrak{g}_2$.

Now, the restriction on the field to be algebraically closed will be removed.

Corollary 3.5. *Let C be a Cayley algebra over a field k of characteristic* 0 *and let* $\mathfrak{h}$ *be a reductive nonabelian subalgebra in* $\mathfrak{g}_2 = \operatorname{Der} C$. *Then either*:

(i) *there exists* $i = 1, \ldots, 6$ *such that* $\mathfrak{h} = \mathfrak{h}^i$ (*for suitable* K, Q, u, w), *or*
(ii) $\mathfrak{h}$ *is three-dimensional simple, there exists a quadratic étale subalgebra K of C annihilated by* $\mathfrak{h}$ *and, as a module for* $\mathfrak{h}$, *C is the direct sum of the trivial module K and two adjoint modules, or*
(iii) $\mathfrak{h}$ *is three-dimensional simple and* C_0 *is irreducible as an* $\mathfrak{h}$*-module*.

Proof. Let $\hat{k}$ be an algebraic closure of k and let $\hat{\mathfrak{h}} = \hat{k} \otimes_k \mathfrak{h}$, $\hat{C} = \hat{k} \otimes_k C$. Then $\hat{\mathfrak{h}}$ is reductive in $\hat{\mathfrak{g}}_2 = \hat{k} \otimes_k \mathfrak{g}_2$, which is identified naturally with $\operatorname{Der}\hat{C}$. If $\hat{\mathfrak{h}}$ acts irreducibly on $\hat{C}_0$, so does $\mathfrak{h}$ on C_0. Now assume that $\hat{\mathfrak{h}} = \hat{\mathfrak{h}}^i$, $i = 1, \ldots, 7$ ($\hat{\mathfrak{h}}^i$ as in Theorem 2.1 for suitable $\hat{K}, \hat{Q}, \hat{u}, \hat{w}$). For $i = 1$, $\hat{C}_0 = \hat{Q}_0 \oplus \hat{Q}\hat{u}$ is the direct sum of two irreducible $\hat{\mathfrak{h}}^1$-modules of different dimensions and, therefore, so is C_0 as an $\mathfrak{h}$-module (the centralizer of the action of $\hat{\mathfrak{h}}$ on $\hat{C}_0$ is $\hat{k} \times \hat{k}$, so the centralizer of the action of $\mathfrak{h}$ on C_0 is either $k \times k$ or a quadratic field extension of k, but this latter option is not possible as $\dim_k C_0$ is odd). If V is the unique three-dimensional irreducible module for $\mathfrak{h}$ in C_0, then $\hat{k} \otimes_k (k1 \oplus V) = \hat{Q}$, and hence $Q := k1 \oplus V$ is a quaternion subalgebra of C and $\mathfrak{h} = \mathfrak{h}^1 = \{d \in \mathfrak{g}_2 : d(Q) \subseteq Q\}$. For $i = 2, 3$, $\hat{Q} = \{x \in \hat{C} : [\hat{\mathfrak{h}}, \hat{\mathfrak{h}}]x = 0\}$ and thus $Q := \{x \in C : [\mathfrak{h}, \mathfrak{h}]x = 0\}$ is a quaternion subalgebra of C with $\hat{Q} = \hat{k} \otimes_k Q$. From here it follows that either $\mathfrak{h} = \mathfrak{h}^2$, for a suitable $K \subseteq Q$, or $\mathfrak{h} = \mathfrak{h}^3$. For $i = 4$ or 5, $[\hat{\mathfrak{h}}, \hat{\mathfrak{h}}]$ decomposes $\hat{C}_0$ into an irreducible module of dimension 3, namely $\hat{Q}_0$ and the sum of two irreducible two-dimensional modules. Hence again there is a unique three-dimensional irreducible $[\mathfrak{h}, \mathfrak{h}]$-submodule V with $Q = k1 \oplus V$ a quaternion subalgebra such that $\hat{Q} = \hat{k} \otimes_k Q$, and $\mathfrak{h} = \mathfrak{h}^4$ or $\mathfrak{h} = \mathfrak{h}^5$. If $i = 6$, $\hat{K} = \{x \in \hat{C} : \hat{\mathfrak{h}}x = 0\}$, so that $K := \{x \in C : \mathfrak{h}x = 0\}$ is a quadratic composition subalgebra of C annihilated by $\mathfrak{h}$ and $\mathfrak{h} = \mathfrak{h}^6$.

We are left with the case $\hat{\mathfrak{h}} = \hat{\mathfrak{h}}^7$. Here $\hat{L} := \hat{k}1 \oplus \hat{k}\hat{u} = \{x \in \hat{C} : \hat{\mathfrak{h}}x = 0\}$ is a quadratic composition subalgebra of $\hat{C}$ and so is $L := \{x \in C : \mathfrak{h}x = 0\}$ in C. Besides, $\hat{L}^\perp = \hat{Q}_0 \oplus \hat{Q}_0\hat{u}$ is the direct sum of two copies of the adjoint module for $\hat{\mathfrak{h}}$, so $L^\perp$ is the sum of two copies of the adjoint module for $\mathfrak{h}$ (because if M and N are two $\mathfrak{h}$-modules such that $\hat{k}\otimes_k M$ and $\hat{k}\otimes_k N$ are isomorphic and completely reducible as $\hat{\mathfrak{h}}$-modules, then M and N are isomorphic, as any isomorphism of $\hat{\mathfrak{h}}$-modules $\hat{k}\otimes_k M \simeq \hat{k}\otimes_k N$ is, in particular, an isomorphism of the $\mathfrak{h}$-modules $\hat{k}\otimes_k M$ and $\hat{k}\otimes_k N$, which are direct sums of copies of the completely reducible modules M and N). □

It is possible to be more explicit in case (ii) of Corollary 3.5 and to show that, under certain restrictions, $\mathfrak{h}$ is $\mathfrak{h}^7$ (for suitable Q and u):

Proposition 3.6. *Let C be a Cayley algebra over a field k of characteristic 0 and let $\mathfrak{h}$ be a three-dimensional simple subalgebra of $\mathfrak{g}_2 = \operatorname{Der} C$, such that $L = \{x \in C : \mathfrak{h}x = 0\}$ is a quadratic composition subalgebra of C and $L^\perp$ is the direct sum of two adjoint modules for $\mathfrak{h}$. If for any $a \in L$ with $n(a) \neq 0$, there is an element $b \in L$ with $0 \neq b^3 \in ka$, then $\mathfrak{h} = \mathfrak{h}^7$ for suitable Q and u.*

Proof. The Lie subalgebra $\mathfrak{h}$ has a basis $\{d_1, d_2, d_3\}$ with $[d_i, d_j] = \varepsilon_{ijk}\mu_k d_k$, $0 \neq \mu_k \in k$, where ε_{ijk} is the totally skewsymmetric tensor with $\varepsilon_{123} = 1$. Let $v_1 \in L^\perp$ with $n(v_1) \neq 0$ and $d_1(v_1) = 0$. Such v_1 exists since over $\hat{k}$, d_1 splits $\hat{L}^\perp$ as $S(0) \oplus S(\alpha) \oplus S(-\alpha)$ for some $0 \neq \alpha \in \hat{k}$, where $S(\mu) = \{x \in \hat{L}^\perp : d_1 x = \mu x\}$ and, by invariance, the restriction of the norm n to $S(0) = \hat{k}\otimes_k\{x \in L^\perp : d_1 x = 0\}$ is nondegenerate. Then the $\mathfrak{h}$-submodule generated by v_1 is isomorphic to the adjoint module, under an isomorphism that maps d_1 to v_1. Thus, there are elements $v_2, v_3 \in L^\perp$ such that

$$d_i v_j = \varepsilon_{ijk}\mu_k v_k$$

for any i, j. Let $\sigma : C \times C \to L$ be the hermitian form defined as in (2.4). Since n is $\mathfrak{h}$-invariant, $n(\ker d_1, d_1(L^\perp)) = 0$, and $L \ker d_1 \subseteq \ker d_1$ because $d_1(L) = 0$. Hence $n(Lv_1, v_2) = 0 = n(Lv_1, v_3)$, so $\sigma(v_1, v_2) = 0 = \sigma(v_1, v_3)$. Similarly, $\sigma(v_2, v_3) = 0$. Also, $n(v_2) = \frac{1}{2}n(v_2, v_2) = \frac{1}{2\mu_2}n(v_2, d_3 v_1) = \frac{-1}{2\mu_2}n(d_3 v_2, v_1) = \frac{\mu_1}{2\mu_2}n(v_1, v_1) = \frac{\mu_1}{\mu_2}n(v_1) \neq 0$ and $n(v_3) = \frac{\mu_1}{\mu_3}n(v_1) \neq 0$. Therefore, $\{v_1, v_2, v_3\}$ is a σ-orthogonal L-basis of $L^\perp$.

As shown in [9, Section 3], the product in C is given by

$$(a + x)(b + y) = (ab - \sigma(x, y)) + (ay + \bar{b}x + x * y)$$

for any $a, b \in L$, $x, y \in L^\perp$, where $x * y$ is an anticommutative product in $L^\perp$ satisfying $a(x * y) = (\bar{a}x) * y = x * (\bar{a}y)$, $\sigma(x * y, z) = \sigma(z * x, y)$ for any $a \in L$ and $x, y, z \in L^\perp$. Now, $v_1 * v_2$ is σ-orthogonal to v_1 and v_2, so that $v_1 * v_2 \in Lv_3$. Using the above properties it follows that there is an element $a \in L$, with $n(a) \neq 0$, such that

$$v_i * v_j = \varepsilon_{ijk}\mu_k a v_k$$

for any i, j. In case $a \in k$, $Q = k1 + kv_1 + kv_2 + kv_3$ is a quaternion subalgebra invariant under $\mathfrak{h}$, $C = Q \oplus Qu$ for any $0 \neq u \in L \cap C_0$, and $\mathfrak{h} = \mathfrak{h}^7$. Also, substituting v_i by $w_i = bv_i$

for any $b \in L$ with $n(b) \neq 0$ we get $w_i * w_j = \varepsilon_{ijk}\mu_k \frac{\bar{b}^3 a}{n(b)} w_k$, so the same conclusion is obtained if there exists $b \in L$ such that $0 \neq b^3 \in ka$, as required. □

Note that the condition in Proposition 3.6 is satisfied for the real octonion division algebra, since L is always isomorphic to $\mathbb{C}$ in this case.

Remarks 3.7. (a) Let us give an example of the situation in item (ii) of Corollary 3.5, where $\mathfrak{h}$ is not of type $\mathfrak{h}^7$. Take the split Cayley algebra C over the rational numbers with a basis as in (3.2). Let $\mathfrak{s}$ be the three-dimensional simple Lie algebra with basis $\{d_1, d_2, d_3\}$ such that $[d_i, d_j] = \varepsilon_{ijk} d_k$ acting on C by means of $d_i e_1 = d_i e_2 = 0$, $d_i \tilde{x}_j = \varepsilon_{ijk}\tilde{x}_k$, $d_i \tilde{y}_j = \varepsilon_{ijk}\tilde{y}_k$, where $\tilde{x}_1 = x_1$, $\tilde{x}_2 = x_2$, $\tilde{x}_3 = 2x_3$ and $\tilde{y}_1 = y_1$, $\tilde{y}_2 = y_2$ and $\tilde{y}_3 = \frac{1}{2} y_3$.Then $\mathfrak{s} \subseteq \mathrm{Der} C$ and $C = \mathbb{Q}e_1 \oplus \mathbb{Q}e_2 \oplus U \oplus V$, where U (respectively, V) is the span of the x_i's (resp. the y_i's). Both U and V are adjoint modules for $\mathfrak{s}$. If Q were a quaternion subalgebra of C invariant under $\mathfrak{s}$, then Q_0 would be an adjoint module for $\mathfrak{s}$, so we could find a $\mu \in \mathbb{Q}$ such that

$$Q = \mathbb{Q}1 \oplus (\oplus_{i=1}^{3} \mathbb{Q}(\tilde{x}_i + \mu\tilde{y}_i)), \tag{3.4}$$

but $(\tilde{x}_1 + \mu\tilde{y}_1)(\tilde{x}_2 + \mu\tilde{y}_2) = y_3 + \mu^2 x_3 = \frac{\mu^2}{2}(\tilde{x}_3 + \frac{4}{\mu^2}\tilde{y}_3)$. Therefore we should have $\frac{4}{\mu^2} = \mu$, which is impossible.

(b) It can be shown that the possibility in item (iii) of Corollary 3.5 can happen if and only if there exists an element $x \in C_0$ such that $n(x) = 15$ (see [5, Teorema 21] for the (very technical) details).

(c) In all the reductive pairs $(\mathfrak{g}_2, \mathfrak{h}^i)$, $1 \leqslant i \leqslant 8$, in Theorem 3.3, $\mathfrak{g}_2 = \mathfrak{h}^i \oplus (\mathfrak{h}^i)^\perp$ (orthogonal relative to the Killing form), and $(\mathfrak{h}^i)^\perp$ is a direct sum of irreducible $\mathfrak{h}^i$-modules, none of which appears in the adjoint representation of $\mathfrak{h}^i$ if $i \neq 7$ (see the last part of the proof of Theorem 2.1). Therefore, for $i \neq 7$, $\mathfrak{m}^i = (\mathfrak{h}^i)^\perp$ is the unique $\mathfrak{h}^i$-invariant complement to $\mathfrak{h}^i$.

However, $(\mathfrak{h}^7)^\perp$ is the direct sum of two copies of the adjoint module and a five-dimensional irreducible module for $\mathfrak{h}^7$ (which is three-dimensional simple). Hence in this case, there is a whole family of $\mathfrak{h}^7$-invariant complements. This will make an important difference for this case in the next section, where an infinite family of nonisomorphic Lie–Yamaguti algebras appears associated to the same reductive pair.

(d) If $\mathfrak{h}$ is an abelian reductive subalgebra of $\mathfrak{g}_2$, then the elements of $\mathfrak{h}$ are semisimple linear transformations of C [12, Chapter III, Theorem 10] and so $\mathfrak{h}$ is contained in a Cartan subalgebra of $\mathfrak{g}_2$ (and all the Cartan subalgebras are conjugate). This determines the abelian reductive subalgebras of $\mathfrak{g}_2$.

4. Description of the Lie–Yamaguti algebras

The aim of this section is the explicit description of the binary and ternary products of the Lie–Yamaguti algebras associated to the reductive pairs $(\mathfrak{g}_2, \mathfrak{h}^i)$, $i = 1, \ldots, 8$ (Theorem 3.3) over an algebraically closed field k of characteristic 0. This assumption on the field will be assumed throughout the section.

Most of the reductive subalgebras $\mathfrak{h}^i$ contain copies of $\mathfrak{sl}_2(k)$. A useful description of the irreducible modules $V(n)$ for $\mathfrak{sl}_2(k)$ (which, for simplicity, will be denoted by V_n) and of the $\mathfrak{sl}_2(k)$-invariant maps $V_n \otimes V_m \to V_p$ is given in terms of the classical *transvections* (see [4] and the references therein). Let us briefly recall the basic features.

Let $k[x, y]$ be the polynomial algebra in two indeterminates x and y, and identify $\mathfrak{sl}_2(k)$ with the following subalgebra of derivations of $k[x, y]$:

$$\operatorname{span}\left\{x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}\right\} \subseteq \operatorname{Der} k[x, y].$$

Let $V_n = k_n[x, y]$ denote the linear space of the degree n homogeneous polynomials, so that V_n is invariant under $\mathfrak{sl}_2(k)$ and this gives, up to isomorphism, the unique $(n + 1)$-dimensional irreducible representation of $\mathfrak{sl}_2(k)$, $\rho_n : \mathfrak{sl}_2(k) \to \operatorname{End}_k(V_n)$.

For any $f \in V_n$ and $g \in V_m$, the *transvection* $(f, g)_q$ is defined by

$$(f, g)_q = \begin{cases} 0 \quad \text{if } q > \min(n, m), \\ \dfrac{(n-q)!}{n!}\dfrac{(m-q)!}{m!}\displaystyle\sum_{i=0}^{q}(-1)^i\binom{q}{i}\frac{\partial^q f}{\partial x^{q-i}\partial y^i}\frac{\partial^q g}{\partial x^i \partial y^{q-i}} & \text{otherwise,}\end{cases}$$

so that $(f, g)_q \in V_{n+m-2q}$. In particular $(f, g)_0 = fg$.

For any $f \in V_n$ and $m, q \geqslant 0$, consider the linear map:

$$T^m_{q,f} : V_m \longrightarrow V_{m+n-2q},$$

$$g \mapsto (f, g)_q.$$

Notice that for $f \in V_2$:

$$T^m_{1,f} = \frac{1}{2m}\rho_m\left(\frac{\partial f}{\partial x}\frac{\partial}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial}{\partial x}\right) \in \operatorname{End}_k(V_m).$$

In particular, $\mathfrak{sl}_2(k) \simeq \mathfrak{sl}(V_1) = \operatorname{span}\{T^1_{1,f} : f \in V_2\}$ and, for $f, g \in V_2$, $[T^1_{1,f}, T^1_{1,g}] = 2T^1_{1,(f,g)_1}$ (see [1, (2.2)]). Thus V_2, with the bracket given by $(\ ,\ )_1$ is isomorphic to $\mathfrak{sl}_2(k) \simeq \mathfrak{sl}(V_1)$ by means of $f \mapsto \frac{1}{2}T^1_{1,f}$.

Let us denote by W_n the degree n homogeneous polynomials in the new indeterminates X, Y. A nice description of the Cayley algebra C over k and of $\mathfrak{g}_2 = \operatorname{Der} C$ is given in [1, Corollary 2.3 and Theorem 3.2]:

$$C = (W_0 \oplus W_2) \oplus (V_1 \otimes_k W_1),$$

$$\mathfrak{g}_2 = (V_2 \oplus W_2) \oplus (V_1 \otimes_k W_3). \tag{4.1}$$

Here $k1 = W_0$, while $K = k1 \oplus kXY$ $(XY \in W_2)$ is a quadratic étale subalgebra and $Q = W_0 \oplus W_2$ is a quaternion subalgebra. The multiplication in $\mathfrak{g}_2$ is given, for any $f, f_1, f_2 \in V_2$, $F, F_1, F_2 \in W_2$, $g, g_1, g_2 \in V_1$ and $G, G_1, G_2 \in W_3$ by

$$\begin{cases} [f_1, f_2] = (f_1, f_2)_1, \quad [F_1, F_2] = (F_1, F_2)_1, \quad [V_2, W_2] = 0, \\ [f, g \otimes G] = \frac{1}{2}(f, g)_1 \otimes G, \quad [F, g \otimes G] = \frac{3}{2} g \otimes (F, G)_1, \\ [g_1 \otimes G_1, g_2 \otimes G_2] = -2(G_1, G_2)_3 g_1 g_2 - 2(g_1, g_2)_1 (G_1, G_2)_2. \end{cases} \tag{4.2}$$

Moreover,

$$\mathfrak{h}^1 = \{d \in \mathfrak{g}_2 : d(Q) \subseteq Q\} = V_2 \oplus W_2,$$

$$\mathfrak{h}^3 = \mathfrak{s}^L = d_{Q_0} = \{d \in \mathfrak{g}_2 : d(Q) = 0\} = V_2, \tag{4.3}$$

and, in consequence,

$$\mathfrak{h}^5 = \mathfrak{s}^R = W_2. \tag{4.4}$$

(Notation as in Theorem 2.1.) Also, with $w = 4\sqrt{-1}XY$ and $u = -x \otimes Y + y \otimes X$,

$$\mathfrak{h}^2 = \{d \in \mathfrak{g}_2 : d(Q) \subseteq Q,\ d(K) = 0\} = V_2 \oplus kXY,$$

$$\mathfrak{h}^4 = \{d \in \mathfrak{g}_2 : \tau_w d = d\tau_w\} = kxy \oplus W_2.$$

Now we are ready to describe the unique $\mathfrak{h}^i$-invariant complement $\mathfrak{m}^i$ to $\mathfrak{h}^i$ in $\mathfrak{g}_2$ and its binary and triple products (1.2) for $i = 1, \ldots, 5$. The proof is obtained by straightforward computations using (4.2) and (1.2).

Theorem 4.1. *With the notations above, the Lie–Yamaguti algebras associated to the reductive pairs* $(\mathfrak{g}_2, \mathfrak{h}^i)$, *for* $i = 1, \ldots, 5$, *are determined as follows*:

(i) $\mathfrak{m}^1 = V_1 \otimes_k W_3$, *with binary and triple products given by*

$$\begin{aligned}
&\mathfrak{m}^1 \cdot \mathfrak{m}^1 = 0,\\
&[g_1 \otimes G_1, g_2 \otimes G_2, g_3 \otimes G_3]\\
&\quad = -(G_1, G_2)_3 (g_1 g_2, g_3)_1 \otimes G_3 - 3(g_1, g_2)_1 g_3 \otimes ((G_1, G_2)_2, G_3)_1.
\end{aligned}$$

(ii) $\mathfrak{m}^2 = \hat{W}_2 \oplus (V_1 \otimes_k W_3)$, *with* $\hat{W}_2 = kX^2 + kY^2$, *and binary product*:

$$\begin{cases}
\hat{W}_2 \cdot \hat{W}_2 = 0,\\
F \cdot (g \otimes G) = \frac{3}{2}\, g \otimes (F, G)_1,\\
(g_1 \otimes G_1) \cdot (g_2 \otimes G_2) = -2(g_1, g_2)_1 (\widehat{G_1, G_2})_2,
\end{cases}$$

(*here* $\hat{F} = F - \frac{\partial^2 F}{\partial X \partial Y} XY$ *for any* $F \in W_2$), *and triple product*:

$$\begin{cases}
[F_1, F_2, F_3] = ((F_1, F_2)_1, F_3)_1,\\
[F_1, F_2, g \otimes G] = \frac{3}{2}\, g \otimes ((F_1, F_2)_1, G)_1,\\
[F, g \otimes G, \mathfrak{m}^2] = 0,\\
[g_1 \otimes G_1, g_2 \otimes G_2, F] = -2(g_1, g_2)_1 \dfrac{\partial^2 (G_1, G_2)_2}{\partial X \partial Y} (XY, F)_1,\\
[g_1 \otimes G_1, g_2 \otimes G_2, g_3 \otimes G_3] = -(G_1, G_2)_3 (g_1 g_2, g_3)_1 \otimes G_3\\
\qquad\qquad -3(g_1, g_2)_1 \dfrac{\partial^2 (G_1, G_2)_2}{\partial X \partial Y}\, g_3 \otimes (XY, G_3)_1.
\end{cases}$$

(iii) $\mathfrak{m}^3 = W_2 \oplus (V_1 \otimes_k W_3)$ *and*

$$\begin{cases} F_1 \cdot F_2 = (F_1, F_2)_1, \\ F \cdot (g \otimes G) = \frac{3}{2}\, g \otimes (F, G)_1, \\ (g_1 \otimes G_1) \cdot (g_2 \otimes G_2) = -2(g_1, g_2)_1 (G_1, G_2)_2, \end{cases}$$

$$\begin{cases} [W_2, \mathfrak{m}^3, \mathfrak{m}^3] = 0 = [\mathfrak{m}^3, \mathfrak{m}^3, W_2], \\ [g_1 \otimes G_1, g_2 \otimes G_2, g_3 \otimes G_3] = -(G_1, G_2)_3 (g_1 g_2, g_3)_1 \otimes G_3. \end{cases}$$

(iv) $\mathfrak{m}^4 = \hat{V}_2 \oplus (V_1 \otimes_k W_3)$, *with* $\hat{V}_2 = kx^2 + ky^2$, *and*

$$\begin{cases} \hat{V}_2 \cdot \hat{V}_2 = 0, \\ f \cdot (g \otimes G) = \frac{1}{2}\, (f, g)_1 \otimes G, \\ (g_1 \otimes G_1) \cdot (g_2 \otimes G_2) = -2(G_1, G_2)_3 \widehat{g_1 g_2} \end{cases}$$

(*here* $\hat{f} = f - \frac{\partial^2 f}{\partial x \partial y} xy$ *for any* $f \in V_2$),

$$\begin{cases} [f_1, f_2, f_3] = ((f_1, f_2)_1, f_3)_1, \\ [f_1, f_2, g \otimes G] = \frac{1}{2}\, ((f_1, f_2)_1, g)_1 \otimes G, \\ [f, g \otimes G, \mathfrak{m}^4] = 0, \\ [g_1 \otimes G_1, g_2 \otimes G_2, f] = -2(G_1, G_2)_3 \dfrac{\partial^2 (g_1 g_2)}{\partial x \partial y} (xy, f)_1, \\ [g_1 \otimes G_1, g_2 \otimes G_2, g_3 \otimes G_3] \\ \quad = -\,(G_1, G_2)_3 \dfrac{\partial^2 (g_1 g_2)}{\partial x \partial y} (xy, g_3)_1 \otimes G_3 \\ \qquad -3(g_1, g_2)_1 g_3 \otimes ((G_1, G_2)_2, G_3)_1. \end{cases}$$

(v) $\mathfrak{m}^5 = V_2 \oplus (V_1 \otimes_k W_3)$ *and*

$$\begin{cases} f_1 \cdot f_2 = (f_1, f_2)_1, \\ f \cdot (g \otimes G) = \frac{1}{2}\, (f, g)_1 \otimes G, \\ (g_1 \otimes G_1) \cdot (g_2 \otimes G_2) = -2(G_1, G_2)_3 g_1 g_2, \end{cases}$$

$$\begin{cases} [V_2, \mathfrak{m}^5, \mathfrak{m}^5] = 0 = [\mathfrak{m}^5, \mathfrak{m}^5, V_2], \\ [g_1 \otimes G_1, g_2 \otimes G_2, g_3 \otimes G_3] = -3(g_1, g_2)_1 g_3 \otimes ((G_1, G_2)_2, G_3)_1. \end{cases}$$

(*In all these equations*, $f, f_1, f_2, f_3 \in V_2$ *or* $\hat{V}_2$, $F, F_1, F_2, F_3 \in W_2$ *or* $\hat{W}_2$, $g, g_1, g_2, g_3 \in V_1$ *and* $G, G_1, G_2, G_3 \in W_3$.)

To describe the Lie–Yamaguti algebra associated to the reductive pair $(\mathfrak{g}_2, \mathfrak{h}^6)$ in Theorem 2.1, the model of $\mathfrak{g}_2$ considered in [1, Remarks after Theorem 5.3] or [12, Chapter IV] is quite useful: Let V be a three-dimensional vector space over k and fix a nonzero alternating trilinear map $\det : V \times V \times V \to k$, which allows us to identify the second exterior power

$\Lambda^2 V$ with the dual vector space V^* ($v \wedge w \mapsto \det(v, w, -)$), and $\Lambda^2 V^*$ with V. Also, $V^* \otimes_k V$ is identified with $\mathrm{End}_k(V)$ ($v^* \otimes v : w \mapsto v^*(w)v$). Then:

$$\mathfrak{g}_2 = \mathfrak{sl}(V) \oplus V \oplus V^*,$$

where

- $\mathfrak{sl}(V)$ is a Lie subalgebra of $\mathfrak{g}_2$,
- $[f, v] = f(v)$, $[f, v^*] = -v^* \circ f$ (composition of maps) for any $f \in \mathfrak{sl}(V)$, $v \in V$ and $v^* \in V^*$ (natural actions of $\mathfrak{sl}(V)$ on V and V^*),
- $[v^*, v] = 3v^* \otimes v - v^*(v) I_V$ ($\in \mathfrak{sl}(V)$), for $v \in V$ and $v^* \in V^*$, and
- $[v, w] = 2v \wedge w$, $[v^*, w^*] = 2v^* \wedge w^*$, for any $v, w \in V$ and $v^*, w^* \in V^*$.

In this model, the reductive subalgebra $\mathfrak{h}^6$ (the unique, up to conjugation, reductive subalgebra isomorphic to $\mathfrak{sl}_3(k)$) can be identified with $\mathfrak{sl}(V)$. Therefore:

Theorem 4.2. *The Lie–Yamaguti algebra associated to the reductive pair* $(\mathfrak{g}_2, \mathfrak{h}^6)$ *is* $\mathfrak{m}^6 = V \oplus V^*$, *with multiplications*:

$$\begin{cases} V \cdot V^* = 0, \\ v \cdot w = 2v \wedge w, \\ v^* \cdot w^* = 2v^* \wedge w^*, \end{cases}$$
$$\begin{cases} [V, V, \mathfrak{m}^6] = 0 = [V^*, V^*, \mathfrak{m}^6], \\ [v, v^*, w] = -3v^*(w)v + v^*(v)w, \\ [v, v^*, w^*] = 3w^*(v)v^* - v^*(v)w^* \end{cases}$$

for any $v, w \in V$ *and* $v^*, w^* \in V^*$.

Now, to deal with the Lie–Yamaguti algebras associated to the reductive pair $(\mathfrak{g}_2, \mathfrak{h}^7)$, the model in [1, Section 6] is instrumental.

Let S be any three-dimensional simple Lie algebra (hence $S \cong \mathfrak{sl}_2(k)$ since k is algebraically closed) and let κ be its Killing form. Consider the orthogonal Lie algebra

$$\mathfrak{so}(S, \kappa) = \{\gamma \in \mathrm{End}_k(S) : \kappa(\gamma(x), y) + \kappa(x, \gamma(y)) = 0 \ \forall x, y \in S\},$$

which coincides with $\mathrm{ad}\, S$ ($\cong S$) and its five-dimensional irreducible module of zero trace symmetric operators:

$$\mathfrak{sym}_0(S, \kappa) = \{\gamma \in \mathrm{End}_k(S) \colon \ \kappa(\gamma(x), y) = \kappa(x, \gamma(y)) \ \forall x, y \in S \text{ and } \mathrm{tr}(\gamma) = 0\}.$$

Let $A = k1 + ka_1 + ka_2$ be the commutative (not associative) k-algebra with

$$1a = a \ \forall a \in A, \quad a_1^2 = \tfrac{3}{4} + a_1, \quad a_2^2 = -\tfrac{3}{4} + a_1, \quad a_1 a_2 = -a_2,$$

endowed with a trace linear form $t : A \to k$ with $t(1) = 1$, $t(a_1) = t(a_2) = 0$, a skewsymmetric bilinear form $\langle\,|\,\rangle$ such that $\langle 1 | A \rangle = 0$, $\langle a_1 | a_2 \rangle = \frac{3}{4}$ and a linear map $l : A \to A$ given by $l(1) = 0$, $l(a_1) = a_2$ and $l(a_2) = a_1$. This is the case $\mu = 1$ in [1, Section 6] (since k is algebraically closed, any $0 \neq \mu \in k$ gives the same result in [1]). Then [1, Theorem 6.1] asserts that

$$\mathfrak{g}_2 = (\mathfrak{so}(S, \kappa) \otimes_k A) \oplus \mathfrak{sym}_0(S, \kappa), \tag{4.5}$$

with the multiplication given by

$$
\begin{aligned}
&[\gamma\otimes c,\delta\otimes d]=[\gamma,\delta]\otimes cd+\langle c|d\rangle(\gamma\delta+\delta\gamma-\tfrac{2}{3}\operatorname{tr}(\gamma\delta)I_S),\\
&[\gamma\otimes c,\varphi]=(\gamma\varphi+\varphi\gamma)\otimes l(c)+t(c)[\gamma,\varphi],\\
&[\varphi,\psi]=[\varphi,\psi]\otimes 1
\end{aligned}
\tag{4.6}
$$

for any $\gamma,\delta\in\mathfrak{so}(S,\kappa)$, $c,d\in A$ and $\varphi,\psi\in\mathfrak{sym}_0(S,\kappa)$. (Notice that the brackets on the right denote the usual bracket in $\mathrm{End}_k(S)$.)

A more convenient basis $\{1,u,v\}$ of A can be chosen with $u=a_1+a_2$ and $v=a_1-a_2$. Then

$$
\begin{aligned}
&u^2=2v,\quad v^2=2u,\quad uv=\tfrac{3}{2},\\
&t(1)=1,\quad t(u)=t(v)=0,\\
&\langle u|v\rangle=-\tfrac{3}{2},\quad \langle 1|u\rangle=0=\langle 1|v\rangle,\\
&l(1)=0,\quad l(u)=u,\quad l(v)=-v.
\end{aligned}
\tag{4.7}
$$

For the Lie algebra S, we may take V_2 with the bracket $(\ ,\ )_1$ (recall that V_2 with this bracket is isomorphic to $\mathfrak{sl}(V_1)$ by means of $f\mapsto\frac{1}{2}T^1_{1,f}$). Then the Killing form is a scalar multiple of $(\ ,\ )_2: V_2\times V_2\to k=V_0$. By Dixmier [4, Lemme 4.3]

$$
\mathrm{End}_k(V_2)=kI_{V_2}\oplus\{T^2_{1,f}: f\in V_2\}\oplus\{T^2_{2,g}: g\in V_4\},
\tag{4.8}
$$

so necessarily

$$
\begin{aligned}
&\mathfrak{so}(S,\kappa)=\{T^2_{1,f}: f\in V_2\}\ (=\mathrm{ad}\,V_2),\ \text{ and}\\
&\mathfrak{sym}_0(S,\kappa)=\{T^2_{2,g}: g\in V_4\}.
\end{aligned}
\tag{4.9}
$$

Lemma 4.3. *For any* $f,f_1,f_2\in V_2$ *and* $g,g_1,g_2\in V_4$:

(1) $[T^2_{1,f_1},T^2_{1,f_2}]=T^2_{1,(f_1,f_2)_1}$,
(2) $[T^2_{1,f},T^2_{2,g}]=2T^2_{2,(f,g)_1}$,
(3) $[T^2_{2,g_1},T^2_{2,g_2}]=-2T^2_{1,(g_1,g_2)_3}$,
(4) $T^2_{1,f_1}T^2_{1,f_2}+T^2_{1,f_2}T^2_{1,f_1}-\frac{2}{3}\operatorname{tr}(T^2_{1,f_1}T^2_{1,f_2})I_{V_2}=T^2_{2,f_1f_2}$,
(5) $T^2_{1,f}T^2_{2,g}+T^2_{2,g}T^2_{1,f}=-T^2_{1,(f,g)_2}$.

Proof. Let us check, for instance, (4). By $\mathfrak{sl}_2(k)$-invariance, and since there exists, up to scalars, a unique $\mathfrak{sl}_2(k)$-invariant linear map $V_2\otimes V_2\to V_4$ (namely, $f_1\otimes f_2\mapsto (f_1,f_2)_0=f_1f_2$), there exists a scalar ξ such that

$$
T^2_{1,f_1}T^2_{1,f_2}+T^2_{1,f_2}T^2_{1,f_1}-\tfrac{2}{3}\operatorname{tr}(T^2_{1,f_1}T^2_{1,f_2})I_{V_2}=\xi T^2_{2,f_1f_2}.
$$

Now, take $f_1=f_2=x^2$ and apply both sides of the equation above to y^2 to get $2(x^2,(x^2,y^2)_1)_1=\xi(x^4,y^2)_2$. But $(x^2,(x^2,y^2)_1)_1=(x^2,xy)_1=\frac{1}{2}x^2$, while $(x^4,y^2)_2=x^2$. Hence $\xi=1$. All the other computations are similar. □

As an immediate consequence of this lemma and the previous arguments one gets:

Corollary 4.4. *The Lie algebra* $\mathfrak{g}_2$ *is, up to isomorphism*:

$$\mathfrak{g}_2 = (V_2 \otimes_k A) \oplus V_4,$$

with multiplication given by:

$$[f_1 \otimes c, f_2 \otimes d] = (f_1, f_2)_1 \otimes cd + \langle c|d\rangle f_1 f_2,$$

$$[f \otimes c, g] = -(f, g)_2 \otimes l(c) + 2t(c)(f, g)_1,$$

$$[g_1, g_2] = -2(g_1, g_2)_3 \otimes 1$$

for any $f, f_1, f_2 \in V_2$, $g, g_1, g_2 \in V_4$ *and where the algebra* A *and the maps* $\langle\,|\,\rangle$, l *and* t *have been defined in* (4.7).

By uniqueness in Corollary 3.4, up to conjugation $\mathfrak{h}^7 = V_2 \otimes 1$ in the Corollary above; but now the $\mathfrak{h}^7$-invariant complements are precisely the subspaces:

$$\mathfrak{m}^7_{\alpha,\beta} = (V_2 \otimes_k A_{\alpha,\beta}) \oplus V_4, \tag{4.10}$$

with $\alpha, \beta \in k$, and where $A_{\alpha,\beta} = k(u - \alpha 1) + k(v - \beta 1)$.

For $\alpha, \beta = 0$, $A_{\alpha,\beta} = \ker t$ and $\mathfrak{m}^7_{0,0}$ will be denoted simply by $\mathfrak{m}^7$.

Theorem 4.5. (a) *The multiplications in the Lie–Yamaguti algebra* $\mathfrak{m}^7_{\alpha,\beta}$ *are given by*

$$\begin{cases} (f_1 \otimes c) \cdot (f_2 \otimes d) = (f_1, f_2)_1 \otimes \pi_{\alpha,\beta}(cd) + \langle c|d\rangle f_1 f_2, \\ (f \otimes c) \cdot g = -(f, g)_2 \otimes \pi_{\alpha,\beta}(l(c)) + 2t(c)(f, g)_1, \\ g_1 \cdot g_2 = 0, \end{cases}$$

$$\begin{cases} [f_1 \otimes c, f_2 \otimes d, f_3 \otimes e] = ((f_1, f_2)_1, f_3)_1 \otimes t_{\alpha,\beta}(cd)e, \\ [f_1 \otimes c, f_2 \otimes d, g] = 2t_{\alpha,\beta}(cd)((f_1, f_2)_1, g)_1, \\ [f_1 \otimes c, g, f_2 \otimes d] = -((f_1, g)_2, f_2)_1 \otimes t_{\alpha,\beta}(l(c))d, \\ [f \otimes c, g_1, g_2] = -2t_{\alpha,\beta}(l(c))((f, g_1)_2, g_2)_1, \\ [g_1, g_2, f \otimes c] = -2((g_1, g_2)_3, f)_1 \otimes c, \\ [g_1, g_2, g_3] = -4((g_1, g_2)_3, g_3)_1 \end{cases}$$

for any $f, f_1, f_2, f_3 \in V_2$, $g, g_1, g_2, g_3 \in V_4$ *and* $c, d, e \in A_{\alpha,\beta}$, *where* $\pi_{\alpha,\beta} : A \to A_{\alpha,\beta}$ *is the projection parallel to* $k1$ ($\pi_{\alpha,\beta}(1) = 0$, $\pi_{\alpha,\beta}|_{A_{\alpha,\beta}} = I_{A_{\alpha,\beta}}$) *and* $t_{\alpha,\beta} : A \to k$ *is the linear map such that* $t_{\alpha,\beta}(1) = 1$, $t_{\alpha,\beta}(u) = \alpha$ *and* $t_{\alpha,\beta}(v) = \beta$.

(b) *In particular, for* $\alpha = \beta = 0$, $t_{\alpha,\beta} = t$, *so* $t_{\alpha,\beta} \circ l = 0$. *In this case* $\mathfrak{m}^7 = \mathfrak{m}^7_{0,0}$ *is the orthogonal complement to* $\mathfrak{h}^7$ *relative to the Killing form of* $\mathfrak{g}_2$, *and the formulae above* *simplify to*

$$\begin{cases} (f_1 \otimes c) \cdot (f_2 \otimes d) = (f_1, f_2)_1 \otimes \pi(cd) + \langle c|d\rangle f_1 f_2, \\ (f \otimes c) \cdot g = -(f, g)_2 \otimes l(c), \\ g_1 \cdot g_2 = 0, \end{cases}$$

$$\begin{cases} [f_1 \otimes c, f_2 \otimes d, f_3 \otimes e] = ((f_1, f_2)_1, f_3)_1 \otimes t(cd)e, \\ [f_1 \otimes c, f_2 \otimes d, g] = 2t(cd)((f_1, f_2)_1, g)_1, \\ [f_1 \otimes c, g, f_2 \otimes d] = 0, \\ [f \otimes c, g_1, g_2] = 0, \\ [g_1, g_2, f \otimes c] = -2((g_1, g_2)_3, f)_1 \otimes c, \\ [g_1, g_2, g_3] = -4((g_1, g_2)_3, g_3)_1 \end{cases}$$

for any $f, f_1, f_2, f_3 \in V_2$, $g, g_1, g_2, g_3 \in V_4$ *and* $c, d, e \in A_{0,0}$, *where* $\pi = \pi_{0,0}$.

(c) *For* $\alpha, \beta, \alpha', \beta' \in k$, *the Lie–Yamaguti algebras* $\mathfrak{m}^7_{\alpha,\beta}$ *and* $\mathfrak{m}^7_{\alpha',\beta'}$ *are isomorphic if and only if* (α', β') *equals one of the following*:

$$(\alpha, \beta), (\omega\alpha, \omega^2\beta), (\omega^2\alpha, \omega\beta), (\beta, \alpha), (\omega\beta, \omega^2\alpha), (\omega^2\beta, \omega\alpha),$$

where $1 \neq \omega \in k$ *is a cube root of* 1; *that is, if and only if* (α', β') *belongs to the orbit of* (α, β) *under the action of the symmetric group* S_3 *on* k^2, *determined by* $(12).(\alpha, \beta) = (\beta, \alpha)$, $(123).(\alpha, \beta) = (\omega\alpha, \omega^2\beta)$.

Proof. The first part is a direct consequence of the Corollary above and of (1.2).

Note that $(V_2 \otimes 1) \oplus V_4$ is a subalgebra of $\mathfrak{g}_2$ and the decomposition $\mathfrak{g}_2 = ((V_2 \otimes 1) \oplus V_4) \oplus V_2 \otimes u \oplus V_2 \otimes v$ is a $\mathbb{Z}_3$-grading. Hence the orthogonal complement to $(V_2 \otimes 1) \oplus V_4$, relative to the Killing form, is $V_2 \otimes_k \ker t = V_2 \otimes u \oplus V_2 \otimes v$. Moreover, $V_2 \otimes 1$ and V_4 are clearly orthogonal relative to any invariant bilinear form of $\mathfrak{g}_2$, since they are not contragredient modules for the subalgebra $\mathfrak{h}^7 = V_2 \otimes 1 \cong \mathfrak{sl}_2(k)$. Hence $(\mathfrak{h}^7)^\perp = (V_2 \otimes_k \ker t) \oplus V_4 = \mathfrak{m}^7$ and part (b) follows.

For the third part, assume that $\varphi : \mathfrak{m}^7_{\alpha,\beta} \to \mathfrak{m}^7_{\alpha',\beta'}$ is an isomorphism of Lie–Yamaguti algebras. Then φ extends to an automorphism (also denoted by φ) of $\mathfrak{g}_2 = \mathfrak{h}^7 \oplus \mathfrak{m}^7_{\alpha,\beta} = \mathfrak{h}^7 \oplus \mathfrak{m}^7_{\alpha',\beta'}$, which is the standard enveloping Lie algebra of both $\mathfrak{m}^7_{\alpha,\beta}$ and $\mathfrak{m}^7_{\alpha',\beta'}$, such that $\varphi(\mathfrak{h}^7) = \mathfrak{h}^7$, that is, $\varphi(V_2 \otimes 1) = V_2 \otimes 1$. Since $\mathfrak{h}^7 \cong \mathfrak{sl}_2(k)$, there is an element s of the special linear group $SL_2(k)$ such that $\varphi|_{\mathfrak{h}^7}$ is given by the natural action of s on V_2. But the maps $(\ ,\ )_p$ are $SL_2(k)$-invariant, so $s \in SL_2(k)$ can be extended to an automorphism φ_s of $\mathfrak{g}_2$ such that

$$\varphi_s(f \otimes c) = s.f \otimes c,$$
$$\varphi_s(g) = s.g$$

for any $f \in V_2$, $c \in A$ and $g \in V_4$, where $s.f$ and $s.g$ denote the action of $SL_2(k)$ on V_2 and V_4. Moreover, φ_s leaves both $\mathfrak{m}^7_{\alpha,\beta}$ and $\mathfrak{m}^7_{\alpha',\beta'}$ invariant. Thus, we may change φ by $\varphi \circ \varphi_s^{-1}$ and hence assume that $\varphi|_{\mathfrak{h}^7}$ is the identity map; that is, $\varphi(f \otimes 1) = f \otimes 1$ for any $f \in V_2$. Then $\varphi|_{\mathfrak{m}^7_{\alpha,\beta}} : \mathfrak{m}^7_{\alpha,\beta} \to \mathfrak{m}^7_{\alpha',\beta'}$ is $\mathfrak{h}^7$-invariant, besides being an isomorphism of Lie–Yamaguti algebras. Since V_2 and V_4 are irreducible modules for $\mathfrak{h}^7$, Schur's lemma shows that there exist a bijective linear map $\lambda : A \to A$ and a nonzero scalar $\mu \in k$ such that $\lambda(1) = 1$, $\lambda(A_{\alpha,\beta}) = A_{\alpha',\beta'}$ and

$$\varphi(f \otimes c) = f \otimes \lambda(c), \quad \varphi(g) = \mu g$$

for any $f \in V_2$, $c \in A$ and $g \in V_4$.

Now, the fact that φ is an automorphism of the Lie algebra $\mathfrak{g}_2$ is equivalent to the following conditions on λ and μ:

$$\lambda(cd) = \lambda(c)\lambda(d),$$

$$\langle \lambda(c)|\lambda(d)\rangle = \mu\langle c|d\rangle,$$

$$\lambda(l(c)) = \mu l(\lambda(c)),$$

$$t(\lambda(c)) = t(c),$$

$$\mu^2 = 1$$

for any $c, d \in A$. But $l(u) = u$ and $l(v) = -v$ by (4.7), so the third condition forces either $\mu = 1$ and $\lambda(u) = \eta u$, $\lambda(v) = \nu v$, or $\mu = -1$ and $\lambda(u) = \eta v$, $\lambda(v) = \nu u$, for some $0 \neq \eta, \nu \in k$. The first condition, with $c = u$ and $d = v$ shows that $\nu = \eta^{-1}$, and with $c = d = u$ that $\eta^2 = \eta^{-1}$ or $\eta^3 = 1$. Conversely, with $\mu = \pm 1$ and $\eta^3 = 1$, the linear map $\lambda : A \to A$ given by $\lambda(1) = 1$ and

$$\begin{cases} \lambda(u) = \eta u, \;\; \lambda(v) = \eta^{-1} v & \text{if } \mu = 1, \\ \lambda(u) = \eta v, \;\; \lambda(v) = \eta^{-1} u & \text{if } \mu = -1, \end{cases}$$

satisfies the conditions above. Finally, since $\lambda(u - \alpha 1), \lambda(v - \beta 1) \in A_{\alpha',\beta'}$, it follows that with $\mu = 1$, $\lambda(u - \alpha 1) = \eta u - \alpha 1 \in A_{\alpha',\beta'}$, so $\eta u - \alpha 1 = \eta(u - \eta^{-1}\alpha 1)$ and $\alpha' = \eta^{-1}\alpha$, and in the same vein, $\beta' = \eta\beta$. The argument for $\mu = -1$ is similar, and this completes the proof. $\square$

The only reductive pair left is $(\mathfrak{g}_2, \mathfrak{h}^8)$. This appears in [4, Section 6], where $\mathfrak{g}_2$ is constructed as

$$B_{5,\lambda,\mu,\nu} = V_2 \oplus V_{10},$$

where $0 \neq \lambda, \mu, \nu \in k$ satisfy $378\lambda\mu = 5\nu^2$ and with the multiplication given by:

$$\begin{aligned}[][f_1 + g_1, f_2 + g_2] = {} & (\lambda(f_1, f_2)_1 + \mu(g_1, g_2)_9) \\ & + (5\lambda(f_1, g_2)_1 + 5\lambda(g_1, f_2)_1 + \nu(g_1, g_2)_5)\end{aligned}$$

for any $f_1, f_2 \in V_2$ and $g_1, g_2 \in V_{10}$.

A word of caution is needed here as this is not exactly what appears in [4]. Actually, in [4, 6.2] no 5's appear multiplying the λ's in the second line, but this is needed to get the

Jacobi identity satisfied for two elements in V_2 and an element in V_{10}. Also, the condition $378\lambda\mu = 5\nu^2$ appears erroneously as $25\lambda\mu = 378\nu^2$ in [4, 6.2].

Now, the map $f + g \mapsto \alpha^{-1} f + \beta^{-1} g$ gives an isomorphism $B_{5,\lambda,\mu,\nu} \cong B_{5,\alpha\lambda,\frac{\beta^2}{\alpha}\mu,\beta\nu}$, so we may take $\lambda = 1$, $\nu = 1$ and $\mu = \frac{5}{378}$ in what follows. Therefore,

$$\mathfrak{g}_2 = V_2 \oplus V_{10},$$

where the multiplication is determined by

$$[f_1 + g_1, f_2 + g_2] = ((f_1, f_2)_1 + \tfrac{5}{378}(g_1, g_2)_9) + (5(f_1, g_2)_1 + 5(g_1, f_2)_1 + (g_1, g_2)_5)$$

for any $f_1, f_2 \in V_2$ and $g_1, g_2 \in V_{10}$.

By uniqueness (Corollary 3.4), we may identify $\mathfrak{h}^8$ with V_2 above and hence:

Theorem 4.6. *The Lie–Yamaguti algebra associated to the reductive pair* $(\mathfrak{g}_2, \mathfrak{h}^8)$ *is* $\mathfrak{m}^8 = V_{10}$ *with multiplications*:

$$g_1 \cdot g_2 = (g_1, g_2)_5,$$

$$[g_1, g_2, g_3] = \tfrac{25}{378}((g_1, g_2)_9, g_3)_1$$

for any $g_1, g_2, g_3 \in V_{10}$.

The binary algebra $(\mathfrak{m}^8, \cdot)$ has been considered in [3].

This finishes our description of the Lie–Yamaguti algebras.

5. Binary products

In this section, several aspects of the Lie–Yamaguti algebras described so far will be looked at, with special attention to the anticommutative algebras $(\mathfrak{m}^i, \cdot)$, $i = 1, \ldots, 8$ over an algebraically closed field k of characteristic 0 (although many of the arguments remain valid over more general fields). This assumption on the field will be assumed throughout in this last section too.

5.1. Simplicity of the binary algebras

First, the simplicity of the algebras $(\mathfrak{m}^i, \cdot)$ will be proved. (Note that Proposition 1.2 shows that all the Lie–Yamaguti algebras involved are simple, but this does not imply the simplicity under the binary product.) To do so, the description of $\mathfrak{g}_2$ in (4.1) will be particularly useful. As in [1, Remarks after Theorem 3.2], a useful $\mathbb{Z}$-grading of $\mathfrak{g}_2$ can be given by assigning a degree 6 to x, -6 to y, 1 to X and -1 to Y. Since $\{x^2, xy, y^2\}$ is a basis of V_2, $\{X^2, XY, Y^2\}$ of W_2, $\{x, y\}$ of V_1 and $\{X^3, X^2Y, XY^2, Y^3\}$ of W_3, a degree is thus assigned to any basis element in $\mathfrak{g}_2$ (for instance, the degree of $x \otimes XY^2$ is $6 + (1 - 2) = 5$).

Then $\mathfrak{g}_2$ decomposes as

$$\begin{aligned}\mathfrak{g}_2 = {} & (\mathfrak{g}_2)_{-12} \oplus (\mathfrak{g}_2)_{-9} \oplus (\mathfrak{g}_2)_{-7} \oplus (\mathfrak{g}_2)_{-5} \oplus (\mathfrak{g}_2)_{-3} \oplus (\mathfrak{g}_2)_{-2} \\ & \oplus (\mathfrak{g}_2)_0 \oplus (\mathfrak{g}_2)_2 \oplus (\mathfrak{g}_2)_3 \oplus (\mathfrak{g}_2)_5 \oplus (\mathfrak{g}_2)_7 \oplus (\mathfrak{g}_2)_9 \oplus (\mathfrak{g}_2)_{12},\end{aligned} \tag{5.1}$$

where $(\mathfrak{g}_2)_0 = kxy \oplus kXY$ (a Cartan subalgebra of $\mathfrak{g}_2$) and $\dim(\mathfrak{g}_2)_i = 1$ for any $i = \pm 2, \pm 3, \pm 5, \pm 7, \pm 9, \pm 12$. (This is just the eigenspace decomposition relative to the element h in a Cartan subalgebra with $\alpha(h) = 2$, $\beta(h) = 3$, where α and β are the short and long roots in a simple system of roots.)

Note that, from (4.3) and (4.4), $\mathfrak{s}^L = V_2 = (\mathfrak{g}_2)_{-12} \oplus [(\mathfrak{g}_2)_{-12}, (\mathfrak{g}_2)_{12}] \oplus (\mathfrak{g}_2)_{12}$, while $\mathfrak{s}^R = W_2 = (\mathfrak{g}_2)_{-2} \oplus [(\mathfrak{g}_2)_{-2}, (\mathfrak{g}_2)_2] \oplus (\mathfrak{g}_2)_2$.

By straightforward computations with (4.2) (or by taking into account the properties of root spaces) we obtain:

Lemma 5.1.

- $[(\mathfrak{g}_2)_i, (\mathfrak{g}_2)_j] = (\mathfrak{g}_2)_{i+j}$ *if* $(\mathfrak{g}_2)_i \neq 0 \neq (\mathfrak{g}_2)_j$ *and* $i + j \neq 0$.
- *For any odd i with* $(\mathfrak{g}_2)_i \neq 0$, $[(\mathfrak{g}_2)_i, (\mathfrak{g}_2)_{-i}] \not\subseteq \mathfrak{s}^L \cup \mathfrak{s}^R$.

As usual, a $\mathbb{Z}_2$-graded nonassociative algebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$ is said to be graded simple if it contains no proper graded ideal of A. The next well-known result will be useful too:

Lemma 5.2. *If* $A = A_{\bar{0}} \oplus A_{\bar{1}}$ *is a graded simple algebra, then either A is simple* (*as an ungraded algebra*) *or* $\dim A_{\bar{0}} = \dim A_{\bar{1}}$.

Proof. Assume that A is not simple and let I be a nontrivial ideal of A. Then both $(I \cap A_{\bar{0}}) \oplus (I \cap A_{\bar{1}})$ and $\pi_{\bar{0}}(I) \oplus \pi_{\bar{1}}(I)$ are graded ideals of A (here π_i denotes the projection onto A_i, $i = \bar{0}, \bar{1}$), with $0 \subseteq (I \cap A_{\bar{0}}) \oplus (I \cap A_{\bar{1}}) \subseteq I \subseteq \pi_{\bar{0}}(I) \oplus \pi_{\bar{1}}(I) \subseteq A$. By the simplicity of A as a graded algebra, $I \cap A_{\bar{0}} = 0 = I \cap A_{\bar{1}}$ and $\pi_{\bar{0}}(I) = A_{\bar{0}}$, $\pi_{\bar{1}}(I) = A_{\bar{1}}$. Hence both $\pi_{\bar{0}}$ and $\pi_{\bar{1}}$ induce linear bijections $I \to A_{\bar{0}}$ and $I \to A_{\bar{1}}$, whence the result. $\square$

Recall that $(\mathfrak{m}^1, \cdot)$ is the trivial algebra, since $(\mathfrak{g}_2, \mathfrak{h}^1)$ is a symmetric pair (that is, the decomposition $\mathfrak{g}_2 = \mathfrak{h}^1 \oplus \mathfrak{m}^1$ is a $\mathbb{Z}_2$-grading).

Theorem 5.3. *The nonassociative algebras* $(\mathfrak{m}^i, \cdot)$, $i = 2, \ldots, 8$, $i \neq 7$, *and the algebras* $(\mathfrak{m}^7_{\alpha,\beta}, \cdot)$, $\alpha, \beta \in k$, *are all simple.*

Proof. For any $i = 2, \ldots, 5$, $\mathfrak{h}^i \subseteq (\mathfrak{g}_2)_{\bar{0}}$ and $\mathfrak{m}^i = (\mathfrak{m}^i \cap (\mathfrak{g}_2)_{\bar{0}}) \oplus (\mathfrak{g}_2)_{\bar{1}}$, so that it is enough, by Lemma 5.2, to prove that $(\mathfrak{m}^i, \cdot)$ is graded simple. Now, Lemma 5.1 implies that any nonzero graded ideal $I = (I \cap (\mathfrak{g}_2)_{\bar{0}}) \oplus (I \cap (\mathfrak{g}_2)_{\bar{1}})$ satisfies $I \cap (\mathfrak{g}_2)_{\bar{1}} \neq 0$, and another application of Lemma 5.1 gives that $\mathfrak{m}^i \cap (\mathfrak{g}_2)_{\bar{0}} \subseteq I$. But $(\mathfrak{g}_2)_{\bar{1}} = (\mathfrak{m}^i \cap (\mathfrak{g}_2)_{\bar{0}}) \cdot (\mathfrak{g}_2)_{\bar{1}}$, so $(\mathfrak{g}_2)_{\bar{1}} \subseteq I$ too, and$I = \mathfrak{m}^i$.

The case $i = 6$ has been treated explicitly in [8] (and can be checked directly too), while [4, Proofs of 2.2 and 2.3] shows that $(\mathfrak{m}^8, \cdot)$ is simple. This can be achieved too using [7, Theorem 3.4], which shows that $\mathrm{Der}(\mathfrak{m}^8, \cdot) \subseteq \mathrm{Lie}(\mathfrak{m}^8, \cdot)$ (the Lie multiplication algebra), so that any ideal of $(\mathfrak{m}^8, \cdot)$ is invariant under its Lie algebra of derivations. Now, the irreducibility of $\mathfrak{m}^8$ as a module for $\mathfrak{h}^8$ (whose action on $\mathfrak{m}^8$ is contained in this algebra of derivations) gives the result.

Finally, $\mathfrak{m}^7_{\alpha,\beta}=(V_2\otimes_k A_{\alpha,\beta})\oplus V_4$ by (4.10) and the first three equations in Theorem 4.5.a) force that any nonzero ideal I of $(\mathfrak{m}^7_{\alpha,\beta},\cdot)$ satisfies $I+V_4=\mathfrak{m}^7_{\alpha,\beta}$. But $I\supseteq I\cdot V_4=\mathfrak{m}^7_{\alpha,\beta}\cdot V_4$, which is an $\mathfrak{h}^7$-module with nonzero projection on V_4, unless $(\alpha,\beta)=(0,0)$. By irreducibility of V_4 as an $\mathfrak{h}^7$-module, and since V_4 appears only once in the decomposition of $\mathfrak{m}^7_{\alpha,\beta}$ as a sum of irreducible $\mathfrak{h}^7$-modules, this shows that, for $(\alpha,\beta)\neq(0,0)$, $V_4\subseteq\mathfrak{m}^7_{\alpha,\beta}\cdot V_4\subseteq I$. On the other hand, if $\alpha=0=\beta$, $I\supseteq I\cdot V_4=\mathfrak{m}^7_{0,0}\cdot V_4=V_2\otimes_k A_{0,0}$ and hence I contains $(V_2\otimes_k A_{0,0})^{\cdot 2}$ (the square under the binary product $\cdot$), which again is a module for $\mathfrak{h}^7$ with nonzero projection on V_4, so it contains V_4. $\square$

5.2. Lie derivation algebras

The computation of the Lie algebra of derivations of our Lie–Yamaguti algebras follows easily from the next auxiliary result, which has its own independent interest. Recall [12, p. 11] that a Lie algebra is said to be *complete* if all its derivations are inner and its center is 0. Simple Lie algebras over fields of characteristic 0 are complete.

Lemma 5.4. *Let* $(\mathfrak{m},\cdot,[\,,\,,\,])$ *be a Lie–Yamaguti algebra such that its standard enveloping Lie algebra* $\mathfrak{g}(\mathfrak{m})$ *is complete. Let* $N(\mathfrak{m})=\{x\in\mathfrak{m}: D(x,\mathfrak{m})=0,\ \ D(\mathfrak{m},\mathfrak{m})(x)=0\}$. *Then*

$$\mathrm{Der}(\mathfrak{m},\cdot,[\,,\,,\,])=D(\mathfrak{m},\mathfrak{m})\oplus\{L_x: x\in N(\mathfrak{m})\}$$

(*direct sum of ideals*), *where* L_x *denotes the left multiplication by* x *in* $(\mathfrak{m},\cdot)$ $(L_x: y\mapsto x\cdot y)$.

Proof. Any $d\in\mathrm{Der}(\mathfrak{m},\cdot,[\,,\,,\,])$ extends to a derivation of $\mathfrak{g}(\mathfrak{m})=D(\mathfrak{m},\mathfrak{m})\oplus\mathfrak{m}$ by means of $d(D(x,y))=[d,D(x,y)]\ (=D(d(x),y)+D(x,d(y)))$ for any $x,y\in\mathfrak{m}$. By hypothesis, there is an element $\hat{d}\in D(\mathfrak{m},\mathfrak{m})$ and an element $x\in\mathfrak{m}$ such that $d=\mathrm{ad}(\hat{d}+x)$. But both d and $\hat{d}$ preserve $D(\mathfrak{m},\mathfrak{m})$ and $\mathfrak{m}$ so that $[x,\mathfrak{m}]\subseteq\mathfrak{m}$ and $[x,D(\mathfrak{m},\mathfrak{m})]\subseteq D(\mathfrak{m},\mathfrak{m})$, conditions which are equivalent, because of (1.3), to $D(x,\mathfrak{m})=0$ and $D(\mathfrak{m},\mathfrak{m})(x)=0$. Conversely, for any $x\in N(\mathfrak{m})$, Eqs. (LY2)–(LY5) in Definition 1.1 imply easily that $L_x\in\mathrm{Der}(\mathfrak{m},\cdot,[\,,\,,\,])$. In particular, this shows that $[L_x,L_y]=L_{x\cdot y}$ for any $y\in\mathfrak{m}$. Also, $[D(u,v),L_x]=L_{D(u,v)(x)}=0$ for any $x\in N(\mathfrak{m})$; so that $D(\mathfrak{m},\mathfrak{m})$ and $L_{N(\mathfrak{m})}$ are ideals of $\mathrm{Der}(\mathfrak{m},\cdot,[\,,\,,\,])$. Finally, if for some $x\in N(\mathfrak{m})$ we have that L_x is a finite sum $\sum_i D(x_i,y_i)$, then $[\sum_i D(x_i,y_i)-x,\mathfrak{m}]=0$ and $\sum_i D(x_i,y_i)-x$ belongs to the center of $\mathfrak{g}(\mathfrak{m})$, which is trivial. Hence $x=0$. $\square$

In all our examples of Lie–Yamaguti algebras $(\mathfrak{m},\cdot,[\,,\,,\,])$ in the previous section, the Lie algebra $D(\mathfrak{m},\mathfrak{m})$ is the corresponding $\mathrm{ad}\,\mathfrak{h}^i|_{\mathfrak{m}}$ by Proposition 1.2. Hence, as a direct consequence of the Lemma above and the arguments in the previous section one gets:

Proposition 5.5.

- $\mathrm{Der}(\mathfrak{m}^i,\cdot,[\,,\,,\,])=\mathrm{ad}\,\mathfrak{h}^i|_{\mathfrak{m}^i}$, *for* $i=1,2,4,6,8$,
- $\mathrm{Der}(\mathfrak{m}^7_{\alpha,\beta},\cdot,[\,,\,,\,])=\mathrm{ad}\,\mathfrak{h}^7|_{\mathfrak{m}^7_{\alpha,\beta}}$,
- $\mathrm{Der}(\mathfrak{m}^i,\cdot,[\,,\,,\,])=\mathrm{ad}\,\mathfrak{h}^1|_{\mathfrak{m}^i}$, *for* $i=3,5$.

A more subtle question is the computation of the Lie algebra of derivations of the binary anticommutative algebras $(\mathfrak{m}, \cdot)$.

Theorem 5.6.

- $\mathrm{Der}(\mathfrak{m}^1, \cdot) = \mathfrak{gl}(\mathfrak{m}^1)$;
- $\mathrm{Der}(\mathfrak{m}^i, \cdot) = \mathrm{ad}\,\mathfrak{h}^i|_{\mathfrak{m}^i}$, *for* $i = 2, 6, 8$, *which is a Lie algebra of type* $Z \oplus A_1$, A_2 *and* A_1, *respectively, where Z denotes a one-dimensional center*;
- $\mathrm{Der}(\mathfrak{m}^3, \cdot) = \mathrm{ad}\,\mathfrak{h}^1|_{\mathfrak{m}^3}$, *of type* $A_1 \oplus A_1$;
- $\mathrm{Der}(\mathfrak{m}^4, \cdot)$ *is a Lie algebra of type* $Z \oplus C_2$;
- $\mathrm{Der}(\mathfrak{m}^5, \cdot)$ *is a Lie algebra of type* $A_1 \oplus C_2$;
- $\mathrm{Der}(\mathfrak{m}^7_{\alpha,\beta}, \cdot) = \mathrm{ad}\,\mathfrak{h}^7|_{\mathfrak{m}^7_{\alpha,\beta}}$ *for any* $\alpha, \beta \in k$.

Proof. For $i = 1$, this is clear since $(\mathfrak{m}^1, \cdot)$ is a trivial algebra, for $i = 6$ this appears in [9].

For $i = 8$, as shown in the proof of Theorem 5.3, $\mathrm{ad}\,\mathfrak{h}^8|_{\mathfrak{m}^8} \subseteq \mathrm{Der}(\mathfrak{m}^8, \cdot) \subseteq \mathrm{Lie}(\mathfrak{m}^8, \cdot)$, which is contained in the orthogonal Lie algebra $\mathfrak{so}(\mathfrak{m}^8, \kappa)$, relative to the restriction to $\mathfrak{m}^8$ of the Killing form κ of $\mathfrak{g}_2$ (note that $\kappa(\mathfrak{h}^8, \mathfrak{m}^8) = 0$). But the restriction of κ to $\mathfrak{m}^8$ is, up to scalars, the unique $\mathfrak{h}^8$-invariant bilinear form on $\mathfrak{m}^8$ by irreducibility. As remarked in [4, Proof of 5.7], $\mathfrak{h}^8$ is maximal in $\mathfrak{so}(\mathfrak{m}^8, \kappa)$, and $\mathrm{Der}(\mathfrak{m}^8, \cdot) \neq \mathrm{Lie}(\mathfrak{m}^8, \cdot)$, since $(\mathfrak{m}^8, \cdot)$ is not a Lie algebra. Hence $\mathrm{ad}\,\mathfrak{h}^8|_{\mathfrak{m}^8} = \mathrm{Der}(\mathfrak{m}^8, \cdot)$ and $\mathrm{Lie}(\mathfrak{m}^8, \cdot) = \mathfrak{so}(\mathfrak{m}^8, \kappa)$.

For $i = 2$ one has $\mathfrak{h}^2 = (\mathfrak{g}_2)_{-12} \oplus (\mathfrak{g}_2)_0 \oplus (\mathfrak{g}_2)_{12}$ and $\mathfrak{m}^2 = \oplus_{i \neq 0, \pm 12} (\mathfrak{g}_2)_i$ in (5.1), so that $\mathfrak{d}^2 = \mathrm{Der}(\mathfrak{m}^2, \cdot)$ is $\mathbb{Z}$-graded. Let $\mathfrak{m}^2_{\bar 0} = \oplus_{i\ \mathrm{even}} \mathfrak{m}^2_i = \mathfrak{m}^2_{-2} \oplus \mathfrak{m}^2_2$ and $\mathfrak{m}^2_{\bar 1} = \oplus_{i\ \mathrm{odd}} \mathfrak{m}^2_i$. For any $d \in \mathfrak{d}^2_j$ for odd j, $d(\mathfrak{m}^2_l) = 0$ for any odd $l \neq \pm 2 - j$. But using Lemma 5.1 one checks that $\oplus_{\substack{l\ \mathrm{odd} \\ l \neq \pm 2 - j}} \mathfrak{m}^2_l$ generates $(\mathfrak{m}^2, \cdot)$. Hence $d(\mathfrak{m}^2) = 0$. Now, let $d \in \mathfrak{d}^2_j$ for even j. If $j \neq 0, \pm 4$, then $d(\mathfrak{m}^2_{\bar 0}) = 0$. Also, any $0 \neq d \in \mathfrak{d}^2_{-4}$ is determined by its action on $\mathfrak{m}^2_9 = (\mathfrak{g}_2)_9$, since the subspace $\mathfrak{m}^2_9$, together with $\mathfrak{m}^2_{-2} \oplus \mathfrak{m}^2_3 \oplus \mathfrak{m}^2_5 \oplus \mathfrak{m}^2_{-7} \oplus \mathfrak{m}^2_{-9}$ (which is annihilated by d) generates $(\mathfrak{m}^2, \cdot)$. Note that $(\mathfrak{g}_2)_2 = kX^2$, $(\mathfrak{g}_2)_{-2} = kY^2$ and $[X^2, Y^2] = (X^2, Y^2)_1 = XY$. Hence we may assume (up to a scalar) that $d|_{\mathfrak{m}^2_9} = (\mathrm{ad}\,Y^2)^2|_{\mathfrak{m}^2_9}$ $(\neq 0)$, so $d(\mathfrak{m}^2_7) = d(Y^2 \cdot \mathfrak{m}^2_9) = Y^2 \cdot d(\mathfrak{m}^2_9)$, since $d(Y^2) \in \mathfrak{m}^2_{-6} = 0$, and thus $d|_{\mathfrak{m}^2_7} = (\mathrm{ad}\,Y^2)^2|_{\mathfrak{m}^2_7}$ too. But $\mathfrak{m}^2_7 = k(x \otimes X^2 Y)$, $d(X^2) = \mu Y^2$ for some $\mu \in k$ and

$$
\begin{aligned}
d(X^2) \cdot (x \otimes XY^2) &= d(X^2 \cdot (x \otimes XY^2)) - X^2 \cdot d(x \otimes X^2 Y) \\
&= [(\mathrm{ad}\,Y^2)^2, \mathrm{ad}\,X^2](x \otimes X^2 Y) \\
&= -(\mathrm{ad}\,XY\,\mathrm{ad}\,Y^2 + \mathrm{ad}\,Y^2\,\mathrm{ad}\,XY)(x \otimes X^2 Y) \quad (\text{as } [X^2, Y^2] = XY) \\
&= 0
\end{aligned}
$$

as XY acts with eigenvalue $\frac{-1}{4}$ on $x \otimes X^2 Y$ and $\frac{1}{4}$ on $[Y^2, x \otimes X^2 Y]$. We conclude that $\mu = 0$ so $d(\mathfrak{m}^2_{\bar 0}) = 0$. Similarly, any $d \in \mathfrak{d}^2_4$ satisfies $d(\mathfrak{m}^2_{\bar 0}) = 0$. If $d \in \mathfrak{d}^2_0$, it acts as a scalar α_l on each $\mathfrak{m}^2_l$, and since by the derivation property $\alpha_2 + \alpha_{-9} = \alpha_{-7}$ and $\alpha_{-2} + \alpha_{-7} = \alpha_{-9}$, it follows that $\alpha_2 = -\alpha_{-2}$ and hence there is an $\alpha \in k$ such that $(d - \alpha\,\mathrm{ad}\,XY)(\mathfrak{m}^2_{\bar 0}) = 0$. Moreover, any $d \in \mathfrak{d}^2_{\bar 0} = \oplus_{j\ \mathrm{even}} \mathfrak{d}^2_j$ with $d(\mathfrak{m}^2_{\bar 0}) = 0$ satisfies the condition that $d|_{\mathfrak{m}^2_{\bar 1}}$ commutes with

$\operatorname{ad}\mathfrak{m}^2_{\bar 0}|_{\mathfrak{m}^2_{\bar 1}}$, and hence with the action of W_2 on $\mathfrak{m}^2_{\bar 1} = V_1 \otimes_k W_3$. By irreducibility of W_3, it follows that $d|_{\mathfrak{m}^2_{\bar 1}} \in \operatorname{End}_k(V_1) \otimes 1$. Also, if $d \in \mathfrak{d}^2_j$, with $j \neq 0$, its trace is 0, and any $d \in \mathfrak{d}^2_0$ with $d(\mathfrak{m}^2_{\bar 0}) = 0$ acts as a scalar α on $x \otimes W_3$ and another scalar β on $y \otimes W_3$. From $0 = d(\mathfrak{m}^2_2)$ it follows that $0 = d((x \otimes X^3) \cdot (y \otimes XY^2))$ and we deduce that $\alpha + \beta = 0$. Therefore $d|_{V_1 \otimes_k W_3} \in \mathfrak{sl}(V_1) \otimes 1 = \operatorname{ad}\mathfrak{s}^L|_{V_1 \otimes_k W_3}$. We conclude that $\mathfrak{d}^2 = \mathfrak{d}^2_0$ is contained in $(k \operatorname{ad} XY + \operatorname{ad}\mathfrak{s}^L)|_{\mathfrak{m}^2}$, whence $\mathfrak{d}^2 = \operatorname{ad}\mathfrak{h}^2|_{\mathfrak{m}^2}$.

Now, consider $\mathfrak{d}^3 = \operatorname{Der}(\mathfrak{m}^3, \cdot)$. Here $\mathfrak{h}^3 = V_2 = [\mathfrak{h}^2, \mathfrak{h}^2]$, and $\mathfrak{m}^3 = W_2 \oplus (V_1 \otimes_k W_3)$, which inherits the $\mathbb{Z}_2$-grading of $\mathfrak{g}_2$. Since $(W_2, \cdot) \cong \mathfrak{sl}_2(k)$, for any $d \in \mathfrak{d}^3_{\bar 0}$, there is an $F \in W_2$ such that $(d - \operatorname{ad} F)|_{\mathfrak{m}^3_{\bar 0}} = 0$, so $(d - \operatorname{ad} F)|_{\mathfrak{m}^3_{\bar 1}}$ is an endomorphism of $V_1 \otimes_k W_3$ commuting with the action of W_2. As before, we conclude that $d - \operatorname{ad} F \in \operatorname{ad}\mathfrak{h}^3|_{\mathfrak{m}^3_{\bar 1}}$ and hence $d \in \operatorname{ad}(\mathfrak{g}_2)_{\bar 0}|_{\mathfrak{m}^3}$. It is also clear that $\operatorname{ad}(\mathfrak{g}_2)_{\bar 0}|_{\mathfrak{m}^3} = \operatorname{ad}\mathfrak{h}^1|_{\mathfrak{m}^3} \subseteq \mathfrak{d}^3$ by Theorem 5.5.Now, if $d \in \mathfrak{d}^3_{\bar 0}$ and $z \in \mathfrak{m}^3_{\bar 1}$, $[d, L_z] = L_{d(z)}$, so $L_{\mathfrak{m}^3_{\bar 1}} \cap \mathfrak{d}^3_{\bar 1}$ is $\mathfrak{d}^3_{\bar 0}$-invariant. But $\mathfrak{m}^3_{\bar 1} = V_1 \otimes_k W_3$ is an irreducible module for $\mathfrak{d}^3_{\bar 0}$, so for any $d \in \mathfrak{d}^3_{\bar 1}$ and $u \in \mathfrak{m}^3_{\bar 0}$, $L_{d(u)} = [d, L_u] = [d, \operatorname{ad} u|_{\mathfrak{m}^3}] \in \mathfrak{d}^3_{\bar 1} \cap L_{\mathfrak{m}^3_{\bar 1}} = 0$ and hence $d(u) = 0$. Thus, $d(\mathfrak{m}^3_{\bar 0}) = 0$, so $d|_{\mathfrak{m}^3_{\bar 1}}$ is a homomorphism from $V_1 \otimes_k W_3$ into W_2 commuting with the action of $\mathfrak{m}^3_{\bar 0}$; so it is 0. The conclusion is that $\mathfrak{d}^3 = \operatorname{ad}\mathfrak{h}^1|_{\mathfrak{m}^3}$.

For $\mathfrak{d}^5 = \operatorname{Der}(\mathfrak{m}^5, \cdot)$, $\mathfrak{h}^5 = W_2$, $\mathfrak{m}^5 = V_2 \oplus (V_1 \otimes_k W_3)$. As for $i = 3$, for any $d \in \mathfrak{d}^3_{\bar 0}$, there is an $f \in V_2$ such that $\tilde{d} = d - \operatorname{ad} f$ annihilates $\mathfrak{m}^5_{\bar 0}$, so that $\tilde{d}$ commutes with the action of V_2 and hence $\tilde{d}|_{\mathfrak{m}^5_{\bar 1}}$ is of the form $1 \otimes \varphi$, for $\varphi \in \operatorname{End}_k(W_3)$. Using (4.2) one concludes easily that $\varphi \in \mathfrak{sp}(W_3, (\ ,\ )_3)$ (the symplectic Lie algebra). The same arguments as for $i = 3$ give that $\mathfrak{d}^5 = \operatorname{ad}\mathfrak{m}^5_{\bar 0}|_{\mathfrak{m}^5} \oplus \mathfrak{sp}(W_3, (\ ,\ )_3)$, a Lie algebra of type $A_1 \oplus C_2$. (Note that [4, 4.3] shows that $\mathfrak{sp}(W_3, (\ ,\ )_3) \cong W_2 \oplus W_6$ under a suitable bracket.)

For $\mathfrak{d}^4 = \operatorname{Der}(\mathfrak{m}^4, \cdot)$, $\mathfrak{h}^4 = kxy \oplus W_2$ and $\mathfrak{m}^4 = (kx^2 \oplus ky^2) \oplus (V_1 \otimes_k W_3)$, so $\mathfrak{m}^4_{\bar 0} = \mathfrak{m}^4_{-12} \oplus \mathfrak{m}^4_{12}$, $\mathfrak{h}^4 = (\mathfrak{g}_2)_{-2} \oplus (\mathfrak{g}_2)_0 \oplus (\mathfrak{g}_2)_2$. Here $\mathfrak{d}^4_{\bar 1} = 0$ since any $d \in \mathfrak{d}^4_j$, for odd j, annihilates all but at most one subspace $\mathfrak{m}^4_l$ for odd l, and these subspaces generate $\mathfrak{m}^4$. On the other hand, as for $i = 2$, any $d \in \mathfrak{d}^4_0$ acts as a scalar α_l on any $\mathfrak{m}^4_l$, and since $\alpha_3 = \alpha_{12} + \alpha_{-9}$ and $\alpha_{-9} = \alpha_{-12} + \alpha_3$, it follows that $\alpha_{-12} = -\alpha_{12}$ so that $(d - \mu \operatorname{ad} xy)|_{\mathfrak{m}^4_{\bar 0}} = 0$ for some $\mu \in k$. Also, any $d \in \mathfrak{d}^4_j$ for even $j \neq 0$, either annihilates $\mathfrak{m}^4_{\bar 1}$, and thus is 0, or annihilates $\mathfrak{m}^4_{\bar 0}$. Besides, any $d \in \mathfrak{d}^4_0$ with $d(\mathfrak{m}^4_{\bar 0}) = 0$ satisfies the condition that $d|_{\mathfrak{m}^4_{\bar 1}}$ is an endomorphism of $\mathfrak{m}^4_{\bar 1} = V_1 \otimes_k W_3$ commuting with the action of V_2 and, as for $i = 5$, we conclude that $\mathfrak{d}^4 = k \operatorname{ad} xy|_{\mathfrak{m}^4} \oplus \mathfrak{sp}(W_3, (\ ,\ )_3)$ (where we identify $\mathfrak{sp}(W_3, (\ ,\ )_3)$ with a subalgebra of $\mathfrak{gl}(\mathfrak{m}^4)$ in the natural way).

The situation for $\mathfrak{d}^7_{\alpha,\beta} = \operatorname{Der}(\mathfrak{m}^7_{\alpha,\beta}, \cdot)$ is a bit more complicated. First, we need the next two results:

(i) Let $\sigma : V_4 \to V_2$ be a linear map such that $(\sigma(g_1), g_2)_2 = (g_1, \sigma(g_2))_2$ for any $g_1, g_2 \in V_4$. Then $\sigma = 0$.

(ii) Let $\tau : V_2 \to V_2$ be a linear map such that $\tau((f, g)_2) = -(\tau(f), g)_2$ for any $f \in V_2$ and $g \in V_4$. Then $\tau = 0$.

These can be checked directly or, using the identification of V_4 with the subspace $\mathfrak{sym}_0(V_2, (.,.)_2)$ in (4.8) and (4.9), they are equivalent to the following easily checked properties about a three-dimensional vector space V endowed with a nondegenerate bilinear form b:

(i') Let $\sigma : \mathfrak{sym}_0(V, b) \to V$ be a linear map such that $f(\sigma(g)) = g(\sigma(f))$ for any $f, g \in \mathfrak{sym}_0(V, b)$, then $\sigma = 0$.
(ii') Let $\tau : V \to V$ be a linear map such that $\tau f + f\tau = 0$ for any $f \in \mathfrak{sym}_0(V, b)$, then $\tau = 0$.

For (i') take $f_u = b(u, -)u$ for any isotropic vector u (recall that the ground field is assumed to be algebraically closed, so there are plenty of isotropic vectors). Then $b(u, \sigma(f_v))u = b(v, \sigma(f_u))v$, so that $b(u, \sigma(f_v)) = 0$ for any linearly independent isotropic vectors u and v. Since there are bases of isotropic vectors linearly independent to a given one, we conclude that $\sigma(f_v) = 0$ for any isotropic vector v, and hence $\sigma = 0$, as the f_v's span $\mathfrak{sym}_0(V, b)$. As for (ii'), τ commutes with $[\mathfrak{sym}_0(V, b), \mathfrak{sym}_0(V, b)] = \mathfrak{so}(V, b)$, so it is a scalar multiple of the identity by Schur's lemma, and this scalar is necessarily 0.

Now, let $d \in \mathrm{Der}(\mathfrak{m}^7_{\alpha,\beta}, \cdot)$ and let $d(g) = \sigma_1(g) \otimes \tilde{u} + \sigma_2(g) \otimes \tilde{v} + \varphi(g)$ for any $g \in V_4$, where $\sigma_1, \sigma_2 : V_4 \to V_2$ and $\varphi : V_4 \to V_4$ are linear maps. Here $\tilde{u} = u - \alpha 1$ and $\tilde{v} = v - \beta 1$ (recall (4.7)). Taking into account the multiplication rules in Theorem 4.5, for any $g_1, g_2 \in V_4$, the coefficient of $\tilde{u}$ in $0 = d(g_1 \cdot g_2) = d(g_1) \cdot g_2 + g_1 \cdot d(g_2)$ is

$$-(\sigma_1(g_1), g_2)_2 + (g_1, \sigma_1(g_2))_2 = 0$$

so, by property (i) above, we conclude that $\sigma_1 = 0$ and, similarly, that $\sigma_2 = 0$. Therefore, V_4 is invariant under $\mathrm{Der}(\mathfrak{m}^7_{\alpha,\beta}, \cdot)$. For any $f \in V_2$, $d(f \otimes \tilde{u}) = \tau_1(f) \otimes \tilde{u} + \tau_2(f) \otimes \tilde{v} + \psi(f)$ for some linear maps $\tau_1, \tau_2 : V_2 \to V_2$ and $\psi : V_2 \to V_4$. The coefficient of $\tilde{v}$ in

$$d((f \otimes \tilde{u}) \cdot g) = d(f \otimes \tilde{u}) \cdot g + (f \otimes \tilde{u}) \cdot d(g) \tag{5.2}$$

is

$$-\tau_2((f, g)_2) = (\tau_2(f), g)_2$$

so, property (ii) above gives $\tau_2 = 0$. Similarly, $d(V_2 \otimes \tilde{v}) \subseteq V_2 \otimes \tilde{v} \oplus V_4$. Therefore, for any $d \in \mathrm{Der}(\mathfrak{m}^7_{\alpha,\beta}, \cdot)$, there are linear maps $\tau_1, \tau_2 : V_2 \to V_2$, $\psi_1, \psi_2 : V_2 \to V_4$ and $\varphi : V_4 \to V_4$ such that

$$d(f \otimes \tilde{u}) = \tau_1(f) \otimes \tilde{u} + \psi_1(f),$$

$$d(f \otimes \tilde{v}) = \tau_2(f) \otimes \tilde{v} + \psi_2(f),$$

$$d(g) = \varphi(g)$$

for any $f \in V_2$ and $g \in V_4$. The coefficient of $\tilde{u}$ in (5.2) is

$$-\tau_1((f, g)_2) = -(\tau_1(f), g)_2 - (f, \varphi(g))_2, \tag{5.3}$$

or $[\tau_1, T^2_{2,g}] = -T^2_{2,\varphi(g)}$. By symmetry, also $[\tau_2, T^2_{2,g}] = -T^2_{2,\varphi(g)}$, so that $[\tau_1 - \tau_2, T^2_{2,g}] = 0$ for any $g \in V_4$. But $\{T^2_{2,g} : g \in V_4\} = \mathfrak{sym}_0(V_2, (.,.)_2)$, which generates $\mathrm{End}_k(V_2)$ as an associative algebra. Thus $\tau_1 - \tau_2$ is a scalar multiple of the identity. Now, one half of the coefficient of $\tilde{v}$ in

$$d((f_1 \otimes \tilde{u}) \cdot (f_2 \otimes \tilde{u})) = d(f_1 \otimes \tilde{u}) \cdot (f_2 \otimes \tilde{u}) + (f_1 \otimes \tilde{u}) \cdot d(f_2 \otimes \tilde{u})$$

gives

$$\tau_2((f_1, f_2)_1) = (\tau_1(f_1), f_2)_1 + (f_1, \tau_1(f_2))_1 \tag{5.4}$$

for any $f_1, f_2 \in V_2$ and, symmetrically,

$$\tau_1((f_1, f_2)_1) = (\tau_2(f_1), f_2)_1 + (f_1, \tau_2(f_2))_1. \tag{5.5}$$

If (5.5) is subtracted from (5.4), we check that the scalar map $\tau_1 - \tau_2$ is 0, or $\tau_1 = \tau_2$ which, by (5.5), is a derivation of $(V_2, (.,.)_1) \simeq \mathfrak{sl}_2(k)$, and hence there exists an element $h \in V_2$ such that $\tau_1 = T^2_{1,h}$. Then, from Corollary 4.4, the new derivation $\hat{d} = d - \operatorname{ad} h \otimes 1|_{\mathfrak{m}^7_{\alpha,\beta}}$ satisfies

$$\hat{d}(f \otimes \tilde{u}) = \psi_1(f), \quad \hat{d}(f \otimes \tilde{v}) = \psi_2(f), \quad \hat{d}(g) = \hat{\varphi}(g)$$

for any $f \in V_2$, $g \in V_4$, for a suitable linear map $\hat{\varphi} : V_4 \to V_4$. In this situation, Eq. (5.3) (with $\tau_1 = 0$ and $\hat{\varphi}$ instead of φ) proves that $T^2_{2,\hat{\varphi}(g)} = 0$ for any $g \in V_4$, and hence that $\hat{\varphi} = 0$. Finally, for any $f \in V_2$ and $g \in V_4$, $\hat{d}(f \otimes \tilde{u}) \cdot g = 0 = (f \otimes \tilde{u}) \cdot \hat{d}(g)$ ($\hat{d}(g) = 0$), while

$$\hat{d}((f \otimes \tilde{u}) \cdot g) = \hat{d}(-(f, g)_2 \otimes \tilde{u} - 2\alpha(f, g)_1) = -\psi_1((f, g)_2).$$

Since $(V_2, V_4)_2 = V_2$, we conclude that $\psi_1 = 0$ and, in the same way, $\psi_2 = 0$. Therefore, $\hat{d} = 0$ and $d \in \operatorname{ad} \mathfrak{h}^7_{\alpha,\beta}$, as required. □

5.3. *Lie multiplication algebras*

Recall that the *Lie multiplication algebra* of a nonassociative algebra A is the Lie subalgebra $\mathrm{Lie}(A)$ of $\mathfrak{gl}(A)$ generated by the operators of left and right multiplications by the elements of A. As before, let us denote by κ both the Killing form of $\mathfrak{g}_2$ and its restriction to $\mathfrak{m}^i$, $i = 1, \ldots, 8$, which is nondegenerate since $\mathfrak{m}^i$ is the orthogonal complement to $\mathfrak{h}^i$ for any i.

Since the anticommutative algebra $(\mathfrak{m}^1, \cdot)$ is trivial, so is its Lie multiplication algebra. For the remaining cases, there is a uniform description.

Theorem 5.7. $\mathrm{Lie}(\mathfrak{m}^i, \cdot) = \mathfrak{so}(\mathfrak{m}^i, \kappa)$, *for any* $i = 2, \ldots, 8$.

Proof. For $i = 6$ this appears in [9, Theorem 4.5] and for $i = 8$ the result has already appeared in the proof of the previous result.

Note that for any $u, v, w \in \mathfrak{m}^i$

$$\kappa(u \cdot v, w) = \kappa([u, v], w) = \kappa(u, [v, w]) = \kappa(u, v \cdot w),$$

so $\mathrm{Lie}(\mathfrak{m}^i, \cdot) \subseteq \mathfrak{so}(\mathfrak{m}^i, \kappa)$. For $i \neq 1, 6, 7, 8$, $\mathfrak{m}^i$ inherits the $\mathbb{Z}_2$-grading of $\mathfrak{g}_2$ in (4.1), thus $\kappa(\mathfrak{m}^i_{\bar{0}}, \mathfrak{m}^i_{\bar{1}}) = 0$ and $\mathfrak{so}(\mathfrak{m}^i, \kappa)$ is generated by $\mathfrak{so}(\mathfrak{m}^i, \kappa)_{\bar{1}}$ (=$\{\varphi \in \mathfrak{so}(\mathfrak{m}^i, \kappa) : \varphi(\mathfrak{m}^i_{\bar{j}}) \subseteq$

$\mathfrak{m}^i_{\overline{j+1}}$, $j=0,1\}$), which is spanned by the linear maps $\kappa(u,.)v-\kappa(v,.)u$ for $u\in\mathfrak{m}^i_{\bar 0}$, $v\in\mathfrak{m}^i_{\bar 1}$. In particular, as a module for $\mathfrak{h}^i$, $\mathfrak{so}(\mathfrak{m}^i,\kappa)_{\bar 1}$ is isomorphic to $\mathfrak{m}^i_{\bar 0}\otimes_k\mathfrak{m}^i_{\bar 1}$. The required result would follow if we could establish that the map

$$\Phi:\mathfrak{m}^i_{\bar 0}\otimes\mathfrak{m}^i_{\bar 1}\longrightarrow\mathfrak{so}(\mathfrak{m}^i,\kappa)_{\bar 1},$$
$$u\otimes v\mapsto [L_u,L_v]$$

is one-to-one. Note also that the $\mathbb{Z}$-grading in (5.1) induces an associated $\mathbb{Z}$-grading on $\mathfrak{so}(\mathfrak{m}^i,\kappa)$ too (preserved by Φ).

For $i=2$, $\mathfrak{m}^2=(kX^2\oplus kY^2)\oplus(V_1\otimes_k W_3)$, $\mathfrak{h}^2=V_2\oplus kXY$ and

$$\mathfrak{m}^2_{\bar 0}\otimes\mathfrak{m}^2_{\bar 1}=\left(\oplus_{j=\pm 2}(\mathfrak{g}_2)_j\right)\otimes\left(\oplus_{j\in\{\pm 3,\pm 5,\pm 7,\pm 9\}}(\mathfrak{g}_2)_j\right)$$

and to show that Φ is one-to-one, it is enough to show that

$$\Phi\left((\mathfrak{g}_2)_{\pm 2}\otimes_k(\mathfrak{g}_2)_{\pm j}\right)\neq 0$$

for any $j=3,5,7,9$, and that

$$\Phi\left((\mathfrak{g}_2)_2\otimes_k(\mathfrak{g}_2)_{j-2}\right)\neq\Phi\left((\mathfrak{g}_2)_{-2}\otimes_k(\mathfrak{g}_2)_{j+2}\right)$$

for $j=\pm 5,\pm 7$. This is obtained by routine verifications. For instance, for $0\neq z_j\in(\mathfrak{g}_2)_j$ for any j,

$$z_2\cdot(z_9\cdot z_{-9})-z_9\cdot(z_2\cdot z_{-9})=-z_9\cdot(z_2\cdot z_{-9})\neq 0$$

by Lemma 5.1, so $\Phi((\mathfrak{g}_2)_2\otimes_k(\mathfrak{g}_2)_9)\neq 0$. Also

$$z_2\cdot(z_5\cdot z_2)-z_5\cdot(z_2\cdot z_2)=z_2\cdot(z_5\cdot z_2)\neq 0,$$

while

$$z_{-2}\cdot(z_9\cdot z_2)-z_9\cdot(z_{-2}\cdot z_2)=0,$$

which shows that $\Phi((\mathfrak{g}_2)_2\otimes_k(\mathfrak{g}_2)_5)\neq\Phi((\mathfrak{g}_2)_{-2}\otimes_k(\mathfrak{g}_2)_9)$.

For $i=4$, $\mathfrak{m}^4=(kx^2\oplus ky^2)\oplus(V_1\otimes_k W_3)$, $\mathfrak{h}^4=kxy\oplus W_2$, and one proceeds similarly.

For $i=3,5,7$ different arguments will be used. First, for $i=3$, $\mathfrak{m}^3=W_2\oplus(V_1\otimes_k W_3)$, $\mathfrak{h}^3=V_2$. As a module for $V_2\cong\mathfrak{sl}_2(k)$, $\mathfrak{m}^3_{\bar 0}=3V(0)$ and $\mathfrak{m}^3_{\bar 1}=4V(1)$, where $V(j)$ denotes the irreducible $\mathfrak{sl}_2(k)$-module of dimension $j+1$ (which, up to isomorphism, is V_j). Hence $\mathfrak{m}^3_{\bar 0}\otimes_k\mathfrak{m}^3_{\bar 1}\cong 12V(1)$ and it is enough to find 12 independent eigenvectors of eigenvalue 1 for the action of $h=-4\,\mathrm{ad}\,xy$ in $L_{\mathfrak{m}^3_{\bar 1}}+[[L_{\mathfrak{m}^3_{\bar 1}},L_{\mathfrak{m}^3_{\bar 1}}],L_{\mathfrak{m}^3_{\bar 1}}]$ $(\subseteq\mathfrak{so}(\mathfrak{m}^3,\kappa)_{\bar 1})$. Note that h acts with eigenvalue 1 on $(\mathfrak{g}_2)_j$ for odd $j>0$ and eigenvalue -1 on $(\mathfrak{g}_2)_j$ for odd $j<0$. Therefore, it is enough to prove that

- $[[L_{(\mathfrak{g}_2)_9},L_{(\mathfrak{g}_2)_7}],L_{(\mathfrak{g}_2)_{-5}}]$ $(\subseteq\mathfrak{so}(\mathfrak{m}^3,\kappa)_{11})$ has dimension 1,
- $L_{(\mathfrak{g}_2)_9}+[[L_{(\mathfrak{g}_2)_9},L_{(\mathfrak{g}_2)_5}],L_{(\mathfrak{g}_2)_{-5}}]$ $(\subseteq\mathfrak{so}(\mathfrak{m}^3,\kappa)_9)$ has dimension 2,
- $L_{(\mathfrak{g}_2)_7}+[[L_{(\mathfrak{g}_2)_3},L_{(\mathfrak{g}_2)_7}],L_{(\mathfrak{g}_2)_{-3}}]+[[L_{(\mathfrak{g}_2)_9},L_{(\mathfrak{g}_2)_7}],L_{(\mathfrak{g}_2)_{-9}}]$ $(\subseteq\mathfrak{so}(\mathfrak{m}^3,\kappa)_7)$ has dimension 3,

- $L_{(\mathfrak{g}_2)_5} + [[L_{(\mathfrak{g}_2)_9}, L_{(\mathfrak{g}_2)_5}], L_{(\mathfrak{g}_2)_{-9}}] + [[L_{(\mathfrak{g}_2)_3}, L_{(\mathfrak{g}_2)_5}], L_{(\mathfrak{g}_2)_{-3}}] (\subseteq \mathfrak{so}(\mathfrak{m}^3, \kappa)_5)$ has dimension 3,
- $L_{(\mathfrak{g}_2)_3} + [[L_{(\mathfrak{g}_2)_3}, L_{(\mathfrak{g}_2)_7}], L_{(\mathfrak{g}_2)_{-7}}] (\subseteq \mathfrak{so}(\mathfrak{m}^3, \kappa)_3)$ has dimension 2, and
- $[[L_{(\mathfrak{g}_2)_3}, L_{(\mathfrak{g}_2)_5}], L_{(\mathfrak{g}_2)_{-7}}] (\subseteq \mathfrak{so}(\mathfrak{m}^3, \kappa)_1)$ has dimension 1,

and all these are routinely checked.

For $i=5$, $\mathfrak{m}^5 = V_2 \oplus (V_1 \otimes_k W_3)$, $\mathfrak{h}^5 = W_2$ and, as a module for $W_2 \cong \mathfrak{sl}_2(k)$, $\mathfrak{m}^5_{\bar{0}} \cong 3V(0)$ and $\mathfrak{m}^5_{\bar{1}} \cong 2V(3)$, so $\mathfrak{m}^5_{\bar{0}} \otimes_k \mathfrak{m}^5_{\bar{1}} \cong 6V(3)$. It is enough to find 6 independent eigenvectors for $h = -4\,\mathrm{ad}\,XY$ with eigenvalue 3 in $\mathfrak{so}(\mathfrak{m}^5, \kappa)_{\bar{1}} \cap \mathrm{Lie}(\mathfrak{m}^5, \cdot)$. Note that here h acts with eigenvalue 3 on $(\mathfrak{g}_2)_9 \oplus (\mathfrak{g}_2)_{-3} = V_1 \otimes X^3$, 1 on $(\mathfrak{g}_2)_7 \oplus (\mathfrak{g}_2)_{-5} = V_1 \otimes X^2Y$, -1 on $(\mathfrak{g}_2)_5 \oplus (\mathfrak{g}_2)_{-7} = V_1 \otimes XY^2$ and -3 on $(\mathfrak{g}_2)_3 \oplus (\mathfrak{g}_2)_{-9} = V_1 \otimes Y^3$. Now

- $L_{(\mathfrak{g}_2)_9} + [[L_{(\mathfrak{g}_2)_9}, L_{(\mathfrak{g}_2)_5}], L_{(\mathfrak{g}_2)_{-5}}] (\subseteq \mathfrak{so}(\mathfrak{m}^3, \kappa)_9)$ has dimension 2,
- $L_{(\mathfrak{g}_2)_{-3}} + [[L_{(\mathfrak{g}_2)_{-3}}, L_{(\mathfrak{g}_2)_{-7}}], L_{(\mathfrak{g}_2)_7}] (\subseteq \mathfrak{so}(\mathfrak{m}^3, \kappa)_{-3})$ has dimension 2,
- $[[L_{(\mathfrak{g}_2)_9}, L_{(\mathfrak{g}_2)_7}], L_{(\mathfrak{g}_2)_5}] (\subseteq \mathfrak{so}(\mathfrak{m}^3, \kappa)_{21})$ has dimension 1, and
- $[[L_{(\mathfrak{g}_2)_{-3}}, L_{(\mathfrak{g}_2)_{-5}}], L_{(\mathfrak{g}_2)_{-7}}] (\subseteq \mathfrak{so}(\mathfrak{m}^3, \kappa)_{-15})$ has dimension 1 too,

thus obtaining the required independent eigenvectors.

Finally, for $i=7$ consider the model in Corollary 4.4 and Theorem 4.5. The decomposition $\mathfrak{m}^7 = (V_2 \otimes (u+v)) \oplus ((V_2 \otimes (u-v)) \oplus V_4)$ is a $\mathbb{Z}_2$-grading and, as a module for $V_2 \otimes 1 \cong \mathfrak{sl}_2(k)$, $\mathfrak{m}^7_{\bar{0}} \otimes_k \mathfrak{m}^7_{\bar{1}} \cong V(2) \otimes_k (V(2) \oplus V(4)) \cong V(6) \oplus 2V(4) \oplus 2V(2) \oplus V(0)$. In order to prove that $\mathfrak{so}(\mathfrak{m}^7, \kappa) = \mathrm{Lie}(\mathfrak{m}^7, \cdot)$, it is enough to check that $L_{\mathfrak{m}^7_{\bar{1}}} + [L_{\mathfrak{m}^7_{\bar{0}}}, L_{\mathfrak{m}^7_{\bar{1}}}] = \mathfrak{so}(\mathfrak{m}^7, \kappa)_{\bar{1}}$. But $[L_{x^2 \otimes (u+v)}, L_{x^4}]$ is a nonzero highest weight vector of weight 6 for $h = -4\,\mathrm{ad}(xy \otimes 1)$, L_{x^4} and $[L_{x^2 \otimes (u+v)}, L_{x^3 y}] - [L_{xy \otimes (u+v)}, L_{x^4}]$ are linearly independent highest weight vectors of weight 4, $[L_{x^2 \otimes (u+v)}, L_{xy \otimes (u-v)}] - [L_{xy \otimes (u+v)}, L_{x^2 \otimes (u-v)}]$ and $L_{x^2 \otimes (u-v)}$ are linearly independent highest weight vectors of weight 2, while the nonzero vector $[L_{x^2 \otimes (u+v)}, L_{y^2 \otimes (u-v)}] - 2[L_{xy \otimes (u+v)}, L_{xy \otimes (u-v)}] + [L_{y^2 \otimes (u+v)}, L_{x^2 \otimes (u-v)}]$ is annihilated by $V_2 \otimes 1$. □

Corollary 5.8. *None of the Lie–Yamaguti algebras* $(\mathfrak{m}^i, \cdot)$, $i = 1, \ldots, 8$, *are homogeneous* (*see* (1.4)).

Proof. The case $i=1$ is obvious, as the binary product is trivial. Hence we may assume $i \geqslant 2$. According to [20], if $(\mathfrak{m}, \cdot, [\ ,\ ,\ ])$ is any finite-dimensional simple homogeneous Lie–Yamaguti algebra, then either:

(1) $(\mathfrak{m}, \cdot)$ is a Lie algebra and $[x, y, z] = (x \cdot y) \cdot z$ for any $x, y, z \in \mathfrak{m}$, or
(2) $(\mathfrak{m}, \cdot)$ is a Malcev algebra and $[x, y, z] = -(x \cdot y) \cdot z - x \cdot (y \cdot z) + y \cdot (x \cdot z)$ for any $x, y, z \in \mathfrak{m}$, or
(3) $(\mathfrak{m}, \cdot)$ satisfies a specific degree 4 identity and $[x, y, z] = \frac{1}{4}(2(x \cdot y) \cdot z - x \cdot (y \cdot z) + y \cdot (x \cdot z))$ for any $x, y, z \in \mathfrak{m}$.

Moreover, in all three cases, $\mathrm{Der}(\mathfrak{m}, \cdot) = \mathrm{Der}(\mathfrak{m}, \cdot, [\ ,\ ,\ ]) = D(\mathfrak{m}, \mathfrak{m})$.

Then, in all three cases, $[L_\mathfrak{m}, L_\mathfrak{m}] \subseteq L_\mathfrak{m} + D(\mathfrak{m}, \mathfrak{m})$ and $D(\mathfrak{m}, \mathfrak{m}) \subseteq L_\mathfrak{m} + [L_\mathfrak{m}, L_\mathfrak{m}]$. Therefore, $\mathrm{Lie}(\mathfrak{m}, \cdot) = L_\mathfrak{m} + D(\mathfrak{m}, \mathfrak{m})$.

But for $i=2,\ldots,8$, $D(\mathfrak{m}^i,\mathfrak{m}^i)=\mathrm{ad}\,\mathfrak{h}^i|_{\mathfrak{m}^i}$ and $L_{\mathfrak{m}^i}+D(\mathfrak{m}^i,\mathfrak{m}^i)$ does not make up all of $\mathfrak{so}(\mathfrak{m}^i,\kappa)=\mathrm{Lie}(\mathfrak{m}^i,\kappa)$ (by dimension count). Hence these algebras are not homogeneous. □

5.4. *Holonomy algebras*

Given a reductive homogeneous space $M\simeq G/H$ with reductive decomposition $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}$ as in (1.1), any G-invariant affine connection on M is uniquely described by a bilinear multiplication $\alpha:\mathfrak{m}\times\mathfrak{m}\to\mathfrak{m}$ such that $\mathrm{Ad}\,H|_{\mathfrak{m}}$ is a subgroup of automorphisms of the (nonassociative) algebra $(\mathfrak{m},\alpha)$. In this way, the space of G-invariant affine connections is in bijection with the space $\mathrm{Hom}_H(\mathfrak{m}\otimes\mathfrak{m},\mathfrak{m})$ (see [16, Chapter X]) or, if H is connected with $\mathrm{Hom}_{\mathfrak{h}}(\mathfrak{m}\otimes\mathfrak{m},\mathfrak{m})$. The computation of this space in all our examples [5] is an exercise about plethysms, that is, it amounts to decomposing $\mathfrak{m}$ and $\mathfrak{m}\otimes\mathfrak{m}$ into direct sums of irreducible submodules.

There are always two distinguished such connections: the *canonical connection*, given by $\alpha=0$, and the *natural connection* (with trivial torsion), given by $\alpha(x,y)=\frac{1}{2}\pi_{\mathfrak{m}}([x,y])=\frac{1}{2}x\cdot y$ for any $x,y\in\mathfrak{m}$. The *holonomy algebra* (the Lie algebra of the holonomy group) is the smallest Lie subalgebra of $\mathfrak{gl}(\mathfrak{m})$ containing the curvature tensors

$$R(x,y)=[\alpha_x,\alpha_y]-\alpha_{x\cdot y}-\mathrm{ad}\,\pi_{\mathfrak{h}}([x,y])|_{\mathfrak{m}} \tag{5.6}$$

for any $x,y\in\mathfrak{m}$, and closed under commutators by the operators of left multiplication α_x $(y\mapsto\alpha_x(y)=\alpha(x,y))$ for any $x\in\mathfrak{m}$.

This holonomy algebra makes sense for any reductive decomposition (1.1) (and hence for any Lie–Yamaguti algebra) over arbitrary fields:

Definition 5.9. Let $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}$ be a reductive decomposition of a Lie algebra and let $\alpha\in\mathrm{Hom}_{\mathfrak{h}}(\mathfrak{m}\otimes_k\mathfrak{m},\mathfrak{m})$. Then the *holonomy algebra* of α: $\mathfrak{hol}(\mathfrak{m},\alpha)$, is the smallest Lie subalgebra of $\mathfrak{gl}(\mathfrak{m})$ containing the *curvature operators* $R(x,y)$ in (5.6) for any $x,y\in\mathfrak{m}$, and closed under commutators by the operators $\{\alpha_x=\alpha(x,.):x\in\mathfrak{m}\}$.

We will finish the paper with the computation of the holonomy algebra in our examples of Lie–Yamaguti algebras.

For the canonical connection ($\alpha_x=0$ for any x), it is clear that $\mathfrak{hol}(\mathfrak{m},0)=\mathrm{ad}\,\pi_{\mathfrak{h}}([\mathfrak{m},\mathfrak{m}])$, which equals $D(\mathfrak{m},\mathfrak{m})$ for the reductive decompositions $\mathfrak{g}(\mathfrak{m})=D(\mathfrak{m},\mathfrak{m})\oplus\mathfrak{m}$ of any Lie–Yamaguti algebra.

For the natural connection, $\alpha_x=\frac{1}{2}L_x$ for any $x\in\mathfrak{m}$, so that

$$R(x,y)=\frac{1}{4}[L_x,L_y]-\frac{1}{2}L_{x\cdot y}-D(x,y)$$

for any $x,y\in\mathfrak{m}$.

In all our reductive pairs $(\mathfrak{g}^i,\mathfrak{h}^i)$, $i=2,\ldots,8$, $D(\mathfrak{m}^i,\mathfrak{m}^i)\subseteq\mathfrak{so}(\mathfrak{m}^i,\kappa)=\mathrm{Lie}(\mathfrak{m}^i,\cdot)$, so $\mathfrak{hol}(\mathfrak{m}^i,\alpha)$ is contained in $\mathrm{Lie}(\mathfrak{m}^i,\cdot)$ and, by its very own definition, $\mathfrak{hol}(\mathfrak{m}^i,\alpha)$ is an ideal of the simple Lie algebra $\mathrm{Lie}(\mathfrak{m}^i,\cdot)=\mathfrak{so}(\mathfrak{m}^i,\kappa)$. Simple case by case considerations show that $\mathfrak{hol}(\mathfrak{m}^i,\alpha)\neq 0$ for any i, and hence:

Proposition 5.10. *For $\alpha(x, y) = \frac{1}{2}x \cdot y$, the holonomy algebras of the reductive decompositions $\mathfrak{g}_2 = \mathfrak{h}^i \oplus \mathfrak{m}^i$, $i = 1, \ldots, 8$ are given by*:

$$\mathfrak{hol}(\mathfrak{m}^i, \alpha) = \begin{cases} \operatorname{ad} \mathfrak{h}^1|_{\mathfrak{m}^1} & \textit{for } i = 1, \\ \mathfrak{so}(\mathfrak{m}^i, \kappa) & \textit{for } i \neq 1. \end{cases}$$

6. Concluding remarks

The large variety of Lie–Yamaguti algebras that appear in this paper suggests that some restrictions have to be imposed in order to obtain general results on these algebras.

A natural restriction is to consider those Lie–Yamaguti algebras $(\mathfrak{m}, \cdot, [\ ,\ ,\ ])$ which are irreducible modules for their *inner derivation Lie algebras* $D(\mathfrak{m}, \mathfrak{m})$ [2]. This irreducibility condition is more restrictive than the simplicity one. Geometrically, these are the Lie–Yamaguti algebras related to the irreducible homogeneous spaces [25]. In work in progress, it has been proved that these *irreducible* Lie–Yamaguti algebras are tightly related to other algebraic systems, like Jordan pairs and Freudenthal triple systems.

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