Available online at www.sciencedirect.com
ScienceDirect

Advances in Mathematics 208 (2007) 834–876

ADVANCES IN Mathematics

www.elsevier.com/locate/aim

Algebras, hyperalgebras, nonassociative bialgebras and loops

José M. Pérez-Izquierdo [1]

Departamento de Matemáticas y Computación, Universidad de La Rioja, 26004 Logroño, Spain

Received 28 July 2004; accepted 3 April 2006

Available online 6 May 2006

Communicated by The Managing Editors

Abstract

Sabinin algebras are a broad generalization of Lie algebras that include Lie, Malcev and Bol algebras as very particular examples. We present a construction of a universal enveloping algebra for Sabinin algebras, and the corresponding Poincaré–Birkhoff–Witt Theorem. A nonassociative counterpart of Hopf algebras is also introduced and a version of the Milnor–Moore Theorem is proved. Loop algebras and universal enveloping algebras of Sabinin algebras are natural examples of these nonassociative Hopf algebras. Identities of loops move to identities of nonassociative Hopf algebras by a linearizing process. In this way, nonassociative algebras and Hopf algebras interlace smoothly.

Keywords: Nonassociative algebras; Loops; Hopf algebras; Hyperalgebras; Poincaré–Birkhoff–Witt Theorem

1. Introduction

Hyperalgebras were introduced by L.V. Sabinin and P.O. Miheev in [22,23]. At the V International Conference "Nonassociative Algebra and Its Applications" celebrated in Oaxtepec (Mexico) from July 27th to August 2nd (2003), and after a public proposal of L. Bokut there was a general agreement in renaming these structures as Sabinin algebras. With the kind agreement[2] *of Professor Miheev, in this paper we shall follow this convention.*

E-mail address: jm.perez@dmc.unirioja.es.
[1] Supported by MCYT (BFM 2001-3239-C03-02) and the Comunidad Autónoma de La Rioja (ANGI 2001/26).
[2] Private correspondence with Professor J. Mostovoy.

0001-8708/$ – see front matter
doi:10.1016/j.aim.2006.04.001

Roughly speaking, a quasigroup is a nonempty set Q with a binary operation $\cdot : Q \times Q \to Q$ such that for any $a \in Q$ the left and right multiplication operators by a are bijective. In case that there exists $e \in Q$, the identity element, with $ea = a = ae$ for any $a \in Q$ then $(Q, \cdot, e)$ is called a loop. Therefore, a loop is the nonassociative counterpart of a group.

However, in the same way that the inverse map is considered as an essential part of the group structure, for any quasigroup or loop it is natural to incorporate the left and right divisions $\backslash : Q \times Q \to Q$ and $/ : Q \times Q \to Q$ given by $a\backslash b = L_a^{-1}(b)$ and $a/b = R_b^{-1}(a)$ in the definition. Hence we arrive at the formal definition of quasigroups and loops:

Definition 1. A quasigroup $(Q, \cdot, \backslash, /)$ is a nonempty set with three binary operations $\cdot$ (multiplication), $\backslash$ (left division) and $/$ (right division) such that

$$a\backslash(ab) = b, \qquad a(a\backslash b) = b, \qquad (ab)/b = a \quad \text{and} \quad (a/b)b = a$$

for any $a, b \in Q$. In case that $a\backslash a = b/b$ for any $a, b \in Q$ we say that the quasigroup is a loop.

The definition of loop is equivalent to imposing the existence of an identity element in the underlying quasigroup. To emphasize the existence of this element it is usual to write $(Q, \cdot, \backslash, /, e)$ for a loop.

One of the most celebrated results in Mathematics is the correspondence between Lie algebras and local Lie groups. However, in [12] Malcev showed that this correspondence may appear even when the associativity is removed. Malcev studied the relationship between local analytic Moufang loops and Malcev algebras and proved that the tangent space on the unit to any local analytic Moufang loop inherits the structure of a Malcev algebra. The converse is a result of Kuz'min [8]. These results were extended by Sabinin and Miheev [21] to local analytic Bol loops and Bol algebras. At this point it came up that two operations on the tangent space were needed to study local Bol loops, and a more general task emerged: to define an algebraic structure on the tangent space of any local analytic loop so that the Lie correspondence holds. This was achieved in [22,23] and the algebraic structures considered there are what now we call Sabinin algebras. In the meantime Akivis algebras arose from the study of multidimensional three-webs and were considered as a potential candidate that fails to provide the third converse Lie theorem. In [1] a summary of results on quasigroups related with webs and Akivis algebras is presented.

A vector space V is called a Sabinin algebra if it is endowed with multilinear operations

$$\langle x_1, x_2, \dots, x_m; y, z\rangle, \quad m \geqslant 0, \quad \text{and}$$

$$\Phi(x_1, x_2, \dots, x_m; y_1, y_2, \dots, y_n), \quad m \geqslant 1,\ n \geqslant 2,$$

which satisfy the identities

$$\langle x_1, x_2, \dots, x_m; y, z\rangle = -\langle x_1, x_2, \dots, x_m; z, y\rangle,$$

$$\langle x_1, x_2, \dots, x_r, a, b, x_{r+1}, \dots, x_m; y, z\rangle - \langle x_1, x_2, \dots, x_r, b, a, x_{r+1}, \dots, x_m; y, z\rangle$$
$$+ \sum_{k=0}^{r} \sum_{\alpha} \big\langle x_{\alpha_1}, \dots, x_{\alpha_k}, \langle x_{\alpha_{k+1}}, \dots, x_{\alpha_r}; a, b\rangle, \dots, x_m; y, z\big\rangle = 0,$$

$$\sigma_{x,y,z}\left(\langle x_1,\dots,x_r,x;y,z\rangle+\sum_{k=0}^{r}\sum_{\alpha}\bigl\langle x_{\alpha_1},\dots,x_{\alpha_k};\langle x_{\alpha_{k+1}},\dots,x_{\alpha_r};y,z\rangle,x\bigr\rangle\right)=0$$

and

$$\Phi(x_1,\dots,x_m;y_1,\dots,y_n)=\Phi(x_{\tau(1)},\dots,x_{\tau(m)};y_{\delta(1)},\dots,y_{\delta(n)}),$$

where α runs the set of all bijections of the type $\alpha:\{1,2,\dots,r\}\to\{1,2,\dots,r\}$, $i\mapsto\alpha_i$, $\alpha_1<\alpha_2<\cdots<\alpha_k$, $\alpha_{k+1}<\cdots<\alpha_r$, $k=0,1,\dots,r$, $r\geqslant 0$, $\sigma_{x,y,z}$ denotes the cyclic sum by x,y,z; $\tau\in S_m$, $\delta\in S_n$ and S_l is the symmetric group on l symbols. The operations $\langle\,;\rangle$ and the so called *multioperator* Φ are independent and sometimes the term "Sabinin algebra" is used for a vector space equipped only with operations $\langle\,;\rangle$ satisfying the corresponding properties. In the basic examples (Lie, Malcev and Bol algebras) the multioperator vanishes.

This unappealing object turned out to be quite natural after the work of Shestakov and Umirbaev [29] where, as a continuation of the deep study of Akivis algebras in [27,28], they showed that in the same way that an associative algebra gives rise to a Lie algebra after skew-symmetrization, over fields of characteristic zero a nonassociative algebra C originates a Sabinin algebra $\text{УШ}(C)$ (formerly denoted by $G(C)$) after adequate manipulations. Moreover, in case that C is a (not necessarily associative) bialgebra then $\mathrm{Prim}(C)$, the primitive elements of C, is a Sabinin subalgebra of $\text{УШ}(C)$.

The basic question to face is the following [29]: *Does any Sabinin algebra appear as a Sabinin subalgebra of* $\text{УШ}(C)$ *for some algebra* C*?*

In this paper we provide an affirmative answer to this question. For any Sabinin algebra $(V,\langle\,;\rangle,\Phi)$ over a field of characteristic zero we construct an algebra, its universal enveloping algebra, $\mathrm{U}((V,\langle\,;\rangle,\Phi))$ such that $(V,\langle\,;\rangle,\Phi)$ is a Sabinin subalgebra of $\text{УШ}(\mathrm{U}((V,\langle\,;\rangle,\Phi)))$. The algebra $\mathrm{U}((V,\langle\,;\rangle,\Phi))$ is a (not necessarily associative) bialgebra and $\mathrm{Prim}(\mathrm{U}((V,\langle\,;\rangle,\Phi)))=V$.

Loosely speaking, our approach derives from the semidirect products of loops by their transassociants [20], a construction that stresses the role of certain maps in the theory of loops. Operators l in Section 5 are imported from that context. From this point of view, free nonassociative algebras do not play any special role in this paper, in detriment of a clear picture of the theory. Many important papers dealing with Milnor–Moore type theorems and nonassociative operations on bialgebras using combinatorial techniques on free nonassociative algebras such as planar binary trees have appeared. Dendriform algebras and their generalizations play an important role in this area (see, for instance, [9–11,18,19] and references therein). A nonassociative analogous of the Baker–Campbell–Hausdorff series has been studied providing formulas for projections on primitives [4]. The dimension of the space of primitive elements of degree n in the free nonassociative algebra on r free generators is known [2]. A vast and sharper generalization of the notion of nonassociative Hopf algebras and primitive operations in terms of operads and props is the topic of [6]. Important results on nilpotency of groups have been extended to loops in [15].

In the present approach to Sabinin algebras a nonassociative counterpart of Hopf algebras comes into play. An H-bialgebra is a bialgebra with two extra operations $\backslash$ and $/$ satisfying

$$\sum x_{(1)}\backslash(x_{(2)}y)=\epsilon(x)y=\sum x_{(1)}(x_{(2)}\backslash y)\quad\text{and}$$
$$\sum(yx_{(1)})/x_{(2)}=\epsilon(x)y=\sum(y/x_{(1)})x_{(2)}.$$

Natural H-bialgebras are the loop algebra of a loop and the universal enveloping algebras of Sabinin algebras. Hopf algebras are exactly the associative and coassociative unital H-bialgebras. The associative law $(xy)z = x(yz)$ is only one example of identity we may impose to an H-bialgebra. However, in working with universal enveloping algebras other not so trivial, but quite well placed in a "Hopf context," identities appear.

The universal enveloping algebras of some special Sabinin algebras have precursors. The generalized alternative nucleus $\mathrm{N_{alt}}(A)$ of an algebra A is defined as

$$\mathrm{N_{alt}}(A) = \{a \in A \mid (a, x, y) = -(x, a, y) = (x, y, a)\ \forall x, y \in A\},$$

where (x, y, z) stands for the associator. Regardless of the algebra A we start with, the generalized alternative nucleus is a Malcev algebra with the commutator product. Moreover, in [17] it is proved that given any Malcev algebra M over a field of characteristic $\neq 2, 3$ there exists a universal enveloping algebra $\mathrm{U}(M)$ such that M is a subalgebra of $\mathrm{N_{alt}}(\mathrm{U}(M))$. This universal enveloping algebra is isomorphic to the universal enveloping algebra constructed in this paper when considering the Malcev algebra as a Sabinin algebra. Probably, the most distinguished property of $\mathrm{U}(M)$ as H-bialgebra is that it satisfies the identity

$$\sum x_{(1)}\big(y(x_{(2)}z)\big) = \sum \big((x_{(1)}y)x_{(2)}\big)z.$$

In fact, if this identity holds in a unital coassociative H-bialgebra then the primitive elements form a Malcev algebra (they lay in the generalized alternative nucleus) and the group-like elements form a Moufang loop. The methods used for Malcev algebras extend to Bol algebras. The left generalized alternative nucleus of an algebra A is defined as

$$\mathrm{LN_{alt}}(A) = \{a \in A \mid (a, x, y) = -(x, a, y)\ \forall x, y \in A\}$$

and it is a Lie triple system with the triple product $[a, b, c] = a(bc) - b(ac) - c(ab - ba)$. Any subspace of V closed by this triple product and the commutator product $[a, b] = ab - ba$ forms a Bol algebra, and any Bol algebra $(V, [\ ,\ ,\ ], [\ ,\ ])$ over a field of characteristic not 2 is obtainable in this way from its universal enveloping algebra $\mathrm{U}(V)$ [16]. Moreover, $\mathrm{U}(V)$ is an H-bialgebra which satisfies the identity

$$\sum a_{(1)}\big(b(a_{(2)}c)\big) = \sum \big(a_{(1)}(ba_{(2)})\big)c.$$

Again, for any coassociative unital H-bialgebra satisfying this identity the primitive elements form a Bol algebra and the group-like elements form a Bol loop.

The universal enveloping algebras of Malcev and Bol algebras are nice examples of H-bialgebras satisfying certain identities, so the corresponding loop algebras are. In this way, nonassociative algebras and Hopf algebras interlace smoothly.

The definition of H-bialgebra is very much oriented to model the existing examples (universal enveloping algebras and loop algebras) and these are naturally cocommutative. Up to some extend, very roughly one might consider a "quantum loop" as a noncocommutative H-bialgebra. A sharper point of view arises from the study of identities. However, concrete examples and formulation deserve further research.

This paper is organized as follows. In Section 2 we prove some basic properties about H-bialgebras. In Section 3 we study in which sense identities on loops give rise to identities on

H-bialgebras allowing us to translate known results on loops to H-bialgebras. We also check that the universal enveloping algebras of Malcev and Bol algebras satisfy the identities previously mentioned. In Section 4 we properly introduce Sabinin algebras and the basic constructions of them from Lie algebras. The approach we adopt shows that the surprising construction of Shestakov and Umirbaev is quite natural indeed. The universal enveloping algebra is constructed in Section 5. The analogous of the Milnor–Moore Theorem is derived in this section. We also compare the existing universal enveloping algebras of Malcev algebras with the ones introduced in this paper. One proof of the Poincaré–Birkhoff–Witt Theorem for Sabinin algebras using Gröbner (or Gröbner–Shirshov) bases is given in Appendix A.

2. Nonassociative bialgebras

A *coalgebra* (C, Δ, ϵ) is a vector space C over a field F equipped with two maps

$$\Delta : C \to C \otimes C \quad \text{and} \quad \epsilon : C \to F$$

such that

$$(\mathrm{Id} \otimes \epsilon)\Delta = \mathrm{Id} = (\epsilon \otimes \mathrm{Id})\Delta,$$

where the natural identification $C \otimes F \cong C \cong F \otimes C$ is assumed. The coalgebra (C, Δ, ϵ) is called *coassociative* if $(\Delta \otimes \mathrm{Id})\Delta = (\mathrm{Id} \otimes \Delta)\Delta$, and it is called *cocommutative* if $\tau\Delta = \Delta$ where $\tau(x \otimes y) = y \otimes x$. It is customary to write

$$\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)} \quad \text{or} \quad \Delta(x) = \sum x_{(1)} \otimes x_{(2)}.$$

A (nonunital) *bialgebra* $(B, \Delta, \epsilon, \cdot)$ is a coalgebra (B, Δ, ϵ) with a bilinear product

$$\begin{aligned} \cdot : C \times C &\to C, \\ (x, y) &\mapsto xy \end{aligned}$$

such that

$$\Delta(xy) = \sum x_{(1)} y_{(1)} \otimes x_{(2)} y_{(2)} \quad \text{and} \quad \epsilon(xy) = \epsilon(x)\epsilon(y).$$

A *unital bialgebra* $(B, \Delta, \epsilon, \cdot, u)$ is a bialgebra $(B, \Delta, \epsilon, \cdot)$ equipped with a linear map $u : F \to B$, the unit, such that

$$1x = x = x1, \quad \Delta(1) = 1 \otimes 1 \quad \text{and} \quad \epsilon(1) = 1,$$

where $1 = u(1) \in B$.

Definition 2. An H-bialgebra $(H, \Delta, \epsilon, \cdot, \backslash, /)$ is a bialgebra $(H, \Delta, \epsilon, \cdot)$ with two extra bilinear operations, the left and right division,

$$\backslash : H \times H \to H, \qquad / : H \times H \to H,$$
$$(x, y) \mapsto x\backslash y, \qquad (x, y) \mapsto x/y$$

such that

$$\sum x_{(1)}\backslash(x_{(2)}y) = \epsilon(x)y = \sum x_{(1)}(x_{(2)}\backslash y) \quad \text{and} \tag{1}$$

$$\sum (yx_{(1)})/x_{(2)} = \epsilon(x)y = \sum (y/x_{(1)})x_{(2)}. \tag{2}$$

A unital H-bialgebra $(H, \Delta, \epsilon, \cdot, u, \backslash, /)$ is a unital bialgebra $(H, \Delta, \epsilon, \cdot, u)$ such that $(H, \Delta, \epsilon, \cdot, \backslash, /)$ is an H-bialgebra.

Proposition 3. *An associative and coassociative unital H-bialgebra is a Hopf algebra. Moreover, in this case $x\backslash y = S(x)y$ and $x/y = xS(y)$, where S denotes the antipode.*

Proof. The natural candidate to be the antipode is $S(x) = x\backslash 1$. In fact, by (1) we have that

$$\sum x_{(1)}S(x_{(2)}) = \epsilon(x)1,$$

and by the associativity and coassociativity

$$\sum x_{(1)}\backslash\big(x_{(2)} \cdot S(x_{(3)})y\big) = \begin{cases} \sum x_{(1)}\backslash(\epsilon(x_{(2)})y) = x\backslash y, \\ \sum \epsilon(x_{(1)})S(x_{(2)})y = S(x)y \end{cases}$$

so

$$S(x)y = x\backslash y.$$

In particular,

$$\sum S(x_{(1)})x_{(2)} = \sum x_{(1)}\backslash x_{(2)} = \sum x_{(1)}\backslash(x_{(2)} \cdot 1) = \epsilon(x)1,$$

hence S is the antipode. The last relation $x/y = xS(y)$ follows easily. □

We make the linear operators to act from the left, however from Section 4 on we will make them to act from the right for coherence with [29].

Given a coalgebra (C, Δ, ϵ), the vector space $\mathrm{Hom}(C, \mathrm{End}(C))$ is a unital algebra with the product given by the convolution

$$(\varphi * \psi)(x) = \sum \varphi(x_{(1)})\psi(x_{(2)})$$

and unit element $\iota : x \mapsto \epsilon(x)\,\mathrm{Id}$. In case that the coalgebra C is coassociative then $\mathrm{Hom}(C, \mathrm{End}(C))$ is an associative algebra. The multiplicative structure of a bialgebra $(B, \Delta, \epsilon, \cdot)$ is determined by the elements of $\mathrm{Hom}(B, \mathrm{End}(B))$

$$L : B \to \mathrm{End}(B), \qquad R : B \to \mathrm{End}(B),$$
$$x \mapsto L_x, \qquad x \mapsto R_x.$$

Although obviously it suffices one of these maps to determine the multiplicative structure, we prefer to present both of them to emphasize their importance on the existence of left and right division respectively.

Proposition 4. *Let $(B, \Delta, \epsilon, \cdot)$ be a coassociative bialgebra. There exist maps $\backslash : B \times B \to B$ and $/ : B \times B \to B$ such that $(B, \Delta, \epsilon, \cdot, \backslash, /)$ is an H-bialgebra if and only if the elements L and R are invertible in* $\mathrm{Hom}(B, \mathrm{End}(B))$. *In that case the operations $\backslash$ and $/$ are uniquely determined.*

Proof. By hypothesis $\mathrm{Hom}(B, \mathrm{End}(B))$ is associative. The element L is invertible in this algebra if and only if there exists $LD : B \to \mathrm{End}(B)$ such that

$$\sum LD(x_{(1)})L(x_{(2)}) = \epsilon(x)\,\mathrm{Id} = \sum L(x_{(1)})LD(x_{(2)}),$$

or equivalently

$$\sum LD(x_{(1)})(x_{(2)}y) = \epsilon(x)y = \sum x_{(1)}LD(x_{(2)})(y),$$

and in that case the inverse LD is unique. Comparing this identity with (1) we obtain the desired result about the existence and uniqueness of $\backslash$. One proceeds with $/$ similarly. □

Recall that a coalgebra is called *connected* if the dimension of the coradical is one.

Proposition 5. *Let $(H, \Delta, \epsilon, \cdot, u)$ be a coassociative unital bialgebra. If the coalgebra (H, Δ, ϵ) is connected then we may define on $(H, \Delta, \epsilon, \cdot, u)$ a (unique) structure of H-bialgebra.*

Proof. Let us show that R, L are invertible in $\mathrm{Hom}(H, \mathrm{End}(H))$. By a result of Takeuchi [30] we only need to check that their restriction to $H_0 = F1$ (the coradical of H) are invertible in $\mathrm{Hom}(H_0, \mathrm{End}(H))$, but this is obvious. □

Proposition 6. *Let $(H, \Delta, \epsilon, \cdot, \backslash, /)$ be a coassociative H-bialgebra. We have that*

$$\Delta(x\backslash y) = \sum x_{(2)}\backslash y_{(1)} \otimes x_{(1)}\backslash y_{(2)} \quad \textit{and} \quad \Delta(x/y) = \sum x_{(1)}/y_{(2)} \otimes x_{(2)}/y_{(1)}.$$

Proof. Consider $A = \mathrm{Hom}(H, \mathrm{End}(H \otimes H))$. With the convolution A is an associative algebra with unit element $\iota : x \mapsto \epsilon(x)\,\mathrm{Id}$ where Id denotes the identity map on $H \otimes H$. Let $V = \mathrm{End}(H \otimes H)$ and define

$$\begin{aligned} &A \times V \to V, \\ &(\varphi, f) \mapsto \varphi f : x \otimes y \mapsto \sum (\varphi_{x_{(1)}} f)(x_{(2)} \otimes y). \end{aligned}$$

With this action V becomes a unital A-module. Let us now define $\varphi \in A$ and $f, g \in V$ by

$$\begin{aligned} &\varphi : x \mapsto \varphi_x = \sum L_{x_{(1)}} \otimes L_{x_{(2)}}, \\ &f : x \otimes y \mapsto \Delta(x\backslash y) \quad \text{and} \\ &g : x \otimes y \mapsto \sum x_{(2)}\backslash y_{(1)} \otimes x_{(1)}\backslash y_{(2)}. \end{aligned}$$

The first formula in the statement will follow once we had proved that φ is invertible in A and that $\varphi f = \varphi g$. To show that φ is invertible we define $\varphi^{-1} : x \mapsto \varphi_x^{-1} : y \otimes z \mapsto \sum x_{(2)} \backslash y \otimes x_{(1)} \backslash z$. We have that

$$\left(\varphi * \varphi^{-1}\right)_x (y \otimes z) = \sum \left(\varphi_{x_{(1)}} \varphi^{-1}_{x_{(2)}}\right)(y \otimes z) = \sum (L_{x_{(1)}} \otimes L_{x_{(2)}})(x_{(4)} \backslash y \otimes x_{(3)} \backslash z)$$
$$= \epsilon(x) y \otimes z$$

and similarly $(\varphi^{-1} * \varphi)_x = \epsilon(x)\,\mathrm{Id}$, therefore φ^{-1} is the inverse of φ in A. Finally,

$$(\varphi g)(x \otimes y) = \sum \varphi_{x_{(1)}} g(x_{(2)} \otimes y) = \sum (x_{(1)} \otimes x_{(2)})(x_{(4)} \backslash y_{(1)} \otimes x_{(3)} \backslash y_{(2)})$$
$$= \sum x_{(1)}(x_{(4)} \backslash y_{(1)}) \otimes x_{(2)}(x_{(3)} \backslash y_{(2)}) = \epsilon(x) y_{(1)} \otimes y_{(2)}$$
$$= \epsilon(x) \Delta(y)$$

and

$$(\varphi f)(x \otimes y) = \sum (x_{(1)} \otimes x_{(2)}) \Delta(x_{(3)} \backslash y) = \sum \Delta\left(x_{(1)}(x_{(2)} \backslash y)\right) = \epsilon(x) \Delta(y).$$

This proves that $\Delta(x \backslash y) = \sum x_{(2)} \backslash y_{(1)} \otimes x_{(1)} \backslash y_{(2)}$. The proof of the second identity in the statement is similar. □

Proposition 7. *In any H-bialgebra* $(H, \Delta, \epsilon, \cdot, \backslash, /)$

$$\epsilon(x \backslash y) = \epsilon(x)\epsilon(y) = \epsilon(x/y).$$

Proof. The first identity follows from

$$\epsilon(x \backslash y) = \epsilon\left(\left(\sum \epsilon(x_{(1)}) x_{(2)}\right) \backslash y\right) = \sum \epsilon(x_{(1)}) \epsilon(x_{(2)} \backslash y) = \sum \epsilon\left(x_{(1)}(x_{(2)} \backslash y)\right)$$
$$= \epsilon\left(\epsilon(x) y\right) = \epsilon(x)\epsilon(y).$$

Similarly $\epsilon(x/y) = \epsilon(x)\epsilon(y)$. □

Corollary 8. *Let* $(H, \Delta, \epsilon, \cdot, \backslash, /)$ *be a coassociative H-bialgebra such that set of group elements*

$$\mathrm{G}(H) = \left\{a \in H \mid \Delta(a) = a \otimes a \text{ and } \epsilon(a) = 1\right\}$$

is nonempty. Then $(\mathrm{G}(H), \cdot, \backslash, /)$ *is a quasigroup.*

Proof. By Propositions 6 and 7, $\mathrm{G}(H)$ is closed under $\backslash$ and $/$. Moreover, (1) and (2) imply that

$$a \backslash (ax) = x = a(a \backslash x) \quad \text{and} \quad (xa)/a = x = (x/a)a$$

for any $a \in \mathrm{G}(H)$, therefore $(\mathrm{G}(H), \cdot, \backslash, /)$ is a quasigroup. □

Corollary 9. *For any coassociative unital H-bialgebra* $(H, \Delta, \epsilon, \cdot, u, \backslash, /)$ *the set* $(\mathrm{G}(H), \cdot, \backslash, /, u)$ *is a loop.*

3. Linearizing identities[3]

The study of varieties of quasigroups is one cornerstone of the theory of quasigroups. Attached to any set of identities for quasigroups there exists a set of "identities" for cocommutative coassociative H-bialgebras. For instance, corresponding to the left Moufang identity

$$x\big(y(xz)\big) = \big((xy)x\big)z$$

on quasigroups we have the "linearization"

$$\sum x_{(1)}\big(y(x_{(2)}z)\big) = \sum \big((x_{(1)}y)x_{(2)}\big)z \tag{3}$$

for H-bialgebras, and corresponding to the right Moufang identity

$$\big((zx)y\big)x = z\big(x(yx)\big)$$

on quasigroups we have the linearization

$$\sum \big((zx_{(1)})y\big)x_{(2)} = \sum z\big(x_{(1)}(yx_{(2)})\big) \tag{4}$$

for H-bialgebras. However, and this is the motivation for this section, it is known that for any quasigroup the left and right Moufang identities are equivalent, so one may wonder whether for any cocommutative coassociative H-bialgebra, (3) and (4) are equivalent. The goal of this section is to prove that if an identity for quasigroups is a consequence of other identities then the linearization of the former is a consequence of the linearizations of the later.

3.1. A linearizing process

In this subsection $\mathcal{F}$ will denote a type of algebras (see [3]) and (C, Δ, ϵ) will be a coassociative and cocommutative coalgebra such that C is an $\mathcal{F}$-algebra and any operation $\{f, k\} \in \mathcal{F}$ induces a homomorphism of coalgebras $f : C^{\otimes k} \to C$ (if $k = 0$ then $f : F \to C$).

Since C is an $\mathcal{F}$-algebra, then for any coassociative coalgebra V, the vector space $\mathrm{Hom}(V, C)$ is an $\mathcal{F}$-algebra by

$$f(\alpha_1, \dots, \alpha_n)(v) = \sum f\big(\alpha_1(v_{(1)}), \dots, \alpha_n(v_{(n)})\big), \quad v \in V,$$

in case that $n \geqslant 1$, and $f(1)(v) = \epsilon(v)f(1)$ if $n = 0$. If we consider the coalgebra $C^{\otimes N}$ $(N \geqslant 1)$ then we have the distinguished maps

$$\epsilon_i(a_1 \otimes \dots \otimes a_N) = \epsilon_i(a_1, \dots, a_N) = \epsilon(a_1) \cdots \widehat{\epsilon(a_i)} \cdots \epsilon(a_N)a_i,$$

where $\widehat{\epsilon(a_i)}$ means that this factor is omitted, and $1 \leqslant i \leqslant N$. Therefore, if $n \leqslant N$ there exists a homomorphism $l_{n,N}$ (the linearizing map) from the term algebra $T(X)$ (see [3]) on $X = \{x_1, x_2, \dots, x_n\}$ over $\mathcal{F}$ to $\mathrm{Hom}(C^{\otimes N}, C)$ sending x_i to ϵ_i.

---

[3] The presentation we adopt is a great simplification of the original arguments. It was written while the author was collaborating with Professor J. Mostovoy.

Definition 10. Given $p, q \in T(X)$ with $X = \{x_1, x_2, \dots, x_n\}$, we will say that C satisfies the linearization of the identity $p \approx q$ if $l_{n,N}(p) = l_{n,N}(q)$ for some $N \geqslant n$.

Note 1. It is convenient to remark that $l_{n,N}(p) = l_{n,N}(q)$ for some $N \geqslant n$ iff $l_{n,N}(p) = l_{n,N}(q)$ for all $N \geqslant n$. So, if we set $l_n = l_{n,n}$ then C satisfies the linearization of the identity $p \approx q$ iff $l_n(p) = l_n(q)$.

Example 11. Consider $\mathcal{F} = \{\{\cdot, 2\}, \{\backslash, 2\}, \{/, 2\}\}$, $p = x_1(x_2(x_1x_3))$ and $q = ((x_1x_2)x_1)x_3$. We have that $l_3(p)(a \otimes b \otimes c) = \sum a_{(1)}(b(a_{(2)}c))$ and $l_3(q)(a \otimes b \otimes c) = \sum((a_{(1)}b)a_{(2)})c$. Thus, C satisfies the linearization of the Moufang identity iff

$$\sum a_{(1)}\big(b(a_{(2)}c)\big) = \sum\big((a_{(1)}b)a_{(2)}\big)c$$

holds on C.

We will denote the set of all coalgebra maps from a coalgebra V to C by $\mathrm{Coalg}(V, C)$. This set is an $\mathcal{F}$-subalgebra of $\mathrm{Hom}(V, C)$.

Lemma 12. *Let V be a coassociative and cocommutative coalgebra. Given $p(x_1, \dots, x_n) \in T(X)$, $n \geqslant 1$, and $\alpha_1, \dots, \alpha_n \in \mathrm{Coalg}(V, C)$ we have that*

$$p(\alpha_1, \dots, \alpha_n) = l_n(p) \circ (\alpha_1 \otimes \dots \otimes \alpha_n) \circ \Delta^n.$$

Proof. We proceed by induction on the length of p, the number of operation symbols in p from $\mathcal{F}$ (see [3] for details). If $p(x_1, \dots, x_n) = x_i$ $(1 \leqslant i \leqslant n)$ then

$$\sum l_n(p)\big(\alpha_1(v_{(1)}) \otimes \dots \otimes \alpha_n(v_{(n)})\big) = \alpha_i(v) = p(\alpha_1, \dots, \alpha_n)(v).$$

In general, given an expression $p(x_1, \dots, x_n) \in T(X)$ with $p(x_1, \dots, x_n) = f(r_1(x_1, \dots, x_n), \dots, r_k(x_1, \dots, x_n))$ for some $r_1, \dots, r_k \in T(X)$ and $\{f, k\} \in \mathcal{F}$ then

$$\begin{aligned}
&\sum l_n(p)\big(\alpha_1(v_{(1)}) \otimes \dots \otimes \alpha_n(v_{(n)})\big) \\
&\overset{\langle 1\rangle}{=} \sum f\big(l_n(r_1), \dots, l_n(r_k)\big)\big(\alpha_1(v_{(1)}) \otimes \dots \otimes \alpha_n(v_{(n)})\big) \\
&\overset{\langle 2\rangle}{=} \sum f\big(l_n(r_1)\big(\alpha_1(v_{(1)}) \otimes \dots \otimes \alpha_n(v_{(n)})\big), \dots, l_n(r_k)\big(\alpha_1(v_{(n(k-1)+1)}) \otimes \dots \otimes \alpha_n(v_{(kn)})\big)\big) \\
&\overset{\langle 3\rangle}{=} \sum f\big(r_1(\alpha_1, \dots, \alpha_n)(v_{(1)}), \dots, r_k(\alpha_1, \dots, \alpha_n)(v_{(k)})\big) \\
&= p(\alpha_1, \dots, \alpha_n)(v),
\end{aligned}$$

where in $\langle 1\rangle$ we have used that l_n is a homomorphism of $\mathcal{F}$-algebras, in $\langle 2\rangle$ we use that $\alpha_1, \dots, \alpha_n$ are coalgebra maps, the structure of $\mathcal{F}$-algebra on $\mathrm{Hom}(V, C)$ and the cocommutativity of V, and $\langle 3\rangle$ follows from the hypothesis of induction. $\square$

Theorem 13. *Let Σ be a set of identities and $p \approx q$ a consequence of Σ. If C satisfies the linearization of all identities in Σ, then C satisfies the linearization of $p \approx q$.*

Proof. Since $p \approx q$ may be derived from a finite subset of Σ (see [3]) we may assume that all the terms involved in the identities we use belong to $T(\{x_1, \dots, x_n\})$ for some n big enough. Then by Lemma 12 we have that $\mathrm{Coalg}(C^{\otimes n}, C)$ satisfies the identities in Σ, so it also satisfies the identity $p \approx q$. Since $(\epsilon_1 \otimes \cdots \otimes \epsilon_n) \circ \Delta^n(c_1 \otimes \cdots \otimes c_n) = c_1 \otimes \cdots \otimes c_n$ for any $c_1, \dots, c_n \in C$, then again Lemma 12 implies that $l_n(p) = l_n(q)$. □

We close this subsection with an example of application of Theorem 13 to obtain new results about H-bialgebras. Many other results may be obtained with the same techniques.

A quasigroup is an $\mathcal{F}$-algebra, with $\mathcal{F} = \{\{\cdot, 2\}, \{\backslash, 2\}, \{/, 2\}\}$, that satisfies the identities

$$a\backslash(ab) \approx b, \quad a(a\backslash b) \approx b, \quad (ba)/a \approx b \quad \text{and} \quad (b/a)a \approx b.$$

By Propositions 6 and 7 any cocommutative and coassociative H-bialgebra is an $\mathcal{F}$-algebra, the operations being homomorphisms of coalgebras, and by definition it satisfies the linearizations of these identities. V. Shcherbacov and V. Izbash [26] (the result was announced in 1993) and independently K. Kunen [7] with the help of the automated deduction tool OTTER proved that the identities

$$\begin{aligned} &M1\colon \quad \big(a(bc)\big)a \approx (ab)(ca), \qquad && M2\colon \quad (ab)(ca) \approx a\big((bc)a\big), \\ &N1\colon \quad \big((ba)c\big)a \approx b\big(a(ca)\big), \qquad && N2\colon \quad \big((ab)a\big)c \approx a\big(b(ac)\big) \end{aligned} \tag{5}$$

are equivalent for quasigroups. Moreover, any of them implies the existence of two-sided unit element, which expressed in terms of identities means that

$$a\backslash a \approx b/b \tag{6}$$

holds, $a\backslash a$ being the identity element for any a. Therefore, they are Moufang loops. For long time it has been known that these loops satisfy

$$a\backslash b \approx \big(a\backslash(c\backslash c)\big)b, \quad b/a \approx b\big(a\backslash(c\backslash c)\big) \quad \text{and} \quad (ab)\backslash(c\backslash c) \approx \big(b\backslash(c\backslash c)\big)\big(a\backslash(c\backslash c)\big),$$

that with the notation $a^{-1} = a\backslash(c\backslash c)$ become the familiar identities

$$a\backslash b \approx a^{-1}b, \quad b/a \approx ba^{-1} \quad \text{and} \quad (ab)^{-1} \approx b^{-1}a^{-1}. \tag{7}$$

Let us now fix a cocommutative and coassociative nonunital H-bialgebra H. The linearizations of (5) are

$$\begin{aligned} M1\colon &\quad \sum \big(a_{(1)}(bc)\big)a_{(2)} = \sum (a_{(1)}b)(ca_{(2)}), \\ M2\colon &\quad \sum (a_{(1)}b)(ca_{(2)}) = \sum a_{(1)}\big((bc)a_{(2)}\big), \\ N1\colon &\quad \sum \big((ba_{(1)})c\big)a_{(2)} = \sum b\big(a_{(1)}(ca_{(2)})\big), \\ N2\colon &\quad \sum \big((a_{(1)}b)a_{(2)}\big)c = \sum a_{(1)}\big(b(a_{(2)}c)\big). \end{aligned} \tag{8}$$

By Theorem 13 and the result from Shcherbacov, Izbash and Kunen, these identities are equivalent on H. Moreover, if H satisfies any of these identities then it also satisfies the linearization of (6)

$$\epsilon(b)\sum a_{(1)}\backslash a_{(2)} = \epsilon(a)\sum b_{(1)}/b_{(2)} \tag{9}$$

and the element $1 = \sum c_{(1)}\backslash c_{(2)}$ with $\epsilon(c) = 1$ is the identity of H. Therefore, H is a unital H-bialgebra. Furthermore, if we define $S(a) = a\backslash 1$ then the linearization of (7) becomes

$$a\backslash b = S(a)b, \quad b/a = bS(a) \quad \text{and} \quad S(ab) = S(b)S(a). \tag{10}$$

In particular,

$$\sum S(a_{(1)})a_{(2)} = \epsilon(a)1 = \sum a_{(1)}S(a_{(2)})$$

and we have an analogue of the antipode of Hopf algebras.

3.2. Envelopes of Malcev and Bol algebras (I)

Recall that a Malcev algebra $(M, [\,,\,])$ over a field of characteristic $\neq 2$ is a vector space with a skew-symmetric product such that

$$\bigl[J(x, y, z), x\bigr] = J\bigl(x, y, [x, z]\bigr),$$

where $J(x, y, z) = [[x, y], z] - [[x, z], y] - [x, [y, z]]$ is the Jacobian of x, y and z. These algebras appear as tangent spaces of smooth Moufang loops.

A Lie triple system $(V, [\,,\,,\,])$ is a vector space with a trilinear operation such that

$$[a, a, b] = 0,$$
$$[a, b, c] + [b, c, a] + [c, a, b] = 0 \quad \text{and}$$
$$\bigl[x, y[a, b, c]\bigr] = \bigl[[x, y, a], b, c\bigr] + \bigl[a, [x, y, b], c\bigr] + \bigl[a, b, [x, y, c]\bigr].$$

A (left) Bol algebra $(V, [\,,\,,\,], [\,,\,])$ is a Lie triple system $(V, [\,,\,,\,])$ with an additional bilinear skew-symmetric operation $[a, b]$ satisfying

$$\bigl[a, b, [c, d]\bigr] = \bigl[[a, b, c], d\bigr] + \bigl[c, [a, b, d]\bigr] + \bigl[c, d, [a, b]\bigr] + \bigl[[a, b], [c, d]\bigr].$$

Left Bol algebras appear as tangent spaces of smooth left Bol loops, that is, loops that satisfy the left Bol identity

$$a\bigl(b(ac)\bigr) = \bigl(a(ba)\bigr)c.$$

Malcev and Bol algebras are very much related since any Malcev algebra (over a field of characteristic $\neq 2, 3$) is a left Bol algebra with its bilinear product and the triple product given by

$$[a,b,c]=\big[[a,b],c\big]-\frac{1}{3}J(a,b,c).$$

In [16], for any left Bol algebra $(V,[\,,\,,\,],[\,,\,])$ over a field of characteristic $\neq 2$ a universal enveloping algebra $\mathrm{U}(V)$ is constructed in such a way that $V\subseteq \mathrm{U}(V)$, the operations in V are recovered as

$$[a,b]=ab-ba,$$
$$[a,b,c]=a(bc)-b(ac)-c(ab-ba)$$

and $V\subseteq \mathrm{LN}_{\mathrm{alt}}(\mathrm{U}(V))=\{a\in \mathrm{U}(V)\mid (a,x,y)=-(x,a,y)\ \forall x,y\in \mathrm{U}(V)\}$.

A similar result was previously proved for Malcev algebras [17]. Given a Malcev algebra $(M,[\,,\,])$ over a field of characteristic $\neq 2,3$ then there exists an algebra $\mathrm{U}(M)$ such that $M\subseteq \mathrm{U}(M)$, the operation on M is recovered as

$$[a,b]=ab-ba$$

and $M\subseteq \mathrm{N}_{\mathrm{alt}}(\mathrm{U}(M))=\{a\in \mathrm{U}(M)\mid (a,x,y)=-(x,a,y)=(x,y,a)\ \forall x,y\in \mathrm{U}(M)\}$. If we consider M as a left Bol algebra then the envelopes as Malcev and Bol algebra are isomorphic. These envelopes are in fact connected cocommutative and coassociative unital bialgebras. Therefore, by Proposition 5 they are H-bialgebras. Moreover,

Theorem 14. *Let $(V,[\,,\,,\,],[\,,\,])$ be a left Bol algebra over a field of characteristic $\neq 2$ and $\mathrm{U}(V)$ its universal enveloping algebra. Then $\mathrm{U}(V)$ satisfies*

$$\sum a_{(1)}\big(b(a_{(2)}c)\big)=\sum \big(a_{(1)}(ba_{(2)})\big)c.$$

Proof. Recall that $\mathrm{U}(V)$ is a filtered algebra $\mathrm{U}(V)=\bigcup_{n=0}^{\infty}\mathrm{U}(V)_n$ with $\mathrm{U}(V)_n=\mathrm{span}\langle a_{i_1}(\cdots(a_{i_{n-1}}a_{i_n})\cdots)\mid a_{i_1},\dots,a_{i_n}\in V,\ n\in\mathbb{N}\rangle$ and the corresponding graded algebra $\mathrm{Gr}(\mathrm{U}(V))$ is commutative and associative.

Given $0\neq a\in \mathrm{U}(V)$, there exists n such that $a\in \mathrm{U}(V)_n$ but $a\notin \mathrm{U}(V)_{n-1}$. We will prove the result using induction on n. For $n=0$ the result is obvious. If $n=1$ then $a\in V$ and the formula follows from the fact that $a\in \mathrm{LN}_{\mathrm{alt}}(\mathrm{U}(V))$. Let us now assume that we have proved the result for any element on $\mathrm{U}(V)_n$ and let $v\in \mathrm{U}(V)_{n+1}$. Since $\mathrm{Gr}(\mathrm{U}(V))$ is commutative and associative we may assume that $v=u\circ a=ua+au$ with $u\in \mathrm{U}(V)_n$ and $a\in V$. We have

$$\begin{aligned}\sum L_{v_{(1)}}L_bL_{v_{(2)}}&=\sum L_{(a\circ u)_{(1)}}L_bL_{(a\circ u)_{(2)}}\\&=\sum L_{a\circ u_{(1)}}L_bL_{u_{(2)}}+L_{u_{(1)}}L_bL_{a\circ u_{(2)}}\\&=\sum (L_a\circ L_{u_{(1)}})L_bL_{u_{(2)}}+L_{u_{(1)}}L_b(L_a\circ L_{u_{(2)}})\\&=\sum L_aL_{u_{(1)}}L_bL_{u_{(2)}}+L_{u_{(1)}}L_aL_bL_{u_{(2)}}+L_{u_{(1)}}L_bL_{u_{(2)}}L_a+L_{u_{(1)}}L_bL_aL_{u_{(2)}}\\&=L_{\sum a\circ(u_{(1)}(bu_{(2)}))+u_{(1)}((a\circ b)u_{(2)})}.\end{aligned}$$

Evaluating these operators on 1 we obtain that $\sum a\circ(u_{(1)}(bu_{(2)}))+u_{(1)}((a\circ b)u_{(2)})=\sum v_{(1)}(bv_{(2)})$, so

$$\sum L_{v_{(1)}} L_b L_{v_{(2)}} = \sum L_{v_{(1)}(bv_{(2)})}$$

as desired. □

A similar result holds for Malcev algebras.

Theorem 15. *Let* $(M, [\,,\,])$ *be a Malcev algebra over a field of characteristic* $\neq 2, 3$ *and* $\mathrm{U}(M)$ *its universal enveloping algebra. Then* $\mathrm{U}(M)$ *satisfies*

$$\sum a_{(1)}\bigl(b(a_{(2)}c)\bigr) = \sum \bigl((a_{(1)}b)a_{(2)}\bigr)c.$$

Finally, we should remark that some properties of $\mathrm{U}(V)$ and $\mathrm{U}(M)$ follow by linearizing the corresponding properties of Bol and Moufang loops as seen in the previous subsection.

4. Sabinin algebras

Recall that a vector space V is called a Sabinin algebra if it is endowed with multilinear operations

$$\langle x_1, x_2, \dots, x_m; y, z\rangle, \quad m \geqslant 0,$$
$$\Phi(x_1, x_2, \dots, x_m; y_1, y_2, \dots, y_n), \quad m \geqslant 1,\ n \geqslant 2,$$

which satisfy the identities

$$\langle x_1, x_2, \dots, x_m; y, z\rangle = -\langle x_1, x_2, \dots, x_m; z, y\rangle,$$

$$\langle x_1, x_2, \dots, x_r, a, b, x_{r+1}, \dots, x_m; y, z\rangle - \langle x_1, x_2, \dots, x_r, b, a, x_{r+1}, \dots, x_m; y, z\rangle$$
$$+ \sum_{k=0}^{r} \sum_{\alpha} \bigl\langle x_{\alpha_1}, \dots, x_{\alpha_k}, \langle x_{\alpha_{k+1}}, \dots, x_{\alpha_r}; a, b\rangle, \dots, x_m; y, z\bigr\rangle = 0,$$

$$\sigma_{x,y,z}\langle x_1, \dots, x_r, x; y, z\rangle + \sum_{k=0}^{r} \sum_{\alpha} \bigl\langle x_{\alpha_1}, \dots, x_{\alpha_k}; \langle x_{\alpha_{k+1}}, \dots, x_{\alpha_r}; y, z\rangle, x\bigr\rangle = 0,$$

and

$$\Phi(x_1, \dots, x_m; y_1, \dots, y_n) = \Phi(x_{\tau(1)}, \dots, x_{\tau(m)}; y_{\delta(1)}, \dots, y_{\delta(n)}),$$

where α runs the set of all bijections of the type $\alpha : \{1, 2, \dots, r\} \to \{1, 2, \dots, r\}$, $i \mapsto \alpha_i$, $\alpha_1 < \alpha_2 < \cdots < \alpha_k$, $\alpha_{k+1} < \cdots < \alpha_r$, $k = 0, 1, \dots, r$, $r \geqslant 0$, $\sigma_{x,y,z}$ denotes the cyclic sum by x, y, z; $\tau \in S_m$, $\delta \in S_n$ and S_l is the symmetric group.

Let $\mathrm{T}(V)$ be the tensor algebra over V endowed with its natural structure of bialgebra, that is, $V \subseteq \mathrm{Prim}(\mathrm{T}(V))$. Looking at $\langle ; \rangle$ as a map

$$\langle ; \rangle : \mathrm{T}(V) \otimes V \otimes V \to V$$

we may write the definition of a Sabinin algebra very shortly as

$$\langle x; a, b\rangle + \langle x; b, a\rangle = 0, \tag{11}$$

$$\big\langle x[a, b]y; c, e\big\rangle + \sum \big\langle x_{(1)} \langle x_{(2)}; a, b\rangle y; c, e\big\rangle = 0, \tag{12}$$

$$\sigma_{a,b,c}\Big(\langle xc; a, b\rangle + \sum \big\langle x_{(1)}; \langle x_{(2)}; a, b\rangle, c\big\rangle\Big) = 0 \quad \text{and} \tag{13}$$

$$\Phi(x_1, \ldots, x_m; y_1, \ldots, y_n) = \Phi(x_{\tau(1)}, \ldots, x_{\tau(m)}; y_{\delta(1)}, \ldots, y_{\delta(n)}), \tag{14}$$

where as always we have used the Sweedler's sigma notation for the comultiplication [29].

4.1. Sabinin algebras from primitive elements (I)

The set of primitive elements of a bialgebra is not a subalgebra in the usual sense, however, in their striking work [29] Shestakov and Umirbaev proved that this set is closed under certain operations and it naturally becomes a Sabinin algebra. We devote this subsection to reviewing the, at this point, surprising construction of Shestakov and Umirbaev. The reader should be aware that some modifications have been made.

Let B be a free unital nonassociative algebra over $X \cup Y \cup \{z\}$ where $X = \{x_1, x_2, \ldots\}$ and $Y = \{y_1, y_2, \ldots\}$. By the universal property of B there exists a unique homomorphism of unital algebras

$$\begin{aligned} \Delta : B &\to B \otimes B, \\ w &\mapsto \sum w_{(1)} \otimes w_{(2)} \end{aligned}$$

such that $X \cup Y \cup \{z\} \subseteq \mathrm{Prim}(B)$. Given $x_1, \ldots, x_m \in X$, $y_1, \ldots, y_n \in Y$, let

$$u = \big((x_1x_2)\cdots\big)x_m \quad \text{and} \quad v = \big((y_1y_2)\cdots\big)y_n.$$

From $q_{0,0}(1, 1, z) = q_{0,n}(1, v, z) = q_{m,0}(u, 1, z) = 0$ one recursively defines $q_{m,n}(u, v, z)$, $m, n \geqslant 1$, by

$$(u, v, z) = \sum u_{(1)} q(u_{(2)}, v_{(2)}, z) \cdot v_{(1)}, \tag{15}$$

where (u, v, z) denotes the associator of u, v and z. So,

$$\begin{aligned} q(x_1, y_1, z) &= (x_1, y_1, z) \quad \text{and} \\ q(u, v, z) &= (u, v, z) - \sum_{|u_{(1)}|+|v_{(1)}| \geqslant 1} u_{(1)} q(u_{(2)}, v_{(2)}, z) \cdot v_{(1)}, \end{aligned}$$

where $|u|$ denotes the total degree of the monomial u.

The importance of this construction is that, as shown in [29], the set of primitive elements of any bialgebra is closed under $[\,,]$ (the usual commutator) and $q_{m,n}(x_1, \ldots, x_m; y_1, \ldots y_n, z)$, $m, n \geqslant 1$. Therefore, given a nonassociative algebra C over a field of characteristic zero one may consider the operations

$$\begin{aligned}\langle y,z\rangle &= -[y,z],\\ \langle x_1,\dots,x_m;y,z\rangle &= -q_{m,1}(x_1,\dots,x_m,y,z)+q_{m,1}(x_1,\dots,x_m,z,y)\end{aligned}$$

and

$$\Phi(x_1,\dots,x_m;y_1,\dots,y_n)=\frac{1}{m!}\frac{1}{n!}\sum_{\tau\in S_m,\,\delta\in S_n} q_{m,n-1}(x_{\tau(1)},\dots,x_{\tau(m)},y_{\delta(1)},\dots,y_{\delta(n)})$$

for all $m\geqslant 1$ and $n\geqslant 2$. Let $\text{УШ}(C)$ denote the vector space C endowed with the operations $\langle\,;\,\rangle$ and Φ. Shestakov and Umirbaev proved in [29] that $\text{УШ}(C)$ is a Sabinin algebra, and in case that C is a bialgebra then $\operatorname{Prim}(C)$ is a Sabinin subalgebra of $\text{УШ}(C)$.

Finally, we also should mention that although we started from a slightly different definition of the primitive operations considered in [29], however the operations $\langle x_1,\dots,x_n;y,z\rangle$ and $\langle y,z\rangle$ remain the same.

4.2. Sabinin algebras from Lie algebras

Sabinin and Miheev showed that given a Lie algebra L, a subalgebra H and a vector space V with $L=H\oplus V$, then V inherits a structure of Sabinin algebra from the product on L. In this subsection we present a different approach to this construction providing a new interpretation of the operations $\langle\,;\,\rangle$. One important example of this construction arises when starting with a unital algebra C, L the Lie algebra generated by the right multiplication operators, $H=\{\varphi\in L\mid 1\varphi=0\}$ and $V=\{R_a\mid a\in C\}$ since we naturally recover the Sabinin algebra $\text{УШ}(C)$ defined by Shestakov and Umirbaev. In the following we will assume that linear maps act from the right.

We first fix some notation. Let L be a Lie algebra, H a subalgebra and V a vector space with $L=H\oplus V$. Given $d_1,\dots,d_n\in L$ we will use the notation

$$\begin{aligned}[][d_1] &= d_1,\\ [d_1,\dots,d_n] &= \bigl[d_1,\bigl[\cdots,[d_{n-1},d_n]\bigr]\bigr],\quad n\geqslant 2,\quad\text{and}\\ \{d_1,\dots,d_n\} &= \pi_V\bigl([d_1,\dots,d_n]\bigr),\quad n\geqslant 1,\end{aligned}$$

where π_V denotes the projection on V parallel to H. Given $a\in V$ we will consider

$$\{\,;a\}:\mathrm{T}(L)\to V$$

defined by

$$\{1;a\}=0,\quad\text{and}\quad \{d_1\cdots d_n;a\}=\{d_1,\dots,d_n,a\},$$

and the projection

$$\begin{aligned}\mathrm{T}(L)&\to \mathrm{U}(L)/H\mathrm{U}(L),\\ d_1\cdots d_n&\mapsto \overline{d_1\cdots d_n},\end{aligned}$$

where $\mathrm{U}(L)$ denotes the universal enveloping algebra of L.

Lemma 16. *Given* $d_1,\dots,d_n\in L$ *and* $w=d_1\cdots d_n\in \mathrm{T}(L)$, *then*

$$\overline{wd}=-\sum_{i=1}^{\infty}\sum(-1)^i\,\overline{w_{(1)}\{w_{(2)};\{\cdots;\{w_{(i)};\{w_{(i+1)}d\}\cdots\}}$$

for any $d\in L$.

Proof. We will proceed by induction on n. If $n=0$ then $w=1$ so the right-hand side of the formula is $-\sum_{i=1}^{\infty}(-1)^i\,\overline{1\{1;\{\cdots,\{1d\}\cdots\}}=-(-\overline{\{d\}})=\bar{d}=\overline{wd}$. Let us assume that the formula holds for any $w=d_1\cdots d_m$ with $m<n$. Given $w=d_1\cdots d_n$, let $w'=d_1\cdots d_{n-1}$, then

$$\begin{aligned}\overline{wd}&=\overline{w'd_nd}=\overline{w'dd_n}+\overline{w'[d_n,d]}\\&=-\sum_{i=1}^{\infty}\sum(-1)^i\,\overline{w'_{(1)}\{w'_{(2)};\{\cdots;\{w'_{(i+1)}d\}\cdots\}d_n}\\&\quad-\sum_{i=1}^{\infty}\sum(-1)^i\,\overline{w'_{(1)}\{w'_{(2)};\{\cdots;\{w'_{(i)};\{w'_{(i+1)}[d_n,d]\}\cdots\}}.\end{aligned}$$

With

$$Z=-\sum_{i=1}^{\infty}\sum(-1)^i\,\overline{w'_{(1)}\{w'_{(2)};\{\cdots;\{w'_{(i)};\{w'_{(i+1)}[d_n,d]\}\cdots\}}\quad\text{and}$$
$$Y=-\sum_{i=1}^{\infty}\sum(-1)^i\,\overline{w'_{(1)}d_n\{w'_{(2)};\{\cdots;\{w'_{(i)};\{w'_{(i+1)}d\}\cdots\}}$$

we have

$$\begin{aligned}\overline{wd}&=Z+Y-\sum_{i=1}^{\infty}\sum(-1)^i\,\overline{w'_{(1)}[\{w'_{(2)};\{\cdots;\{w'_{(i)};\{w'_{(i+1)}d\}\cdots\},d_n]}\\&=Z+Y-\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum(-1)^{i+j}\,\overline{w'_{(1)}\{w'_{(2)};\cdots;}\\&\qquad\overline{\{w'_{(j)};\{w'_{(j+1)}d_n\{w'_{(j+2)};\cdots;\{w'_{(i+j+1)}d\}\cdots\}}\\&=Z+Y-\sum_{k=2}^{\infty}\sum_{j=1}^{k-1}\sum(-1)^k\,\overline{w'_{(1)}\{w'_{(2)};\{\cdots;}\\&\qquad\overline{\{w'_{(j+1)}d_n;\{w'_{(j+2)};\{\cdots;\{w'_{(k+1)}d\}\cdots\}}\\&=-\sum_{k=1}^{\infty}\sum_{j=1}^{k-1}\sum(-1)^k\,\overline{w'_{(1)}\{w'_{(2)};\cdots;\{w'_{(j)}d_n;\{\cdots;\{w'_{(k+1)}d\}\cdots\}}\end{aligned}$$

$$= -\sum_{k=1}^{\infty}(-1)^k \sum \overline{w_{(1)}\{w_{(2)}; \cdots; \{w_{(j)}; \{\cdots; \{w_{(k+1)}d\}\cdots\}}$$

as desired. □

Given $w \in \mathrm{T}(L)$ we define

$$\langle w; d\rangle = \sum_{i=1}^{\infty} \sum (-1)^i \{w_{(1)}; \{\cdots; \{w_{(i)}d\}\cdots\} \in V \tag{16}$$

for any $d \in L$. So, in $\mathrm{U}(L)/H\mathrm{U}(L)$ the relation

$$\overline{wd} = -\sum \overline{w_{(1)}\langle w_{(2)}; d\rangle} \tag{17}$$

holds. Given $x_1, \ldots, x_n, a, b \in V$ with $n \geqslant 1$, let us define

$$\begin{aligned} \langle x_1 \cdots x_n; a, b\rangle &= \bigl\langle x_1 \cdots x_n; [a, b]\bigr\rangle \quad \text{and} \\ \langle 1; a, b\rangle &= \langle a, b\rangle = \bigl\langle 1; [a, b]\bigr\rangle. \end{aligned} \tag{18}$$

Proposition 17. *The vector space V with the operations $\langle\,;\,\rangle$ defined by* (18) *is a Sabinin algebra.*

Proof. First, in order to prove (13), we observe that in $\mathrm{U}(L)/H\mathrm{U}(L)$ we have

$$\begin{aligned} &-\sum \overline{x_{(1)}\langle x_{(2)}c; a, b\rangle} - \sum \overline{x_{(1)}\langle x_{(2)}; \langle x_{(3)}; a, b\rangle, c\rangle} \\ &= \overline{xc[a, b]} + \sum \overline{x_{(1)}c\langle x_{(2)}; a, b\rangle} + \sum \overline{x_{(1)}[\langle x_{(2)}; a, b\rangle, c]} \\ &= \overline{xc[a, b]} + \sum \overline{x_{(1)}\langle x_{(2)}; a, b\rangle c} = \overline{xc[a, b]} - \overline{x[a, b]c} = -\overline{x[[a, b], c]} \end{aligned}$$

so the cyclic sum on a, b, c of this element vanishes.

We may use induction on the degree of $x \in \mathrm{T}(L)$ to prove (13). If $x = 1$ then $\sigma_{a,b,c}(\overline{\langle xc; a, b\rangle} + \sum \overline{\langle x_{(1)}; \langle x_{(2)}; a, b\rangle, c\rangle}) = 0$, so

$$\sigma_{a,b,c}\Bigl(\langle xc; a, b\rangle + \sum \bigl\langle x_{(1)}; \langle x_{(2)}; a, b\rangle, c\bigr\rangle\Bigr) \in H\mathrm{U}(L) \cap V.$$

Since by the Poincaré–Birkhoff–Witt Theorem for Lie algebras $H\mathrm{U}(L) \cap V = 0$ then we obtain the result in this case. Let us assume that we have proved (13) for any element of degree lower than $|x|$,

$$\begin{aligned} 0 &= \sigma_{a,b,c}\Bigl(\sum \overline{x_{(1)}\langle x_{(2)}c; a, b\rangle} + \sum \overline{x_{(1)}\langle x_{(2)}; \langle x_{(3)}; a, b\rangle, c\rangle}\Bigr) \\ &= \sigma_{a,b,c}\Bigl(\overline{\langle xc; a, b\rangle} + \sum \overline{\langle x_{(1)}; \langle x_{(2)}; a, b\rangle, c\rangle}\Bigr) \\ &\quad + \sigma_{a,b,c}\Bigl(\sum_{|x_{(2)}|<|x|} x_{(1)}\Bigl(\langle x_{(2)}c; a, b\rangle + \sum \bigl\langle x_{(2)}; \langle x_{(3)}; a, b\rangle, c\bigr\rangle\Bigr)\Bigr) \end{aligned}$$

$$= \sigma_{a,b,c}\Bigl(\overline{\langle xc; a, b\rangle} + \sum \overline{\langle x_{(1)}; \langle x_{(2)}; a, b\rangle, c\rangle}\Bigr)$$

and the result follows as in the case $x = 1$.

Given $y = y_1 \cdots y_r \in \mathrm{T}(L)$ with $y_1, \dots, y_r \in V$,

$$\begin{aligned}
&\sum \overline{x_{(1)} y_{(1)} \langle x_{(2)}[a, b] y_{(2)}; c, d\rangle} + \sum \overline{x_{(1)}[a, b] y_{(1)} \langle x_{(2)} y_{(2)}; c, d\rangle} \\
&\quad = -\overline{x[a, b] y[c, d]} = -\bigl(\cdots \bigl(\overline{x[a, b]} y_1\bigr) \cdots\bigr) y_r[c, d] \\
&\quad = \sum \overline{x_{(1)} \langle x_{(2)}; a, b\rangle y[c, d]} \\
&\quad = -\sum \overline{x_{(1)} \langle x_{(2)}; a, b\rangle y_{(1)} \langle x_{(3)} y_{(2)}; c, d\rangle} - \sum \overline{x_{(1)} y_{(1)} \langle x_{(2)} \langle x_{(3)}; a, b\rangle y_{(2)}; c, d\rangle} \\
&\quad = \sum \overline{x_{(1)}[a, b] y_{(1)} \langle x_{(2)} y_{(2)}; c, d\rangle} - \sum \overline{x_{(1)} y_{(1)} \langle x_{(2)} \langle x_{(3)}; a, b\rangle y_{(2)}; c, d\rangle}
\end{aligned}$$

so

$$\sum \overline{x_{(1)} y_{(1)} \langle x_{(2)}[a, b] y_{(2)}; c, d\rangle} = -\sum \overline{x_{(1)} y_{(1)} \langle x_{(2)} \langle x_{(3)}; a, b\rangle y_{(2)}; c, d\rangle}$$

and we can conclude (12) by induction on $|x| + |y|$. □

Example 18 *(Shestakov–Umirbaev).* Let C be an algebra with 1, L the Lie algebra generated by the right multiplication operators on C, and $H = \{\varphi \in L \mid 1\varphi = 0\}$. Since $L = H \oplus V$ with $V = \mathrm{span}\langle R_x \mid x \in C\rangle$, then the structure of Sabinin algebra on V is inherited by C by

$$\langle R_{x_1} \cdots R_{x_n}; R_a, R_b\rangle = R_{\langle x_1, \dots, x_n; a, b\rangle}.$$

If we denote $R_{x_1} \cdots R_{x_n}$ by R then in $\mathrm{U}(L)/H\mathrm{U}(L)$ we have

$$\overline{R[R_a, R_b]} = -\sum \overline{R_{(1)} \langle R_{(2)}; R_a, R_b\rangle}.$$

Since $H\mathrm{U}(L)$ kills the unit element then we may evaluate this equality on 1 to obtain

$$(ua)b - (ub)a = -\sum u_{(1)} \langle u_{(2)}; a, b\rangle \tag{19}$$

with $u = ((x_1 x_2) \cdots) x_n$, so

$$(u, a, b) - (u, b, a) + u[a, b] = -\sum u_{(1)} \langle u_{(2)}; a, b\rangle.$$

Using that

$$\langle 1; a, b\rangle = 1\langle R_1; R_a, R_b\rangle = -1\{R_a, R_b\} = -1[R_a, R_b] = -[a, b]$$

then we finally obtain that

$$(u, b, a) - (u, a, b) = \sum_{|u_{(2)}| > 1} u_{(1)} \langle u_{(2)}; a, b\rangle.$$

This is the relation (17) in [29], so $\langle ; \rangle$ is the same Sabinin algebra structure on V as the one in [29].

Given a Lie algebra $L = H \oplus V$ with H a subalgebra, one may induce multilinear operations $\langle ; \rangle$ on V by the following recurrence:

$$\{xab\} + \sum \big\{x_{(1)} \langle x_{(2)}; a, b\rangle\big\} = 0. \tag{20}$$

The Lie algebra L is called a *Lie envelope* of $(V, \langle ; \rangle)$. Sabinin and Miheev [24] proved that $(V, \langle ; \rangle)$ is a Sabinin algebra and that any Sabinin algebra admits a Lie envelope. This construction of $(V, \langle ; \rangle)$ agrees with the one in (18).

Proposition 19. *Let* $\langle ; \rangle$ *be defined by* (20)*, then*

$$\langle x; a, b\rangle = \sum_{i=1}^{\infty} \sum (-1)^i \big\{x_{(1)}; \big\{\cdots; \{x_{(i)}ab\} \cdots\big\}.$$

Proof. For $x = 1$ the statement follows from

$$\langle 1; a, b\rangle + \{ab\} = 0.$$

Let us assume that the statement is true for any element of degree $< |x|$ then

$$\begin{aligned}
\langle x; a, b\rangle &= -\{xab\} - \sum_{|x_{(1)}| \neq 1} \big\{x_{(1)} \langle x_{(2)}; a, b\rangle\big\} \\
&= -\{xab\} - \sum_{|x_{(1)}| \neq 1} \sum_{i=1}^{\infty} (-1)^i \big\{x_{(1)} \big\{x_{(2)}; \big\{\cdots; \{x_{(i+1)}ab\} \cdots\big\} \\
&= -\{xab\} + \sum_{|x_{(1)}| \neq 1} \sum_{i=2}^{\infty} (-1)^i \big\{x_{(1)} \big\{x_{(2)}; \big\{\cdots; \{x_{(i)}ab\} \cdots\big\} \\
&= -\{xab\} + \sum_{i=2}^{\infty} \sum (-1)^i \big\{x_{(1)}; \big\{x_{(2)}; \big\{\cdots; \{x_{(i)}ab\} \cdots\big\} \\
&= \sum_{i=1}^{\infty} \sum (-1)^i \big\{x_{(1)}; \big\{\cdots; \{x_{(i)}ab\} \cdots\big\}. \qquad \square
\end{aligned}$$

5. A universal enveloping algebra for Sabinin algebras

Recall that given a nonassociative algebra U over a field of characteristic zero one may consider a structure of Sabinin algebra on U, denoted by $\text{УШ}(U)$, where the operations are given by

$$\begin{aligned}
\langle y, z\rangle &= -[y, z], \\
\langle x_1, \ldots, x_m; y, z\rangle &= -q(x_1, \ldots, x_m, y, z) + q(x_1, \ldots, x_m, z, y)
\end{aligned}$$

and

$$\Phi(x_1,\dots,x_m;y_1,\dots,y_n)=\frac{1}{m!}\frac{1}{n!}\sum_{\tau\in S_m,\,\delta\in S_n} q(x_{\tau(1)},\dots,x_{\tau(m)},y_{\delta(1)},\dots,y_{\delta(n)})$$

(subindices on q are omitted for simplicity). In case that U is a bialgebra then $\mathrm{Prim}(U)$ is a Sabinin subalgebra of УШ(U). The main problem posed in [29] is stated as follows: *Does any Sabinin algebra appears as a Sabinin subalgebra of* УШ(U) *for some algebra U?* In this section we will provide an affirmative answer to this problem. For any Sabinin algebra $(V,\langle\,;\,\rangle,\Phi)$ over a field of characteristic zero we will construct a connected H-bialgebra $\mathrm{U}(V)$ so that $(V,\langle\,;\,\rangle,\Phi)=(\mathrm{Prim}(\mathrm{U}(V)),\langle\,;\,\rangle,\Phi)$.

Let us first discuss how any H-bialgebra U is recovered from $V=\mathrm{Prim}(U)$ by means of the operations $\langle\,;\,\rangle$ and Φ since this will ultimately determine our strategy in the construction of $\mathrm{U}(V)$. By Theorem 3.2 and Corollary 3.3 in [29], given a basis $a_1,a_2,\dots,a_\alpha,\dots$ of V then the set of right-normed words of type $a_{i_1}a_{i_2}\cdots a_{i_k}$ where $i_1\leqslant i_2\leqslant\cdots\leqslant i_k$, $k\geqslant 0$ forms a (Poincaré–Birkhoff–Witt) basis of the algebra U. Although the factors in $a_{i_1}a_{i_2}\cdots a_{i_k}$ do not commute and it would be misleading to interpret them as elements in $S(V)$, the symmetric algebras on V, however they obey the relation $(xa)b-(xb)a=-\sum x_{(1)}\langle x_{(2)};a,b\rangle$ for any $x\in U$ and $a,b\in V$. Thus, a natural candidate to play the role of U is the quotient $\tilde S(V)$ of the tensor algebra $\mathrm{T}(V)$ over V by the ideal

$$\mathrm{span}\Big\langle x[a,b]y+\sum x_{(1)}\langle x_{(2)};a,b\rangle y \;\Big|\; x,y\in\mathrm{T}(V),\ a,b\in V\Big\rangle.$$

In Section 5.1 we prove that for any Sabinin algebra $(V,\langle\,;\,\rangle,\Phi)$ this vector space always admits a Poincaré–Birkhoff–Witt-type basis and we will use it as the underlying vector space in our construction of $\mathrm{U}(V)$.

A more delicate task is the definition of an adequate product on $\tilde S(V)$. Conceptually this is quite simple since, using a Poincaré–Birkhoff–Witt basis, it amounts to expressing the product of two right-normed words on U as a linear combination of right-normed words, and this is easily obtained by induction on the degree of y by

$$x(ya)=(xy)a-\sum x_{(1)}q(x_{(2)},y_{(2)},a)\cdot y_{(1)}.$$

However, as the reader may guess, there are two main obstacles in this approach. The first one is that the product so obtained might depend on the chosen basis and the previous identity fail to hold for arbitrary x, y and a. The second obstacle is that $q(x,y,a)$ are not the native operations on V, and even worst, to the best of our knowledge no axioms for these operations are known.

To overcome the first obstacle and getting a coherent product on $\tilde S(V)$ from V, $\langle\,;\,\rangle$ and q (we will come back to the multioperator Φ later), we may define an extension l of the operations q that fortunately admits a simple axiomatic. To this end it is useful to interpret maps d in $E=\mathrm{Hom}(U,V)$ as acting on U by $xd=-\sum x_{(1)}\langle x_{(2)};d\rangle$, where $\langle x;d\rangle$ stands for the image of x by d. For instance, the map $\tau_a: x\mapsto -\epsilon(x)a$ originates $x\tau_a=xa$ the right multiplication by a. An easy computation shows that $xdd'-xd'd=x[d,d']$ where

$$\big\langle x;[d,d']\big\rangle=\sum\big\langle x_{(1)};\langle x_{(2)};d\rangle,\langle x_{(3)};d'\rangle\big\rangle-\big\langle x_{(1)}\langle x_{(2)};d\rangle;d'\big\rangle+\big\langle x_{(1)}\langle x_{(2)};d'\rangle;d\big\rangle$$

which makes E a Lie algebra acting on U. The identity $(xy)a = x(ya) + \sum x_{(1)}q(x_{(2)}, y_{(2)}, a) \cdot y_{(1)}$ can be written in an operator form as $(xy)\tau_a = x(y\tau_a) + xl(y_{(1)}, \tau_a)y_{(2)}$ with $\langle x; l(y, \tau_a)\rangle = q(x, y, a)$. This suggests the possibility of extending l to $l : C \otimes E \to H$, with $H = \{d \in E \mid \langle 1; d\rangle = 0\}$, so that

$$(xy)d = x(yd) + \sum xl(y_{(1)}, d)y_{(2)} \tag{21}$$

holds. This extension always exits, it is unique and easily obtained because U is an H-bialgebra. Clearly, from the "equation" (21) on the "indeterminate" l we find that

$$\bigl\langle x; l(y, d)\bigr\rangle = -\sum x_{(1)} \backslash \bigl(\bigl((x_{(2)}y_{(2)})d - x_{(2)}(y_{(2)}d)\bigr)/y_{(1)}\bigr) \tag{22}$$

is the only solution. Two obvious relations for l arising from (21) are

$$\begin{aligned} l(1, d) &= d + \tau_{\langle 1; d\rangle}, \\ l\bigl(v, [d, d']\bigr) &= l(vd, d') - l(vd', d) + \sum \bigl[l(v_{(1)}, d), l(v_{(2)}, d')\bigr]. \end{aligned} \tag{23}$$

Can we recover the product on U by starting with V, $\langle\,;\,\rangle$ and l? The answer now is fortunately positive. Given a Sabinin algebra $(V, \langle\,;\,\rangle)$ and a map $l : U \otimes E \to H$ (where we assume the identification $U = \tilde{S}(V)$) satisfying (23) there is only one way of defining a unital product on U so that (21) holds. The multiplication of two right-normed words is inductively determined in any Poincaré–Birkhoff–Witt basis by $x(ya) = (xy)a - \sum xl(y_{(1)}, \tau_a) \cdot y_{(2)}$. This is carried out in Section 5.2, and the algebras so obtained are denoted by $(\tilde{S}(V), l)$.

Although we have succeed in constructing U from V, $\langle\,;\,\rangle$ and l, our second obstacle remains, namely, the native operations on V are $\langle\,;\,\rangle$ and Φ. The map l that turned out to be very useful in establishing a coherent multiplication is unnatural now. Even more, different choices of Φ in $(V, \langle\,;\,\rangle, \Phi)$ should eventually produce different universal enveloping algebras $\mathrm{U}(V)$, all of them sharing the same $(V, \langle\,;\,\rangle)$. In other words, fixed V and $\langle\,;\,\rangle$ it seems that many different maps l satisfying (23) are possible. So, how many choices of l are allowed? The reader may guess: basically so many as multioperators Φ. In fact, l and Φ should be related by $\langle a^m; l(b^{n-1}, \tau_b)\rangle = q(a^m, b^{n-1}, b) = \Phi(a, \dots, a, b, \dots, b)$, and it is not hard to realize that there is at most one l satisfying this condition and (23). In Section 5.3 we determine the degrees of freedom available in l.

The strategy is now clear. Given $(V, \langle\,;\,\rangle, \Phi)$ a Sabinin algebra over a field of characteristic zero, construct $\tilde{S}(V)$ and the unique l satisfying (23) and $\langle a^m; l(b^{n-1}, \tau_b)\rangle = \Phi(a, \dots, a, b, \dots, b)$. The universal enveloping algebra $\mathrm{U}(V)$ you are looking for is $(\tilde{S}(V), l)$. This is done in Section 5.4.

Before digging into details, let us remark that in the course of our discussion we have shown that any cocommutative, coassociative connected unital H-bialgebra U admits a description as an algebra $(\tilde{S}(V), l)$ with l given by the very concise formula (22). The explicit computation of V or l in concrete examples such as for instance free nonassociative unital algebras seems however a painful task.

We should also notice that the construction of $(\tilde{S}(V), l)$ is characteristic free when l is given. Characteristic zero is used to get a particular l that fulfills the requirement $\langle a^m; l(b^{n-1}, \tau_b)\rangle = \Phi(a, \dots, a, b, \dots, b)$ needed in the universal enveloping algebra $\mathrm{U}(V)$. The main reason to relegate the multioperator Φ to the end is that we feel certain "arbitrariness" in its definition. In fact, the multioperator we are using is different from the multioperator considered in [29], so we have tried to minimize the impact of this election in our exposition.

5.1. A Poincaré–Birkhoff–Witt Theorem for Sabinin algebras

Let $(V, \langle ; \rangle)$ be a Sabinin algebra and

$$\tilde{S}(V) = \mathrm{T}(V)/\operatorname{span}\Big\langle x[a,b]y + \sum x_{(1)}\langle x_{(2)}; a, b\rangle y \;\Big|\; x, y \in \mathrm{T}(V),\ a, b \in V\Big\rangle.$$

The aim of this subsection is to prove that for any basis $\{a_i\}_{i\in\Lambda}$ of V with Λ a totally ordered set then

$$\{\overline{a_{i_1}\cdots a_{i_n}} \mid i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n \text{ and } n \in \mathbb{N}\}$$

is a basis of $\tilde{S}(V)$.

The name Poincaré–Birkhoff–Witt Theorem for this result is justified by the following example.

Example 20. Any Lie algebra L is a Sabinin algebra with

$$\langle x_1, \dots, x_n; a, b\rangle = \begin{cases} 0, & n \geqslant 1, \\ -[a,b], & n = 0. \end{cases}$$

For this structure of Sabinin algebra we have

$$\tilde{S}(L) = \mathrm{T}(L)\big/\big\langle x(ab - ba)y - x[a,b]y \mid x, y \in \mathrm{T}(L),\ a, b \in L\big\rangle = \mathrm{U}(L)$$

and the Poincaré–Birkhoff–Witt Theorem for Sabinin algebras specializes to the usual Poincaré–Birkhoff–Witt Theorem for Lie algebras.

There are several ways to achieve our goal, and it is a matter of taste which one to follow. We derive the result very quickly from the usual Poincaré–Birkhoff–Witt Theorem for Lie algebras using a Lie envelope of $(V, \langle ; \rangle)$ (see paragraph previous to Proposition 19). However we also present in Appendix A a proof based on Gröbner bases so that the result for Lie algebras becomes a particular case.

Theorem 21 *(Poincaré–Birkhoff–Witt). Let $\{a_i \mid i \in \Lambda\}$ be a totally ordered basis of V. Then $\{\overline{a_{i_1}\cdots a_{i_n}} \mid i_1 \leqslant \cdots \leqslant i_n$ and $n \geqslant 0\}$ is a basis of $\tilde{S}(V)$.*

Proof. Let $E = H \oplus V$ be a Lie envelope. By Proposition 19, (16) and (17) in $\mathrm{U}(E)/H\mathrm{U}(E)$ we have that $\overline{x[a,b]} = -\sum \overline{x_{(1)}\langle x_{(2)}; a, b\rangle}$. Thus the natural map from $\mathrm{T}(V)$ to $\mathrm{U}(E)/H\mathrm{U}(E)$ factors through $\tilde{S}(V)$. By the usual Poincaré–Birkhoff–Witt Theorem the image of $\{\overline{a_{i_1}\cdots a_{i_n}} \mid i_1 \leqslant \cdots \leqslant i_n$ and $n \geqslant 0\}$ is linearly independent in $\mathrm{U}(E)/H\mathrm{U}(E)$, so this set is linearly independent in $\tilde{S}(V)$. Since it obviously generates the whole $\tilde{S}(V)$ then it is a basis. □

The vector space $\tilde{S}(V)$ is filtered by $\tilde{S}(V)_n = \operatorname{span}\langle \bar{x} \in \tilde{S}(V) \mid |x| \leqslant n\rangle$, where $|x|$ stands for the degree of x, and we may define the degree $|\bar{x}|$ of a nonzero element $\bar{x} \in \tilde{S}(V)$ as the minimum n such that $\bar{x} \in \tilde{S}(V)_n$ (the degree of $\bar{0}$ is set to $-\infty$ as usual). We may consider the map

$$\phi : \mathrm{T}(V) \to \mathrm{Gr}\big(\tilde{S}(V)\big),$$
$$x_1 \otimes \cdots \otimes x_n \mapsto \overline{x_1 \cdots x_n} + \tilde{S}(V)_{n-1},$$

where $\mathrm{Gr}(\tilde{S}(V)) = \bigoplus_{n=0}^{\infty} \tilde{S}(V)_n / \tilde{S}(V)_{n-1}$ denotes the graded vector space associated to the filtration $\tilde{S}(V) = \bigcup_{n=0}^{\infty} \tilde{S}(V)_n$. By definition of $\tilde{S}(V)$ this map induces a map

$$\phi : S(V) \to \mathrm{Gr}\big(\tilde{S}(V)\big)$$

from the symmetric algebra on V to $\mathrm{Gr}(\tilde{S}(V))$. The Poincaré–Birkhoff–Witt Theorem amounts to saying that this map is an isomorphism.

Since in $\mathrm{T}(V)$

$$\begin{aligned}
&\Delta\Big(x[a,b]y + \sum x_{(1)}\langle x_{(2)}; a,b\rangle y\Big) \\
&\quad = \sum_{(x),(y)} \Big(x_{(2)}[a,b]y_{(2)} + \sum_{(x_{(2)})} x_{(2)}\langle x_{(3)}; a,b\rangle y_{(2)}\Big) \otimes x_{(1)}y_{(1)} \\
&\qquad + \sum_{(x),(y)} x_{(1)}y_{(1)} \otimes \Big(x_{(2)}[a,b]y_{(2)} + \sum_{(x_{(2)})} x_{(2)}\langle x_{(3)}; a,b\rangle y_{(2)}\Big)
\end{aligned}$$

then $\tilde{S}(V)$ inherits the structure of coalgebra from $\mathrm{T}(V)$, i.e.

$$\Delta(\bar{x}) = \sum \bar{x}_{(1)} \otimes \bar{x}_{(2)} = \sum \overline{x_{(1)}} \otimes \overline{x_{(2)}} \quad \text{and} \quad \epsilon(\bar{x}) = \epsilon(x). \tag{24}$$

Using the Poincaré–Birkhoff–Witt Theorem it is easy to check that $\tilde{S}(V)$ and $S(V)$ are isomorphic coalgebras.

5.2. *Algebras* $(\tilde{S}(V), l)$

To avoid annoying notation, given $\bar{x} \in \tilde{S}(V)$ and $a \in V$ we will write $\bar{x}a$ instead of $\overline{xa}$. After defining the product on $(\tilde{S}(V), l)$ this notation will become even more natural. Given $d \in \mathrm{Hom}(\tilde{S}(V), V)$ we will use the notation $\langle \bar{x}; d\rangle$ for the image of x by d. As mentioned at the beginning of this section, it will be natural in our context to consider for any $a \in V$ the map

$$\begin{aligned}
\tau_a : \tilde{S}(V) &\to V, \\
\bar{1} &\mapsto -a, \\
\bar{x} &\mapsto 0, \quad |\bar{x}| \geqslant 1.
\end{aligned}$$

Proposition 22. *The algebra* $E = (\mathrm{Hom}(\tilde{S}(V), V), [\,,\,])$ *is a Lie algebra with the product*

$$\langle \bar{x}; [d, d'] \rangle = \sum \langle \bar{x}_{(1)}; \langle \bar{x}_{(2)}; d\rangle, \langle \bar{x}_{(3)}; d'\rangle \rangle - \langle \bar{x}_{(1)} \langle \bar{x}_{(2)}; d\rangle; d'\rangle + \langle \bar{x}_{(1)} \langle \bar{x}_{(2)}; d'\rangle; d\rangle,$$

and $\tilde{S}(V)$ *is a right* E*-module with the action* $\bar{x}d = -\sum \bar{x}_{(1)}\langle \bar{x}_{(2)}; d\rangle$.

Proof. Since

$$\begin{aligned}
\bar{x}dd' - \bar{x}d'd &= -\sum \big(\bar{x}_{(1)}\langle \bar{x}_{(2)}; d\rangle\big)d' + \sum \big(\bar{x}_{(1)}\langle \bar{x}_{(2)}; d'\rangle\big)d \\
&= \sum \big(\bar{x}_{(1)}\langle \bar{x}_{(2)}; d\rangle\big)\langle \bar{x}_{(3)}; d'\rangle + \sum \bar{x}_{(1)}\big\langle \bar{x}_{(2)}\langle \bar{x}_{(3)}; d\rangle; d'\big\rangle \\
&\quad - \sum \big(\bar{x}_{(1)}\langle \bar{x}_{(2)}, d'\rangle\big)\langle \bar{x}_{(3)}; d\rangle - \sum \bar{x}_{(1)}\big\langle \bar{x}_{(2)}\langle \bar{x}_{(3)}; d'\rangle; d\big\rangle \\
&= -\sum \bar{x}_{(1)}\big\langle \bar{x}_{(2)}; \langle \bar{x}_{(3)}; d\rangle, \langle x_{(4)}; d'\rangle\big\rangle + \sum \bar{x}_{(1)}\big\langle \bar{x}_{(2)}\langle \bar{x}_{(3)}; d\rangle; d'\big\rangle \\
&\quad - \sum \bar{x}_{(1)}\big\langle \bar{x}_{(2)}\langle \bar{x}_{(3)}; d'\rangle; d\big\rangle \\
&= -\sum \bar{x}_{(1)}\big\langle \bar{x}_{(2)}; [d, d']\big\rangle \\
&= \bar{x}[d, d'],
\end{aligned}$$

then the map

$$\begin{aligned}
\rho : E &\to \mathrm{End}_F\big(\tilde{S}(V)\big)^{(-)}, \\
d \mapsto d &: \bar{x} \mapsto -\sum \bar{x}_{(1)}\langle \bar{x}_{(2)}; d\rangle
\end{aligned}$$

is a homomorphism of Lie algebras. Moreover, given a map $d \in E$ such that $\sum \bar{x}_{(1)}\langle \bar{x}_{(2)}; d\rangle = 0$ for any $\bar{x} \in \tilde{S}(V)$ then it follows by induction on $|x|$, the degree of x, that $\langle \bar{x}; d\rangle = 0$. Hence ρ is injective and the result follows. □

Now we proceed to construct the algebras $(\tilde{S}(V), l)$. This is done using a fixed Poincaré–Birkhoff–Witt basis $\{\bar{a}_I\}_I$ of $\tilde{S}(V)$. The independence of this basis is a consequence of Proposition 26.

Given $l : \tilde{S}(V) \otimes E \to H$ satisfying

$$\begin{aligned}
l(\bar{1}, d) &= d + \tau_{\langle 1; d\rangle}, \\
l\big(\bar{y}, [d, d']\big) &= l(\bar{y}d, d') - l(\bar{y}d', d) + \sum \big[l(\bar{y}_{(1)}, d), l(\bar{y}_{(2)}, d')\big],
\end{aligned} \tag{25}$$

we define a product on $\tilde{S}(V)$ by

$$\begin{aligned}
\bar{x}\bar{1} &= \bar{x}, \\
\bar{x}\bar{a} &= \overline{xa}, \\
\bar{x}\bar{a}_I &= (\bar{x}\bar{a}_{I'})\bar{a}_{i_n} - \sum \bar{x}l(\bar{a}_{I'(1)}, \tau_{a_{i_n}})\bar{a}_{I'(2)},
\end{aligned}$$

where $\bar{a}_I = \overline{a_{i_1} \cdots a_{i_n}}$ with $i_1 \leqslant \cdots \leqslant i_n$ and $\bar{a}_{I'} = \overline{a_{i_1} \cdots a_{n-1}}$. We will denote this algebra by $(\tilde{S}(V), l)$.

Proposition 23. *The element $\bar{1}$ is the unit element of $(\tilde{S}(V), l)$.*

Proof. By definition, $\bar{a}\bar{1} = \bar{a}$ and $\bar{1}\bar{a} = \bar{a}$. Using induction we have

$$\bar{1}\bar{a}_I = (\bar{1}\bar{a}_{I'})\bar{a}_{i_n} - \sum \bar{1}l(\bar{a}_{I'(1)}, \tau_{a_{i_n}})\bar{a}_{I'(2)} = \bar{a}_I$$

as desired. □

Formally we denote by $u : F \to \tilde{S}(V)$ the unit map $1 \mapsto \bar{1}$. Recall that in (24) we saw that $\tilde{S}(V)$ is a coassociative, cocommutative connected coalgebra.

Proposition 24. *$(\tilde{S}(V), \Delta, \epsilon, \cdot, u)$ is a coassociative, cocommutative connected unital bialgebra and $V \subseteq \mathrm{Prim}(\tilde{S}(V))$.*

Proof. We will show that $\Delta(\bar{x}\bar{a}_I) = \Delta(\bar{x})\Delta(\bar{a}_I)$ by induction on the degree $|I|$ of I (the case $|I| = 0$ is trivial). By the very definition of the product on $(\tilde{S}(V), l)$ and the action of E on $\tilde{S}(V)$ we have

$$\begin{aligned}
\Delta(\bar{x}\bar{a}_I) &= \Delta(\bar{x})\Delta(\bar{a}_{I'}) \cdot \Delta(\bar{a}_{i_n}) - \sum \Delta\big(\bar{x}l(\bar{a}_{I'(1)}, \tau_{a_{i_n}})\big)\Delta(\bar{a}_{I'(2)}) \\
&= \sum \bar{x}_{(1)}\bar{a}_{I'(1)} \cdot \bar{a}_{i_n} \otimes \bar{x}_{(2)}\bar{a}_{I'(2)} + \sum \bar{x}_{(1)}\bar{a}_{I'(1)} \otimes \bar{x}_{(2)}\bar{a}_{I'(2)} \cdot \bar{a}_{i_n} \\
&\quad - \sum \bar{x}_{(1)}\big\langle \bar{x}_{(2)}; l(\bar{a}_{I'(1)}, \tau_{a_{i_n}})\big\rangle \cdot \bar{a}_{I'(2)} \otimes \bar{x}_{(3)}\bar{a}_{I'(3)} \\
&\quad - \sum \bar{x}_{(1)}\bar{a}_{I'(1)} \otimes \bar{x}_{(2)}\big\langle \bar{x}_{(3)}; l(\bar{a}_{I'(2)}, \tau_{a_{i_n}})\big\rangle \cdot \bar{a}_{I'(3)} \\
&= \sum \bar{x}_{(1)}(\bar{a}_{I'(1)}\bar{a}_{i_n}) \otimes \bar{x}_{(2)}\bar{a}_{I'(2)} + \sum \bar{x}_{(1)}\bar{a}_{I'(1)} \otimes \bar{x}_{(2)}(\bar{a}_{I'(2)}\bar{a}_{i_n}) \\
&= \Delta(\bar{x})\Delta(\bar{a}_I).
\end{aligned}$$

Similarly one obtains that $\epsilon(\bar{x}\bar{a}_I) = \epsilon(\bar{x})\epsilon(\bar{a}_I)$. □

The following corollary follows from Proposition 5.

Corollary 25. *There exist unique $\backslash$ and $/$ such that $(\tilde{S}(V), \Delta, \epsilon, \cdot, u, \backslash, /)$ is an H-bialgebra.*

Notice that the left and right divisions are uniquely determined by (1) and (2) and the coalgebra structure indeed. For instance, $\bar{1}\backslash\bar{x} = \bar{x} = \bar{x}/\bar{1}$, $\bar{a}\backslash\bar{x} = -\bar{a}\bar{x}$ and $\bar{x}/\bar{a} = -\bar{x}\bar{a}$ for any $a \in V$ and $\bar{x} \in \tilde{S}(V)$.

Proposition 26. *In $(\tilde{S}(V), l)$ we have that*

$$(\bar{x}\bar{y})d = \bar{x}(\bar{y}d) + \sum \bar{x}l(\bar{y}_{(1)}, d)\bar{y}_{(2)} \quad \forall d \in E.$$

Proof. We may assume that $\bar{y} = \bar{a}_I$ and use induction on $|I|$. If $|I| = 0$ then $\bar{y} = \bar{1}$ and

$$\bar{x}(\bar{1}d) + \bar{x}l(\bar{1}, d)\bar{1} = -\bar{x}\tau_{\langle \bar{1}; d\rangle} + \bar{x}(d + \tau_{\langle \bar{1}; d\rangle}) = \bar{x}d$$

which proves the statement in this case. In general, we have that

$$\begin{aligned}
(\bar{x}\bar{a}_I)d &\overset{\langle 1\rangle}{=} (\bar{x}\bar{a}_{I'})\tau_{a_{i_n}} d - \sum \bigl(\bar{x}l(\bar{a}_{I'(1)}, \tau_{a_{i_n}})\bar{a}_{I'(2)}\bigr)d \\
&\overset{\langle 2\rangle}{=} (\bar{x}\bar{a}_{I'})d\tau_{a_{i_n}} + (\bar{x}\bar{a}_{I'})[\tau_{a_{i_n}}, d] - \sum \bar{x}l(\bar{a}_{I'(1)}, \tau_{a_{i_n}})(\bar{a}_{I'(2)}d) \\
&\quad - \sum \bar{x}l(\bar{a}_{I'(1)}, \tau_{a_{i_n}})l(\bar{a}_{I'(2)}, d)\bar{a}_{I'(3)} \\
&\overset{\langle 3\rangle}{=} (\bar{x}\bar{a}_{I'})d\tau_{a_{i_n}} + \bar{x}(\bar{a}_I d) - \bar{x}(\bar{a}_{I'}d\tau_{a_{i_n}}) + \sum \bar{x}l\bigl(\bar{a}_{I'(1)}, [\tau_{a_{i_n}}, d]\bigr)\bar{a}_{I'(2)} \\
&\quad - \sum \bar{x}l(\bar{a}_{I'(1)}, \tau_{a_{i_n}})(\bar{a}_{I'(2)}d) - \sum \bar{x}l(\bar{a}_{I'(1)}, \tau_{a_{i_n}})l(\bar{a}_{I'(2)}, d)\bar{a}_{I'(3)} \\
&\overset{\langle 4\rangle}{=} (\bar{x}\bar{a}_{I'})d\tau_{a_{i_n}} + \bar{x}(\bar{a}_I d) - \bar{x}(\bar{a}_{I'}d\tau_{a_{i_n}}) + \sum \bar{x}l(\overline{a_{I'(1)}a_{i_n}}, d)\bar{a}_{I'(2)} \\
&\quad - \sum \bar{x}l(\bar{a}_{I'(1)}d, \tau_{a_{i_n}})\bar{a}_{I'(2)} - \sum \bar{x}l(\bar{a}_{I'(1)}, d)l(\bar{a}_{I'(2)}, \tau_{a_{i_n}})\bar{a}_{I'(3)} \\
&\quad - \sum \bar{x}l(\bar{a}_{I'(1)}, \tau_{a_{i_n}})(\bar{a}_{I'(2)}d) \\
&\overset{\langle 5\rangle}{=} \bar{x}(\bar{a}_I d) + \sum \bar{x}l(\bar{a}_{I(1)}, d)\bar{a}_{I(2)} + \bigl(\bar{x}(\bar{a}_{I'}d)\bigr)\tau_{a_{i_n}} - \bar{x}(\bar{a}_{I'}d\tau_{a_{i_n}}) \\
&\quad - \sum \bar{x}l\bigl((\bar{a}_{I'}d)_{(1)}, \tau_{a_{i_n}}\bigr)(\bar{a}_{I'}d)_{(2)},
\end{aligned}$$

where $\langle 1\rangle$ follows by definition, $\langle 2\rangle$ and $\langle 3\rangle$ by induction, $\langle 4\rangle$ by the properties imposed to l and in $\langle 5\rangle$ we have used that

$$\begin{aligned}
(\bar{x}\bar{a}_{I'})d\tau_{a_{i_n}} &= \Bigl(\bar{x}(\bar{a}_{I'}d) + \sum \bar{x}l(\bar{a}_{I'(1)}, d)\bar{a}_{I'(2)}\Bigr)\tau_{a_{i_n}} \\
&= \bigl(\bar{x}(\bar{a}_{I'}d)\bigr)\tau_{a_{i_n}} + \sum \bar{x}l(\bar{a}_{I'(1)}, d)\cdot \bar{a}_{I'(2)}\bar{a}_{i_n} + \sum \bar{x}l(a_{I'(1)}, d)l(\bar{a}_{I'(2)}, \tau_{a_{i_n}})\bar{a}_{I'(3)} \\
&= \bigl(\bar{x}(\bar{a}_{I'}d)\bigr)\tau_{a_{i_n}} + \sum \bar{x}l(\bar{a}_{I(1)}, d)\cdot \bar{a}_{I(2)} - \sum \bar{x}l(\bar{a}_{I'(1)}\tau_{a_{i_n}}, d)\bar{a}_{I'(2)} \\
&\quad + \sum \bar{x}l(a_{I'(1)}, d)l(\bar{a}_{I'(2)}, \tau_{a_{i_n}})\bar{a}_{I'(3)}.
\end{aligned}$$

In case that $d = \tau_a$ with $a \geqslant a_{i_n}$ then the statement follows from the definition of the product, and the previous computation is superfluous. In case that $d \in H$ then by the hypothesis of induction $(\bar{x}(\bar{a}_{I'}d))\tau_{a_{i_n}} - \bar{x}(\bar{a}_{I'}d\tau_{a_{i_n}}) - \sum \bar{x}l((\bar{a}_{I'}d)_{(1)}, \tau_{a_{i_n}})(\bar{a}_{I'}d)_{(2)} = 0$ and the result follows. Finally, in case that $d = \tau_a$ with $a < a_{i_n}$ then up to terms of lower degree we may order the monomial $\bar{a}_{I'}d$ falling in one of the previous cases. □

Proposition 27. *In $(\tilde{S}(V), l)$ we have that*

$$q(\bar{u}, \bar{v}, \bar{a}) = -\bigl\langle \bar{u}; l(\bar{v}, \tau_a)\bigr\rangle.$$

Proof. Since

$$(\bar{u}, \bar{v}, \bar{a}) = \begin{cases} \sum \bar{u}_{(1)} q(\bar{u}_{(2)}, \bar{v}_{(1)}, \bar{a}) \cdot \bar{v}_{(2)} & \text{(by definition of } q\text{)}, \\ (\bar{u}\bar{v})\tau_a - \bar{u}(\bar{v}\tau_a) = \sum \bar{u} l(\bar{v}_{(1)}, \tau_a)\bar{v}_{(2)}, \end{cases}$$

then the statement follows trivially for $\bar{v} = \bar{1}$. Now we may proceed by induction on $|\bar{v}|$ to prove that $-\sum \bar{u}_{(1)} \langle \bar{u}_{(2)}; l(\bar{v}, \bar{a})\rangle = \sum \bar{u}_{(1)} q(\bar{u}_{(2)}, \bar{v}, \bar{a})$:

$$\begin{aligned} \sum \bar{u} l(\bar{v}_{(1)}, \tau_a)\bar{v}_{(2)} = (\bar{u}, \bar{v}, \bar{a}) &= \sum \bar{u}_{(1)} q(\bar{u}_{(2)}, \bar{v}_{(1)}, \bar{a}) \cdot \bar{v}_{(2)} \\ &= \sum \bar{u}_{(1)} q(\bar{u}_{(2)}, \bar{v}, \bar{a}) + \sum_{|\bar{v}_{(2)}| \geqslant 1} \bar{u}_{(1)} q(\bar{u}_{(2)}, \bar{v}_{(1)}, \bar{a}) \cdot \bar{v}_{(2)} \\ &= \sum \bar{u}_{(1)} q(\bar{u}_{(2)}, \bar{v}, \bar{a}) + \sum_{|\bar{v}_{(2)}| \geqslant 1} \bar{u} l(\bar{v}_{(1)}, \tau_a)\bar{v}_{(2)} \end{aligned}$$

implies that $\bar{u} l(\bar{v}, \bar{a}) = \sum \bar{u}_{(1)} q(\bar{u}_{(2)}, \bar{v}, \bar{a})$. Therefore,

$$-\sum \bar{u}_{(1)} \bigl\langle \bar{u}_{(2)}; l(\bar{v}, \bar{a}) \bigr\rangle = \sum \bar{u}_{(1)} q(\bar{u}_{(2)}, \bar{v}, \bar{a}).$$

From this relation one easily obtains the result by induction on the degree of $\bar{u}$. □

Note 2. It is worthwhile to notice that although in the previous proof we used induction, we might avoid it just by computing $\sum \bar{u}_{(1)} \backslash ((\bar{u}_{(2)}, \bar{v}_{(1)}, \bar{a})/\bar{v}_{(2)})$ in two different ways as in the proof. One leads to $q(\bar{u}, \bar{v}, \bar{a})$ while the other leads to $-\langle \bar{u}; l(\bar{u}, \tau_a)\rangle$.

We recover the operations $\langle\, ;\, \rangle$ as expected by the Shestakov–Umirbaev construction.

Proposition 28. *In $(\tilde{S}(V), l)$ we have that if $|u| \geqslant 1$ then*

$$\langle \bar{u}; \bar{a}, \bar{b} \rangle = q(\bar{u}, \bar{b}, \bar{a}) - q(\bar{u}, \bar{a}, \bar{b}) \quad \textit{and} \quad \langle \bar{a}, \bar{b} \rangle = -[\bar{a}, \bar{b}].$$

Proof. We first note that since $l(\bar{1}, \tau_a) = 0$ then

$$l\bigl(\bar{1}, [\tau_a, \tau_b]\bigr) = l(\bar{a}, \tau_b) - l(\bar{b}, \tau_a).$$

So,

$$\begin{aligned} q(\bar{u}, \bar{b}, \bar{a}) - q(\bar{u}, \bar{a}, \bar{b}) &= -\bigl\langle \bar{u}; l(\bar{b}, \tau_a) \bigr\rangle + \bigl\langle \bar{u}; l(\bar{a}, \tau_b) \bigr\rangle \\ &= \bigl\langle \bar{u}; l\bigl(\bar{1}, [\tau_a, \tau_b]\bigr) \bigr\rangle = \bigl\langle \bar{u}; [\tau_a, \tau_b] + \tau_{\langle \bar{1}; [\tau_a, \tau_b] \rangle} \bigr\rangle \\ &\overset{\langle 1 \rangle}{=} \bigl\langle \bar{u}; [\tau_a, \tau_b] \bigr\rangle = \langle \bar{u}; \bar{a}, \bar{b} \rangle, \end{aligned}$$

where $\langle 1 \rangle$ follows because $|\bar{u}| \geqslant 1$ and the definition of τ_a. □

5.3. Degrees of freedom of l

Let $\hat{\Phi} : \tilde{S}(V) \otimes \tilde{S}(V) \to V$ be an arbitrary linear map, that will ultimately collect the degrees of freedom of l, verifying

$$\hat{\Phi}(\bar{1}, \bar{y}) = 0 \quad \forall \bar{y},$$
$$\hat{\Phi}(\bar{x}, \bar{a}) = \hat{\Phi}(\bar{x}, \bar{1}) = 0 \quad \forall a \in V,\ \bar{x} \in \tilde{S}(V),$$

and $B = \{a_i\}_{i\in\Lambda}$ an ordered basis of V. While the first requirement about $\hat{\Phi}$ will be necessary to ensure that the image of l lays in H, the second is superfluous and does not play any special role in the following.

From $\hat{\Phi}$ and the ordered basis B we define another map

$$l : \tilde{S}(V) \otimes E \to H$$

as follows:

1. $l(\bar{1}, d) = d + \tau_{\langle 1;d\rangle}$,
2. $l(\bar{a}_I, \tau_a) = \hat{\Phi}(-, \overline{a_I a}) \quad$ if $a_{i_n} \leqslant a$,
3. $l(\bar{a}_I, d) = l(\bar{a}_{I'} d, \tau_{a_{i_n}}) + l(\bar{a}_{I'}, [\tau_{a_{i_n}}, d]) - \sum [l(\bar{a}_{I'(1)}, \tau_{a_{i_n}}), l(\bar{a}_{I'(2)}, d)]$
   if $d \in H$ or $d = \tau_a$ with $a < a_{i_n}$. (26)

We should notice that in this later case,

$$l(\bar{a}_{I'}, [\tau_{a_{i_n}}, d]) = l(\bar{a}_{I'} \tau_{a_{i_n}}, d) - l(\bar{a}_{I'} d, \tau_{a_{i_n}}) + \sum [l(\bar{a}_{I'(1)}, \tau_{a_{i_n}}), l(\bar{a}_{I'(2)}, d)]. \tag{27}$$

In case that $d = \tau_a$ with $a \geqslant a_{i_n}$ then it also follows by skew-symmetry.

Proposition 29. *For any $d, d' \in E$ and $\bar{y} \in \tilde{S}(V)$ we have*

$$l(\bar{y}, [d, d']) = l(\bar{y} d, d') - l(\bar{y} d', d) + \sum [l(\bar{y}_{(1)}, d), l(\bar{y}_{(2)}, d')].$$

Proof. We may assume that $\bar{y} = \bar{a}_I$ and use induction on $|I|$. If $|I| = 0$ then $\bar{y} = \bar{1}$ and

$$l(\bar{1}, [d, d']) = l(\bar{1} d, d') - l(\bar{1} d', d) + [l(\bar{1}, d), l(\bar{1}, d')]$$

follows trivially if $d, d' \in H$, and by the remark otherwise.

Now we will deal with the general case. We have by (27) and the Jacobi identity

$$\begin{aligned} l(\bar{a}_I, [d, d']) &= l(\bar{a}_{I'} \tau_{a_{i_n}}, [d, d']) = l(\bar{a}_{I'}, [\tau_{a_{i_n}}, [d, d']]) + l(\bar{a}_{I'}[d, d'], \tau_{a_{i_n}}) \\ &\quad - \sum [l(\bar{a}_{I'(1)}, \tau_{a_{i_n}}), l(\bar{a}_{I'(2)}, [d, d'])] \\ &= l(\bar{a}_{I'}, [[\tau_{a_{i_n}}, d], d']) + l(\bar{a}_{I'}, [d, [\tau_{a_{i_n}}, d']]) + l(\bar{a}_{I'} d d', \tau_{a_{i_n}}) \\ &\quad - l(\bar{a}_{I'} d' d, \tau_{a_{i_n}}) - \sum [l(\bar{a}_{I'(1)}, \tau_{a_{i_n}}), l(\bar{a}_{I'(2)}, [d, d'])]. \end{aligned}$$

Now, applying the hypothesis of induction to this last expression we get

$$\begin{aligned} l\big(\bar a_I,[d,d']\big) &= l\big(\bar a_{I'}[\tau_{a_{i_n}},d],d'\big) - l\big(\bar a_{I'}d',[\tau_{a_{i_n}},d]\big) + \sum\big[l\big(\bar a_{I'(1)},[\tau_{a_{i_n}},d]\big),l(\bar a_{I'(2)},d')\big] \\ &\quad + l\big(\bar a_{I'}d,[\tau_{a_{i_n}},d']\big) - l\big(\bar a_{I'}[\tau_{a_{i_n}},d'],d\big) + \sum\big[l(\bar a_{I'(1)},d),l\big(\bar a_{I'(2)},[\tau_{a_{i_n}},d']\big)\big] \\ &\quad + l(\bar a_{I'}dd',\tau_{a_{i_n}}) - l(\bar a_{I'}d'd,\tau_{a_{i_n}}) - \sum\big[l(\bar a_{I'(1)},\tau_{a_{i_n}}),l\big(\bar a_{I'(2)},[d,d']\big)\big]. \end{aligned}$$

Again, by (27) and the hypothesis of induction,

$$\begin{aligned} l\big(\bar a_I,[d,d']\big) &= l(\bar a_I d,d') - l(\bar a_{I'}d\tau_{a_{i_n}},d') + l\big(\bar a_{I'}d,[\tau_{a_{i_n}},d']\big) + l(\bar a_{I'}dd',\tau_{a_{i_n}}) \\ &\quad - l(\bar a_I d',d) + l(\bar a_{I'}d'\tau_{a_{i_n}},d) - l\big(\bar a_{I'}d',[\tau_{a_{i_n}},d]\big) - l(\bar a_{I'}d'd,\tau_{a_{i_n}}) \\ &\quad + \Big(\sum\big[l(\bar a_{I'(1)}\bar a_{i_n},d),l(\bar a_{I'(2)},d')\big] - \sum\big[l(\bar a_{I'(1)}d,\tau_{a_{i_n}}),l(\bar a_{I'(2)},d')\big] \\ &\quad + \sum\big[\big[l(\bar a_{I'(1)},\tau_{a_{i_n}}),l(\bar a_{I'(2)},d)\big],l(\bar a_{I'(3)},d')\big]\Big) \\ &\quad + \Big(\sum\big[l(\bar a_{I'(1)},d),l(\bar a_{I'(2)}\tau_{a_{i_n}},d')\big]\sum\big[l(\bar a_{I'(1)},d),l(\bar a_{I'(2)}d',\tau_{a_{i_n}})\big] \\ &\quad + \sum\big[l(\bar a_{I'(1)},d),\big[l(\bar a_{I'(2)},\tau_{a_{i_n}}),l(\bar a_{I'(3)},d')\big]\big]\Big) \\ &\quad + \Big(-\sum\big[l(\bar a_{I'(1)},\tau_{a_{i_n}}),l(\bar a_{I'(2)}d,d')\big] + \sum\big[l(\bar a_{I'(1)},\tau_{a_{i_n}}),l(\bar a_{I'(2)}d',d)\big] \\ &\quad - \sum\big[l(\bar a_{I'(1)},\tau_{a_{i_n}}),\big[l(\bar a_{I'(2)},d),l(\bar a_{I'(3)},d')\big]\big]\Big). \end{aligned}$$

The cyclic sum on $d, d', \tau_{a_{i_n}}$ of $\sum[[l(\bar a_{I'(1)},\tau_{a_{i_n}}),l(\bar a_{I'(2)},d)],l(\bar a_{I'(3)},d')]$ in this last term vanishes by the Jacobi identity. So, collecting together some terms we obtain

$$\begin{aligned} l\big(\bar a_I,[d,d']\big) &= l(\bar a_I d,d') - l(\bar a_I d',d) + \sum\big[l(\bar a_{I(1)},d),l(\bar a_{I(2)},d')\big] + l\big(\bar a_{I'}d,[\tau_{a_{i_n}},d']\big) \\ &\quad - l(\bar a_{I'}d\tau_{a_{i_n}},d') + l(\bar a_{I'}dd',\tau_{a_{i_n}}) - \sum\big[l\big((\bar a_{I'}d)_{(1)},\tau_{a_{i_n}}\big),l\big((\bar a_{I'}d)_{(2)},d'\big)\big] \\ &\quad - l\big(\bar a_{I'}d',[\tau_{a_{i_n}},d]\big) + l(\bar a_{I'}d'\tau_{a_{i_n}},d) - l(\bar a_{I'}d'd,\tau_{a_{i_n}}) \\ &\quad + \sum\big[l\big((\bar a_{I'}d')_{(1)},\tau_{a_{i_n}}\big),l\big((\bar a_{I'}d')\big)_{(2)},d\big]. \end{aligned}$$

If $d \in H$ or $d = \tau_a$ with $a \leqslant a_{i_n}$ and also $d' \in H$ or $d' = \tau_b$ with $b \leqslant a_{i_n}$, then we may use the hypothesis of induction and (27) to conclude that

$$l\big(\bar a_{I'}d,[\tau_{a_{i_n}},d']\big) - l(\bar a_{I'}d\tau_{a_{i_n}},d') + l(\bar a_{I'}dd',\tau_{a_{i_n}}) - \sum\big[l\big((\bar a_{I'}d)_{(1)},\tau_{a_{i_n}}\big),l\big((\bar a_{I'}d)_{(2)},d'\big)\big] = 0$$

and

$$\begin{aligned} &l\big(\bar a_{I'}d',[\tau_{a_{i_n}},d]\big) + l(\bar a_{I'}d'\tau_{a_{i_n}},d) - l(\bar a_{I'}d'd,\tau_{a_{i_n}}) + \sum\big[l\big((\bar a_{I'}d')_{(1)},\tau_{a_{i_n}}\big),l\big((\bar a_{I'}d')_{(2)},d\big)\big] \\ &\quad = 0, \end{aligned}$$

therefore, the statement follows. Otherwise, i.e. d or d' is of the form τ_a with $a \geqslant a_{i_n}$ then the statement is a direct consequence of (27). □

5.4. Sabinin algebras from primitive elements (II): A universal enveloping algebra for Sabinin algebras

Theorem 30. *Let $(V, \langle ; \rangle, \Phi)$ be a Sabinin algebra over a field of characteristic zero. There exist a unital algebra $\mathrm{U}(V, \langle ; \rangle, \Phi)$ and a monomorphism of Sabinin algebras $\iota : V \to$ УШ$(\mathrm{U}(V, \langle ; \rangle, \Phi))$ such that for any unital algebra C and any homomorphism of Sabinin algebras $\varphi : V \to$ УШ(C) there exists a unique homomorphism of unital algebras $\bar{\varphi} : \mathrm{U}(V, \langle ; \rangle, \Phi) \to C$ with $\varphi = \bar{\varphi} \circ \iota$.*

Proof. We will construct an algebra $(\tilde{S}(V), l)$ with the property that $\iota : V \to$ УШ$((\tilde{S}(V), l))$ is a monomorphism of Sabinin algebras. The universal property will follow easily.

By the Poincaré–Birkhoff–Witt Theorem we may look at V as contained in $\tilde{S}(V)$. We will fix an ordered basis $\{a_i\}_{i \in \Lambda}$ of V.

By Proposition 27, in order to obtain that

$$\Phi(\bar{x}_1, \dots, \bar{x}_m; \bar{y}_1, \dots, \bar{y}_n) = \frac{1}{m!} \frac{1}{n!} \sum_{\tau \in S_m, \delta \in S_n} q_{m,n-1}(\bar{x}_{\tau(1)}, \dots, \bar{x}_{\tau(m)}, \bar{y}_{\delta(1)}, \dots, \bar{y}_{\delta(n)})$$

the map l should satisfy

$$-\frac{1}{m!} \frac{1}{n!} \sum_{\tau \in S_m, \delta \in S_n} \big\langle \overline{x_{\tau(1)} \cdots x_{\tau(m)}}; l(\overline{y_{\delta(1)} \cdots y_{\delta(n-1)}}, \tau_{y_{\delta(n)}}) \big\rangle = \Phi(\bar{x}_1, \dots, \bar{x}_m; \bar{y}_1, \dots, \bar{y}_n). \tag{28}$$

The map l is defined through an auxiliary map $\hat{\Phi}$ as in (26). We set $\hat{\Phi}(-, \bar{1}) = \hat{\Phi}(-, \bar{y}_1) = 0$ and assume that we have defined $\hat{\Phi}(-, \overline{a_{i_1} \cdots a_{i_n}})$ ($n \geqslant 2$) so that the corresponding l in (26) satisfies (28). Modulo the subalgebra of H generated by $l(\tilde{S}(V)_{n-1}, E)$ (which by (27) at this point is already constructed) we have that

$$\begin{aligned} -\frac{1}{(n+1)!} \sum_{\delta \in S_{n+1}} l(\overline{a_{i_{\delta(1)}} \cdots a_{i_{\delta(n)}}}, \tau_{a_{i_{\delta(n+1)}}}) &\equiv -l(\overline{a_{i_1} \cdots a_{i_n}}, \tau_{a_{i_{n+1}}}) \\ &= -\hat{\Phi}(-, \overline{a_{i_1} \cdots a_{i_{n+1}}}), \end{aligned}$$

therefore, we may choose $\hat{\Phi}(-, \overline{a_{i_1} \cdots a_{i_{n+1}}})$ so that $\hat{\Phi}(\bar{1}, \overline{a_{i_1} \cdots a_{i_{n+1}}}) = 0$ and the corresponding l in (26) satisfies

$$\begin{aligned} &-\frac{1}{m!} \frac{1}{(n+1)!} \sum_{\tau \in S_m, \delta \in S_{n+1}} \big\langle \overline{x_{\tau(1)} \cdots x_{\tau(m)}}; l(\overline{a_{i_{\delta(1)}} \cdots a_{i_{\delta(n)}}}, \tau_{a_{i_{\delta(n+1)}}}) \big\rangle \\ &\quad = \Phi(\bar{x}_1, \dots, \bar{x}_m; \bar{a}_{i_1}, \dots, \bar{a}_{i_{n+1}}) \end{aligned}$$

as desired (the argument also shows how to set the initial step $n = 2$ of the induction). This together with Proposition 28 proves that $\iota : V \to$ УШ$((\tilde{S}(V), l))$ is a monomorphism of Sabinin algebras.

Now we proceed to prove the universal property. Consider a unital algebra C and a homomorphism of Sabinin algebras $\varphi : V \to \text{УШ}(C)$. Since in C we have

$$\begin{aligned}(ua)b - (ub)a - u[a,b] &= (u,a,b) - (u,b,a)\\ &= -\sum_{|u_{(2)}|\geqslant 1} u_{(1)}\big(q(u_{(2)},b,a) - q(u_{(2)},a,b)\big)\\ &= -\sum_{|u_{(2)}|\geqslant 1} u_{(1)}\langle u_{(2)}; a, b\rangle\end{aligned}$$

then $(ub)a - (ua)b = -\sum u_{(1)}\langle u_{(2)}; a, b\rangle$. Therefore, there exists a well defined linear map

$$\begin{aligned}&\bar{\varphi} : \tilde{S}(V) \to C,\\ &\overline{x_1 \cdots x_n} \mapsto \big(\big(\varphi(x_1)\varphi(x_2)\big)\cdots\big)\varphi(x_n).\end{aligned}$$

We have to show that this map is a homomorphism of algebras. Since $\tilde{S}(V)$ is spanned by $\{\bar{a}^m \mid a \in V,\ m \in \mathbb{N}\}$ we only have to check that $\bar{\varphi}(\bar{x}\bar{b}^n) = \bar{\varphi}(\bar{x})\bar{\varphi}(\bar{b}^n)$. We use induction on n. If $n = 0$ the result is obvious. In general we may assume that $\bar{x} = \bar{a}^m$ so

$$\begin{aligned}\bar{\varphi}\big(\bar{a}^m\bar{b}^n\big) &= \bar{\varphi}\Big(\big(\bar{a}^m\bar{b}^{n-1}\big)\bar{b} - \sum \bar{a}^m_{(1)} q\big(\bar{a}^m_{(2)}, \bar{b}^{n-1}_{(2)}, \bar{b}\big)\cdot \bar{b}^{n-1}_{(1)}\Big)\\ &= \bar{\varphi}\left(\big(\bar{a}^m\bar{b}^{n-1}\big)\bar{b} - \sum_{i,j}\binom{m}{i}\binom{n-1}{j}\bar{a}^{m-i} q\big(\bar{a}^i, \bar{b}^j, \bar{b}\big)\cdot \bar{b}^{n-1-j}\right)\\ &= \bar{\varphi}\left(\big(\bar{a}^m\bar{b}^{n-1}\big)\bar{b} - \sum_{i,j}\binom{m}{i}\binom{n-1}{j}\bar{a}^{m-i}\Phi(a,\ldots,a;b,\ldots,b,b)\cdot \bar{b}^{n-1-j}\right)\\ &= \big(\bar{\varphi}(\bar{a})^m\bar{\varphi}(\bar{b})^{n-1}\big)\bar{\varphi}(\bar{b})\\ &\quad - \sum_{i,j}\binom{m}{i}\binom{n-1}{j}\bar{\varphi}(\bar{a})^{m-i}\bar{\varphi}\big(\Phi(a,\ldots,a;b,\ldots,b,b)\big)\cdot\bar{\varphi}(\bar{b})^{n-1-j}\\ &= \bar{\varphi}(\bar{a})^m\bar{\varphi}(\bar{b})^n = \bar{\varphi}\big(\bar{a}^m\big)\bar{\varphi}\big(\bar{b}^n\big)\end{aligned}$$

and the result follows. □

Corollary 31 *(Milnor–Moore). Over a field of characteristic zero any cocommutative connected unital H-bialgebra H is isomorphic to the universal enveloping algebra* U(Prim(H)) *of the Sabinin algebra* Prim(H).

Proof. Since Prim(H) is a Sabinin subalgebra of УШ(H) then by the universal property of U(Prim(H)) there exists a homomorphism of unital algebras $\bar{\varphi} : \mathrm{U}(\mathrm{Prim}(H)) \to H$ extending the inclusion $\varphi : \mathrm{Prim}(H) \hookrightarrow H$. By results in [29], for instance, this map is injective. Moreover, the arguments in the proof of Theorem 5.6.5 in [14] work verbatim in this case to obtain the bijectivity of $\bar{\varphi}$. □

5.5. Envelopes of Malcev and Bol algebras (II)

Let $(V, [\,,\,,\,], [\,,\,])$ be a right Bol algebra over a field of characteristic $\neq 2$, and $U = \mathrm{U}(V)$ its universal enveloping algebra as defined in [16]. We have that $V \subseteq \mathrm{RN}_{\mathrm{alt}}(U)$ and the ternary and binary products are recovered as $[a, b, c] = cb \cdot a - ca \cdot b - [b, a]c$ and $[a, b] = ab - ba$. We define on V the operations

$$\begin{aligned} \langle 1; a, b\rangle &= -[a, b], \\ \langle c; a, b\rangle &= [a, b, c] - \big[[a, b], c\big] \quad \text{and} \\ \langle xc; a, b\rangle &= \sum \big\langle x_{(1)}; c, \langle x_{(2)}; a, b\rangle\big\rangle \quad \text{if } |x| \geqslant 1. \end{aligned}$$

Proposition 32. $(V, \langle\,;\,\rangle)$ *is a Sabinin subalgebra of* $\text{УШ}(\mathrm{U}(V))$.

Proof. By (19) it is enough to prove that $x[R_a, R_b] = -\sum x_{(1)}\langle x_{(2)}; a, b\rangle$. We will use induction on the degree of x. If $x = 1$ then the result is obvious. If $x = c$ for some $c \in V$ then $c[R_a, R_b] = ca \cdot b - cb \cdot a = -[a, b, c] - [b, a]c = -c[b, a] - [a, b, c] - [[b, a], c] = c[a, b] - \langle c; a, b\rangle$. In the general case we have

$$\begin{aligned} (xc)[R_a, R_b] &= -(x)\big[[R_a, R_b], R_c\big] + (x)[R_a, R_b]R_c \\ &= -x[a, b, c] - \sum x_{(1)}\langle x_{(2)}; a, b\rangle \cdot c, \end{aligned}$$

where the last equality follows from the relation $[[R_a, R_b], R_c] = R_{[a,b,c]}$ valid for any $a, b, c \in \mathrm{RN}_{\mathrm{alt}}(U)$ (see [16]) and the hypothesis of induction. Thus,

$$\begin{aligned} (xc)[R_a, R_b] &= -x[a, b, c] - \sum x_{(1)}c \cdot \langle x_{(2)}; a, b\rangle + \sum x_{(1)}[R_c, R_{\langle x_{(2)}; a, b\rangle}] \\ &= -x[a, b, c] - \sum x_{(1)}c \cdot \langle x_{(2)}; a, b\rangle - \sum x_{(1)}\big\langle x_{(2)}; c, \langle x_{(3)}; a, b\rangle\big\rangle. \end{aligned}$$

Since by definition we have that

$$\begin{aligned} \sum x_{(1)}\langle x_{(2)}c; a, b\rangle &= \langle c; a, b\rangle + \sum_{|x_{(2)}| \geqslant 1} x_{(1)}\langle x_{(2)}c; a, b\rangle \\ &= \langle c; a, b\rangle - \sum_{|x_{(1)}| < |x|} x_{(1)}\big\langle x_{(2)}; c, \langle x_{(3)}; a, b\rangle\big\rangle \\ &= \langle c; a, b\rangle + x\big[c, [a, b]\big] - \sum x_{(1)}\big\langle x_{(2)}; c, \langle x_{(3)}; a, b\rangle\big\rangle \end{aligned}$$

then

$$\begin{aligned} (xc)[R_a, R_b] &= -x[a, b, c] - \sum x_{(1)}c \cdot \langle x_{(2)}; a, b\rangle \\ &\quad - \sum x_{(1)}\langle x_{(2)}c; a, b\rangle + x\langle c; a, b\rangle - x\big[c, [a, b]\big] \\ &= -\sum (xc)_{(1)}\big\langle (xc)_{(2)}; a, b\big\rangle \end{aligned}$$

as desired. $\square$

Proposition 33. *The multioperator inherited by V from* $\mathrm{УШ}(\mathrm{U}(V))$ *is zero.*

Proof. By the definition of Φ it is enough to prove that $(x, a^n, a) = 0\ \forall a \in V$. Recall that if $a \in \mathrm{RN}_{\mathrm{alt}}(U)$ then $R_{ya+ay} = R_y R_a + R_a R_y$ (see [16]). So if we assume proved that $R_{a^n} = R_a^n$ then $a^n a = aa^{n-1} \cdot a = aR_a^{n-1} R_a = aR_a^n = aR_{a^n} = aa^n$, so $2R_{a^{n+1}} = R_{aa^n + a^n a} = R_{a^n} R_a + R_a R_{a^n} = 2R_a^{n+1}$. Therefore, $R_a^n = R_{a^n}$ for any n. In particular, $(x, a^n, a) = xR_{a^n} R_a - xR_{a^{n+1}} = 0$. □

A Sabinin algebra $(V, ())$ is called homogeneous if there exists a bilinear operation $\diamond : V \times V \to V$ such that for any $l \geqslant 2$

$$(x_0 x_1 \cdots x_l) = (x_0 \diamond x_1 \cdots x_l) + \cdots + (x_1 \cdots x_0 \diamond x_l) - x_0 \diamond (x_1 \cdots x_l). \tag{29}$$

The structure of homogeneous Sabinin algebras is determined by the bilinear operations (xy) and $x \diamond y$. From Differential Geometry [13] it is very well known that Malcev algebras (over fields of characteristic $\neq 2, 3$) constitute a natural example of homogeneous Sabinin algebras where the maps $()$ and $\diamond$ are proportional. Any Malcev algebra $(M, [\,,\,])$ is seen as an homogeneous Sabinin algebra by setting $(xy) = [x, y]$ and $x \diamond y = -\frac{1}{3}[x, y]$.

Since any Malcev algebra is a particular example of Bol algebra where $[a, b, c] = [[a, b], c] - \frac{1}{3} J(a, b, c)$ and $J(a, b, c)$ denotes the Jacobian of a, b and c, then we may wonder about the relationship between the structure induced by the universal enveloping algebra $\mathrm{U}(M)$ and the previously known (29). Surprisingly they are essentially the same.

For any Malcev algebra $(M, [\,,\,])$, the structure of Sabinin algebra on M induced by its universal enveloping algebra is given by

$$\langle 1; a, b\rangle = -[a, b],$$
$$\langle c; a, b\rangle = -\frac{1}{3} J(a, b, c),$$
$$\langle xc; a, b\rangle = \sum \big\langle x_{(1)}; c, \langle x_{(2)}; a, b\rangle \big\rangle.$$

The structure of $M^{\mathrm{opp}} = (M, [\,,\,]')$ (the opposite of $(M, [\,,\,])$, which is isomorphic to M) for being an homogeneous Sabinin algebra is given by

$$a \diamond b = -\frac{1}{3}[a, b]' = \frac{1}{3}[a, b],$$
$$(ab) = (1ab) = [a, b]' = -[a, b],$$
$$(x_0 x_1 \cdots x_l) = (x_0 \diamond x_1 \cdots x_l) + \cdots + (x_1 \cdots x_0 \diamond x_l) - x_0 \diamond (x_1 \cdots x_l).$$

For instance,

$$\begin{aligned}(cab) &= (c \diamond ab) + (ac \diamond b) - c \diamond (ab)\\ &= -\frac{1}{3}\big[[c, a], b\big] - \frac{1}{3}\big[a, [c, b]\big] + \frac{1}{3}\big[c, [a, b]\big]\\ &= -\frac{1}{3} J(a, b, c) = \langle c; a, b\rangle.\end{aligned}$$

Proposition 34. *We have that* $\langle x_0 \cdots x_{l-2}; x_{l-1}, x_l\rangle = (x_0 \cdots x_l)$.

Proof. We only have to check that $(xcab) = \sum (x_{(1)}c(x_{(2)}ab))$ if $|x| \geqslant 1$. If $x = d \in M$ then the relation becomes

$$\begin{aligned}
0 &= (dcab) - \big(dc(ab)\big) - \big(c(dab)\big) \\
&= (dcab) - \frac{1}{3}J\big(d, c, [a, b]\big) - \frac{1}{3}\big[c, J(d, a, b)\big] \\
&= (dcab) - 2\big(d, c, [a, b]\big) - 2\big[c, (d, a, b)\big] \\
&= -\frac{2}{3}\big([d, c], a, b\big) - \frac{2}{3}\big(c, [d, a], b\big) - \frac{2}{3}\big(c, a, [d, b]\big) + \frac{2}{3}\big[d, (c, a, b)\big] \\
&\quad - 2\big(d, c, [a, b]\big) - 2\big[c, (d, a, b)\big].
\end{aligned}$$

Since $J(a, b, c) = 6(a, b, c)$ in $\mathrm{U}(M)$, we should check that $-3J(d, c, [a, b]) - 3[c, J(d, a, b)] - J([d, c], a, b) - J(c, [d, a], b) - J(c, a, [d, b]) + [d, J(c, a, b)] = 0$ on any Malcev algebra. With the notation $J(a, b, c) = a\Delta(b, c)$ as in [25] this equality is equivalent to $2[-\mathrm{ad}_d, \Delta(a, b)] = 2\Delta([d, a], b) + 2\Delta(a, [d, b]) - 6\Delta(d, [a, b]) + 6(-\mathrm{ad}_{J(d,a,b)})$. Using that $2[-\mathrm{ad}_d, \Delta(a, b)] = -3\Delta(d, [a, b]) + \Delta(a, [b, d]) + \Delta(b, [d, a])$, formula (2.35) in [25], then we only need to prove that $0 = \Delta([d, a], b) + \Delta(a, [d, b]) - \Delta(d, [a, b]) + 2(-\mathrm{ad}_{J(d,a,b)})$ which is exactly the identity (2.32) in [25].

Assume that we have proved that $(xab) = \sum (x'_{(1)}x_l(x'_{(2)}ab))$ with $x = x_1 \cdots x_l$ and $x' = x_1 \cdots x_{l-1}$. Then using a dot to separate arguments and using the notation $(x_0 \diamond x \cdot ab) = (x_0 \diamond x_1 \cdot x_2 \cdots x_l ab) + \cdots + (x_1 x_2 \cdots x_0 \diamond x_l \cdot ab)$ we have

$$\begin{aligned}
(x_0 xab) &= (x_0 \diamond x \cdot ab) + (x \cdot x_0 \diamond a \cdot b) + (xa \cdot x_0 \diamond b) - x_0 \diamond (xab) \\
&= \sum \big(x_0 \diamond x'_{(1)} \cdot x_l(x'_{(2)}ab)\big) + \sum \big(x'_{(1)} \cdot x_0 \diamond x_l \cdot (x'_{(2)}ab)\big) \\
&\quad + \sum \big(x'_{(1)}x_l(x_0 \diamond x'_{(2)} \cdot ab)\big) + \sum \big(x'_{(1)}x_l(x'_{(2)} \cdot x_0 \diamond a \cdot b)\big) \\
&\quad + \sum \big(x'_{(1)}x_l(x'_{(2)}a \cdot x_0 \diamond b)\big) - \sum x_0 \diamond \big(x'_{(1)}x_l(x'_{(2)}ab)\big) \\
&= \sum \big(x_0 x'_{(1)}x_l(x'_{(2)}ab)\big) + \sum \big(x'_{(1)}x_l(x_0 x'_{(2)}ab)\big) \\
&= \sum \big((x_0 x')_{(1)}x_l\big((x_0 x')_{(2)}ab\big)\big)
\end{aligned}$$

as desired. □

Corollary 35. *Let* $(M, [\,,\,])$ *be a Malcev algebra and* $(M, (\,))$ *the homogeneous Sabinin algebra defined by* (29) *with* $(xy) = [x, y]$, $x \diamond y = -\frac{1}{3}[x, y]$ *and trivial multioperator. Then the universal enveloping algebra* $\mathrm{U}((M, (\,)))$ *of the Sabinin algebra* $(M, (\,))$ *is isomorphic* (*as H-bialgebra*) *to the universal enveloping algebra* $\mathrm{U}(M)$ *of the Malcev algebra* $(M, [\,,\,])$.

Proof. Consider $M^{\mathrm{opp}} = (M, [\,,\,]')$ the opposite algebra of $(M, [\,,\,])$. By Propositions 34 and 33 we have a monomorphism of Sabinin algebras $(M, (\,)) \to \text{УШ}(\mathrm{U}(M^{\mathrm{opp}}))$. By the universal property of $\mathrm{U}((M, (\,)))$ this map extends to a homomorphism $\mathrm{U}((M, (\,))) \to \mathrm{U}(M^{\mathrm{opp}})$. Since this map obviously sends the Poincaré–Birkhoff–Witt basis of $\mathrm{U}((M,()))$ to the Poincaré–Birkhoff–Witt basis of $\mathrm{U}(M^{\mathrm{opp}})$ then it is an isomorphism as bialgebras. Since the left and right division on $\mathrm{U}((M,()))$ and $\mathrm{U}(M^{\mathrm{opp}})$ are determined by the bialgebra structure (they both are connected coalgebras) then this map is an isomorphism as H-bialgebras indeed. Finally, it is easy to prove that the isomorphism $x \mapsto -x$ form M^{opp} to M extends to an isomorphism of H-bialgebras between $\mathrm{U}(M^{\mathrm{opp}})$ and $\mathrm{U}(M)$. □

Acknowledgment

I would like to thank the referee for suggestions that improved the readability of the paper.

Appendix A. An alternative proof of the Poincaré–Birkhoff–Witt Theorem for Sabinin algebras

Let us present a proof of the Poincaré–Birkhoff–Witt Theorem for Sabinin algebras based on Gröbner bases so that the usual Poincaré–Birkhoff–Witt Theorem for Lie algebras becomes a particular instance. *We restrict ourselves to the case* $\dim V \leqslant \aleph_0$.

In the following we will use the word monomial as synonymous of (associative) monomial in the basis $\{a_1, a_2, \ldots\}$ of V. It will be convenient to order these monomials by the *deglex* order. In this order $u < v$ if and only if either $\deg(u) < \deg(v)$ or $\deg(u) = \deg(v)$ and $u = wa_iu'$, $v = wa_jv'$ with $i < j$. The main feature of this order to our purposes is that it is multiplicative and it satisfies the descending chain condition allowing us to use induction (see [5] from which we borrow the main ideas in our arguments). The leading monomial $LM(f)$ of a nonzero element f in $\mathrm{T}(V)$ is defined in the obvious way, but it will always be taken with coefficient 1.

Let

$$I = \operatorname{span}\Big\langle x[a,b]y + \sum x_{(1)}\langle x_{(2)}; a,b\rangle y \;\Big|\; x,y \in \mathrm{T}(V),\ a,b \in V\Big\rangle$$

with $[a,b] = ab - ba$ and

$$G = \Big\{x[a,b] + \sum x_{(1)}\langle x_{(2)}; a,b\rangle \;\Big|\; x \text{ is a monomial ordered in a nondecreasing way and } a > b\Big\}.$$

Lemma 36. *The right ideal I of* $\mathrm{T}(V)$ *is generated by G.*

Proof. Given a monomial w we define

$$\hat{I}_w = \operatorname{span}\big\langle gu \mid g \in G,\ u \text{ is a monomial, and } LM(g)u \leqslant w\big\rangle.$$

By definition $\hat{I}_w \subseteq I$ (in fact $\hat{I}_w$ is contained in the right ideal generated by G) for any w so, we will finish once we prove the following claim:

$$f = x[a,b] + \sum x_{(1)}\langle x_{(2)}; a,b\rangle \in \hat{I}_{LM(f)}. \tag{A.1}$$

This element might not belong to G since no order is assumed on x. We will prove this claim by using induction on the degree of x. Without lost of generality we may assume that $a > b$. If x is ordered in a nondecreasing way, and this happens if $\deg(x) \leqslant 1$ for instance, then f belongs to G and the result follows. Assume that we have proved the claim for any x with $\deg(x) < n$. Given any monomial x of degree n we define $O(x) = \max\{\deg(x') \mid x = x'y$ for some monomials x' and y with x' ordered$\}$. We will use backwards induction (starting from n) on $O(x)$ to show that the claim also holds if $\deg(x) = n$. If $O(x) = n$ then x is ordered and we know that the claim is true in this case. Therefore we may assume that $x = x'dcx''$ with $d > c$ and x' ordered. We have that

$$
\begin{aligned}
f &= x'dcx''[a,b] + \sum x_{(1)}\langle x_{(2)}; a, b\rangle \\
&= x'cdx''[a,b] + x'[d,c]x''[a,b] + \sum x_{(1)}\langle x_{(2)}; a, b\rangle.
\end{aligned}
$$

Since $O(x'cdx'') > O(x'dcx'')$ then by the backwards induction $x'cdx''[a,b] + \sum (x'cdx'')_{(1)} \times \langle (x'cdx'')_{(2)}; a, b\rangle \in \hat{I}_{LM(x'cdx''[a,b])} \subseteq \hat{I}_{LM(f)}$ (note that $\hat{I}_{w'} \subseteq \hat{I}_w$ if $w' \leqslant w$). Thus, modulo $\hat{I}_{LM(f)}$,

$$
\begin{aligned}
f &\equiv -\sum (x'cdx'')_{(1)}\big\langle (x'cdx'')_{(2)}; a, b\big\rangle + x'[d,c]x''[a,b] + \sum x_{(1)}\langle x_{(2)}; a, b\rangle \\
&\overset{\langle 1\rangle}{=} -\sum x'_{(1)}cdx''_{(1)}\langle x'_{(2)}x''_{(2)}; a, b\rangle - \sum x'_{(1)}cx''_{(1)}\langle x'_{(2)}dx''_{(2)}; a, b\rangle \\
&\quad - \sum x'_{(1)}dx''_{(1)}\langle x'_{(2)}cx''_{(2)}; a, b\rangle - \sum x'_{(1)}x''_{(1)}\langle x'_{(2)}cdx''_{(2)}; a, b\rangle \\
&\quad + x'[d,c]x''[a,b] + \sum x_{(1)}\langle x_{(2)}; a, b\rangle \\
&\overset{\langle 2\rangle}{=} \sum x'_{(1)}[d,c]x''_{(1)}\langle x'_{(2)}x''_{(2)}; a, b\rangle + \sum x'_{(1)}x''_{(1)}\big\langle x'_{(2)}[d,c]x''_{(2)}; a, b\big\rangle + x'[d,c]x''[a,b],
\end{aligned}
$$

where $\langle 1\rangle$ follows by expanding $\sum (x'cdx'')_{(1)}\langle (x'cdx'')_{(2)}; a, b\rangle$ (recall that c and d are primitive elements) and $\langle 2\rangle$ follows by expanding $\sum x_{(1)}\langle x_{(2)}; a, b\rangle$ and simplifying terms. Since $x'[d,c] + \sum x'_{(1)}\langle x'_{(2)}; d, c\rangle \in G$ then by the very definition of $\hat{I}_w$ we obtain that $x'[d,c]x''[a,b] + \sum x'_{(1)}\langle x'_{(2)}; d, c\rangle x''[a,b] \in \hat{I}_{LM(f)}$. Therefore, modulo $\hat{I}_{LM(f)}$,

$$
\begin{aligned}
f &\equiv \sum x'_{(1)}[d,c]x''_{(1)}\langle x'_{(2)}x''_{(2)}; a, b\rangle + \sum x'_{(1)}x''_{(1)}\big\langle x'_{(2)}[d,c]x''_{(2)}; a, b\big\rangle \\
&\quad - \sum x'_{(1)}\langle x'_{(2)}; d, c\rangle x''[a,b] \\
&\overset{\langle 3\rangle}{\equiv} \sum x'_{(1)}[d,c]x''_{(1)}\langle x'_{(2)}x''_{(2)}; a, b\rangle + \sum x'_{(1)}x''_{(1)}\big\langle x'_{(2)}[d,c]x''_{(2)}; a, b\big\rangle \\
&\quad + \sum x'_{(1)}\langle x'_{(2)}; d, c\rangle x''_{(1)}\langle x'_{(3)}x''_{(2)}; a, b\rangle + \sum x'_{(1)}x''_{(1)}\big\langle x'_{(2)}\langle x'_{(3)}; d, c\rangle x''_{(2)}; a, b\big\rangle,
\end{aligned}
$$

where in $\langle 3\rangle$ we have used that the claim holds for elements of degree $< n$ and that $\hat{I}_{LM(x'_{(1)}\langle x'_{(2)}; d, c\rangle x''[a,b])} \subseteq \hat{I}_{LM(f)}$ (compare the degree of the subindices). Finally, since

$$\sum x'_{(1)}[d,c]x''_{(1)}\langle x'_{(2)}x''_{(2)};a,b\rangle + \sum x'_{(1)}\langle x'_{(2)};d,c\rangle x''_{(1)}\langle x'_{(3)}x''_{(2)};a,b\rangle$$
$$= \sum\Big(x'_{(2)}[d,c] + \sum x'_{(2)}\langle x'_{(3)};d,c\rangle\Big)x''_{(1)}\langle x'_{(1)}x''_{(2)};a,b\rangle$$

and $x'_{(2)}[d,c] + \sum x'_{(2)}\langle x'_{(3)};d,c\rangle \in G$ then modulo $\hat{I}_{LM(f)}$ this sum vanishes. So,

$$f \equiv \sum x'_{(1)}x''_{(1)}\big\langle x'_{(2)}[d,c]x''_{(2)};a,b\big\rangle + \sum x'_{(1)}x''_{(1)}\big\langle x'_{(2)}\langle x'_{(3)};d,c\rangle x''_{(2)};a,b\big\rangle$$

which by (12) gives $f \in \hat{I}_{LM(f)}$. □

Theorem 37 *(Poincaré–Birkhoff–Witt). Let $\{a_1, a_2, \ldots\}$ be a basis of V. Then*

$$\{\overline{a_{i_1}\cdots a_{i_n}} \mid i_1 \leqslant \cdots \leqslant i_n \text{ and } n \geqslant 0\}$$

is a basis of $\tilde{S}(V)$.

Proof. As in [5] it is enough to prove that whenever $LM(g)u = LM(g')v$ then $gu - g'v \in \operatorname{span}\langle g''u \mid g'' \in G, u$ is a monomial and $LM(g'')u < LM(g)u\rangle$. For short we will denote $LM(g)u$ by t and $\operatorname{span}\langle g''u \mid g'' \in G, u$ is a monomial and $LM(g'')u < LM(g)u\rangle$ by I_t.

Let x, x' be two ordered monomials, $a > b$, $a' > b'$ and u, v such that

$$xabu = x'a'b'v.$$

We have that either $xab = x'a'b'$ and $u = v$ (there is nothing to prove in this case) or xab is a prefix of $x'a'b'$ with $xab \neq x'a'b'$. We will focus on this later case. Since $a > b$ and x' is ordered then $x' = xa$ and $a' = b$, so $u = b'v$. That is,

$$x' = xa, \quad a' = b \quad \text{and} \quad u = b'v \quad \text{with } a > b > b' \text{ and the factors of } x \text{ are } \leqslant a.$$

On the other hand,

$$gu - g'v = \Big(x[a,b] + \sum x_{(1)}\langle x_{(2)};a,b\rangle\Big)u - \Big(x'[a',b'] + \sum x'_{(1)}\langle x'_{(2)};a',b'\rangle\Big)v$$
$$= xab'bv - xbab'v + \sum x_{(1)}\langle x_{(2)};a,b\rangle b'v - \sum x_{(1)}a\langle x_{(2)};b,b'\rangle v$$
$$- \sum x_{(1)}\langle x_{(2)}a;b,b'\rangle v.$$

Therefore, if we prove that the element C defined by

$$xab'b - xbab' + \sum x_{(1)}\langle x_{(2)};a,b\rangle b' - \sum x_{(1)}a\langle x_{(2)};b,b'\rangle - \sum x_{(1)}\langle x_{(2)}a;b,b'\rangle$$

belongs to $I_{xabb'}$ then $gu - g'v$ will belong to I_{xabu} as desired.

Since

$$C = xab'b - xbab' + \sum x_{(1)}\langle x_{(2)}; a, b\rangle b' - \sum x_{(1)}\langle x_{(2)}; b, b'\rangle a$$
$$+ \sum x_{(1)}\big[\langle x_{(2)}; b, b'\rangle, a\big] - \sum x_{(1)}\langle x_{(2)}a; b, b'\rangle$$

and $x[b,b'] + \sum x_{(1)}\langle x_{(2)}; b, b'\rangle$, $\sum x_{(1)}[\langle x_{(2)}; b, b'\rangle, a] + \sum x_{(1)}\langle x_{(2)}; \langle x_{(3)}; b, b'\rangle, a\rangle \in \operatorname{span}\langle G\rangle$ then modulo $I_{xabb'}$ we have

$$\begin{aligned}
C &\equiv xab'b - xbab' + \sum x_{(1)}\langle x_{(2)}; a, b\rangle b' + x[b,b']a - \sum x_{(1)}\big\langle x_{(2)}; \langle x_{(3)}; b, b'\rangle, a\big\rangle \\
&\quad - \sum x_{(1)}\langle x_{(2)}a; b, b'\rangle \\
&= xab'b - xbab' + \sum x_{(1)}b'\langle x_{(2)}; a, b\rangle - \sum x_{(1)}\big[b', \langle x_{(2)}; a, b\rangle\big] + x[b,b']a \\
&\quad - \sum x_{(1)}\big\langle x_{(2)}; \langle x_{(3)}; b, b'\rangle, a\big\rangle - \sum x_{(1)}\langle x_{(2)}a; b, b'\rangle.
\end{aligned}$$

By (A.1) $(xb')[a,b] + \sum (xb')_{(1)}\langle (xb')_{(2)}; a, b\rangle \in \hat{I}_{xb'ab}$ (the leading monomial of this element is $xb'ab$) and since $b' < a$ then by definition $\hat{I}_{xb'ab} \subseteq I_{xabb'}$, so

$$\begin{aligned}
C &\equiv xab'b - xbab' - xb'[a,b] - \sum x_{(1)}\langle x_{(2)}b'; a, b\rangle - \sum x_{(1)}\big[b', \langle x_{(2)}; a, b\rangle\big] \\
&\quad + x[b,b']a - \sum x_{(1)}\big\langle x_{(2)}; \langle x_{(3)}; b, b'\rangle, a\big\rangle - \sum x_{(1)}\langle x_{(2)}a; b, b'\rangle.
\end{aligned}$$

Since $\sum x_{(1)}[b', \langle x_{(2)}; a, b\rangle] + \sum x_{(1)}\langle x_{(2)}; b', \langle x_{(3)}; a, b\rangle\rangle \in \operatorname{span}\langle G\rangle$ then

$$\begin{aligned}
C &\equiv xab'b - xbab' - xb'[a,b] - \sum x_{(1)}\langle x_{(2)}b'; a, b\rangle + \sum x_{(1)}\big\langle x_{(2)}; b', \langle x_{(3)}; a, b\rangle\big\rangle \\
&\quad + x[b,b']a - \sum x_{(1)}\big\langle x_{(2)}; \langle x_{(3)}; b, b'\rangle, a\big\rangle - \sum x_{(1)}\langle x_{(2)}a; b, b'\rangle \\
&= xab'b - xbab' - xb'[a,b] + x[b,b']a + \sum x_{(1)}\langle x_{(2)}b; b', a\rangle \\
&\quad + \sum x_{(1)}\big\langle x_{(2)}; \langle x_{(3)}; b', a\rangle, b\big\rangle,
\end{aligned}$$

where the equality follows from (13). Collecting terms we obtain

$$C \equiv x[a,b']b - xb[a,b'] + \sum x_{(1)}\langle x_{(2)}b; b', a\rangle + \sum x_{(1)}\big\langle x_{(2)}; \langle x_{(3)}; b', a\rangle, b\big\rangle.$$

By (A.1) we have that $xb[a,b'] - \sum x_{(1)}\langle x_{(2)}b; b', a\rangle + \sum x_{(1)}b\langle x_{(2)}; a, b'\rangle \in \hat{I}_{xbab'} \subseteq I_{xabb'}$ so

$$C \equiv x[a,b']b + \sum x_{(1)}b\langle x_{(2)}; a, b'\rangle + \sum x_{(1)}\big\langle x_{(2)}; \langle x_{(3)}; b', a\rangle, b\big\rangle.$$

Again, by (A.1) $x[a,b']b + \sum x_{(1)}\langle x_{(2)}; a, b'\rangle b \in \hat{I}_{xab'b} \subseteq I_{xabb'}$, thus

$$\begin{aligned}C &\equiv -\sum x_{(1)}\langle x_{(2)}; a, b'\rangle b + \sum x_{(1)} b\langle x_{(2)}; a, b'\rangle + \sum x_{(1)}\big\langle x_{(2)}; \langle x_{(3)}; b', a\rangle, b\big\rangle \\ &= \sum x_{(1)}\big[b, \langle x_{(2)}; a, b'\rangle\big] + \sum x_{(1)}\big\langle x_{(2)}; b, \langle x_{(3)}; a, b'\rangle\big\rangle \\ &\equiv 0,\end{aligned}$$

where the last congruence is a consequence of

$$\sum x_{(1)}\big[b, \langle x_{(2)}; a, b'\rangle\big] + x_{(1)}\big\langle x_{(2)}; b, \langle x_{(3)}; a, b'\rangle\big\rangle \in \operatorname{span}\langle G\rangle. \qquad \square$$

Appendix B. Some complementary results

Proposition 26 can be generalized a little bit to get

Theorem 38. *The map l extends to $l : \tilde{S}(V) \otimes \mathrm{U}(E) \to \mathrm{U}(H)$ with*

$$(\bar{x}\bar{y})D = \sum \bar{x}l(\bar{y}_{(1)}, D_{(1)})(\bar{y}_{(2)}D_{(2)})$$

for any $D \in \mathrm{U}(E)$ and $\bar{x}, \bar{y} \in \tilde{S}(V)$.

Proof. If we define $l(\bar{y}, 1) = \epsilon(\bar{y})1$ then Proposition 26 says that

$$(\bar{x}\bar{y})d = \sum \bar{x}l(\bar{y}_{(1)}, d_{(1)}) \cdot \bar{y}_{(2)}d_{(2)}$$

and

$$(\bar{x}\bar{y})1 = \sum \bar{x}l(\bar{y}_{(1)}, 1) \cdot \bar{y}_{(2)}1.$$

We may extend these formulas to $l : \tilde{S}(V) \otimes \mathrm{T}(E) \to \mathrm{U}(H)$ by

$$l(\bar{y}, dD) = \sum l(\bar{y}_{(1)}, d)l(\bar{y}_{(2)}, D) + l(\bar{y}d, D)$$

for any $d \in E$ and $D \in \mathrm{T}(E)$. First we claim that

$$l(\bar{y}, DD') = \sum l(\bar{y}_{(1)}, D_{(1)})l(\bar{y}_{(2)}D_{(2)}, D'). \tag{B.1}$$

To show this equation we proceed by induction on $|D|$:

$$\begin{aligned}l\big(\bar{y}, (dD)D'\big) &= \sum l(\bar{y}_{(1)}, d)l(\bar{y}_{(2)}, DD') + l(\bar{y}d, DD') \\ &= \sum l(\bar{y}_{(1)}, d)l(\bar{y}_{(2)}, D_{(1)})l(\bar{y}_{(3)}D_{(2)}, D') + \sum l(\bar{y}_{(1)}d, D_{(1)})l(\bar{y}_{(2)}D_{(2)}, D') \\ &\quad + \sum l(\bar{y}_{(1)}, D_{(1)})l(\bar{y}_{(2)}dD_{(2)}, D') \\ &= \sum l(\bar{y}_{(1)}, dD_{(1)})l(\bar{y}_{(2)}D_{(2)}, D') + \sum l(\bar{y}_{(1)}, D_{(1)})l(\bar{y}_{(2)}dD_{(2)}, D') \\ &= \sum l\big(\bar{y}_{(1)}(dD)_{(1)}\big)l\big(\bar{y}_{(2)}(dD)_{(2)}, D'\big).\end{aligned}$$

We also claim that

$$(\bar{x}\bar{y})D = \sum \bar{x}l(\bar{y}_{(1)}, D_{(1)}) \cdot \bar{y}_{(2)}D_{(2)}.$$

Again by induction, with $D = dD'$,

$$\begin{aligned}(\bar{x}\bar{y})D = (\bar{x}\bar{y})dD' &= \sum \big(\bar{x}l(\bar{y}_{(1)}, d_{(1)}) \cdot \bar{y}_{(2)}d_{(2)}\big)D' \\ &= \sum \bar{x}l(\bar{y}_{(1)}, d_{(1)})l(\bar{y}_{(2)}d_{(2)}, D'_{(1)}) \cdot \bar{y}_{(3)}d_{(3)}D'_{(2)} \\ &= \sum \bar{x}l(\bar{y}_{(1)}, d_{(1)}D'_{(1)}) \cdot \bar{y}_{(2)}d_{(2)}D'_{(2)} \\ &= \sum \bar{x}l(\bar{y}_{(1)}, D_{(1)}) \cdot \bar{y}_{(2)}D_{(2)}.\end{aligned}$$

So, to prove the theorem it suffices to check that l induces a corresponding map $l : \tilde{S}(V) \otimes \mathrm{U}(E) \to \mathrm{U}(H)$, i.e.

$$l(\bar{y}, Ddd'D') - l(\bar{y}, Dd'dD') = l\big(\bar{y}, D[d, d']D'\big).$$

By (B.1) it its enough to show that

$$l(\bar{y}, dd'D') - l(\bar{y}, d'dD') - l\big(\bar{y}, [d, d']D'\big) = 0.$$

Since

$$\begin{aligned}l(\bar{y}, dd'D') &= \sum l(\bar{y}_{(1)}, d)l(\bar{y}_{(2)}, d'D') + l(\bar{y}d, d'D') \\ &= \sum l(\bar{y}_{(1)}, d)l(\bar{y}_{(2)}, d')l(\bar{y}_{(3)}, D') + \sum l(\bar{y}_{(1)}, d)l(\bar{y}_{(2)}d', D') \\ &\quad + \sum l(\bar{y}_{(1)}d, d')l(\bar{y}_{(2)}, D') + \sum l(\bar{y}_{(1)}, d')l(\bar{y}_{(2)}d, D') + l(\bar{y}dd', D')\end{aligned}$$

then

$$\begin{aligned}l(\bar{y}, dd'D') - l(\bar{y}, d'dD') &= \sum \big[l(\bar{y}_{(1)}, d), l(\bar{y}_{(2)}, d')\big]l(\bar{y}_{(3)}, D') + \sum l(\bar{y}_{(1)}d, d')l(\bar{y}_{(2)}, D') \\ &\quad - \sum l(\bar{y}_{(1)}d', d)l(\bar{y}_{(2)}, D') + l\big(\bar{y}[d, d'], D'\big) \\ &= \sum l\big(\bar{y}_{(1)}, [d, d']\big)l(\bar{y}_{(2)}, D') + l\big(\bar{y}[d, d'], D'\big) \\ &= l\big(\bar{y}, [d, d']D\big)\end{aligned}$$

as desired. □

The decomposition $E = H \oplus \bar{V}$, with $\bar{V} = \{\tau_a \mid a \in V\}$, of the Lie algebra $E = \mathrm{Hom}(\tilde{S}(V), V)$ introduced in Proposition 22 induces on $\bar{V}$ the structure of Sabinin algebra. In fact, we can prove that E is a Lie envelope of $(V, \langle\,;\,\rangle)$ in the sense of (20).

Proposition 39. *The map*

$$\tau : V \to \bar{V}, \qquad a \mapsto \tau_a$$

is an isomorphism of Sabinin algebras.

Proof. Clearly τ is bijective. The structure of Sabinin algebra on $\bar{V}$ is determined by the relation

$$\overline{wd} = -\sum \overline{w_{(1)}\langle w_{(2)}; d\rangle} \quad \forall w \in \mathrm{U}(E),\ d \in E, \tag{B.2}$$

on $\mathrm{U}(E)/H\mathrm{U}(E)$, and by definition given $\tau_x = \tau_{a_1} \cdots \tau_{a_n} \in \mathrm{T}(\bar{V})$,

$$\langle \tau_x; \tau_a, \tau_b\rangle = \big\langle \tau_x; [\tau_a, \tau_b]\big\rangle.$$

Since $\langle \tau_x; \tau_a, \tau_b\rangle$ belongs to $\bar{V}$ then there exists $\langle x; a, b\rangle' \in V$ such that

$$\langle \tau_x; \tau_a, \tau_b\rangle = \tau_{\langle x;a,b\rangle'}.$$

Using the action of E on $\tilde{S}(V)$ and the fact that H kills $\bar{1}$ then by (B.2) with $w = \tau_x$ and $d = [\tau_a, \tau_b]$ we obtain

$$\bar{1}\tau_x[\tau_a, \tau_b] = \begin{cases} \overline{xab} - \overline{xba} = -\sum \overline{x_{(1)}}\langle x_{(2)}; a, b\rangle, \\ -\sum \bar{1}\tau_{x_{(1)}}\langle \tau_{x_{(2)}}; [\tau_a, \tau_b]\rangle = -\sum \overline{x_{(1)}}\langle x_{(2)}; a, b\rangle', \end{cases}$$

from which $\langle x; a, b\rangle' = \langle x; a, b\rangle$. Therefore, $\langle \tau_{a_1} \cdots \tau_{a_n}; \tau_a, \tau_b\rangle = \tau_{\langle a_1 \cdots a_n; a, b\rangle}$. □

Corollary 40. *The map*

$$\psi : \mathrm{U}(E)/H\mathrm{U}(E) \to \tilde{S}(V), \qquad \bar{D} \mapsto \bar{1}D$$

is an isomorphism of E-modules.

Proof. Since $\bar{1}\tau_{a_1} \cdots \tau_{a_n} = \overline{a_1 \cdots a_n}$ then by the Poincaré–Birkhoff–Witt Theorem ψ sends a basis of $\mathrm{U}(E)/H\mathrm{U}(E)$ to a basis of $\tilde{S}(V)$, so ψ is an linear isomorphism. Furthermore,

$$(\bar{\tau}_x d)\psi = \bar{1}\tau_x d = \bar{x}d = (\bar{\tau}_x)\psi d$$

for any $d \in E$ implies that ψ is an isomorphism as modules. □

References

[1] M.A. Akivis, V.V. Goldberg, Local algebras of a differential quasigroup, Bull. Amer. Math. Soc. 43 (2006) 207–226.
[2] M.R. Bremner, I.R. Hentzel, L.A. Peresi, Dimension formulas for free Akivis algebras and primitive elements in free nonassociative algebras, Comm. Algebra 33 (11) (2005) 4063–4081.
[3] S. Burris, H. Sankappanaver, A Course in Universal Algebra. The Millennium Edition, http://thoralf.uwaterloo.ca/htdocs/UALG/univ-algebra.pdf.
[4] L. Gerritzen, R. Holtkamp, Hopf co-addition for free magma algebras and the non-associative Hausdorff series, J. Algebra 265 (1) (2003) 264–284.
[5] W.A. de Graaf, Lie Algebras: Theory and Algorithms, North-Holland, Amsterdam, 2000.
[6] R. Holtkamp, On Hopf algebra structures over free operads, Adv. Math., in press.
[7] K. Kunen, Moufang quasigroups, J. Algebra 183 (1) (1996) 231–234.
[8] E.N. Kuz'min, On a relation between Malcev algebras and analytic Moufang loops, Algebra Logika 10 (1) (1971) 3–22; English translation: Algebra Logic 10 (1972) 1–14.
[9] P. Leroux, Ennea-algebras, J. Algebra 281 (1) (2004) 287–302.
[10] J.-L. Loday, Dialgebras, in: Dialgebras and Related Operads, in: Lectures Notes in Math., vol. 1763, Springer, Berlin, 2001, pp. 7–66.
[11] J.-L., Loday, Scindement d'associativité et algèbres de Hopf, in: Actes des Journées Mathématiques à la Mémoire de Jean Leray, in: Sémin. Congr., vol. 9, Soc. Math. France, Paris, 2004, pp. 155–172.
[12] A.I. Malcev, Analytic loops, Mat. Sb. (N.S.) 36 (78) (3) (1955) 569–575.
[13] P.O. Miheev, Pseudoautomorphisms of smooth loops, Algebras Groups Geom. 8 (2) (1991) 131–144.
[14] S. Montgomery, Hopf Algebras and Their Actions on Rings, CBMS Reg. Conf. Ser. Math., vol. 82, Amer. Math. Soc., 1993.
[15] J. Mostovoy, Nilpotency and dimension series for loops, preprint, math.GR/0510161.
[16] J.M. Pérez-Izquierdo, An envelope for Bol algebras, J. Algebra 284 (2) (2005) 480–493.
[17] J.M. Pérez-Izquierdo, I.P. Shestakov, An envelope for Malcev algebras, J. Algebra 272 (2004) 379–393.
[18] M. Ronco, Primitive elements in a free dendriform algebra, in: New Trends in Hopf Algebra Theory, La Falda, 1999, in: Contemp. Math., vol. 267, Amer. Math. Soc., 2000, pp. 245–263.
[19] M. Ronco, Eulerian idempotents and Milnor–Moore theorem for certain non-cocommutative Hopf algebras, J. Algebra 254 (1) (2002) 152–172.
[20] L.V. Sabinin, Smooth Quasigroups and Loops, Math. Appl., vol. 492, Kluwer Academic Publishers, 1999.
[21] L.V. Sabinin, P.O. Miheev, Analytic Bol loops, in: Webs and Quasigroups, Kalinin. Gos. Univ., Kalinin, 1982, pp. 102–109.
[22] L.V. Sabinin, P.O. Miheev, On local analytic loops and their corresponding hyperalgebras, in: Proceedings of 9th Conference of Young Researchers, VINITI Press, Moscow, 1986, pp. 34–54.
[23] L.V. Sabinin, P.O. Miheev, Infinitesimal theory of local analytic loops, Soviet Math. Dokl. 36 (3) (1988) 545–548.
[24] L.V. Sabinin, P.O. Miheev, Quasigroups and differential geometry, in: Quasigroups and Loops: Theory and Applications, in: Sigma Ser. Pure Math., vol. 8, Heldermann, Berlin, 1990, pp. 357–430.
[25] A.A. Sagle, Malcev algebras, Trans. Amer. Math. Soc. 101 (1961) 426–458.
[26] V. Shcherbacov, V. Izbash, On quasigroups with Moufang identity, Bul. Acad. Ştiinţe Repub. Mold. Mat. 27 (2) (1998) 109–116.
[27] I.P. Shestakov, Linear representability of Akivis algebras, Dokl. Akad. Nauk 368 (1) (1999) 21–23 (in Russian).
[28] I.P. Shestakov, Every Akivis algebra is linear, Geom. Dedicata 77 (2) (1999) 215–223.
[29] I.P. Shestakov, U.U. Umirbaev, Free Akivis algebras, primitive elements, and hyperalgebras, J. Algebra 250 (2) (2002) 533–548.
[30] M. Takeuchi, Free Hopf algebras generated by coalgebras, J. Math. Soc. Japan 23 (1971) 561–582.