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Jacobi transplantation revisited

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Abstract A transplantation theorem for Jacobi series proved by Muckenhoupt is reinvestigated by means of a suitable variant of Calderón–Zygmund operator theory. An essential novelty of our paper is weak type (1,1) estimate for the Jacobi transplantation operator, located in a fairly general weighted setting. Moreover, L^p estimates are proved for a class of weights that contains the class admitted in Muckenhoupt's theorem.

Keywords Jacobi polynomials · Transplantation · Local Calderón–Zygmund operators · Weighted norm inequalities · Darboux type formula

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1 Introduction

The principal purpose of this paper is to reinvestigate the transplantation theorem for Jacobi series proved by Muckenhoupt [9, Theorem (1.14)] from the Calderón–

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Zygmund (frequently abbreviated to CZ) theory point of view. A transplantation theorem for Jacobi expansions was first obtained by Askey [2]. Then Muckenhoupt substantially enhanced Askey's result in several directions: by considering the largest possible range of Jacobi parameters, admitting fairly general class of weights for L^p estimates (a class which is different from the usual A_p class), introducing a shift in the order parameter of Jacobi orthonormalized polynomials, adding a multiplier sequence, and, eventually, by assuming moment conditions. Recently Miyachi [8] extended Muckenhoupt's result to the setting of weighted Hardy spaces $H^p_{a,b}$, $0 < p \le 1, a, b \in \mathbb{R}$.

In our approach we partly benefit from the work done by Muckenhoupt, but on the other hand we apply a new technique. The benefit we have in mind is best seen in Proposition 3.2 below (needless to say, this is not the single place): the growth estimates (3.3) were furnished by Muckenhoupt. Since the method we use relies on suitably established local version of CZ theory, we also need a gradient estimate; it is contained in (3.4) of Proposition 3.2 (and does not appear in [9]). Another ingredient that essentially distinguishes our approach from that of Muckenhoupt is the following: having kernels (and their appropriate estimates) of operators that approximate the Jacobi transplantation operator, we define a kernel associated with that operator in the sense of CZ theory. Such a definition requires proving the existence of a limit of the approximating kernels, and to achieve this we use a (first order) Darboux type formula for Jacobi polynomials, cf. [9, (2.12)]. Similar formula of order 2 is necessary for justifying the abovementioned gradient estimate. Thus, believing that it may be of independent interest, we derive a general higher order Darboux type formula for Jacobi polynomials. The related reasoning relies on a uniform asymptotic representation for Jacobi polynomials proved recently by Wong and Zhao [12].

The technique we apply brings an immediate advantage since weighted weak type (1,1) and L^p estimates for the transplantation operator then follow, essentially by a variant of the CZ operator theory we establish; weak type (1,1) estimates do not appear in [9]. The class of admissible weights in our L^p result (Theorem 2.5) is at least as large as the class allowed in Muckenhoupt's theorem [9, Theorem (1.14)]. The assumptions imposed on weights in the weak type $(1, 1)$ result (Theorem 2.6) are rather involved, but this is a price being paid for generality. However, these assumptions take a much simpler form after specifying to double power weights, in which case a consistence with the L^p weight conditions can be easily observed. It should be also mentioned that the fact that an operator is a CZ operator has further well-known consequences: it maps H^1 into L^1 and L^∞ into BMO (in the latter case the operator has to be appropriately redefined, to be precise). Finally, we mention that a similar technique was used recently by two of the three authors in [5] in the setting of Fourier–Bessel expansions.

Given $\alpha, \beta \in (-1, \infty)$, consider the orthonormalized Jacobi polynomials

$$\phi_n^{(\alpha,\beta)}(x) = t_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos x) \sin^{\alpha+1/2}(x/2) \cos^{\beta+1/2}(x/2), \quad n \in \mathbb{N}$$

(seemingly more appropriate name *Jacobi functions* would be confusing), where

$$t_n^{(\alpha,\beta)} = \left(\frac{(2n + \alpha + \beta + 1)\Gamma(n + 1)\Gamma(n + \alpha + \beta + 1)}{\Gamma(n + \alpha + 1)\Gamma(n + \beta + 1)} \right)^{1/2} \tag{1.1}$$

(for $n = 0$ the product $(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)$ must be replaced by $\Gamma(\alpha+\beta+2)$). The functions $\{\phi_n^{(\alpha,\beta)}\}_{n\in\mathbb{N}}$ form a complete orthonormal system in $L^2((0, \pi), dx)$. They are also the eigenfunctions of the symmetric in $L^2((0,\pi),dx)$ differential operator

$$\mathcal{L}_{\alpha,\beta} = -\frac{d^2}{dx^2} - \left(\frac{1/4-\alpha^2}{4\sin^2(x/2)} + \frac{1/4-\beta^2}{4\cos^2(x/2)}\right),$$

cf. [11, (4.24.2)], and

$$\mathcal{L}_{\alpha,\beta}\phi_n^{(\alpha,\beta)} = \left(n+\frac{\alpha+\beta+1}{2}\right)^2\phi_n^{(\alpha,\beta)}.$$

Given (α,β) and (γ,δ) with $\alpha,\beta,\gamma,\delta\in(-1,\infty)$, we define the transplantation operator $T=T^{(\alpha,\beta),(\gamma,\delta)}$ on $L^2((0,\pi),dx)$ by the convergent in $L^2((0,\pi),dx)$ series

$$Tf = \sum_{n=0}^{\infty}\langle f,\phi_n^{(\gamma,\delta)}\rangle\phi_n^{(\alpha,\beta)}.$$

Clearly, T is an L^2 isometry which becomes the identity operator when $(\alpha,\beta)=(\gamma,\delta)$.

For the sake of convenience we now state a simplified version of Muckenhoupt's transplantation theorem; in [9, Theorem (1.14)] we choose $s=d=M=N=0$ and $g(n)\equiv 1$.

Theorem 1.1 (Muckenhoupt) *Let* $\alpha,\beta,\gamma,\delta\in(-1,\infty)$, $1<p<\infty$, *and* $w(x)$ *be a weight on* $(0,\pi)$ *such that*

$$\left(\int_u^v\left[w(x)x^{\alpha+1/2}(\pi-x)^{\beta+1/2}\right]^p dx\right)^{1/p}\left(\int_u^v\left[w(x)^{-1}x^{\gamma+1/2}(\pi-x)^{\delta+1/2}\right]^{p'} dx\right)^{1/p'}$$
$$\le C(v-u)v^{\alpha+\gamma+1}(\pi-u)^{\beta+\delta+1},\quad 0\le u<v\le\pi. \tag{1.2}$$

Then, given $f\in L^p(w)$ *and* $0<r<1$, *the series*

$$T_rf(x) = \sum_{n=0}^{\infty} r^n\langle f,\phi_n^{(\gamma,\delta)}\rangle\phi_n^{(\alpha,\beta)}(x)$$

converges for every $x\in(0,\pi)$, *the inequality*

$$\left(\int_0^\pi\left|T_rf(x)w(x)\right|^p dx\right)^{1/p} \le C\left(\int_0^\pi |f(x)w(x)|^p dx\right)^{1/p}$$

holds with C *independent of* r *and* f, *and there is a function* $Tf\in L^p(w)$ *such that* T_rf *converges to* Tf *in* $L^p(w)$ *as* $r\to 1^-$. *If it is also assumed that*

$$\int_0^\pi\left[w(x)^{-1}x^{\alpha+1/2}(\pi-x)^{\beta+1/2}\right]^{p'} dx<\infty, \tag{1.3}$$

then

$$\langle Tf,\phi_n^{(\alpha,\beta)}\rangle = \langle f,\phi_n^{(\gamma,\delta)}\rangle.$$

Note that if either $w_{a,b}(x)=x^a(\pi-x)^b$ or $w_{a,b}(x)=\sin^a(x/2)\cos^b(x/2)$ is a double power weight, a,b real, then for such a weight Condition (1.2) is equivalent to

$$-\alpha-1/2-1/p<a<\gamma+3/2-1/p,\quad -\beta-1/2-1/p<b<\delta+3/2-1/p, \tag{1.4}$$

see [9, Corollary 17.11], whereas Condition (1.3) holds if and only if $a < \alpha + 3/2 - 1/p$ and $b < \beta + 3/2 - 1/p$. Note also that for $\alpha = \beta = \gamma = \delta = -1/2$, Condition (1.2) becomes simply the usual A_p condition for w^p.

Throughout the paper we use a fairly standard notation. Thus, for a weight w on $(0, \pi)$ (a nonnegative measurable function such that $w(x) < \infty$, x-a.e.) we write $L^p(w)$ and $L^{1,\infty}(w)$ to denote the weighted L^p and the weighted weak L^1 spaces that consist of all functions f on $(0, \pi)$ for which

$$\|f\|_{L^p(w)} = \left(\int_0^\pi |f(x)w(x)|^p \, dx \right)^{1/p} < \infty,$$

or

$$\|f\|_{L^{1,\infty}(w)} = \sup_{t>0} \left(t \int_{\{0<x<\pi : |f(x)|>t\}} w(x) \, dx \right) < \infty,$$

respectively. If $w \equiv 1$, we simplify the notation by writing L^p or $L^{1,\infty}$. We write $\langle f, g \rangle$ for $\int_0^\pi f(x)\overline{g(x)} \, dx$ provided the integral converges, and $f \sim \sum c_n \phi_n^{(\alpha,\beta)}$ to indicate that the last series represents the expansion of f with respect to the system $\{\phi_n^{(\alpha,\beta)}\}$; in particular, this means that the integrals $\langle f, \phi_n^{(\alpha,\beta)} \rangle$ defining the Fourier–Jacobi coefficients c_n do exist.

2 Preliminaries and statement of results

We shall use a variant of the local Calderón–Zygmund theory established in [10, Sect. 4] (see also [5, Sect. 3]), suited to the present setting. For the sake of convenience and completeness we state the relevant definitions and results. Let $\Delta = \{(x, x) : x \in (0, \pi)\}$.

Definition 2.1 A *double local standard kernel* is a kernel $K\colon (0, \pi) \times (0, \pi) \backslash \Delta \mapsto \mathbb{C}$ supported in the region (see Fig. 1 below)

$$\mathcal{D} = \left\{ (x, y) \in (0, \pi) \times (0, \pi) \backslash \Delta : \max\left(\frac{x}{2}, \frac{3x - \pi}{2}\right) < y < \min\left(\frac{3x}{2}, \frac{x + \pi}{2}\right) \right\},$$

and satisfying on $\mathcal{D}$ the standard estimates

$$\begin{aligned} |K(x, y)| &\leq C|x - y|^{-1}, \\ |\nabla_{x,y} K(x, y)| &\leq C(x - y)^{-2}. \end{aligned}$$

Notice that the region $\mathcal{D}$ is "local" both near $(0, 0)$ and (π, π), which motivates the usage of the word *double* in the above definition as well as in other related places.

Definition 2.2 An operator $\mathcal{T}$ is a *double local Calderón–Zygmund operator* if

(1) $\mathcal{T}$ is bounded on $L^2(0, \pi)$;
(2) there exists a double local standard kernel K associated with $\mathcal{T}$ such that

$$\langle \mathcal{T} f, g \rangle = \int_0^\pi \int_{\max(x/2, 3x/2 - \pi/2)}^{\min(3x/2, x/2 + \pi/2)} K(x, y) f(y) \overline{g(x)} \, dy \, dx$$

for all $f, g \in C_c^\infty(0, \pi)$ with disjoint supports.

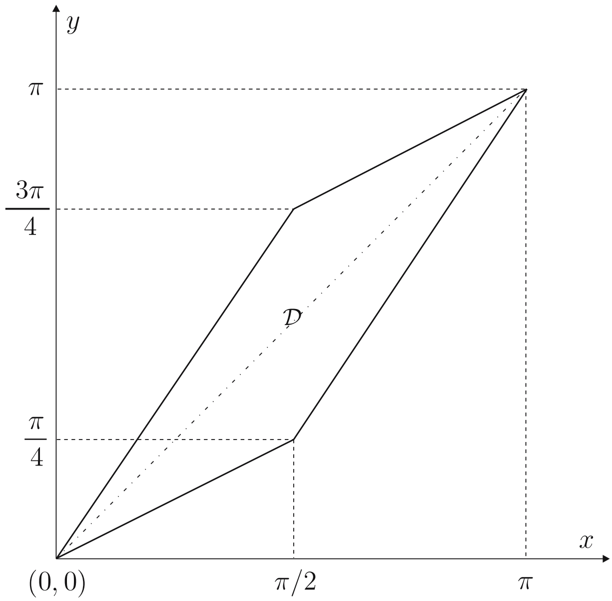

Fig. 1 The region $\mathcal{D}$

Definition 2.3 Let $1 \le p < \infty$ and w be a weight on $(0,\pi)$. We say that w^p satisfies the *double local A_p condition* if

$$\sup_{0\le u<v\le\min(2u,u/2+\pi/2)\le\pi} \frac{1}{v-u}\left(\int_u^v w^p\right)^{1/p}\left(\int_u^v w^{-p'}\right)^{1/p'} < \infty \tag{2.1}$$

(if $p = 1$, then the second integral is understood as $\operatorname{ess\,sup}_{(u,v)} w^{-1}$; p' denotes the conjugate of p, $1/p+1/p' = 1$). We then write $w^p \in A_p^{2\mathrm{loc}}(0,\pi)$ (or shortly $w^p \in A_p^{2\mathrm{loc}}$) and call the left hand side of (2.1) the *double local A_p norm* of w^p.

Arguments parallel to those used in [10, Sect. 4], cf. also [5, Sect. 3], allow to prove the following result, which in fact is the principal tool in our treatment of the Jacobi transplantation operator.

Theorem 2.4 *Assume that $\mathcal{T}$ is a double local Calderón–Zygmund operator and let w be a weight on $(0,\pi)$ such that $w^p \in A_p^{2\mathrm{loc}}$.*

(a) *If $1 < p < \infty$, then $\mathcal{T}$ extends to a bounded linear operator on $L^p(w)$;*
(b) *If $p = 1$, then $\mathcal{T}$ extends to a bounded linear operator from $L^1(w)$ to $L^{1,\infty}(w)$.*

Moreover, the corresponding L^p and weak type constants depend on w only through the double local A_p norm of w^p.

Let $1 < p < \infty$. Given a weight function $w(x)$ on $(0,\pi)$, consider the following two conditions:

$$\sup_{0<r<\pi}\left(\int_r^\pi\left(\frac{w(x)(\pi-x)^{\beta+1/2}}{x^{\gamma+3/2}}\right)^p dx\right)^{1/p}\left(\int_0^r\left(\frac{x^{\gamma+1/2}}{w(x)(\pi-x)^{\beta+3/2}}\right)^{p'} dx\right)^{1/p'} < \infty, \tag{2.2}$$

$$\sup_{0<r<\pi}\left(\int_0^r\left(\frac{w(x)x^{\alpha+1/2}}{(\pi-x)^{\delta+3/2}}\right)^p dx\right)^{1/p}\left(\int_r^\pi\left(\frac{(\pi-x)^{\delta+1/2}}{w(x)x^{\alpha+3/2}}\right)^{p'} dx\right)^{1/p'} < \infty. \quad (2.3)$$

Condition (2.2) is necessary and sufficient for the weighted Hardy's inequality

$$\int_0^\pi \left|U(x)\int_0^x f(t)\,dt\right|^p dx \le C\int_0^\pi |f(x)V(x)|^p\,dx \quad (2.4)$$

to hold, with $U(x) = (\pi-x)^{\beta+1/2}x^{-(\gamma+3/2)}w(x)$ and $V(x) = (\pi-x)^{\beta+3/2}x^{-(\gamma+1/2)}w(x)$. Similarly, Condition (2.3) is necessary and sufficient for

$$\int_0^\pi \left|U(x)\int_x^\pi f(t)\,dt\right|^p dx \le C\int_0^\pi |f(x)V(x)|^p\,dx \quad (2.5)$$

to hold, with $U(x) = (\pi-x)^{-(\delta+3/2)}x^{\alpha+1/2}w(x)$ and $V(x) = (\pi-x)^{-(\delta+1/2)}x^{\alpha+3/2}w(x)$. These facts follow directly from [1, Theorems A, B].

Finally, note that if a weight w on $(0,\pi)$ satisfies any of the conditions (2.1), (2.2), or (2.3), then either $w = 0$, x-a.e. or $w(x) > 0$, x-a.e (here the convention $0\cdot\infty = 0$ is used).

Theorem 2.5 *Let $\alpha,\beta,\gamma,\delta \in (-1,\infty)$ and $T = T^{(\alpha,\beta),(\gamma,\delta)}$. Assume that $1 < p < \infty$ and $w(x)$ is a weight that satisfies the conditions (2.1), (2.2) and (2.3). Then*

$$\left(\int_0^\pi \left|Tf(x)w(x)\right|^p dx\right)^{1/p} \le C\left(\int_0^\pi |f(x)w(x)|^p\,dx\right)^{1/p} \quad (2.6)$$

for all $f \in L^2 \cap L^p(w)$. Consequently, T extends uniquely to a bounded linear operator on $L^p(w)$. Using the same symbol T to denote this extension and assuming, in addition, that $w(x)$ satisfies (1.3), we have

$$Tf \sim \sum_{n=0}^\infty \langle f,\phi_n^{(\gamma,\delta)}\rangle\phi_n^{(\alpha,\beta)}, \quad f \in L^p(w). \quad (2.7)$$

When $w(x) = w_{a,b}(x)$ is a double power weight, a,b real, then it can be verified that the double local A_p condition holds for all $a,b \in \mathbb{R}$, Condition (2.2) is satisfied if and only if $a < \gamma + 3/2 - 1/p$ and $b > -\beta - 1/2 - 1/p$, and Condition (2.3) is satisfied if and only if $a > -\alpha - 1/2 - 1/p$ and $b < \delta + 3/2 - 1/p$. Consequently, we see that Theorem 2.5 and Theorem 1.1 allow exactly the same range of double power weights, cf. (1.4). Moreover, it can be shown that in fact (1.2) implies (2.1), (2.2) and (2.3), hence the last three conditions are together no more restrictive than that from Theorem 1.1; details are provided in Sect. 5. The question whether our conditions are essentially weaker than (1.2) seems to be tricky and remains open.

In order to treat the weak type (1,1) inequalities for the transplantation operator, for given weight functions $U(x)$, $V(x)$ on $(0,\pi/2)$ and η real, consider the following two conditions:

$$\sup_{0<r<\pi/2} r^{-\eta} \int_r^{\pi/2} \left(\chi_{(-\infty,0]}(\eta) + \chi_{(0,\infty)}(\eta) \left(\frac{r}{x}\right)^{\eta+\mu} \right) U(x)\, dx \cdot \operatorname*{ess\,sup}_{x\in(0,r)} \frac{1}{V(x)} < \infty, \tag{2.8}$$

$$\sup_{0<r<\pi/2} r^{-\eta} \int_0^{r} \left(\chi_{[0,\infty)}(\eta) + \chi_{(-\infty,0)}(\eta) \left(\frac{r}{x}\right)^{\eta-\mu} \right) U(x)\, dx \cdot \operatorname*{ess\,sup}_{x\in(r,\pi/2)} \frac{1}{V(x)} < \infty; \tag{2.9}$$

in (2.8) and (2.9) we assume that there exists a positive μ such that the corresponding quantities are finite.

Let P_η, Q_η, η real, denote the Hardy operators acting on functions defined on $(0,\pi/2)$,

$$P_\eta f(x) = x^{-\eta} \int_0^x f(t)\, dt, \quad Q_\eta f(x) = x^{-\eta} \int_x^{\pi/2} f(t)\, dt, \qquad 0 < x < \frac{\pi}{2}.$$

Condition (2.8) is necessary and sufficient for the two-weight inequality

$$\int_{\{0<x<\pi/2\,:\,|P_\eta f(x)|>\lambda\}} U(x)\, dx \le \frac{C}{\lambda} \int_0^{\pi/2} |f(x)| V(x)\, dx, \quad \lambda > 0, \tag{2.10}$$

to hold, cf. [1, Theorems 1, 2]. Condition (2.9) is necessary and sufficient for the inequality (2.10) with P_η replaced by Q_η, cf. [1, Theorems 4, 5].

It is easily seen that when $U(x) = x^A$ and $V(x) = x^B$ are power weights, A, B real, then: for $\eta \le 0$, (2.8) is satisfied if and only if $B \le 0$ and $A + 1 \ge B + \eta$ (with the last $\ge$ replaced by $>$ in case $A = -1$); for $\eta > 0$, (2.8) is satisfied if and only if $B \le 0$ and $A + 1 \ge B + \eta$; for $\eta \ge 0$, (2.9) is satisfied if and only if $A + 1 > 0$ and $A+1 \ge \max(B,0)+\eta$; finally, for $\eta < 0$, (2.9) holds if and only if $A+1 \ge \max(B,0)+\eta$.

Theorem 2.6 *Let $\alpha,\beta,\gamma,\delta \in (-1,\infty)$ and $T = T^{(\alpha,\beta),(\gamma,\delta)}$. Assume that $w(x)$ is a weight from $A_1^{2\mathrm{loc}}(0,\pi)$ satisfying*:

(i) *Condition (2.8) with $\eta = \gamma + 3/2$, $U(x) = w(x)$, $V(x) = w(x)x^{-(\gamma+1/2)}$ and with $\eta = \delta + 3/2$, $U(x) = w(\pi - x)$, $V(x) = w(\pi - x)x^{-(\delta+1/2)}$;*
(ii) *Condition (2.9) with $\eta = -(\alpha + 1/2)$, $U(x) = w(x)$, $V(x) = w(x)x^{\alpha+3/2}$ and with $\eta = -(\beta + 1/2)$, $U(x) = w(\pi - x)$, $V(x) = w(\pi - x)x^{\beta+3/2}$;*
(iii) *Conditions (2.8) and (2.9) with $\eta = -(\beta + 1/2)$, $U(x) = w(\pi - x)$, $V(x) = w(x)x^{-(\gamma+1/2)}$ and with $\eta = -(\alpha+1/2)$, $U(x) = w(x)$, $V(x) = w(\pi - x)x^{-(\delta+1/2)}$.*

Then

$$\int_{\{0<x<\pi\,:\,|Tf(x)|>\lambda\}} w(x)\, dx \le \frac{C}{\lambda} \int_0^{\pi} |f(x)| w(x)\, dx, \quad \lambda > 0,$$

for all $f \in L^2 \cap L^1(w)$. Consequently, T extends to a bounded linear operator from $L^1(w)$ to $L^{1,\infty}(w)$.

Assumptions imposed on a weight $w(x)$ in the above theorem are relatively complicated, but this is the price of generality. Moreover, the numerous conditions do not seem to be easily compressible in a way that allows to preserve the straightforwardness of their verification for concrete weights. For the sake of convenience, for each of the conditions (i)–(iii) we now determine those $a, b \in \mathbb{R}$, for which a double power weight $w_{a,b}(x)$ satisfies the assumed condition. Using the comment preceding Theorem 2.6 we see that, with $w(x) = w_{a,b}(x)$, (i) holds if and only if $a \le \gamma + 1/2$, $b \le \delta + 1/2$, (ii) if and only if $a \ge -(\alpha + 3/2)$ ($>$ if $\alpha = -1/2$), $b \ge -(\beta + 3/2)$ ($>$ if $\beta = -1/2$), (iii) if and only if $a \le \gamma + 1/2$ and $b \ge a - (\beta + \gamma + 2)$, $b > -1$ in case $\beta \le -1/2$, and either $a \ge \gamma + 1/2$ and $b \ge a - (\beta + \gamma + 2)$ or $a < \gamma + 1/2$ and $b \ge -(\beta + 3/2)$, $b \le \delta + 1/2$ and $a \ge b - (\alpha + \delta + 2)$, $a > -1$ in case $\alpha \le -1/2$, and either $b \ge \delta + 1/2$ and $a \ge b - (\alpha + \delta + 2)$ or $b < \delta + 1/2$ and $a \ge -(\alpha + 3/2)$. Then it is not hard to observe that (i) and (ii) imply (iii), therefore, taking into account that the double local A_1 condition holds for all $a, b \in \mathbb{R}$, a weight $w_{a,b}(x)$ satisfies the assumptions of Theorem 2.6 if and only if

$$-(\alpha + 3/2) \le a \le \gamma + 1/2, \quad -(\beta + 3/2) \le b \le \delta + 1/2,$$

with the first lower inequality replaced by $<$ in case $\alpha = -1/2$, and with the same replacement concerning the second lower inequality in case $\beta = -1/2$. Notice that this is consistent with the strong type range described in (1.4).

3 Kernel estimates

To associate with T a kernel (in the sense of CZ theory) we consider the operator

$$T_r f(x) = \sum_{n=0}^{\infty} r^n \langle f, \phi_n^{(\gamma,\delta)} \rangle \phi_n^{(\alpha,\beta)}(x), \quad f \in L^2((0,\pi), dx), \ \ x \in (0,\pi),$$

$0 < r < 1$, which is an integral operator with the kernel $L_r = L_r^{(\alpha,\beta),(\gamma,\delta)}$ given by

$$L_r(x,y) = \sum_{n=0}^{\infty} r^n \phi_n^{(\alpha,\beta)}(x) \phi_n^{(\gamma,\delta)}(y), \quad x, y \in (0,\pi), \tag{3.1}$$

that is

$$T_r f(x) = \int_0^{\pi} L_r(x,y) f(y)\, dy, \quad x \in (0,\pi).$$

The theorem below provides a Darboux type formula of higher order for Jacobi polynomials, the main tool in establishing relevant estimates of $L_r(x,y)$. The proof of this result is given in the Appendix (Sect. 6).

Theorem 3.1 *Let $\alpha, \beta \in (-1,\infty)$. Given $q \in \mathbb{N}$, there exist bounded measurable functions $A_k(x)$ and $B_k(x)$, $k = 0, \dots, q$, on $[0,\pi]$ such that*

$$\phi_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{q} \left(\frac{A_k(x)}{(n \sin x)^k} \sin(nx) + \frac{B_k(x)}{(n \sin x)^k} \cos(nx) \right) + \mathcal{O}\big((n \sin x)^{-q-1}\big), \tag{3.2}$$

uniformly in $n \in \mathbb{N} \backslash \{0\}$ and $x \in (0,\pi)$. Moreover, the statement remains valid if, given an integer d, n on the right of (3.2) *is replaced by $n + d$ (if $d < 0$ then* (3.2) *is assumed to hold for $n \ge -d + 1$).*

The case $q = 1$ in Theorem 3.1 corresponds to the classical Darboux formula for Jacobi polynomials used by Muckenhoupt, see [9, (2.12)]; here, however, for the sake of simplicity of justification, we do not postulate more regularity of $A_k(x)$ and $B_k(x)$ than will be needed. On the other hand, although we will use (3.2) only with $q \le 2$, the more general statement we give seems to be of independent interest and does not require any extra arguments.

Recall that the Poisson kernel

$$P_r(t) = \frac{1}{2} + \sum_{n=1}^{\infty} r^n \cos(nt) = \frac{1-r^2}{2(1-2r\cos t + r^2)}$$

and the conjugate Poisson kernel

$$Q_r(t) = \sum_{n=1}^{\infty} r^n \sin(nt) = \frac{r \sin t}{1-2r\cos t + r^2},$$

satisfy for $0 < |t| < 3\pi/2$ and $0 < r < 1$: $|P_r(t)| \le C|t|^{-1}$, $|\frac{d}{dt}P_r(t)| \le C|t|^{-2}$, and the same estimates are true if Q_r replaces P_r.

Proposition 3.2 *Let $\alpha, \beta, \gamma, \delta \in (-1, \infty)$, $0 < r < 1$ and $0 < x, y < \pi$. Then*

$$|L_r(x,y)| \le C \begin{cases} \dfrac{y^{\gamma+1/2}(\pi - x)^{\beta+1/2}}{x^{\gamma+3/2}(\pi - y)^{\beta+3/2}}, & 0 < y \le \max\left(\frac{x}{2}, \frac{3x-\pi}{2}\right), \\ |x-y|^{-1}, & \max\left(\frac{x}{2}, \frac{3x-\pi}{2}\right) < y < \min\left(\frac{3x}{2}, \frac{x+\pi}{2}\right), \\ \dfrac{x^{\alpha+1/2}(\pi - y)^{\delta+1/2}}{y^{\alpha+3/2}(\pi - x)^{\delta+3/2}}, & \min\left(\frac{3x}{2}, \frac{x+\pi}{2}\right) \le y < \pi, \end{cases} \tag{3.3}$$

and

$$\left|\nabla_{x,y} L_r(x,y)\right| \le C|x-y|^{-2}, \quad \max\left(\frac{x}{2}, \frac{3x-\pi}{2}\right) < y < \min\left(\frac{3x}{2}, \frac{x+\pi}{2}\right); \tag{3.4}$$

in both cases C is independent of r, x and y. Moreover, if $\alpha, \beta, \gamma, \delta \in [-1/2, \infty)$ then the middle estimate in (3.3) *holds globally, i.e. for all $0 < x, y < \pi$, and if $\alpha, \beta, \gamma, \delta \in [1/2, \infty)$ the same is true for the gradient estimate in* (3.4).

Proof The third estimate in (3.3) is a dual form of the first one (see (3.5) below), thus we only need to verify the first two estimates in (3.3). In order to show the bound in the first line of (3.3) note that if $0 < x, y < 3\pi/4$, the right-hand side there is comparable with $x^{-(\gamma+3/2)} y^{\gamma+1/2}$ and the required estimate is included in [9, Theorem 7.1]. In the case $\pi/4 < x, y < \pi$, the same expression is comparable with $(\pi - y)^{-(\beta+3/2)}(\pi - x)^{\beta+1/2}$, and the required bound is obtained by using again [9, Theorem 7.1] and the identities

$$L_r^{(\alpha,\beta),(\gamma,\delta)}(x,y) = L_r^{(\beta,\alpha),(\delta,\gamma)}(\pi - x, \pi - y) = L_r^{(\gamma,\delta),(\alpha,\beta)}(y,x). \tag{3.5}$$

To finish showing the first bound, consider the remaining case when $3\pi/4 \le x < \pi$ and $0 < y \le \pi/4$. In this square the bound is equivalent to $(\pi - x)^{\beta+1/2} y^{\gamma+1/2}$, and the desired inequality is contained in [9, Theorem 5.1] (there, and also in [9, Theorem 7.1], we take $s = 0$, $d = 0$ and $g(n) \equiv 1$).

To show the middle bound in (3.3) note that for $0 < x, y < 3\pi/4$, it is a straightforward consequence of the asymptotic estimate given in [9, Theorem 8.3],

$$\left|\sum_{n=0}^{\infty} r^n \phi_{n+d}^{(\alpha,\beta)}(x)\phi_n^{(\gamma,\delta)}(y) - u(x)v(y)Q_r(x-y)\right| \le \frac{C}{x}\log\left(\frac{2x}{|x-y|}\right) + CP_r(x-y), \tag{3.6}$$

(where d is an integer, $u(x)$ and $v(y)$ are bounded functions) taken with $d = 0$. To be precise, in [9, Theorem 8.3] it is assumed that $0 < y \le 2x \le 3y$, but the same reasoning shows that (3.6) holds for $0 < x \le 2y \le 3x$. In the case $\pi/4 < x, y < \pi$, we use in addition (3.5).

We now pass to showing (3.4). By symmetry reasons we may concentrate our attention on $\frac{\partial}{\partial x}L_r$. Then it is enough to prove the inequality

$$\left|\frac{\partial L_r(x,y)}{\partial x}\right| \le C(x-y)^{-2}, \tag{3.7}$$

with the additional assumption $0 < x, y < 3\pi/4$; analogous estimates in the considered region but with the assumption $\pi/4 < x, y < \pi$ follow by means of (3.5).

By using the identity (cf. [11, (4.21.7)])

$$\frac{dP_n^{(\alpha,\beta)}(x)}{dx} = \frac{n+\alpha+\beta+1}{2}P_{n-1}^{(\alpha+1,\beta+1)}(x),$$

(we put $P_{-1}^{(\alpha+1,\beta+1)}(x) \equiv 0$ and, consequently, $\phi_{-1}^{(\alpha+1,\beta+1)}(x) \equiv 0$) we get

$$\begin{aligned}\frac{d\phi_n^{(\alpha,\beta)}(x)}{dx} &= -(n+\alpha+\beta+1)\frac{t_n^{(\alpha,\beta)}}{t_{n-1}^{(\alpha+1,\beta+1)}}\phi_{n-1}^{(\alpha+1,\beta+1)}(x)\\ &\quad + \frac{(2\alpha+1)\cos^2(x/2) - (2\beta+1)\sin^2(x/2)}{2\sin x}\phi_n^{(\alpha,\beta)}(x).\end{aligned}$$

Then we split accordingly

$$\frac{\partial}{\partial x}L_r(x,y) = -L_r^1(x,y) + L_r^2(x,y),$$

where

$$L_r^1(x,y) = \sum_{n=1}^{\infty} r^n(n+\alpha+\beta+1)\frac{t_n^{(\alpha,\beta)}}{t_{n-1}^{(\alpha+1,\beta+1)}}\phi_{n-1}^{(\alpha+1,\beta+1)}(x)\phi_n^{(\gamma,\delta)}(y)$$

and

$$L_r^2(x,y) = \frac{(2\alpha+1)\cos^2(x/2) - (2\beta+1)\sin^2(x/2)}{2\sin x}L_r(x,y).$$

It is clear that (here and later on the restrictions imposed on x and y are in force)

$$\left|\frac{(2\alpha+1)\cos^2(x/2) - (2\beta+1)\sin^2(x/2)}{2\sin x}\right| \le Cx^{-1} \le C|x-y|^{-1}.$$

Thus, using the middle bound in (3.3), we arrive at $|L_r^2(x,y)| \le C(x-y)^{-2}$.

The estimate of $L_r^1(x,y)$ is much more involved. By the very definition of $t_n^{(\alpha,\beta)}$ given in (1.1) it follows that

$$(n+\alpha+\beta+1)\frac{t_n^{(\alpha,\beta)}}{t_{n-1}^{(\alpha+1,\beta+1)}} = An + B + \frac{C}{n} + \mathcal{O}(n^{-2}), \quad n \geq 1, \tag{3.8}$$

for some constants A, B, and C. Consequently, $|L_r^1(x,y)|$ is bounded, up to a multiplicative constant, by the sum of the absolute values of

$$L_r^{1,j}(x,y) = \sum_{n=1}^{\infty} r^n n^j \phi_{n-1}^{(\alpha+1,\beta+1)}(x)\phi_n^{(\gamma,\delta)}(y),$$

for $j = -1, 0, 1$, and the remainder sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2}\left|\phi_{n-1}^{(\alpha+1,\beta+1)}(x)\right| \left|\phi_n^{(\gamma,\delta)}(y)\right|.$$

Now, for x and y fixed, take $M = [3x^{-1}]/3 \simeq [3y^{-1}]/3$ (here $[\,\cdot\,]$ denotes the *entier* function). By using the bound (see for instance [9, (2.8)])

$$\left|\phi_n^{(\alpha,\beta)}(x)\right| \leq C \begin{cases} \big((n+1)x\big)^{\alpha+1/2}, & 0 < x \leq 1/(n+1), \\ 1, & 1/(n+1) < x < \pi - 1/(n+1), \\ \big((n+1)(\pi - x)\big)^{\beta+1/2}, & \pi - 1/(n+1) \leq x < \pi, \end{cases} \tag{3.9}$$

we estimate the remainder

$$\sum_{n=1}^{\infty} \frac{1}{n^2}\left|\phi_{n-1}^{(\alpha+1,\beta+1)}(x)\right| \left|\phi_n^{(\gamma,\delta)}(y)\right| \leq C\left(x^{\alpha+3/2}y^{\gamma+1/2}\sum_{n=1}^{M-1} n^{\alpha+\gamma} + \sum_{n=M}^{\infty}\frac{1}{n^2}\right).$$

The very last series is bounded by a constant. Since the sum $\sum_{n=1}^{M-1} n^{\alpha+\gamma}$ is bounded by C, $C\log M$ or $CM^{\alpha+\gamma+1}$, depending on whether $-2 < \alpha+\gamma < -1$, $\alpha+\gamma = -1$ or $\alpha+\gamma > -1$, the bound

$$x^{\alpha+3/2}y^{\gamma+1/2}\sum_{n=1}^{M-1} n^{\alpha+\gamma} \leq Cx^{\alpha+\gamma+2}\sum_{n=1}^{M-1} n^{\alpha+\gamma} \leq C$$

follows. Thus, the remainder sum is bounded by a constant which is obviously enough for our purpose. We now continue with estimating $|L_r^{1,j}(x,y)|$, $j = -1, 0, 1$.

Assuming M to be as above and applying again (3.9) we see that $|L_r^{1,-1}(x,y)|$ is bounded, up to a multiplicative constant, by the expression

$$\sum_{n=1}^{M-1}\frac{1}{n}(nx)^{\alpha+3/2}(ny)^{\gamma+1/2} + \left|\sum_{n=M}^{\infty}\frac{r^n}{n}\phi_{n-1}^{(\alpha+1,\beta+1)}(x)\phi_n^{(\gamma,\delta)}(y)\right|.$$

The first sum is clearly controlled by C, thus we only need to estimate the second one. To do this we apply Theorem 3.1 with $q = 0$ and either $d = 1$ or $d = 0$, first to $\phi_{n-1}^{(\alpha+1,\beta+1)}(x)$ and then to $\phi_n^{(\gamma,\delta)}(y)$. In what follows writing $|\sum a_n \left\{\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\right\}(nx)|$ means that in fact the sum of two absolute values of the series with the sine and the cosine respectively, appear. Analogously, writing $|\sum a_n \left\{\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\right\}(nx)\left\{\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\right\}(ny)|$ means the sum of absolute values of four series, each combination of the sine and the cosine is allowed but, in any given series, the same combination must occur for all n. With this notation,

$$\begin{aligned}\left|\sum_{n=M}^{\infty}\frac{r^n}{n}\phi_{n-1}^{(\alpha+1,\beta+1)}(x)\phi_n^{(\gamma,\delta)}(y)\right| &\le C\left(\left|\sum_{n=M}^{\infty}\frac{r^n}{n}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(nx)\phi_n^{(\gamma,\delta)}(y)\right|+\frac{1}{x}\sum_{n=M}^{\infty}\frac{1}{n^2}\right)\\ &\le C\left(\left|\sum_{n=M}^{\infty}\frac{r^n}{n}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(nx)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(ny)\right|+\left(\frac{1}{x}+\frac{1}{y}\right)\sum_{n=M}^{\infty}\frac{1}{n^2}\right)\\ &\le C\log\left(\frac{2x}{|x-y|}\right),\end{aligned}$$

where in the last step we used [4, (5.3)] to control the first summand. This estimate is sufficient for concluding $|L_r^{1,-1}(x,y)|\le C(x-y)^{-2}$.

Using (3.6) with $d=-1$, we simply have $|L_r^{1,0}(x,y)|\le C|x-y|^{-1}\le C(x-y)^{-2}$.

It remains to estimate $|L_r^{1,1}(x,y)|$. To proceed, we decompose the relevant sum into

$$J_1=\sum_{n=0}^{M-1}r^n n\phi_{n-1}^{(\alpha+1,\beta+1)}(x)\phi_n^{(\gamma,\delta)}(y),\quad J_2=\sum_{n=M}^{\infty}r^n n\phi_{n-1}^{(\alpha+1,\beta+1)}(x)\phi_n^{(\gamma,\delta)}(y).$$

The absolute value of J_1, in view of (3.9), is bounded by

$$Cx^{\alpha+3/2}y^{\gamma+1/2}\sum_{n=1}^{M-1}n^{\alpha+\gamma+3}\le Cx^{\alpha+\gamma+2}M^{\alpha+\gamma+4}\le Cx^{-2}\le C(x-y)^{-2}.$$

In order to analyze J_2 we apply Theorem 3.1 with $q=2$ and $d=1$ to $\phi_{n-1}^{(\alpha+1,\beta+1)}(x)$ and obtain

$$|J_2|\le C\left(\sum_{k=0}^{2}\left|\sum_{n=M}^{\infty}\frac{r^n}{(nx)^k}n\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(nx)\phi_n^{(\gamma,\delta)}(y)\right|+\sum_{n=M}^{\infty}n\frac{1}{(nx)^3}\left|\phi_n^{(\gamma,\delta)}(y)\right|\right).$$

The very last sum, by means of (3.9), is easily seen to be bounded by $C(x-y)^{-2}$. Thus it remains to bound each of the terms

$$I_k=\left|\sum_{n=M}^{\infty}\frac{r^n}{(nx)^k}n\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(nx)\phi_n^{(\gamma,\delta)}(y)\right|,\quad k=0,1,2,$$

by $C(x-y)^{-2}$. Observe that (3.2) taken with $q=2$ and applied to $\phi_n^{(\gamma,\delta)}(y)$ gives

$$I_k\le C\left(\sum_{j=0}^{2}O_{k,j}+\sum_{n=M}^{\infty}\frac{1}{(nx)^k}n\frac{1}{(ny)^3}\right),$$

with

$$O_{k,j}=\left|\sum_{n=M}^{\infty}\frac{r^n}{(nx)^k(ny)^j}n\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(nx)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(ny)\right|.$$

Again, the remainder sum, by using (3.9), is easily seen to be bounded by $C(x-y)^{-2}$. Thus, we are reduced to proving that $O_{k,j}\le C(x-y)^{-2}$ for $k,j\in\{0,1,2\}$. We distinguish four cases. If $k+j>2$, then

$$O_{k,j} \le Cx^{-k}y^{-j}\sum_{n=M}^{\infty} n^{1-k-j} \le Cx^{-(k+j)}M^{2-k-j} \le Cx^{-2} \le C(x-y)^{-2}.$$

If $k+j=2$, then using [4, (5.3)], we obtain

$$O_{k,j} \le x^{-k}y^{-j}\log\left(\frac{2x}{|x-y|}\right) \le Cx^{-(k+j)+1}|x-y|^{-1} \le C(x-y)^{-2}.$$

If $k+j=1$, we analyze only the sum corresponding to $\sin(nx)\sin(ny)$; in the other cases the reasoning is similar. It is clear that

$$\begin{aligned} O_{k,j} &= x^{-k}y^{-j}\left|\sum_{n=M}^{\infty} r^n \sin(nx)\sin(ny)\right| \\ &\le \frac{C}{x}\left(\left|\sum_{n=0}^{\infty} r^n \sin(nx)\sin(ny)\right| + \left|\sum_{n=0}^{M-1} r^n \sin(nx)\sin(ny)\right|\right) \\ &\le \frac{C}{x}\big(|P_r(x-y)| + |P_r(x+y)| + M\big), \end{aligned}$$

and the desired bound follows. Finally, if $k+j=0$, as in the previous case, we treat only the sum related to $\sin(nx)\sin(ny)$. We write

$$\begin{aligned} \left|\sum_{n=M}^{\infty} r^n n\sin(nx)\sin(ny)\right| &\le C\left(\left|\sum_{n=0}^{\infty} r^n n\sin(nx)\sin(ny)\right| + \left|\sum_{n=0}^{M-1} r^n n\sin(nx)\sin(ny)\right|\right) \\ &\le C\left(\left|\frac{dQ_r}{dt}(x-y)\right| + \left|\frac{dQ_r}{dt}(x+y)\right| + M^2\right), \end{aligned}$$

and again the bound by $C(x-y)^{-2}$ follows. The proof of the bound of $|J_2|$ and thus also of the estimate (3.4) is finished.

In order to justify the last statement of the proposition observe that the quantities on the right in the first and third lines of (3.3) can be further bounded by $|x-y|^{-1}$ when $\alpha,\beta,\gamma,\delta \in [-1/2,\infty)$, hence the estimate $|L_r(x,y)| \le C|x-y|^{-1}$ holds globally. For $\alpha,\beta,\gamma,\delta \in [1/2,\infty)$ also the inequality $|\nabla_{x,y}L_r(x,y)| \le C(x-y)^{-2}$ holds in the whole square. Indeed, considering $\frac{\partial}{\partial x}L_r$, using (3.3) and taking into account different parts of the square separately, we get

$$|L_r^2(x,y)| \le C\frac{|L_r(x,y)|}{|\sin x|} \le C(x-y)^{-2}$$

(note that the assumption $\alpha,\beta \ge 1/2$ is essential here, whereas the assumption concerning γ and δ comes into play when dealing with $\frac{\partial}{\partial y}L_r$). On the other hand, the bound $|L_r^1(x,y)| \le C(x-y)^{-2}$ can be obtained with the aid of

$$\left|L_r^1(x,y)\right| \le C\begin{cases} \dfrac{y^{\gamma+1/2}(\pi-x)^{\beta+3/2}}{x^{\gamma+5/2}(\pi-y)^{\beta+7/2}}, & 0<y\le \max\left(\frac{x}{2},\frac{3x-\pi}{2}\right), \\ (x-y)^{-2}, & \max\left(\frac{x}{2},\frac{3x-\pi}{2}\right)<y<\min\left(\frac{3x}{2},\frac{x+\pi}{2}\right), \\ \dfrac{x^{\alpha+3/2}(\pi-y)^{\delta+1/2}}{y^{\alpha+7/2}(\pi-x)^{\delta+5/2}}, & \min\left(\frac{3x}{2},\frac{x+\pi}{2}\right)\le y<\pi, \end{cases}$$

by noting that the quantities on the right-hand side in the first and third lines above are bounded by $C(x-y)^{-2}$. It remains to comment on the last estimate. Essentially, we repeat the arguments from the first two paragraphs of the proof of Proposition 3.2. Then the bound in the first line above is deduced in the way analogous to that of (3.3); apart from (3.5), we use either [9, Theorem 7.1] or [9, Theorem 5.1] taken with $s=d=-1$ and $g(n)=(n+\alpha+\beta+1)t_n^{(\alpha,\beta)}/t_{n-1}^{(\alpha+1,\beta+1)}$. □

Proposition 3.3 *Let $\alpha,\beta,\gamma,\delta\in(-1,\infty)$ and $L_r=L_r^{(\alpha,\beta),(\gamma,\delta)}$ be given by (3.1). Then for every $x\neq y$, $0<x,y<\pi$, the limit*

$$L(x,y)=\lim_{r\to 1^-}L_r(x,y)$$

exists. Moreover, $|L(x,y)|$ is bounded by the right-hand side of (3.3); similarly, $|\nabla L(x,y)|$ is bounded by the right-hand side of (3.4). In addition, the statement made in the last sentence of Proposition 3.2 applies to $|L(x,y)|$ as well.

Proof Once we prove the existence of the limit, the bounds of $|L(x,y)|$ and $|\nabla L(x,y)|$ are direct consequences of the corresponding bounds from Proposition 3.2. To be precise, justifying the estimate of $|\nabla L(x,y)|$ requires also the identity

$$\frac{\partial}{\partial x}\Big(\lim_{r\to 1^-}L_r(x,y)\Big)=\lim_{r\to 1^-}\frac{\partial}{\partial x}L_r(x,y) \tag{3.10}$$

and similarly for $\frac{\partial}{\partial y}$. Assuming for a moment that $\lim_{r\to 1^-}L_r(x,y)$ exists, what is still needed for proving (3.10) is the fact that for each fixed y, $0<y<\pi$, the convergence on the right of (3.10) is locally uniform in $x\neq y$. Using the splitting $\frac{\partial}{\partial x}L_r=-L_r^1+L_r^2$ that appears in the proof of Proposition 3.2 it is sufficient to check that for a given y, $0<y<\pi$, the convergence of $L_r^1(x,y)$ and $L_r^2(x,y)$ as $r\to 1^-$ is locally uniform in x. For $L_r^2(x,y)$, or rather for $L_r(x,y)$, this will be explained along the lines of the proof of the existence of $\lim_{r\to 1^-}L_r(x,y)$; see the lines that follow. For $L_r^1(x,y)$ the argument is essentially the same, hence we do not provide any details (a look into the proof of Proposition 3.2 is helpful).

Using (3.2) with $q=1$ we expand the functions $\phi_n^{(\alpha,\beta)}(x)$ and $\phi_n^{(\gamma,\delta)}(y)$, $n=1,2,\ldots$,

$$\phi_n^{(\alpha,\beta)}(x)=A_0(x)\sin(nx)+B_0(x)\cos(nx)+\frac{A_1(x)}{n}\sin(nx)+\frac{B_1(x)}{n}\cos(nx)+H_n(x),$$

and similarly

$$\phi_n^{(\gamma,\delta)}(y)=C_0(y)\sin(ny)+D_0(y)\cos(ny)+\frac{C_1(y)}{n}\sin(ny)+\frac{D_1(y)}{n}\cos(ny)+K_n(y),$$

where $|H_n(x)|\le C(n\sin x)^{-2}$, $|K_n(y)|\le C(n\sin y)^{-2}$ (C independent of n and $x,y\in(0,\pi)$), and A_i,B_i,C_i,D_i are locally bounded functions. Denoting by $\widetilde{L}_r(x,y)$ the series as in (3.1) but with the summation starting from $n=1$, by exploiting the above expansions we obtain

$$\widetilde{L}_r(x,y)=\sum_{j,l=0}^{1}O_{j,l}(r,x,y)+J_1(r,x,y)+J_2(r,x,y)+G(r,x,y).$$

Here the terms $O_{j,l}$ capture the part that comes from the main parts of the abovementioned expansions and are sums of terms of the form

$$E_j(x)F_l(y)\sum_{n=1}^{\infty} r^n n^{-j-l}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(nx)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(ny),$$

and E,F may replace any of the letters A,B,C,D, the term J_1 gathers the part that comes from the main parts of the second expansion and the remainder of the first one, hence it is a sum of terms of the form

$$F_j(y)\sum_{n=1}^{\infty} r^n n^{-j}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(ny)H_n(x),\quad j=0,1,$$

J_2 acts as J_1 but with the position of the both expansions switched, hence it is a sum of terms of the form

$$E_i(x)\sum_{n=1}^{\infty} r^n n^{-j}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(nx)K_n(y),\quad i=0,1,$$

and, eventually, G captures the part that comes from the both remainders,

$$G(r,x,y)=\sum_{n=1}^{\infty} r^n H_n(x)K_n(y).$$

Due to the bounds on H_n and K_n, it is evident that each of the series in the terms entering into either J_1 or J_2, or in $G_r(x,y)$, but with the factor r^n removed, is absolutely convergent since, either $|H_n(x)|\le C_x n^{-2}$ or $|K_n(y)|\le C_y n^{-2}$ (or both) applies. Thus the corresponding expressions converge as $r\to 1^-$. In addition, the convergence is locally uniform in x. It is therefore sufficient to analyze the $O_{j,l}$ terms. Given $j,l\in\{0,1\}$ we have to verify that

$$\lim_{r\to 1^-}\sum_{n=1}^{\infty} r^n n^{-j-l}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}\big(n(x\pm y)\big)$$

converges for $x\pm y\notin\{0,\pi\}$, and the convergence is locally uniform in x. If $j+l=0$ then we deal with either $P_r(x\pm y)$ or $Q_r(x\pm y)$ hence the convergence is obvious; if $j+l=1$ then the series also takes a compact form (see [13, p. 2]) and the required convergence follows; if $j+l=2$ the claim is obvious. The proof of the proposition is completed. □

Finally, we show that the kernel $L(x,y)$ is associated with $T=T^{(\alpha,\beta),(\gamma,\delta)}$ in the sense of Calderón–Zygmund theory.

Proposition 3.4 *Let $f,g\in C_c^{\infty}(0,\pi)$ have disjoint supports. Then*

$$\langle Tf,g\rangle=\int_0^{\pi}\int_0^{\pi} L(x,y)f(y)\overline{g(x)}\,dy\,dx. \tag{3.11}$$

Proof Let $g=\sum_{n=0}^{\infty}\langle f,\phi_n^{(\alpha,\beta)}\rangle\phi_n^{(\alpha,\beta)}$. By Parseval's identity

$$\langle Tf,g\rangle=\sum_{n=0}^{\infty}\langle f,\phi_n^{(\gamma,\delta)}\rangle\overline{\langle f,\phi_n^{(\alpha,\beta)}\rangle}. \tag{3.12}$$

We will show that the right-hand sides of (3.11) and (3.12) coincide. Denoting by $F(x,y)$ the function from Proposition 3.2 that majorizes $|L_r(x,y)|$ it is clear that

$$\int_0^\pi\int_0^\pi |F(x,y)f(y)\overline{g(x)}|\,dy\,dx < \infty.$$

Therefore the dominated convergence theorem justifies the second equality in the following chain of identities

$$\begin{aligned}\int_0^\pi\int_0^\pi L(x,y)f(y)\overline{g(x)}\,dy\,dx &= \int_0^\pi\int_0^\pi \lim_{r\to1^-} L_r(x,y)f(y)\overline{g(x)}\,dy\,dx\\ &= \lim_{r\to1^-}\int_0^\pi\int_0^\pi L_r(x,y)f(y)\overline{g(x)}\,dy\,dx\\ &= \lim_{r\to1^-}\int_0^\pi T_rf(x)\overline{g(x)}\,dx\\ &= \lim_{r\to1^-}\sum_{n=0}^\infty r^n\langle f,\phi_n^{(\gamma,\delta)}\rangle\overline{\langle f,\phi_n^{(\alpha,\beta)}\rangle}.\end{aligned}$$

The fourth identity is a consequence of Parseval's identity. Finally, since the series $\sum_{n=0}^\infty\langle f,\phi_n^{(\gamma,\delta)}\rangle\overline{\langle f,\phi_n^{(\alpha,\beta)}\rangle}$ converges, the last limit equals the right-hand side of (3.12). □

Remark 3.5 For $(\alpha,\beta)=(-1/2,-1/2)$ and $(\gamma,\delta)=(1/2,1/2)$ we have

$$L(x,y)=\frac{\sqrt{2}-1}{\pi}\sin y+\frac{1}{\pi}\cos y\frac{\sin y}{\cos y-\cos x}.$$

This is because $\phi_n^{(-1/2,-1/2)}(\theta)=(2/\pi)^{1/2}\cos(n\theta)$ for $n>0$, $\phi_0^{(-1/2,-1/2)}(\theta)=1/\sqrt{\pi}$, $\phi_n^{(1/2,1/2)}(\theta)=\sqrt{2/\pi}\sin((n+1)\theta)$, and as a direct calculation shows,

$$\begin{aligned}L_r(x,y) &= \frac{\sqrt{2}}{\pi}\sin y+\frac{2}{\pi}\sum_{n=1}^\infty r^n\cos(nx)\sin\big((n+1)y\big)\\ &= \frac{\sqrt{2}}{\pi}\sin y+\frac{1}{\pi}\bigg[\cos y\sum_{n=1}^\infty r^n\Big(\sin\big(n(x+y)\big)-\sin\big(n(x-y)\big)\Big)\\ &\quad+\sin y\sum_{n=1}^\infty r^n\Big(\cos\big(n(x-y)\big)+\cos\big(n(x+y)\big)\Big)\bigg].\end{aligned}$$

Summing the above series leads to expressions involving the Poisson and the conjugate Poisson kernels, then passing to the limit with $r\to1^-$ gives

$$L(x,y)=\frac{\sqrt{2}}{\pi}\sin y+\frac{1}{\pi}\cos y\left(\frac{1}{\tan(x/2+y/2)}-\frac{1}{\tan(x/2-y/2)}\right)-\frac{1}{\pi}\sin y,$$

and finally an application of trigonometric identities does the job. The fact that $L(x,y)$ is a C^1 function on $(0,1)\times(0,1)\setminus\Delta$ and satisfies the estimates of Proposition 3.3 now follows by a straightforward inspection.

The case $\alpha=\beta=-1/2, \gamma=\delta=1/2$ is not the only one when the kernel $L^{(\alpha,\beta),(\gamma,\delta)}$ may be computed explicitly. However, even in such special cases the results of this section are meaningful. The following example is to some extent instructive.

Assume that $\gamma=\beta$ and $\delta=\alpha$. We shall compute $L^{(\alpha,\beta),(\beta,\alpha)}$. Using the identity

$$\phi_n^{(\alpha,\beta)}(\pi-x)=(-1)^n\phi_n^{(\beta,\alpha)}(x), \tag{3.13}$$

which in turn is a consequence of the relation $P_n^{(\alpha,\beta)}(-x)=(-1)^nP_n^{(\beta,\alpha)}(x)$ (cf. [11, (4.1.3)]), we get

$$L_r^{(\alpha,\beta),(\beta,\alpha)}(x,y)=\sum_{n=0}^{\infty}r^n\phi_n^{(\alpha,\beta)}(x)\phi_n^{(\beta,\alpha)}(y)=\sum_{n=0}^{\infty}(-r)^n\phi_n^{(\alpha,\beta)}(x)\phi_n^{(\alpha,\beta)}(\pi-y).$$

Now the well-known formula of Bailey [3] for the Jacobi–Poisson kernel can be applied, and this leads to

$$\begin{aligned}L_r^{(\alpha,\beta),(\beta,\alpha)}(x,y)=\;&\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\frac{1+r}{(1-r)^{\alpha+\beta+2}}\big(\Phi(x,y)\big)^{\alpha+1/2}\big(\Phi(y,x)\big)^{\beta+1/2}\\&\times F_4\bigg(\frac{\alpha+\beta+2}{2},\frac{\alpha+\beta+3}{2},\alpha+1,\beta+1;\frac{-r\Phi^2(x,y)}{(1-r)^2},\frac{-r\Phi^2(y,x)}{(1-r)^2}\bigg);\end{aligned}$$

here $\Phi(x,y)=2\sin(x/2)\cos(y/2)$, and F_4 denotes Appel's hypergeometric function of two variables, cf. [6, Sect. 5.7] (to be precise, here by F_4 we mean the analytic continuation of the defining series in [6, Sect. 5.7, (9)]). Next, we transform the above expression by means of the formula (cf. [6, Sect. 5.11, (9)] after correcting an obvious misprint concerning the power of $-y$ in the second summand there)

$$\begin{aligned}F_4(a,b,c,d;X,Y)=\;&\frac{\Gamma(d)\Gamma(b-a)}{\Gamma(d-a)\Gamma(b)}(-Y)^{-a}F_4\Big(a,a+1-d,c,a+1-b;\frac{X}{Y},\frac{1}{Y}\Big)\\&+\frac{\Gamma(d)\Gamma(a-b)}{\Gamma(d-b)\Gamma(a)}(-Y)^{-b}F_4\Big(b+1-d,b,c,b+1-a;\frac{X}{Y},\frac{1}{Y}\Big),\end{aligned}$$

valid when the variables of all F_4 functions are not in $[1,\infty)$. This, under assumption $\Phi(x,y)<\Phi(y,x)$ (or equivalently $x<y$), allows to pass to the limit as $r\to1^-$. Indeed, the second summand then vanishes, and Appel's function in the first one reduces to the Gauss hypergeometric function ${}_2F_1$ (cf. [6, Chap. 2] or [7, Chap. 9]) since its second argument tends to 0. The final result, after some simplifications with the aid of the duplication formula for the gamma function (cf. [7, (1.2.3)]), is

$$\begin{aligned}L^{(\alpha,\beta),(\beta,\alpha)}(x,y)=\;&\frac{2\Gamma((\alpha+\beta+2)/2)}{\Gamma(\alpha+1)\Gamma((\beta-\alpha)/2)}\big(\Phi(x,y)\big)^{\alpha+1/2}\big(\Phi(y,x)\big)^{-(\alpha+3/2)}\\&\times{}_2F_1\bigg(\frac{\alpha+\beta+2}{2},\frac{\alpha-\beta+2}{2};\alpha+1;\Big(\frac{\Phi(x,y)}{\Phi(y,x)}\Big)^2\bigg).\end{aligned}$$

By the symmetry, for $x>y$ we have $L^{(\alpha,\beta),(\beta,\alpha)}(x,y)=L^{(\beta,\alpha),(\alpha,\beta)}(y,x)$. In the above formula we consider $\Gamma(z)^{-1}$ to be a continuous function with the sequence of isolated zeroes in $0,-1,-2,\ldots$ Hence, if $\alpha=\beta+2k$, $k=1,2,\ldots$, then $L^{(\alpha,\beta),(\beta,\alpha)}(x,y)=0$ on $0<x<y$ and, moreover, $L^{(\alpha,\beta),(\beta,\alpha)}(x,y)$ is continuous as a function considered on

the region $0 < y \le x$. Similarly, if $\beta = \alpha + 2k$, $k = 1, 2, \dots$, then $L^{(\alpha,\beta),(\beta,\alpha)}(x, y) = 0$ on $0 < y < x$ and $L^{(\alpha,\beta),(\beta,\alpha)}(x, y)$ is continuous on $0 < x \le y$. This is because, in the first case, $(\beta - \alpha + 2)/2 \in \{0, -1, -2, \dots\}$ which means that ${}_2F_1((\alpha + \beta + 2)/2, (\beta - \alpha + 2)/2; \beta + 1; t)$ is a polynomial in t and the same is true, in the second case, for ${}_2F_1((\alpha + \beta + 2)/2, (\alpha - \beta + 2)/2; \alpha + 1; t)$. If $\alpha \neq \beta + 2k$ for all integers k then the kernel has a singularity along the diagonal.

Noteworthy, there is a striking coincidence between $L^{(\alpha,\beta),(\beta,\alpha)}$ and the Hankel transform transplantation kernel $K_{\alpha\beta}$ (see [10, Sect. 3]). Namely, we have

$$L^{(\alpha,\beta),(\beta,\alpha)}(x, y) = K_{\alpha\beta}\big(\Phi(x, y), \Phi(y, x)\big).$$

This connection indicates that no significant simplifications can be expected in the analysis of the Jacobi transplantation operator: even in this special case, with the exact expression for the kernel, obtaining standard estimates is not a trivial task (see [10, Proposition 3.2]).

4 Proofs of the main results

Recall that $\mathcal{D}$ stands for the region defined in Sect. 2, see Fig. 1. Let T_i, $i = 1, 2$, denote the integral operators

$$T_i f(x) = \int_0^\pi \chi_{\mathcal{D}_i}(x, y) L(x, y) f(y)\, dy,$$

where $\mathcal{D}_i$ denote the two components of the complement of $\mathcal{D}$ in $(0, \pi) \times (0, \pi)$,

$$\begin{aligned} \mathcal{D}_1 &= \big\{(x, y) : 0 < x < \pi,\ 0 < y \le \max(x/2, (3x - \pi)/2)\big\}, \\ \mathcal{D}_2 &= \big\{(x, y) : 0 < x < \pi,\ \min(3x/2, (x + \pi)/2) \le y < \pi\big\}. \end{aligned}$$

It is straightforward that the operator

$$\widetilde{T} = T - T_1 - T_2$$

is bounded in L^2. Indeed, L^2-boundedness of T_1 and T_2 follows by taking $w \equiv 1$ and $p = 2$ in (2.4) and (2.5) and using the estimates of $L(x, y)$, see Proposition 3.3. This fact together with Propositions 3.3 and 3.4 shows that $\widetilde{T}$ is a double local Calderón–Zygmund operator with the associated kernel $\chi_{\mathcal{D}}(x, y) L(x, y)$.

Proof of Theorem 2.5 Using the estimate of $|L(x, y)|$ by the right-hand side of (3.3) and applying weighted Hardy's inequality (2.4), we obtain

$$\begin{aligned} \int_0^\pi |T_1 f(x) w(x)|^p\, dx &= \int_0^\pi \left| w(x) \int_0^{\max(x/2, (3x - \pi)/2)} L(x, y) f(y)\, dy \right|^p dx \\ &\le C \int_0^\pi \left(w(x) \frac{(\pi - x)^{\beta + 1/2}}{x^{\gamma + 3/2}} \int_0^x \frac{y^{\gamma + 1/2}}{(\pi - y)^{\beta + 3/2}} |f(y)|\, dy \right)^p dx \\ &\le C \int_0^\pi |f(x) w(x)|^p\, dx. \end{aligned}$$

Similarly, using weighted Hardy's inequality (2.5), we get

$$\begin{aligned}\int_0^\pi |T_2 f(x)w(x)|^p\,dx &= \int_0^\pi \left| w(x) \int_{\min(3x/2,(x+\pi)/2)}^{\pi} L(x,y)f(y)\,dy\right|^p dx\\ &\le C\int_0^\pi \left(w(x)\frac{x^{\alpha+1/2}}{(\pi-x)^{\delta+3/2}} \int_x^\pi \frac{(\pi-y)^{\delta+1/2}}{y^{\alpha+3/2}}|f(y)|\,dy\right)^p dx\\ &\le C\int_0^\pi |f(x)w(x)|^p\,dx.\end{aligned}$$

Now the desired L^p inequality for $\widetilde{T}$ is a consequence of item (a) in Theorem 2.4.

In order to prove (2.7) we first note that the existence of $\langle f, \phi_n^{(\gamma,\delta)}\rangle$, $f \in L^p(w)$, is a direct consequence of Conditions (2.2) and (2.3). Indeed, we can use Hölder's inequality, the estimate (3.9) (with (γ,δ) replacing (α,β)) and either the fact that the second term in (2.2) for $r = \pi/2$ is finite to check that $\int_0^{\pi/2} |f(x)\phi_n^{(\gamma,\delta)}(x)|\,dx < \infty$, or the fact that the second term in (2.3) is finite to verify that $\int_{\pi/2}^{\pi} |f(x)\phi_n^{(\gamma,\delta)}(x)|\,dx < \infty$. In a similar way, with the aid of Hölder's inequality, it may be seen that Condition (1.3) implies that $\langle f, \phi_n^{(\alpha,\beta)}\rangle$ exists for $f \in L^p(w)$, thus the existence of $\langle Tf, \phi_n^{(\alpha,\beta)}\rangle$ for any $f \in L^p(w)$ is ensured. Next, observe that another application of Hölder's inequality gives for $g \in L^p(w)$

$$\int_0^\pi |g(x)\phi_n^{(\alpha,\beta)}(x)|\,dx \le \|g\|_{L^p(w)}\|\phi_n^{(\alpha,\beta)}\|_{L^{p'}(w^{-1})}.$$

It follows by (3.9) and Condition (1.3) that the right-hand side above is finite. Therefore, for any fixed $n \in \mathbb{N}$, the mapping $g \mapsto \langle g, \phi_n^{(\alpha,\beta)}\rangle$ is a bounded functional on $L^p(w)$. We now fix $f \in L^p(w)$ and choose a sequence $f_k \in L^2 \cap L^p(w)$ such that $f_k \to f$ in $L^p(w)$, $k \to \infty$. Then, by the very definition, $Tf = \lim_{k\to\infty} Tf_k$ in $L^p(w)$. Thus

$$\langle Tf, \phi_n^{(\alpha,\beta)}\rangle = \lim_{k\to\infty}\langle Tf_k, \phi_n^{(\alpha,\beta)}\rangle,$$

which gives (2.7) in view of the identity $\langle Tf_k, \phi_n^{(\alpha,\beta)}\rangle = \langle f_k, \phi_n^{(\gamma,\delta)}\rangle$ and the fact that $f \mapsto \langle f, \phi_n^{(\gamma,\delta)}\rangle$ is also a bounded functional on $L^p(w)$ (more precisely, the last statement requires $\|\phi_n^{(\gamma,\delta)}\|_{L^{p'}(w^{-1})} < \infty$, but this is again guaranteed by Conditions (2.2) and (2.3)).

This finishes justifying (2.7), hence Theorem 2.5 is proved. □

Proof of Theorem 2.6 Denote by $\mathcal{E}_i$, $i = 1, \ldots, 8$, the triangles decomposing $(0,\pi) \times (0,\pi)$ according to Fig. 2 below. Taking into account the estimates of the kernel $L(x,y)$ asserted in Proposition 3.3, we see that

$$|Tf(x)| \le C\Big(\sum_{i=1}^{8} K_i f(x) + |\widetilde{T}f(x)|\Big),$$

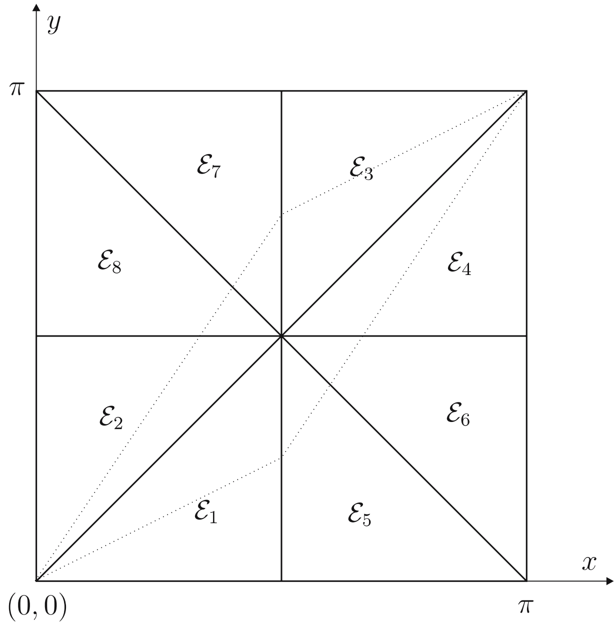

Fig. 2 Decomposition into $\mathcal{E}_i$ (*dots* indicate the region $\mathcal{D}$)

where C is independent of f, and

$$K_i f(x) = \int_0^\pi \chi_{\mathcal{E}_i}(x,y) K_i(x,y)|f(y)|\,dy, \quad i = 1,\ldots,8,$$

with the kernels

$$\begin{aligned}
K_1(x,y) &= x^{-(\gamma+3/2)} y^{\gamma+1/2},\\
K_2(x,y) &= x^{\alpha+1/2} y^{-(\alpha+3/2)},\\
K_3(x,y) &= (\pi-x)^{-(\delta+3/2)}(\pi-y)^{\delta+1/2},\\
K_4(x,y) &= (\pi-x)^{\beta+1/2}(\pi-y)^{-(\beta+3/2)},\\
K_5(x,y) &= K_6(x,y) = (\pi-x)^{\beta+1/2} y^{\gamma+1/2},\\
K_7(x,y) &= K_8(x,y) = x^{\alpha+1/2}(\pi-y)^{\delta+1/2}.
\end{aligned}$$

The weighted weak type $(1,1)$ inequality for each K_i, $i = 1,\ldots,8$, is obtained in a straightforward manner by means of Hardy's inequality (2.10) or its dual (i.e. with P_η replaced by Q_η), choosing appropriately the parameters and using the corresponding condition imposed on a weight $w(x)$. For example, for K_4 we write

$$\int_{\{0<x<\pi\,:\,K_4 f(x)>\lambda\}} w(x)\,dx = \int_{\left\{\frac{\pi}{2}<x<\pi\,:\,(\pi-x)^{\beta+1/2}\int_{\pi/2}^{x}(\pi-y)^{-(\beta+3/2)}|f(y)|\,dy>\lambda\right\}} w(x)\,dx$$

$$= \int\limits_{\left\{0<x<\frac{\pi}{2}\,:\,x^{\beta+1/2}\int_{\pi/2}^{\pi-x}(\pi-y)^{-(\beta+3/2)}|f(y)|\,dy>\lambda\right\}} w(\pi-x)\,dx$$

$$= \int\limits_{\left\{0<x<\frac{\pi}{2}\,:\,x^{\beta+1/2}\int_{x}^{\pi/2}y^{-(\beta+3/2)}|f(\pi-y)|\,dy>\lambda\right\}} w(\pi-x)\,dx.$$

The last expression is, by weighted Hardy's inequality for $Q_{-(\beta+1/2)}$ applied with $U(x) = w(\pi - x)$ and $V(x) = w(\pi - x)x^{\beta+3/2}$ (notice that this is legitimate by the second part of condition (ii) imposed on $w(x)$), estimated by

$$\frac{C}{\lambda}\int_0^{\pi/2} |f(\pi - x)|w(\pi - x)\,dx \le \frac{C}{\lambda}\int_0^{\pi} |f(x)|w(x)\,dx.$$

The remaining K_i, $i \le 8$, are treated in a similar way.

Finally, the operator $\widetilde{T}$ is a double local CZ operator, hence it satisfies the desired weighted weak type $(1, 1)$ estimate by item (b) of Theorem 2.4 (the corresponding weight is assumed to verify the double local A_1 condition). □

5 Final comments

We first comment the fact that (1.2) implies each of the conditions: (2.1), (2.2) and (2.3). The first implication is easy: under the restriction $0 \le u < v \le \min(2u, (u+\pi)/2) \le \pi$ that appears in (2.1), for $u < x < v$ one has $1/2 < x/v < 1$ and $1 < (\pi - x)/(\pi - v) < 2$ hence (2.1) follows.

To check that (1.2) implies (2.2) note that the latter is equivalent to the conjunction of the conditions:

$$\sup_{0<r<3\pi/4}\left(\int_r^{3\pi/4}\left(\frac{w(x)}{x^{\gamma+3/2}}\right)^p dx\right)^{1/p}\left(\int_0^r\left(\frac{x^{\gamma+1/2}}{w(x)}\right)^{p'} dx\right)^{1/p'} < \infty \tag{5.1}$$

and

$$\sup_{\pi/4<r<\pi}\left(\int_r^{\pi}\left(w(x)(\pi-x)^{\beta+1/2}\right)^p dx\right)^{1/p}\left(\int_{\pi/4}^r\left(\frac{1}{w(x)(\pi-x)^{\beta+3/2}}\right)^{p'} dx\right)^{1/p'} < \infty. \tag{5.2}$$

An application of [9, Lemma (9.19)] taken with $s = 0$, $q = p$, $a = \alpha+1/2$, $b = \gamma+1/2$, then shows that (5.1) is implied by

$$\left(\int_u^v \left(w(x)x^{\alpha+1/2}\right)^p dx\right)^{1/p}\left(\int_u^v\left(\frac{x^{\gamma+1/2}}{w(x)}\right)^{p'} dx\right)^{1/p'} \le C(v-u)v^{\alpha+\gamma+1},\quad 0\le u < v\le 3\pi/4. \tag{5.3}$$

The change of variables $y = \pi - x$, $s = \pi - r$, gives the equivalent form of (5.2):

$$\sup_{0<s<3\pi/4} \left(\int_0^s \left(\frac{1}{w(\pi-y)y^{\beta+3/2}} \right)^{p'} dy \right)^{1/p'} \left(\int_s^{3\pi/4} \left(w(\pi-y)y^{\beta+1/2} \right)^p dy \right)^{1/p} < \infty.$$

Again an application of [9, Lemma (9.19)] taken with $s = 0$, $q = p$, $a = \beta + 1/2$, $b = \delta + 1/2$, and with $w(\pi - y)$ in place of $w(x)$, shows that the above inequality, and thus (5.2), is implied by

$$\left(\int_u^v \left(\frac{y^{\delta+1/2}}{w(\pi-y)} \right)^{p'} dy \right)^{1/p'} \left(\int_u^v \left(w(\pi-y)y^{\beta+1/2} \right)^p dy \right)^{1/p} \\ \le C(v-u)v^{\beta+\delta+1}, \quad 0 \le u < v \le 3\pi/4. \tag{5.4}$$

The inequalities (5.3) and (5.4) are easily seen to be consequences of (1.2). The verification that (1.2) implies (2.3) goes along similar lines.

As the second comment we point out that more general versions (in the spirit of the general Muckenhoupt's theorem [9, Theorem (1.14)]) of our main results, Theorem 2.5 and Theorem 2.6, are possible. Such generalizations would be applicable, for instance, in studying conjugacy and Riesz transforms for Jacobi expansions. Let $g(n)$, $n \ge 0$, be a sequence satisfying the smoothness condition, cf. [9, (1.4)],

$$g(n) = \sum_{j=0}^{J-1} c_j n^{-j} + \mathcal{O}(n^{-J}), \quad n \ge 1, \tag{5.5}$$

where J is sufficiently large and $c_0, c_1, \dots, c_{J-1}$, are fixed. Then, given an integer d, we define the generalized transplantation operator $T = T_{d,g}^{(\alpha,\beta),(\gamma,\delta)}$ by

$$Tf = \sum_{n=0}^{\infty} g(n) \langle f, \phi_n^{(\gamma,\delta)} \rangle \phi_{n+d}^{(\alpha,\beta)}, \quad f \in L^2((0,\pi), dx),$$

with the convention that $\phi_n^{(\alpha,\beta)} \equiv 0$ for $n < 0$. Clearly, T is an L^2-bounded operator. Accordingly, we consider the kernel

$$L_{r,d,g}(x,y) = \sum_{n=0}^{\infty} r^n g(n) \phi_{n+d}^{(\alpha,\beta)}(x) \phi_n^{(\gamma,\delta)}(y), \quad x, y \in (0,\pi).$$

A thorough analysis of the arguments used in the proof of Proposition 3.2 shows that the estimates (3.3) and (3.4) remain valid for the kernel $L_{r,d,g}$ provided

$$J \ge 5 + \max(\alpha, -1/2) + \max(\beta, -1/2) + \max(\gamma, -1/2) + \max(\delta, -1/2).$$

Here are the details. The initial comments concerning (3.3) remain valid as well for $L_{r,d,g}$ replacing L_r: first, the third estimate is a dual form of the first one (if $g(n)$ satisfies (5.5) so does $g'(n) = g(n + d)$); second, the first bound is included in [9, Theorems 7.1 and 5.1] since the assumptions on J imposed there are weaker than the present assumption; third, the modified (by the sequence $g(n)$) version of (3.6), implying the middle bound, remains valid for $J \ge 2$, cf. [9, Theorem 8.3]. Comments concerning

(3.4) are as follows. The suitable splitting $\partial_x L_{r,d,g} = -L^1_{r,d,g} + L^2_{r,d,g}$ is in force, with $L^2_{r,d,g}$ essentially the same as L^2_r, and $L^1_{r,d,g}$ modified to

$$L^1_{r,d,g}(x,y) = \sum_{n=1}^{\infty} r^n g(n)(n+d+\alpha+\beta+1)\frac{t^{(\alpha,\beta)}_{n+d}}{t^{(\alpha+1,\beta+1)}_{n+d-1}}\phi^{(\alpha+1,\beta+1)}_{n+d-1}(x)\phi^{(\gamma,\delta)}_n(y).$$

The estimate of $L^1_{r,d,g}$ (the one of $L^2_{r,d,g}$ is immediate) requires the analogue of (3.8) with the left-hand side first changed by replacing n by $n+d$ and then attaching the factor $g(n)$; this analogue is possible since $J \geq 3$. Consequently, the bound of $|L^1_{r,d,g}(x,y)|$ relies on estimating $|L^{1,j}_{r,d,g}(x,y)|$, $j = -1,0,1$, (where, in their defining identities, $\phi^{(\alpha+1,\beta+1)}_{n-1}(x)$ are replaced by $\phi^{(\alpha+1,\beta+1)}_{n+d-1}(x)$ and the factor $g(n)$ is inserted) and the corresponding remainder (this time only with $\phi^{(\alpha+1,\beta+1)}_{n-1}(x)$ replaced by $\phi^{(\alpha+1,\beta+1)}_{n+d-1}(x)$). With these changes the further reasoning is essentially identical to that performed for $L^1_r(x,y)$; notice that a generalization of the Darboux type formula needed here is already provided by the last statement of Theorem 3.1.

Similarly, investigating the proof of Proposition 3.3 reveals that the limit $L_{d,g}(x,y) = \lim_{r\to 1^-} L_{r,d,g}(x,y)$, $x \neq y$, exists, and satisfies (3.3) and (3.4) as well. Finally, minor modifications of the argument from the proof of Proposition 3.4 show that $T = T^{(\alpha,\beta),(\gamma,\delta)}_{d,g}$ also satisfies (3.11).

Consequently, with the above assumption on J, Theorems 2.5 and 2.6 hold, with $T^{(\alpha,\beta),(\gamma,\delta)}_{d,g}$ replacing $T^{(\alpha,\beta),(\gamma,\delta)}$ ((2.7) must be adjusted appropriately, to be precise).

6 Appendix: proof of the Darboux type formula for Jacobi polynomials

Throughout this section we assume that $\alpha, \beta \in (-1,\infty)$ and $q \in \mathbb{N}$ are fixed, and $n \geq 1$. The main tool in proving Theorem 3.1 will be the following uniform asymptotic formula for Jacobi polynomials obtained recently by Wong and Zhao [12]: for $x \in (0,\pi/2]$,

$$\left(\sin\frac{x}{2}\right)^{\alpha}\left(\cos\frac{x}{2}\right)^{\beta} P^{(\alpha,\beta)}_n(\cos x) = J_\alpha(Nx)\sum_{k=0}^{q}\frac{c_k(x)}{N^k} + J_{\alpha+1}(Nx)\sum_{k=0}^{q}\frac{d_k(x)}{N^k} + \delta_q(n,x);$$

here J_α denotes the Bessel function of the first kind of order α, $N = n + (\alpha+\beta+1)/2$, the coefficients $c_k(x)$ and $d_k(x)$ are bounded continuous functions of $x \in (0,\pi/2]$, and the remainder $\delta_q(n,x)$ satisfies

$$|\delta_q(n,x)| \leq \frac{\Lambda_q}{N^{q+1}}\big(|J_\alpha(Nx)| + |J_{\alpha+1}(Nx)|\big) \tag{6.1}$$

with a constant Λ_q independent of x and n; see [12, (2.29), Theorems 4.1, 5.1]. Thus for $x \in (0,\pi/2]$

$$\phi^{(\alpha,\beta)}_n(x) = t^{(\alpha,\beta)}_n\sqrt{\frac{\sin x}{2}}\Big(J_\alpha(Nx)\sum_{k=0}^{q}\frac{c_k(x)}{N^k} + J_{\alpha+1}(Nx)\sum_{k=0}^{q}\frac{d_k(x)}{N^k} + \delta_q(n,x)\Big). \tag{6.2}$$

We shall also need the following asymptotics of the Bessel function (cf. [7, (5.11.6)])

$$\sqrt{z}J_\nu(z) = \sum_{k=0}^{q} \Big(\frac{A_{k,\nu}}{z^k} \sin z + \frac{B_{k,\nu}}{z^k} \cos z \Big) + \mathcal{O}\big(n^{-q-1}\big), \quad z \geq 1. \tag{6.3}$$

Finally, we will make use of the fact that the normalizing coefficients $t_n^{(\alpha,\beta)}$ satisfy

$$\frac{t_n^{(\alpha,\beta)}}{\sqrt{N}} = \sum_{k=0}^{q} \frac{a_k}{n^k} + \mathcal{O}\big(n^{-q-1}\big), \tag{6.4}$$

with N defined above and some constants a_k independent of n; this follows from the exact expression (1.1) (compare with [9, (2.4)]).

Proof of Theorem 3.1 We first observe that it is enough to prove the Darboux type formula only for $x \in (0, \pi/2]$, since then the result for all $x \in (0, \pi)$ follows immediately by a symmetry argument and the identity (3.13). Further, we may assume that $Nx \geq 1$; otherwise the Darboux formula has minor significance since the error term dominates (in the sense of the absolute value) all the remaining terms, including $\phi_n^{(\alpha,\beta)}$ itself, as is readily deduced from the estimate (3.9).

Thus, assume that $x \in (0, \pi/2]$ and $Nx \geq 1$. Combining (6.2) with (6.3) gives

$$\begin{aligned}
\phi_n^{(\alpha,\beta)}(x) = \;& \frac{t_n^{(\alpha,\beta)}}{\sqrt{N}} \sqrt{\frac{\sin x}{2x}} \Bigg\{ \Bigg[\sum_{k=0}^{q} \Big(\frac{A_{k,\alpha}}{(Nx)^k} \sin Nx + \frac{B_{k,\alpha}}{(Nx)^k} \cos Nx \Big) + \mathcal{O}\big((Nx)^{-q-1}\big) \Bigg] \\
& \times \sum_{k=0}^{q} \frac{c_k(x)}{N^k} + \Bigg[\sum_{k=0}^{q} \Big(\frac{A_{k,\alpha+1}}{(Nx)^k} \sin Nx + \frac{B_{k,\alpha+1}}{(Nx)^k} \cos Nx \Big) + \mathcal{O}\big((Nx)^{-q-1}\big) \Bigg] \\
& \times \sum_{k=0}^{q} \frac{d_k(x)}{N^k} + \sqrt{Nx}\,\delta_q(n,x) \Bigg\}.
\end{aligned}$$

Next, writing $\sin Nx$ and $\cos Nx$ as linear combinations with bounded functions as coefficients of $\sin nx$ and $\cos nx$, and using, among others, the fact that $(\sin x)/x$ is comparable with 1 for $x \in (0, \pi/2]$, we obtain

$$\begin{aligned}
\phi_n^{(\alpha,\beta)}(x) = \;& \frac{t_n^{(\alpha,\beta)}}{\sqrt{N}} \Bigg[\sum_{k=0}^{q} \Big(\frac{A_k(x)}{(N \sin x)^k} \sin nx + \frac{B_k(x)}{(N \sin x)^k} \cos nx \Big) \\
& + \mathcal{O}\big((Nx)^{-q-1}\big) + \sqrt{Nx}\,\delta_q(n,x) \Bigg];
\end{aligned} \tag{6.5}$$

here $A_k(x), B_k(x)$ are of course bounded functions of x. Now observe that by (6.1) and $\sqrt{z}J_\nu(z) = \mathcal{O}(1)$, $z \geq 1$, (this is (6.3) specified to $q = 0$)

$$\big|\sqrt{Nx}\,\delta_q(n,x)\big| \leq \frac{\Lambda_q}{N^{q+1}} = \mathcal{O}\big((Nx)^{-q-1}\big).$$

Moreover, since Nx is comparable with $n \sin x$ when $Nx \geq 1$ and $x \in (0, \pi/2]$, both error terms in (6.5) can be written as $\mathcal{O}((n \sin x)^{-q-1})$. Thus

$$\phi_n^{(\alpha,\beta)}(x) = \frac{t_n^{(\alpha,\beta)}}{\sqrt{N}} \Bigg[\sum_{k=0}^{q} \Big(\frac{A_k(x)}{(N \sin x)^k} \sin nx + \frac{B_k(x)}{(N \sin x)^k} \cos nx \Big) + \mathcal{O}\big((n \sin x)^{-q-1}\big) \Bigg]. \tag{6.6}$$

Observe that (6.4) may be slightly weakened to

$$\frac{t_n^{(\alpha,\beta)}}{\sqrt{N}} = \sum_{k=0}^{q} \frac{a_k (\sin x)^k}{(n \sin x)^k} + \mathcal{O}\big((n \sin x)^{-q-1}\big),$$

whereas for each fixed $k = 0, \ldots, q$,

$$\begin{aligned}\frac{1}{(N \sin x)^k} &= \frac{1}{(n \sin x)^k}\Big(1 + \frac{\alpha+\beta+1}{2n}\Big)^{-k} \\ &= \frac{1}{(n \sin x)^k}\Big[\sum_{j=0}^{q-k} \frac{d_j}{n^j} + \mathcal{O}\big(n^{-q+k-1}\big)\Big] \\ &= \frac{1}{(n \sin x)^k} \sum_{j=0}^{q-k} \frac{d_j (\sin x)^j}{(n \sin x)^j} + \mathcal{O}\big((n \sin x)^{-q-1}\big).\end{aligned}$$

Plugging these two expansions into (6.6) we arrive at

$$\phi_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{q} \Big(\frac{A_k(x)}{(n \sin x)^k} \sin nx + \frac{B_k(x)}{(n \sin x)^k} \cos nx\Big) + \text{error terms},$$

with some new bounded functions $A_k(x), B_k(x)$, and all error terms easily seen to be captured by $\mathcal{O}((n \sin x)^{-q-1})$. This finishes the proof of (3.2).

Finally, we comment on necessary modifications to be done in the preceding lines in order to obtain the variant of (3.2) with n on the right replaced by $n+d$; recall that we assume $n \geq -d+1$ in the case $d < 0$. First, it is clear that writing $\sin Nx$ and $\cos Nx$ as linear combinations with bounded functions as coefficients of $\sin((n+d)x)$ and $\cos((n+d)x)$, we obtain an analogue of (6.5) with $\sin nx$ and $\cos nx$ replaced by $\sin((n+d)x)$ and $\cos((n+d)x)$; moreover both error terms in (6.5) are $\mathcal{O}(((n+d) \sin x)^{-q-1})$. Second, it is also straightforward that in the asymptotics of $t_n^{(\alpha,\beta)}/\sqrt{N}$ and $1/(N \sin x)^k$, $k = 0, \ldots, q$, that follow (6.6), on their right-hand sides n may be replaced by $n+d$. Plugging the two modified asymptotics into (6.6) gives the aforementioned variant of (3.2). The proof of Theorem 3.1 is completed. □

Remark 6.1 In fact the functions $A_k(x)$ and $B_k(x)$ appearing in Theorem 3.1 can be even chosen to be analytic on $[0, \pi]$, but proving this requires a more detailed analysis.

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