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Maximal subalgebras of Jordan superalgebras

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A B S T R A C T

The maximal subalgebras of the finite-dimensional simple special Jordan superalgebras over an algebraically closed field of characteristic 0 are studied. This is a continuation of a previous paper by the same authors about maximal subalgebras of simple associative superalgebras, which is instrumental here.

1. Introduction

Finite-dimensional simple Jordan superalgebras over an algebraically closed field of characteristic zero were classified by Kac in 1977 [13], with one missing case that was later described by Kantor in 1990 [14]. More recently Racine and Zelmanov [21] gave a classification of finite-dimensional simple Jordan superalgebras over arbitrary fields of characteristic different from 2 whose even part is semisimple. Later, in 2002, Martínez and Zelmanov [16] completed the remaining cases, where the even part is not semisimple.

Here we are interested in describing the maximal subalgebras of the finite-dimensional simple special Jordan superalgebras with semisimple even part over an algebraically closed field of characteristic zero. Precedents of this work are the papers of Dynkin in 1952 (see [2,3]), where the maximal subgroups of some classical groups and the maximal subalgebras of semisimple Lie algebras are classified, the papers of Racine (see [19,20]), who classifies the maximal subalgebras of finite-dimensional central simple algebras belonging to one of the following classes: associative, associative with involution, alternative and special and exceptional Jordan algebras; and the paper by the first author in 1986 (see [4]), solving the same question for central simple Malcev algebras.

In a previous work [5], the authors described the maximal subalgebras of finite-dimensional central simple superalgebras which are either associative or associative with superinvolution. The results obtained there will be useful in what follows. The maximal subalgebras of the ten-dimensional Kac Jordan superalgebra are determined in [6].

First of all, let us recall some basic facts. A *superalgebra* over a field F is just a $\mathbb{Z}_2$-graded algebra $A = A_{\bar 0} \oplus A_{\bar 1}$ over F (so $A_\alpha A_\beta \subseteq A_{\alpha+\beta}$ for $\alpha, \beta \in \mathbb{Z}_2$). An element a in A_α ($\alpha = \bar 0, \bar 1$) is said to be *homogeneous* of degree α and the notation $\bar a = \alpha$ is used. A superalgebra is said to be *nontrivial* if $A_{\bar 1} \neq 0$ and *simple* if $A^2 \neq 0$ and A contains no proper graded ideal.

An *associative superalgebra* is just a superalgebra that is associative as an ordinary algebra. Here are some important examples:

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(a) $A = M_n(F)$, the algebra of $n \times n$ matrices over F, where

$$A_{\bar 0} = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a \in M_r(F), b \in M_s(F) \right\},$$

$$A_{\bar 1} = \left\{ \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix} : c \in M_{r\times s}(F), d \in M_{s\times r}(F) \right\},$$

with $r + s = n$. This superalgebra is denoted by $M_{r,s}(F)$.

(b) The subalgebra $A = A_{\bar 0} \oplus A_{\bar 1}$ of $M_{n,n}(F)$, with

$$A_{\bar 0} = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in M_n(F) \right\}, \qquad A_{\bar 1} = \left\{ \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} : b \in M_n(F) \right\}.$$

This superalgebra is denoted by $Q_n(F)$.

Over an algebraically closed field, these two previous examples exhaust the simple finite-dimensional associative superalgebras, up to isomorphism.

(c) The *Grassmann superalgebra*:

$$G = alg\langle 1, e_1, e_2, \ldots : e_i^2 = 0 = e_ie_j + e_je_i \forall i, j = 1, 2, \ldots\rangle$$

over a field F, with the grading $G = G_{\bar 0} \oplus G_{\bar 1}$, where $G_{\bar 0}$ is the vector space spanned by the products of an even number of e_i's, while $G_{\bar 1}$ is the vector subspace spanned by the products of an odd number of e_i's. (The product of zero e_i's is, by convention, equal to 1.)

Following standard conventions, given a superalgebra $A = A_{\bar 0} \oplus A_{\bar 1}$, the graded tensor product $G \otimes A$, where G is the Grassmann superalgebra, becomes a superalgebra with the product given by $(g\otimes a)(h\otimes b) = (-1)^{\bar a \bar h} gh\otimes ab$ for homogeneous elements $g, h \in G$ and $a, b \in A$, and grading given by $(G \otimes A)_{\bar 0} = G_{\bar 0} \otimes A_{\bar 0} \oplus G_{\bar 1} \otimes A_{\bar 1}$, $(G \otimes A)_{\bar 1} = G_{\bar 0} \otimes A_{\bar 1} \oplus G_{\bar 1} \otimes A_{\bar 0}$. Its even part $G(A) = (G \otimes A)_{\bar 0}$ is called the *Grassmann envelope* of the superalgebra A. Moreover, the superalgebra A is said to be a superalgebra in a fixed variety if $G(A)$ is an ordinary algebra (over $G_{\bar 0}$) in this variety. In particular, A is a Jordan superalgebra if and only if $G(A)$ is a Jordan algebra.

It then follows that over fields of characteristic $\neq 2, 3$, a superalgebra $J = J_{\bar 0} \oplus J_{\bar 1}$ is a Jordan superalgebra if and only if for any homogeneous elements a, b, c in J:

$$L_ab = (-1)^{\bar a \bar b} L_ba,$$

where L_a denotes the multiplication by a, and

$$\begin{aligned} L_aL_bL_c + (-1)^{\bar a\bar b + \bar a\bar c + \bar b\bar c} L_cL_bL_a + (-1)^{\bar b\bar c} L_{(ac)b} &= L_{ab}L_c + (-1)^{\bar b\bar c} L_{ac}L_b + (-1)^{\bar a\bar b + \bar a\bar c} L_{bc}L_a \\ &= (-1)^{\bar a\bar b} L_bL_aL_c + (-1)^{\bar a\bar c + \bar b\bar c} L_cL_aL_b + L_{a(bc)} \\ &= (-1)^{\bar a\bar c + \bar b\bar c} L_cL_{ab} + (-1)^{\bar a\bar b} L_bL_{ac} + L_aL_{bc}. \end{aligned} \tag{1.1}$$

Let A be a superalgebra. A *superinvolution* is a graded linear map $*: A \to A$ such that $x^{**} = x$, and $(xy)^* = (-1)^{\bar x\bar y} y^*x^*$, for any homogeneous elements x, y in A.

The simplest examples of Jordan superalgebras over a field of characteristic $\neq 2$ are the following:

(i) Let $A = A_{\bar 0} + A_{\bar 1}$ be an associative superalgebra. Replace the associative product in A with the new one: $x \circ y = \frac{1}{2}(xy + (-1)^{\bar x\bar y} yx)$. With this product A becomes a Jordan superalgebra, denoted by A^+.

(ii) Let A be an associative superalgebra with superinvolution $*$. Then the subspace of hermitian elements $H(A, *) = \{a \in A : a^* = a\}$ is a subalgebra of A^+.

In fact, if a Jordan superalgebra J is a subalgebra of A^+ for an associative superalgebra A, J is said to be *special*. Otherwise J is said to be *exceptional*. Any graded Jordan homomorphism $\sigma: J \to A^+$ is called a *specialization*. So J is special if there exists a faithful specialization of J. Otherwise, J is exceptional. Both examples (i) and (ii) given above are examples of special Jordan superalgebras.

A specialization $u: J \to U^+$ into an associative superalgebra U is said to be *universal* if the subalgebra of U generated by $u(J)$ is U, and for any arbitrary specialization $\varphi: J \to A^+$, there exists a homomorphism of associative superalgebras $\chi: U \to A$ such that $\varphi = \chi \circ u$. The superalgebra U is called the *universal enveloping algebra* of J.

In what follows, and unless otherwise stated, only finite-dimensional Jordan superalgebras over an algebraically closed field of characteristic zero will be considered.

The restriction on the characteristic is necessary because Lie theoretical methods are used. Both the methods and the results are not valid in general in prime characteristic (see, for instance, Example 5.2).

We recall the classification of the nontrivial simple Jordan superalgebras given by Kac [13] and completed by Kantor [14].

(1) $J = K_3$, the *Kaplansky superalgebra*:

$$J_{\bar 0} = Fe, \qquad J_{\bar 1} = Fx + Fy, \qquad e^2 = e, \quad e \cdot x = \frac{1}{2}x, \qquad e \cdot y = \frac{1}{2}y, \quad x \cdot y = e.$$

(2) The one-parameter family of superalgebras $J = D_t$, with $t \in F \setminus \{0\}$:

$$J_{\bar 0} = Fe + Ff, \qquad J_{\bar 1} = Fu + Fv$$
$$e^2 = e, \qquad f^2 = f, \qquad e \cdot f = 0, \qquad e \cdot u = \frac{1}{2}u, \qquad e \cdot v = \frac{1}{2}v, \qquad f \cdot u = \frac{1}{2}u,$$
$$f \cdot v = \frac{1}{2}v, \quad u \cdot v = e + tf.$$

Note that $D_t \cong D_{1/t}$, for any $t \neq 0$.

(3) $J = K_{10}$, the *Kac superalgebra*. This is a ten-dimensional Jordan superalgebra with six-dimensional even part (see [7,15,1] or [6] for details).

(4) Let $V = V_{\bar 0} \oplus V_{\bar 1}$ be a graded vector space over F, and let $(\, , \,)$ be a nondegenerate supersymmetric bilinear superform on V, that is, a nondegenerate bilinear map which is symmetric on $V_{\bar 0}$, skewsymmetric on $V_{\bar 1}$, and $V_{\bar 0}$ and $V_{\bar 1}$ are orthogonal relative to $(\, , \,)$. Now consider $J_{\bar 0} = Fe + V_{\bar 0}, J_{\bar 1} = V_{\bar 1}$ with $e \cdot x = x, v \cdot w = (v, w)e$, for any $x \in J$ and $v, w \in V$. This superalgebra J is called the *superalgebra of a superform*. If $\dim V_{\bar 0} = 1$ and $\dim V_{\bar 1} = 2$, the superalgebra of a superform is isomorphic to D_t with $t = 1$.

(5) A^+, with A a finite-dimensional simple associative superalgebra, that is, either $A = M_{r,s}(F)$ or $A = Q_n(F)$. Note that $M_{1,1}(F)^+$ is isomorphic to D_{-1}.

(6) $H(A, *)$, where A and $*$ are of one of the following types:

(i)
$$A = M_{n,n}(F), \qquad *\colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} d^t & -b^t \\ c^t & a^t \end{pmatrix}.$$

(ii)
$$A = M_{n,2m}(F), \qquad *\colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a^t & c^t q \\ -q^t b^t & q^t d^t q \end{pmatrix}, \quad \text{where } q = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}.$$

The first one is called the *transpose superinvolution* and $H(A, *)$ is denoted then by $p(n)$, and the second one the *orthosymplectic superinvolution* and $H(A, *)$ is denoted in this case by $\mathrm{osp}_{n,2m}$. The isomorphisms $D_{-2} \cong D_{-1/2} \cong \mathrm{osp}_{1,2}$ are easy to prove.

(7) Let G be the Grassmann superalgebra. Consider the following product in G:

$$\{f, g\} = \sum_{i=1}^{n} (-1)^{\bar f} \frac{\partial f}{\partial e_i} \frac{\partial g}{\partial e_i},$$

and build the vector space, sum of two copies of G: $J = G + Gx$, with the product in J given by

$$a(bx) = (ab)x, \qquad (bx)a = (-1)^{\bar a}(ba)x, \qquad (ax)(bx) = (-1)^{\bar b}\{a, b\}.$$

Finally take the following grading in J: $J_{\bar 0} = G_{\bar 0} + G_{\bar 1}x, J_{\bar 1} = G_{\bar 1} + G_{\bar 0}x$. This superalgebra is called the *Kantor double of the Grassmann algebra* or the *Kantor superalgebra*.

The ten-dimensional Kac superalgebra and the Kantor superalgebra are the unique exceptional superalgebras in the above list (see [18,23]). Note that the Kaplansky superalgebra is the unique nonunital simple superalgebra.

Let J be a nonunital Jordan superalgebra, the unital hull of J is defined to be $H_F(J) = J + F \cdot 1$, where 1 is the formal identity and J is an ideal inside $H_F(J)$. In [25] Zelmanov determined a classification theorem for finite-dimensional semisimple Jordan superalgebras.

Theorem 1.1 (*E. Zelmanov*). *Let J be a finite-dimensional Jordan superalgebra over a field F of characteristic not 2. Then J is semisimple if and only if J is a direct sum of simple Jordan superalgebras and unital hulls $H_K(J_1 \oplus \cdots \oplus J_r) = (J_1 \oplus \cdots \oplus J_r) + K \cdot 1$ where J_i are nonunital simple Jordan superalgebras over an extension K of F.*

The maximal subalgebras of the Kac Jordan superalgebra (type (3) above) have been determined in [6]. Our purpose in this paper is to describe the maximal subalgebras of the simple special Jordan superalgebras (types (1), (2), (4), (5) and (6)). This is achieved completely for the simple Jordan superalgebras of types (1), (2) and (4). For types (5) and (6) the results are not completed and some questions arose.

In what follows the word subalgebra will always be used in the graded sense, so any subalgebra is graded.

First note that any maximal subalgebra B in a simple unital Jordan superalgebra J, with identity element 1, contains the identity element. Indeed, if $1 \notin B$, the algebra generated by B and 1: $B + F \cdot 1$, is the whole J by maximality. So B is a nonzero graded ideal of J, a contradiction with J being simple. Therefore $1 \in B$.

The paper is organized as follows. Section 2 deals with the easy problem of determining the maximal subalgebras of the Kaplansky superalgebra, the superalgebras D_t and the Jordan superalgebras of superforms. Then Section 3 will collect some known results on universal enveloping algebras and will put them in a way suitable for our purposes. Sections 4 and 5 will be devoted, respectively, to the description of the maximal subalgebras of the simple Jordan superalgebras A^+ and $H(A, *)$, for a simple finite-dimensional associative algebra A, and a superinvolution $*$.

2. The easy cases

Let us first describe the maximal subalgebras of the simple Jordan superalgebras of types (1), (2), and (4) in Section 1. The result, whose proof is straightforward, is valid in prime characteristic ($\neq 2$) too.

Theorem 2.1. (i) *Let $J = K_3$ be the Kaplansky superalgebra. A subalgebra M of J is maximal if and only if $M = J_{\bar 0} \oplus M_{\bar 1}$ where $M_{\bar 1}$ is a vector subspace of $J_{\bar 1}$ with $\dim M_{\bar 1} = 1$.*

(ii) *Let $J = D_t$ with $t \neq 0$. A subalgebra M of J is maximal if and only if either $M = J_{\bar 0} \oplus M_{\bar 1}$ where $M_{\bar 1}$ is a vector subspace of $J_{\bar 1}$ with $\dim M_{\bar 1} = 1$, or if $t = 1$, $M = F \cdot 1 + J_{\bar 1}$.*

(iii) *Let J be the Jordan superalgebra of a nondegenerate bilinear superform. A subalgebra M of J is maximal if and only if either $M = J_{\bar 0} \oplus M_{\bar 1}$ where $M_{\bar 1}$ is a vector subspace and $\dim M_{\bar 1} = \dim J_{\bar 1} - 1$, or $M = (F \cdot 1 + M_{\bar 0}) \oplus J_{\bar 1}$ where $M_{\bar 0}$ is a vector subspace and $\dim M_{\bar 0} = \dim V_{\bar 0} - 1$.*

Note that item (ii) in Theorem 2.1 cover the maximal subalgebras of $M_{1,1}(F)^+ \cong D_{-1}$ and of $\mathrm{osp}_{1,2} \cong D_{-2}$.

3. Universal enveloping algebras

In order to determine the maximal subalgebras of the remaining simple special Jordan superalgebras, some previous results are needed.

Given an associative superalgebra A and a subalgebra B of the Jordan superalgebra A^+, B' will denote the (associative) subalgebra of A generated by B.

Proposition 3.1. *There is no unital subalgebra B of the Jordan superalgebra $Q_n(F)^+$ ($n \geq 2$), isomorphic to D_t ($t \neq 0$), and with $B' = Q_n(F)$.*

Proof. Write $A = Q_n(F)$, and take a basis $\{e, f, u, v\}$ of $B \cong D_t$ as in Section 1. Since B is a unital subalgebra, $e + f = 1_A$. Therefore, as $e^2 = e, f^2 = f$ and $ef = fe = (1_A - e)e = 0$, we may assume also that

$$e = \begin{pmatrix} I_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_s & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_m \end{pmatrix}.$$

Consider the Peirce decomposition associated to the idempotents e and f, and note that $u, v \in A_{\bar 1} \cap (Q_n(F)^+)_{1/2}(e) \cap (Q_n(F)^+)_{1/2}(f)$. Hence

$$u = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ 0 & a & 0 & 0 \\ b & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 0 & 0 & 0 & c \\ 0 & 0 & d & 0 \\ 0 & c & 0 & 0 \\ d & 0 & 0 & 0 \end{pmatrix},$$

for some $a, c \in M_{s\times m}(F)$, $b, d \in M_{m\times s}(F)$. But this contradicts the assumption that B' is equal to A, because, for instance,

$$\begin{pmatrix} 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \notin B', \quad \text{for} \quad 0 \neq x \in M_{s\times s}(F).$$

This finishes the proof. □

Now, if $Q_n(F)$ is replaced by $M_{p,q}(F)$, some knowledge of the universal enveloping algebra of D_t is needed.

I. P. Shestakov determined $U(D_t)$ (see [17]), which is intimately related to the orthosymplectic Lie superalgebra $\mathrm{osp}(1, 2)$, that is, the superalgebra whose elements are the skewsymmetric matrices of $M_{1,2}(F)$ relative to the orthosymplectic superinvolution, with Lie bracket $[a, b] = ab - (-1)^{\bar a \bar b} ba$:

$$\mathrm{osp}(1, 2) = \left\{ \begin{pmatrix} 0 & \beta & \alpha \\ -\alpha & \gamma & \mu \\ \beta & \nu & -\gamma \end{pmatrix} : \alpha, \beta, \mu, \gamma, \nu \in F \right\}.$$

The following elements in $\mathrm{osp}(1, 2)$, which form a basis, will be considered throughout:

$$h = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$x = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

These verify $[h, e] = 2e$, $[h, f] = -2f$, $[h, x] = x$, $[h, y] = -y$, $[e, y] = x$, $[f, x] = y$, $[x, x] = -2e$, $[y, y] = 2f$, $[x, y] = xy + yx = h$.

Note that $\mathrm{osp}_{n,2m}$ denotes the orthosymplectic Jordan superalgebra, while $\mathrm{osp}(n, 2m)$ denotes the orthosymplectic Lie superalgebra.

Then $U(D_t)$ is given by

Theorem 3.2 (*I. Shestakov [22]*). *If $t \neq 0, \pm 1$, then the universal associative enveloping of D_t is $(U(D_t), \iota)$ where $U(D_t) = U(\mathrm{osp}(1, 2))/\ \mathrm{ideal}\langle (xy - yx)^2 + (xy - yx) + \frac{t}{(1+t)^2}\rangle$ and*

$$
\begin{aligned}
&\iota : D_t \longrightarrow U(D_t)\\
&e \longmapsto \iota(e) = \frac{1}{t-1}(t1 + (1+t)\overline{(xy - yx)}),\\
&f \longmapsto \iota(f) = \frac{1}{1-t}(1 + (1+t)\overline{(xy - yx)}),\\
&u \longmapsto \iota(u) = 2\bar{x},\\
&v \longmapsto \iota(v) = -(1+t)\bar{y},
\end{aligned}
$$

where $\bar{z}$ denotes the class of $z \in \mathrm{osp}(1, 2)$ modulo the ideal generated by $(xy - yx)^2 + (xy - yx) + \frac{t}{(1+t)^2}$.

Here $U(\mathrm{osp}(1, 2))$ denotes the universal enveloping algebra of the Lie superalgebra $\mathrm{osp}(1, 2)$ (see [12, section 1.1.3]).

Note that the element $a = \overline{xy - yx} \in U(D_t)$ satisfies $a^2 + a + \frac{t}{(1+t)^2} = 0$, hence if $a' = -(1+t)a$, $a'^2 - (1+t)a' + t = 0$ and in this way the original version of Shestakov's Theorem is recovered.

The even part of $\mathrm{osp}(1, 2)$, which is the span of the elements h, e, f above, is isomorphic to the three-dimensional simple Lie algebra $sl(2, F)$, so given any finite-dimensional irreducible $U(\mathrm{osp}(1, 2))$-module V, by restriction V is also a module for $sl(2, F)$. The well-known representation theory of $sl(2, F)$ shows that h acts diagonally on V (see [11, 7.2 Corollary]), its eigenvalues constitute a sequence of integers, symmetric relative to 0, and hence V is the direct sum of the subspaces $V_m = \{v \in V : h \cdot v = mv\}$ with $m \in \mathbb{Z}$.

By finite dimensionality, there exists a largest nonnegative integer m with $V_m \neq 0$. Pick a nonzero element $v \in V_m$ (a highest weight vector). Changing the parity in V if necessary, this element v can be assumed to be even.

Since $h(ev) = [h, e]v + e(hv) = (m + 2)ev$, it follows that $ev = 0$, and since $h(xv) = [h, x]v + x(hv) = (m + 1)xv$, it follows that $xv = 0$ too. Let $\mathfrak{g} = \mathrm{osp}(1, 2)$, then $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$, where $\mathfrak{g}_+ = Fe + Fx$, $\mathfrak{h} = Fh$, and $\mathfrak{g}_- = Ff + Fy$, and let $W = W_0 = Fw$ be the module over $\mathfrak{h} + \mathfrak{g}_+$ given by $hw = mw$, $ew = 0$, and $xw = 0$. The map $W \longrightarrow V$ such that $\lambda w \longmapsto \lambda v$ for any $\lambda \in F$ is a homomorphism of $(\mathfrak{h} + \mathfrak{g}_+)$-modules, which can be extended to a homomorphism of $\mathfrak{g}$-modules (that is, of $U(\mathrm{osp}(1, 2))$-modules) as follows:

$$
\begin{aligned}
&\varphi : U(\mathfrak{g}) \otimes_{U(\mathfrak{h}+\mathfrak{g}_+)} W \longrightarrow V\\
&a \otimes w \longmapsto av.
\end{aligned}
$$

Since V is an irreducible $\mathrm{osp}(1, 2)$-module, φ is onto. We denote by $U(m)$ the $U(\mathfrak{g})$-module $U(\mathfrak{g}) \otimes_{U(\mathfrak{h}+\mathfrak{g}_+)} W$ and identify the element $1 \otimes w$ with w. Then:

$$
\begin{aligned}
&hy^i w = (m - i)y^i w, \qquad fy^i w = y^{i+2} w,\\
&xy^{2i} w = -iy^{2i-1} w, \qquad xy^{2i+1} w = (m - i)y^{2i} w,\\
&ey^{2i} w = i(m - i + 1)y^{2i-2} w, \qquad ey^{2i+1} w = i(m - i)y^{2i-1} w,
\end{aligned}
$$

and hence it follows that the set $\{w, yw, y^2w, \ldots\}$ spans the vector space $U(m)$. We remark that $I_m = span\langle y^{2m+1}w, y^{2m+2}w, \ldots\rangle$ is a proper submodule of $U(m)$, and because V is irreducible and the weights of the elements $y^{2m+i}w$ are all different from m, it follows that $\varphi(I_m) \neq V$, so by irreducibility $\varphi(I_m) = 0$. Thus the set $\{v, yv, y^2v, \ldots, y^{2m}v\}$ spans the vector space V. Again, the theory of modules for $sl(2, F)$ shows that $v, y^2v, \ldots, y^{2m}v$ are all nonzero (see [11, 7.2]), and hence so are the elements $yv, y^3v, \ldots, y^{2m-1}v$. Note that the elements $v, yv, y^2v, \ldots, y^{2m}v$ are linearly independent, as they belong to different eigenspaces relative to the action of h. We conclude that $\{v, yv, y^2v, \ldots, y^{2m}v\}$ is a basis of V.

Denote V by $V(m)$ and write $e_i = y^i v$. Then,

$$
\begin{aligned}
&V(m)_{\bar{0}} = \langle e_0, e_2, \ldots, e_{2m}\rangle,\\
&V(m)_{\bar{1}} = \langle e_1, e_3, \ldots, e_{2m-1}\rangle.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&(xy - yx)e_{2i} = (m - i)e_{2i} + ie_{2i} = me_{2i},\\
&(xy - yx)e_{2i+1} = xe_{2i+2} - (m - i)e_{2i+1} = -(m + 1)e_{2i+1},
\end{aligned}
$$

and so the minimal polynomial of the action of $xy - yx$ is $(X - m)(X + (m + 1)) = X^2 + X - m(m + 1)$, and therefore the finite-dimensional irreducible $U(\mathrm{osp}(1, 2))$-modules coincide with the irreducible modules for $U(\mathrm{osp}(1, 2))/\ \mathrm{ideal}\langle (xy - yx)^2 + (xy - yx) - m(m + 1)\rangle$.

Therefore, if V is a finite-dimensional irreducible $U(D_t)$-module ($t \neq 0, \pm 1$), then by Shestakov's Theorem (Theorem 3.2), V is an irreducible module for $\mathrm{osp}(1,2)$ in which the minimal polynomial of the action of $xy - yx$ divides $X^2 + X + \frac{t}{(1+t)^2}$. From our above discussion, there must exist a natural number m such that $\frac{t}{(1+t)^2} = -m(m+1)$, that is, either $t = -\frac{m}{m+1}$ or $t = -\frac{m+1}{m}$. Thus,

Corollary 3.3 (*C. Martínez, E. Zelmanov [17, Theorem 5.3]*). *The universal enveloping algebra $U(D_t)$ ($t \neq 0, \pm 1$) has a finite-dimensional irreducible module if and only if there exists a natural number m such that either $t = -\frac{m}{m+1}$ or $t = -\frac{m+1}{m}$. In this case, up to parity exchange, its unique irreducible module is $V(m)$ (that is, the irreducible module for $U(\mathrm{osp}(1,2))$ annihilated by the ideal generated by $(xy - yx)^2 + (xy - yx) - m(m+1)$).*

Something can be added here:

Proposition 3.4. *Up to scalars, the module $V(m)$ has a unique nonzero even bilinear form $(\cdot \mid \cdot)$ such that ρ_x and ρ_y, the multiplication operators by x and y, are supersymmetric, that is, $(zv|w) = (-1)^{|v|}(v|zw)$ for any $v, w \in V_{\bar 0} \cup V_{\bar 1}$ with $z = x, y$.*

Proof. If ρ_x, ρ_y are supersymmetric then $\rho_{[x,x]} = 2\rho_x^2$, $\rho_{[y,y]} = 2\rho_y^2$, and $\rho_{[x,y]} = \rho_x\rho_y + \rho_y\rho_x$ are skewsymmetric, that is, ρ_e, ρ_f, and ρ_h are skewsymmetric. But ρ_h being skewsymmetric implies that $(V_{(\alpha)}|V_{(\beta)}) = 0$ if $\alpha + \beta \neq 0$, where $V_{(\alpha)} = \{v \in V(m) : hv = \alpha v\}$, because $(hV_{(\alpha)}|V_{(\beta)}) = -(V_{(\alpha)}|hV_{(\beta)})$, and therefore $(\alpha + \beta)(V_{(\alpha)}|V_{(\beta)}) = 0$. Hence we can check that $(\cdot \mid \cdot)$ is determined by $(e_0|e_{2m})$, as

$$(e_1|e_{2m-1}) = (ye_0|e_{2m-1}) = (e_0|ye_{2m-1}) = (e_0|e_{2m}).$$

So, up to scalars, it can be assumed that $(e_0|e_{2m}) = 1$.
Using that ρ_y is supersymmetric, recursively we get

$$(e_{2r}|e_{2(m-r)}) = (-1)^r,$$
$$(e_{2r+1}|e_{2(m-r)-1}) = (-1)^r$$

and $(e_i|e_j) = 0$ otherwise. Now it can be checked that ρ_x is supersymmetric too. □

Note that $(\cdot \mid \cdot)$ is supersymmetric if m is even and superskewsymmetric if m is odd. In the latter case, one can consider $V(m)^{op}$ with the supersymmetric bilinear superform given by $(u|v)' = (-1)^{|u|}(u|v)$ where $|u|$ denotes the parity in $V(m)$.

Consider again the finite-dimensional irreducible $U(D_t)$-module ($t = -\frac{m}{m+1}$ or $t = -\frac{m+1}{m}$) $V = V(m)$, with the bilinear superform in the proposition above. It is known that this determines a superinvolution in $A = \mathrm{End}_F(V)$ such that every homogeneous element $f \in \mathrm{End}_F(V)$ is mapped to f^* verifying $(fv, w) = (-1)^{\bar f \bar v}(v, f^*w)$. Note that, since ρ_x and ρ_y are supersymmetric, D_t is thus embedded in $H(\mathrm{End}_F(V), *)$ as follows:

$$\begin{aligned}
D_t &\longrightarrow H(\mathrm{End}_F(V), *)\\
e &\longmapsto \frac{1}{t-1}(t\rho_{Id} + (1+t)(\rho_x\rho_y - \rho_y\rho_x))\\
f &\longmapsto \frac{1}{1-t}(\rho_{Id} + (1+t)(\rho_x\rho_y - \rho_y\rho_x))\\
u &\longmapsto 2\rho_x\\
v &\longmapsto -(1+t)\rho_y.
\end{aligned}$$

Moreover, unless $t \neq -2, -1/2$ (that is, unless $m = 1$), by dimension count, one has $D_t \subsetneq H(\mathrm{End}_F(V), *)$.

The conclusion of all these arguments is the following:

Proposition 3.5. *Let V be a nontrivial finite-dimensional vector superspace and let B be a unital subalgebra of the simple Jordan superalgebra $\mathrm{End}_F(V)^+$, isomorphic to D_t ($t \neq 0, \pm 1$), and such that $B' = \mathrm{End}_F(V)$. Then one of the following situations holds:*

(i) *either $t = -\frac{m}{m+1}$ or $t = -\frac{m+1}{m}$ for an even number m, such that $V \cong V(m)$, and through this isomorphism $B \subseteq H(\mathrm{End}_F(V), *)$ where $*$ is the superinvolution associated to the bilinear superform of Proposition 3.4,*

(ii) *or $t = -\frac{m}{m+1}$ or $t = -\frac{m+1}{m}$ for an odd number m such that $V \cong V(m)^{op}$ and through this isomorphism $D_t \subseteq H(\mathrm{End}_F(V), \diamond)$, where $\diamond$ is the superinvolution associated to the bilinear superform $(.\mid.)'$.*

Proof. The hypotheses imply that there is a surjective homomorphism of associative algebra $U(D_t) \to \mathrm{End}_F(V)$, so V becomes an irreducible module for $U(D_t)$ and the arguments above apply. □

Since the superalgebra $\mathrm{End}_F(V)$, for a superspace V, is isomorphic to $M_{p,q}(F)$, for $p = \dim V_{\bar 0}$, $q = \dim V_{\bar 1}$, the next result follows:

Corollary 3.6. *The simple Jordan superalgebra $M_{p,q}(F)^+$ contains a unital subalgebra B, isomorphic to D_t ($t \neq 0, \pm 1$), and such that $B' = M_{p,q}(F)$, if and only if $q = p \pm 1$ and either $t = -\frac{p}{q}$, or $t = -\frac{q}{p}$.*

Proposition 3.1 and Corollary 3.6 give all the possibilities for embeddings of the Jordan superalgebra D_t ($t \neq 0, \pm 1$) as unital subalgebras in A^+, in such a way that the associative subalgebra generated by D_t is the whole A, A being a simple associative superalgebra. For these cases, one always has $D_t \subseteq H(A, *)$, for a suitable superinvolution. By dimension count, equality is only possible here if $t = -2$ (or $t = -\frac{1}{2}$). This corresponds to the isomorphism $D_{-2} \cong \mathrm{osp}_{1,2}$.

For later use, let us recall the following results on universal enveloping algebras of some other Jordan superalgebras (see [17, Theorems 1.1, 2.1 and 4.1]):

Theorem 3.7 (*C. Martínez and E. Zelmanov*).

(i) *The universal enveloping algebra of* $p(2)$ *is isomorphic to* $M_{2,2}(F[t])$, *where* $F[t]$ *is the polynomial algebra in the variable* t.
(ii) *The universal enveloping algebra of* $M_{1,1}(F)$ *is* $(U(D), u)$ *with*

$$U(D) = \begin{pmatrix} F[z_1, z_2] + F[z_1, z_2]a & 0 \\ 0 & F[z_1, z_2] + F[z_1, z_2]a \end{pmatrix} \oplus \begin{pmatrix} 0 & F[z_1, z_2] + F[z_1, z_2]a^{-1}z_2 \\ F[z_1, z_2]z_1 + F[z_1, z_2]a & 0 \end{pmatrix}$$

where z_1, z_2 *are variables,* a *is a root of* $X^2 + X - z_1z_2 \in F[z_1, z_2]$, *and* $u : M_{1,1}(F) \to U(D)^+$ *is given by*

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \mapsto \begin{pmatrix} \alpha_{11} & \alpha_{12} + \alpha_{21}a^{-1}z_2 \\ \alpha_{12}z_1 + \alpha_{21}a & \alpha_{22} \end{pmatrix}.$$

Theorem 3.8 (*C. Martínez and E. Zelmanov*).

(i) $U(M^+_{m,n})(F) \cong M_{m,n}(F) \oplus M_{m,n}(F)$ *for* $(m, n) \neq (1, 1)$;
(ii) $U(Q^+_n(F)) = Q_n(F) \oplus Q_n(F)$, $n \geq 2$;
(iii) $U(\mathrm{osp}_{m,n}(F)) \cong M_{m,n}(F)$, $(m, n) \neq (1, 2)$;
(iv) $U(p(n)) \cong M_{n,n}(F)$, $n \geq 3$.

4. Maximal subalgebras of A^+

Let B be a maximal subalgebra of A^+, A being a simple associative superalgebra (so A is isomorphic to either $M_{p,q}(F)$ or $Q_n(F)$, for some p and q, or n). If $B' \neq A$ then $B' \subseteq C$ with C a maximal subalgebra of the associative superalgebra A, and then $C^+ = B$ by maximality. Therefore a maximal subalgebra of A^+ is of one of the following types, either:

(i) $B' = A$ and B is semisimple, or
(ii) $B = C^+$ with C a maximal subalgebra of A as associative superalgebra, or
(iii) $B' = A$ and B is not semisimple.

4.1. $B' = A$ and B semisimple

Let us assume first that B is a maximal subalgebra of the simple superalgebra A^+, with $B' = A$ and B semisimple.

For the moment being, let us drop the maximality condition, so let us suppose that B is just a semisimple subalgebra of A^+ with $B' = A$. By Theorem 1.1, $B = \sum_{i=1}^{r}(J_{i1} \oplus \cdots \oplus J_{ir_i} + Fe_i) \oplus M_1 \oplus \cdots \oplus M_t$ where $M_1, \ldots, M_t$ are simple Jordan superalgebras and J_{ij} are Kaplansky superalgebras.

We claim that B has neither direct summands M_i isomorphic to the Kaplansky superalgebra K_3 nor direct summands of the type $(J_{i1} \oplus \cdots \oplus J_{ir_i} + Fe_i)$. Indeed, otherwise A^+ would contain a subalgebra isomorphic to K_3. Let e be its nonzero even idempotent and x, y odd elements with $x \cdot y = e$. Then, in the associative superalgebra A (which is isomorphic to either $M_{p,q}(F)$ or $Q_n(F)$, and hence there is a trace form), one has $\mathrm{trace}(e) = \mathrm{trace}(x \cdot y) = \frac{1}{2}\,\mathrm{trace}(xy - yx) = 0$. However, any nonzero idempotent in a matrix algebra over a field of characteristic 0 has nontrivial trace, a contradiction.

Therefore, $B = M_1 \oplus \cdots \oplus M_t$, where the M_i's are unital simple Jordan superalgebras.

Consider now the identity element f_i of each M_i. Then $B = f_1Bf_1 \oplus \cdots \oplus f_tBf_t$. If $t > 1$, it follows that $B' \subset f_1Af_1 \oplus (1-f_1)A(1-f_1) \subsetneq A$, a contradiction. Hence B is simple and, therefore, is isomorphic to one of the following special superalgebras: D_t, $H(D, *)$ (for a simple associative superalgebra D with superinvolution $*$), the superalgebra of a superform, or D^+ for a simple associative superalgebra D. (Recall that K_{10} and the Kantor superalgebra are exceptional superalgebras.)

If B were the superalgebra of a superform over a vector superspace V, let $x, y \in V_{\bar{1}}$ such that $x \cdot y = 1_A$. Then $x \cdot y = \frac{1}{2}(xy - yx) = 1_A$, and again $\mathrm{trace}(x \cdot y) = 0 \neq \mathrm{trace}(1_A)$, a contradiction that shows that $V_{\bar{1}} = 0$. But then $B \subseteq A_{\bar{0}}$ and $B' \subseteq A_{\bar{0}} \neq A$, contrary to our hypotheses.

Now, in case B is isomorphic to D_t ($t \neq 0$), Proposition 3.1 shows that A is not isomorphic to $Q_n(F)$ and Corollary 3.6 and Proposition 3.5 show that either $t = 1$, and hence there are odd elements x, y such that $x \cdot y = \frac{1}{2}(xy - yx) = 1_A$, so the same argument as in the previous paragraph applies, or B is never a maximal subalgebra of $A \cong M_{p,q}(F)$ unless $t = -2$ (or $-\frac{1}{2}$). In this case B is isomorphic to $H(D, *)$ for a suitable $(D, *)$.

Therefore:

Lemma 4.1. *Let B be a subalgebra of the Jordan superalgebra A^+, where A is a finite-dimensional simple associative superalgebra over an algebraically closed field F of characteristic 0. If $B' = A$ and B is semisimple, then either B is isomorphic to D_t ($t \neq 0, 1, -1, -2, -\frac{1}{2}$), or $B = D^+$ or $B = H(D, *)$, for a simple associative superalgebra D and a superinvolution $*$. Moreover, if B is a maximal subalgebra of A^+, then the first possibility does not hold.*

Our next goal consists in proving that, in case $B = D^+$ or $B = H(D, *)$, one has that D is isomorphic to A. For this, the following result (see [8]) will be used:

Theorem 4.2 (*C. Gómez-Ambrosi*). *Let S be a unital associative superalgebra with superinvolution $*$. Assume that the following conditions hold:*

(i) *S has at least three symmetric orthogonal idempotents.*
(ii) *If $S = \sum_{i=1}^{n} S_{ij}$ is the Peirce decomposition related to them, then $S_{ij}S_{ji} = S_{ii}$ holds for $i, j = 1, \ldots, n$,*

and let $\phi\colon H(S, *) \to (A, \cdot)^+$ be a homomorphism of Jordan superalgebras, for an associative superalgebra $(A, \cdot)$. Then ϕ can be extended uniquely to an associative homomorphism $\varphi\colon S \to A$.

We shall proceed in several steps, where the assumptions are that B is just a semisimple subalgebra of A^+ with $B' = A$:
(a) Assume first that $B = H(D, *)$ for a simple associative superalgebra with involution $(D, *)$. Let us denote the multiplication in D by $\diamond$. The inclusion map $\iota\colon B = H(D, *) \to (A, \cdot)^+$ is a Jordan homomorphism. Then (Section 1), D is isomorphic to $M_{p,q}(F)$, for suitable p, q, and $*$ corresponds to either the transpose involution or an orthosymplectic involution. If neither D is a quaternion superalgebra (isomorphic to $M_{1,1}(F)$), nor $H(D, *)$ is isomorphic to $p(2)$ or $\mathrm{osp}_{1,2}$, then D satisfies the hypotheses of Theorem 4.2 and, therefore, $\iota : B \to A$ can be extended to an associative homomorphism $\tau : D \to A$. But the subalgebra B' generated by B in A is the whole A. Hence τ is onto and, as D is simple, it is one-to-one too. Therefore D is isomorphic to A. Thus, we are left with three cases:
(a.1) If $H(D, *)$ is isomorphic to $\mathrm{osp}_{1,2}$ then, since $\mathrm{osp}_{1,2}$ is isomorphic to D_{-2}, $H(D, *)$ is isomorphic to D_{-2}.
(a.2) If D, with multiplication $\diamond$, is isomorphic to $M_{1,1}(F)^+$, with superinvolution $*$ as in (6)(i) in Section 1, then $H(D, *)$ is isomorphic to $F1 + Fu$, with $u^2 = 0$. Thus, the universal enveloping algebra of $H(D, *)$ is $F[u]$, the ring of polynomials over F on the variable u, and there exists an associative homomorphism $\varphi : F[u] \to A$, which extends $\iota : B \to A$. Again, φ is onto since $B' = A$. Therefore A should be commutative, a contradiction.
(a.3) Finally, if $H(D, *)$ is isomorphic to $p(2)$, Theorem 3.7 shows that its universal enveloping algebra is isomorphic to $M_{2,2}(F[t])$, where $F[t]$ is the polynomial algebra on the indeterminate t. As before, this gives a surjective homomorphism $\phi : M_{2,2}(F[t]) \to A$. Recall that A is isomorphic either to $M_{p,q}(F)$ or to $Q_n(F) = M_n(F) \oplus M_n(F)u$ ($u^2 = 1$). Let e_1, e_2, e_3, e_4 be primitive orthogonal idempotents of $M_{2,2}(F)$, with e_1+e_2 and e_3+e_4 being the unital elements in the two simple direct summands of the even part. Since the restriction of ϕ to $M_{2,2}(F)$ is injective because $M_{2,2}(F)$ is simple, the images $\phi(e_1), \phi(e_2), \phi(e_3), \phi(e_4)$ are nonzero orthogonal idempotents in $A_{\bar{0}}$ with $\sum_{i=1}^{4} \phi(e_i) = 1_A$. Write $U = M_{2,2}(F[t])$ and consider the Peirce decomposition of U relative to e_1, e_2, e_3, e_4,: $U = \sum U_{ij}$, and the Peirce decomposition of A relative to $\phi(e_1), \phi(e_2), \phi(e_3), \phi(e_4)$: $A = \sum A_{ij}$. Since U_{ii} is isomorphic to $F[t]$, it follows that A_{ii} is commutative (as a quotient of $F[t]$) for any $i = 1, 2, 3, 4$. Therefore either $p+q = 4$ or $n = 4$, that is $A \cong Q_4(F)$. Consider now the restriction $\phi|_{M_{2,2}(F[t])_{\bar{0}}} : M_{2,2}(F[t])_{\bar{0}} \to A$. If $A \cong M_{p,q}(F)$, with $p + q = 4$ one has that $\phi(M_{2,2}(F[t])_{\bar{0}}) = \phi(M_2(F[t])) \oplus \phi(M_2(F[t])) = A_{\bar{0}} \cong M_p(F) \oplus M_q(F)$, and therefore $p = 2$ and $q = 2$, and $D \cong M_{2,2}(F) = A$. If $A \cong Q_4(F)$, then $(M_2(F[t]) \times 0)$ is an ideal of $M_{2,2}(F[t])_{\bar{0}}$, and so $\phi(M_2(F[t]) \times 0)$ is an ideal of $A_{\bar{0}} \cong M_4(F)$. Since $M_4(F)$ is simple and $\phi(e_1), \phi(e_2)$ are nonzero idempotents, it follows that $\phi(M_2(F[t]) \times 0) = A_{\bar{0}}$, and so $\phi(e_1) + \phi(e_2) = 1_A$, that is a contradiction because $\phi(e_1) + \phi(e_2) + \phi(e_3) + \phi(e_4) = 1$, with $\phi(e_3), \phi(e_4)$ nonzero orthogonal idempotents.
(b) Assume now that $B = D^+$ for a simple associative superalgebra D. Consider the opposite superalgebra D^{op} defined on the same vector space as D, but with the multiplication given by $a \diamond b = (-1)^{\bar{a}\bar{b}} b \cdot a$, and the direct sum $D \oplus D^{op}$, which is endowed with the superinvolution $-\colon D \oplus D^{op} \to D \oplus D^{op}$, such that $\overline{(x, a)} = (a, x)$. Note that if $e_1, e_2, \ldots, e_n$ are orthogonal idempotents in D, then $(e_1, e_1), (e_2, e_2), \ldots, (e_n, e_n)$ are also orthogonal idempotents in $D \oplus D^{op}$, and the Peirce spaces are given by $(D \oplus D^{op})_{ij} = D_{ij} \oplus (D^{op})_{ji}$. So if D satisfies conditions (i) and (ii) in Theorem 4.2, then so does $D \oplus D^{op}$. Since D^+ is isomorphic to $H(D \oplus D^{op}, -)$, there is a homomorphism of Jordan superalgebras $\phi\colon H(D \oplus D^{op}, -) \to A^+$.
(b.1) Suppose that D is not isomorphic to $M_{1,1}(F)$, nor to $Q_2(F)$, then from Theorem 4.2, ϕ can be extended to an associative homomorphism $\varphi\colon D \oplus D^{op} \to A$. As before, φ is onto because $B' = A$, so $D \oplus D^{op}/Ker\varphi$ is isomorphic to A and either $Ker\varphi \cong D$ or $Ker\varphi \cong D^{op}$, because A is simple. Hence either $D \cong A$ or $D^{op} \cong A$, that is, $\dim D = \dim A$, a contradiction.
(b.2) If D is isomorphic to $M_{1,1}(F)$ (that is, D is a quaternion superalgebra), consider the universal enveloping algebra $(U(D), u)$ of D^+ (see Theorem 3.7). The Jordan homomorphism $\iota\colon D \to A^+$ extends to an associative homomorphism $\varphi\colon U(D) \to A$ such that $\varphi \circ u = \iota$. But $B' = A$, and hence it follows that φ is onto and, therefore, $U(D)/Ker\varphi \cong A$. Recall that F, the ground field, is assumed to be algebraically closed, so either $A \cong Q_n(F)$ or $A \cong M_{p,q}(F)$. But $(U(D)/Ker\varphi)_{\bar{0}}$ is commutative, so $A_{\bar{0}}$ is commutative and therefore either $A \cong Q_1(F)$ or $A \cong M_{1,1}(F)$, a contradiction to D being isomorphic to $M_{1,1}(F)$.
(b.3) Otherwise D is isomorphic to $Q_2(F)$, and hence the universal enveloping algebra $(U(D), u)$ of D^+ is isomorphic to $D \oplus D$ (see Theorem 3.8). Hence there is a surjective homomorphism $\varphi\colon U(D) \to A$ which extends ι. As before, φ is onto and so $U(D)/Ker\varphi \cong A$. But A is simple, so $Ker\varphi \cong D$ and $A \cong D$, a contradiction.

Therefore, Lemma 4.1 can be improved to:

Lemma 4.3. *Let A be a finite-dimensional simple associative superalgebra over F, and let B be a semisimple subalgebra of A^+ with $B' = A$, then either B is isomorphic to D_t $(t \neq 0, \pm 1, -2, -\frac{1}{2})$, or B equals $H(A, *)$, for a superinvolution $*$. Moreover, if B is a maximal subalgebra of A^+, then $B = H(A, *)$ for a superinvolution $*$ of A.*

Thus, if B is a maximal subalgebra of A, which is semisimple and satisfies $B' = A$, Lemma 4.3 shows that B coincides with the subalgebra of hermitian elements of A relative to a suitable superinvolution. The converse also holds:

Theorem 4.4. *Let A be a finite-dimensional simple associative superalgebra over an algebraically closed field of characteristic zero, and let B be a semisimple subalgebra of A^+ such that $B' = A$. Then B is a maximal subalgebra of A^+ if and only if there is a superinvolution $*$ in A such that $B = H(A, *)$.*

Proof. The only thing left is to show that if A is a finite-dimensional simple associative superalgebra endowed with a superinvolution $*$, then $H(A, *)$ is a maximal subalgebra of A^+.

Our hypotheses on the ground field imply that, up to isomorphism, we are left with the next two possibilities:

(i) $A = M_{n,n}(F)$, and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d^t & -b^t \\ c^t & a^t \end{pmatrix}$.

(ii) $A = M_{n,2m}(F)$, and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} a^t & c^t q \\ -q^t b^t & q^t d^t q \end{pmatrix}$, where $q = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$.

Note that $A = H \oplus K$, where $H = H(A, *)$ and K is the set of skewsymmetric elements of $(A, *)$.

(i) In the first case

$$H = \left\{ \begin{pmatrix} a & b \\ c & a^t \end{pmatrix} : c \text{ symmetric}, b \text{ skewsymmetric} \right\},$$

$$K = \left\{ \begin{pmatrix} a & b \\ c & -a^t \end{pmatrix} : b \text{ symmetric}, c \text{ skewsymmetric} \right\},$$

and to check that $H(A, *)$ is a maximal subalgebra of A^+ it suffices to prove that $\mathrm{Jalg}\langle H, x\rangle = A^+$ for any nonzero homogeneous element $x \in K$, where $\mathrm{Jalg}\langle S\rangle$ denotes the subalgebra generated by S.

If $0 \neq x \in K_{\bar 0}$ then

$$x = \begin{pmatrix} a & 0 \\ 0 & -a^t \end{pmatrix}$$

with $a \in M_n(F)$ and so

$$\begin{pmatrix} a & 0 \\ 0 & -a^t \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & a^t \end{pmatrix} = \begin{pmatrix} 2a & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{Jalg}\langle H, x\rangle.$$

We claim that if $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{Jalg}\langle H, x\rangle$, then $\begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{Jalg}\langle H, x\rangle$, for any $u \in M_n(F)$. Similarly, if $\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \in \mathrm{Jalg}\langle H, x\rangle$, then $\begin{pmatrix} 0 & 0 \\ 0 & u \end{pmatrix} \in \mathrm{Jalg}\langle H, x\rangle$, for any $u \in M_n(F)$. Actually, since $M_n(F)^+$ is simple and the ideal generated by a in $M_n(F)^+$ is the vector subspace spanned by $\{\langle L_{b_1} \dots L_{b_m}(a) : m \in \mathbb{N}, b_1, \dots, b_m \in M_n(F)\rangle\}$ (L_b denotes the left multiplication by b in $M_n(F)^+$), it is enough to realize that

$$\begin{pmatrix} L_{b_1} \dots L_{b_m}(a) & 0 \\ 0 & 0 \end{pmatrix} = L_{\begin{pmatrix} b_1 & 0 \\ 0 & b_1^t \end{pmatrix}} \dots L_{\begin{pmatrix} b_m & 0 \\ 0 & b_m^t \end{pmatrix}} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{Jalg}\langle H, x\rangle.$$

So, if $0 \neq x \in K_{\bar 0}$, then $A_{\bar 0} \subseteq \mathrm{Jalg}\langle H, x\rangle$. In order to prove that $A_{\bar 1} \subseteq \mathrm{Jalg}\langle H, x\rangle$, note that

$$\begin{pmatrix} 0 & 0 \\ I_n & 0 \end{pmatrix} \in H,$$

and since

$$\begin{pmatrix} 0 & 0 \\ I_n & 0 \end{pmatrix} \circ \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ d & 0 \end{pmatrix}$$

it follows that

$$\begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix} \in \mathrm{Jalg}\langle H, x\rangle \quad \text{for any } u \in M_n(F).$$

It remains to prove that

$$\begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix} \in \mathrm{Jalg}\langle H, x\rangle \quad \text{for any } u \in M_n(F),$$

and the above implies that

$$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in \mathrm{Jalg}\langle H, x\rangle$$

for any nonzero skewsymmetric matrix b. But

$$\left(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & 0 \\ 0 & M_n(F) \end{pmatrix}\right) \circ \begin{pmatrix} M_n(F) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & M_n(F)bM_n(F) \\ 0 & 0 \end{pmatrix} \subseteq \mathrm{Jalg}\langle H, x\rangle$$

and $M_n(F)bM_n(F)$ is a nonzero ideal of the simple algebra $M_n(F)$, so it is the whole $M_n(F)$ and

$$\begin{pmatrix} 0 & M_n(F) \\ 0 & 0 \end{pmatrix} \subseteq \mathrm{Jalg}\langle H, x\rangle.$$

Therefore, $\mathrm{Jalg}\langle H, x\rangle = A^+$ for any nonzero element $x \in K_{\bar{0}}$.
Now, if $0 \neq x \in K_{\bar{1}}$, then

$$x = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$$

with b a symmetric and c a skewsymmetric $n \times n$-matrix respectively. Let $y \in H_{\bar{1}}$,

$$y = \begin{pmatrix} 0 & \bar{b} \\ \bar{c} & 0 \end{pmatrix}$$

with $\bar{b}$ skewsymmetric and $\bar{c}$ symmetric, such that $x \circ y \neq 0$. Since $0 \neq x \circ y \in K_{\bar{0}}$ we are back to the 'even' case, and so $\mathrm{Jalg}\langle H, x\rangle = A^+$.

(ii) In the second case (orthosymplectic superinvolution), $A = M_{n,2m}(F)$ and

$$H(A, *) = \left\{ \begin{pmatrix} a & b \\ -q^t b^t & d \end{pmatrix} : a \text{ symmetric}, d = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{11}^t \end{pmatrix}, d_{12}, d_{21} \text{ skewsymmetric} \right\},$$

$$K(A, *) = \left\{ \begin{pmatrix} a & b \\ q^t b^t & d \end{pmatrix} : a \text{ skewsymmetric}, d = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & -d_{11}^t \end{pmatrix}, d_{12}, d_{21} \text{ symmetric} \right\}.$$

We claim that $\mathrm{Jalg}\langle H, x\rangle = A^+$ for any nonzero homogeneous element $x \in K$. If $0 \neq x \in K_{\bar{1}}$, then

$$x = \begin{pmatrix} 0 & b \\ q^t b^t & 0 \end{pmatrix}$$

and so

$$x + \begin{pmatrix} 0 & b \\ -q^t b^t & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2b \\ 0 & 0 \end{pmatrix} \in \mathrm{Jalg}\langle H, x\rangle$$

with $b \in M_{n\times 2m}(F)$. Suppose that $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = \sum_{i=1,j=n+1}^{n,n+2m} \lambda_{ij} e_{ij}$ with $\lambda = \lambda_{pq} \neq 0$, where, as usual, e_{ij} denotes the matrix whose (i, j)-entry is 1 and all the other entries are 0, then

$$\left(e_{pp} \circ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\right) \circ (e_{qq} + e_{q\pm m,q\pm m}) = \frac{1}{4}(\lambda e_{pq} + \lambda_{p,q\pm m} e_{p,q\pm m}) \in \mathrm{Jalg}\langle H, x\rangle,$$

where $q \pm m$ means $q + m$ if $q \in \{n + 1, \ldots, n + m\}$ and $q - m$ if $q \in \{n + m + 1, \ldots, n + 2m\}$.

Assume that $n > 1$ and consider the element $(e_{qk} - q^t e_{kq}) \in H(A, *)$ with $k \in \{1, \ldots, n\}$ and $k \neq p$, then it follows that $2(e_{qk} - q^t e_{kq}) \circ e_{pq} = e_{pk} \in \mathrm{Jalg}\langle H, x\rangle$ with $p, k \in \{1, \ldots, n\}$ and $k \neq p$. Therefore we have found an element $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{Jalg}\langle H, x\rangle$ with $a \in M_n(F)$ and $a \notin H(M_n(F), t)$ (t denotes the usual transpose involution). Since $H(M_n(F), t)$ is maximal subalgebra of $M_n(F)^+$ (see [19, Theorem 6]) we obtain that

$$\mathrm{Jalg}\langle H(M_n(F), t), a\rangle = M_n(F)^+$$

and so

$$\begin{pmatrix} M_n(F) & 0 \\ 0 & 0 \end{pmatrix} \subseteq \mathrm{Jalg}\langle H, x\rangle.$$

Besides, for any skewsymmetric matrix $a \in M_n(F)$ and for every $b \in M_{n\times 2m}(F)$ one has

$$\left[\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & b \\ -q^t b^t & 0 \end{pmatrix}\right] + \frac{1}{2}\begin{pmatrix} 0 & ab \\ -q^t (ab)^t & 0 \end{pmatrix} = \begin{pmatrix} 0 & ab \\ 0 & 0 \end{pmatrix} \in \mathrm{Jalg}\langle H, x\rangle,$$

and thus $\begin{pmatrix} 0 & M_{n\times 2m}(F) \\ 0 & 0 \end{pmatrix} \subseteq \mathrm{Jalg}\langle H, x\rangle$, because it is easy to check that

$$K(M_n(F), t)M_{n\times 2m}(F) = M_{n\times 2m}(F).$$

But also

$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & -b^t q^t \\ b & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & -(ba)^t q^t \\ ba & 0 \end{pmatrix} \in \mathrm{Jalg}\langle H, x\rangle$$

and hence

$$\begin{pmatrix} 0 & 0 \\ M_{2m\times n}(F) & 0 \end{pmatrix} \subseteq \mathrm{Jalg}\langle H, x\rangle \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & M_{2m}(F) \end{pmatrix} \subseteq \mathrm{Jalg}\langle H, x\rangle.$$

Finally, if $n = 1$ then $\lambda e_{1j} + \mu e_{1,j\pm m} \in \mathrm{Jalg}\langle H, x\rangle$, with $j+m$ for $j \in \{n+1, \dots, n+m\}$, and $j-m$ for $j \in \{n+m+1, \dots, n+2m\}$. Now it is clear that

$$\begin{pmatrix} M_n(F) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix} \subseteq H(A, *) \subseteq \mathrm{Jalg}\langle H, x\rangle.$$

Taking $e_{j1} - e_{1,j\pm m} \in H$ one has

$$2(\lambda e_{1j} + \mu e_{1,j\pm m}) \circ (e_{j1} - e_{1,j\pm m}) = \lambda e_{11} + \lambda e_{jj} \in \mathrm{Jalg}\langle H, x\rangle.$$

Therefore, $e_{jj} \in \mathrm{Jalg}\langle H, x\rangle$.

Write $e_{jj} = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$ for a suitable $a \in M_{2m}(F)$. Then $a \notin H(M_{2m}(F), *)$ with $*$ the involution determined by the skewsymmetric bilinear form with matrix $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, and from the ungraded case (see [19, Theorem 6]) we deduce that

$$\mathrm{Jalg}\langle H(M_{2m}(F), *), a\rangle = M_{2m}(F)^+$$

and therefore $\begin{pmatrix} 0 & 0 \\ 0 & M_{2m}(F) \end{pmatrix} \subseteq \mathrm{Jalg}\langle H, x\rangle$. Now it is easy to check that since

$$\begin{pmatrix} 0 & b \\ -q^t b^t & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & 0 \\ 0 & M_{2m}(F) \end{pmatrix} \subseteq \mathrm{Jalg}\langle H, x\rangle$$

then $\begin{pmatrix} 0 & M_{1,2m}(F) \\ M_{1,2m}(F) & 0 \end{pmatrix} \subseteq \mathrm{Jalg}\langle H, x\rangle$ also in this case.

If x is now a nonzero homogeneous even element then

$$x = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

for a skewsymmetric matrix a and a matrix $b = -q^t b^t q$. Consider

$$y = x \circ \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \in \mathrm{Jalg}\langle H, x\rangle,$$

and

$$z = \begin{pmatrix} 0 & c \\ -q^t c^t & 0 \end{pmatrix}$$

such that $cb \neq 0$. Then

$$y \circ z = \frac{1}{2}\begin{pmatrix} 0 & cb \\ -bq^t c^t & 0 \end{pmatrix} \in \mathrm{Jalg}\langle H, x\rangle \cap K_{\bar 1}$$

and the 'odd' case applies. □

4.2. $B = C^+$, $C \leq_{\max} A$

Let us assume now that $B = C^+$ for a maximal subalgebra C of the simple associative superalgebra A. It has to be proved that C^+ is a maximal subalgebra of A^+.

Two different cases appear according to the classification of simple associative superalgebras (see [24]):

(1) A is simple as an (ungraded) algebra, that is, A is isomorphic to $M_{p,q}(F)$, for some p, q. In this case, [5, Theorem 2.2] shows that either $C = eAe + eAf + fAf$ with e, f even orthogonal idempotents in A such that $e + f = 1$, or $C = C_A(u)$ (centralizer of u), with $u \in A_{\bar 1}$ and $u^2 = 1$.

(2) A is not simple as an algebra, and hence it is isomorphic to $Q_n(F)$ for some n. Then $A = A_{\bar 0} + A_{\bar 0}u$ with $u \in Z(A)_{\bar 1}$, $u^2 = 1$ and $A_{\bar 0}$ is a simple algebra. In this case, [5, Theorem 2.5] shows that either $C = C_{\bar 0} + C_{\bar 0}u$ with $C_{\bar 0}$ a maximal subalgebra of $A_{\bar 0}$, or $C = A_{\bar 0}$, or $A_{\bar 0} = D_{\bar 0} + D_{\bar 1}$ is a $\mathbb{Z}_2$-graded algebra and $C = D_{\bar 0} + D_{\bar 1}u$.

(1.a) Assume that A is simple as an algebra, and that there are even orthogonal idempotents e, f such that $C = eAe + eAf + fAf$. Take an element $a_\alpha \in A_\alpha \setminus C_\alpha$, so one has that $fa_\alpha e \neq 0$. Now the element $(e \circ a_\alpha) \circ f = \frac{1}{4}(ea_\alpha f + fa_\alpha e)$ lies in $\mathrm{Jalg}\langle C^+, a_\alpha\rangle$. Since $(fAf \circ fa_\alpha e) \circ eAe = fAfa_\alpha eAe$, and $Afa_\alpha eA = A$, because A is simple, it follows that $fAe \subseteq \mathrm{Jalg}\langle C^+, a_\alpha\rangle$, and therefore C^+ is a maximal subalgebra of A^+. So we have that in this case this condition is also sufficient to be a maximal subalgebra of A^+.
(1.b) If A is simple as an algebra, but $C = C_A(u)$, for an element $u \in A_{\bar 1}$ with $u^2 = 1$, let V be the irreducible A-module (unique, up to isomorphism), so that A can be identified with $\mathrm{End}_F(V)$. Then u lies in $\mathrm{End}(V)_{\bar 1}$, and if $\{v_1, \ldots, v_s\}$ is a basis of the F-vector space $V_{\bar 1}$, it follows that $\{u(v_1), \ldots, u(v_s)\}$ is an F-basis of $V_{\bar 0}$, and so $p = q$ and, since $u^2 = 1$, the coordinate matrix of u in this basis is

$$u = \begin{pmatrix} 0 & I_s \\ I_s & 0 \end{pmatrix}.$$

Therefore $C_A(u) = Q_p(F)$, and then one can check easily that $Q_p(F)$ is maximal in $M_{p,p}(F)$.
(2.a) Assume now that A is not simple as an algebra, so $A = A_{\bar 0} + A_{\bar 0}u$, with $u \in Z(A)_{\bar 1}$, $u^2 = 1$ and $A_{\bar 0}$ a simple algebra, and that $C = C_{\bar 0} + C_{\bar 0}u$, with $C_{\bar 0}$ a maximal subalgebra of $A_{\bar 0}$. As for the ungraded case (see [19, page 192]) it follows that $\mathrm{Jalg}\langle C_{\bar 0}^+, a_{\bar 0}\rangle = A_{\bar 0}^+$ for any $a_{\bar 0} \in A_{\bar 0} \setminus C_{\bar 0}$. Thus $A_{\bar 0} \subseteq \mathrm{Jalg}\langle C^+, a_{\bar 0}\rangle$. Moreover since $1 \in C_{\bar 0}$, then $u \in C$ and it follows that $b_{\bar 0} \circ u = \frac{1}{2}(b_{\bar 0}u + ub_{\bar 0}) = b_{\bar 0}u \in \mathrm{Jalg}\langle C^+, a_{\bar 0}\rangle$ for any $b_{\bar 0} \in A_{\bar 0}$. Thus $A_{\bar 0}u \subseteq \mathrm{Jalg}\langle C^+, a_{\bar 0}\rangle$ and $\mathrm{Jalg}\langle C^+, a_{\bar 0}\rangle = A^+$. Now take an element $a_{\bar 1} \in A_{\bar 1} \setminus C_{\bar 1}$. Then $a_{\bar 1} = a_{\bar 0}u$ with $a_{\bar 0} \in A_{\bar 0} \setminus C_{\bar 0}$. Since u lies in C, it follows that $a_{\bar 1} \circ u = a_{\bar 0} \in \mathrm{Jalg}\langle C^+, a_{\bar 1}\rangle$, with $a_{\bar 0} \in A_{\bar 0} \setminus C_{\bar 0}$ and the 'even' case applies.
(2.b) If A is not simple as an algebra and $C = A_{\bar 0}$, let b be any odd element: $b \in A_{\bar 1} = A_{\bar 0}u$. Thus $b = b_{\bar 0}u$, for some $b_{\bar 0} \in A_{\bar 0}$. Then $a_{\bar 0} \circ b = (a_{\bar 0} \circ b_{\bar 0})u$, so $\mathrm{Jideal}\langle b_0\rangle u \subseteq \mathrm{Jalg}\langle A_{\bar 0}^+, b\rangle$ (where $\mathrm{Jideal}\langle b_{\bar 0}\rangle$ denotes the ideal generated by $b_{\bar 0}$ in the Jordan algebra $A_{\bar 0}^+$). By simplicity of $A_{\bar 0}^+$, $A_{\bar 0}u \subseteq \mathrm{Jalg}\langle A_{\bar 0}^+, b\rangle$, that is, C^+ is a maximal subalgebra of A^+.
(2.c) Finally, assume that A is not simple as an algebra, and $A_{\bar 0}$ (which is isomorphic to $M_p(F)$ for some p) is $\mathbb{Z}_2$-graded: $A_{\bar 0} = D_{\bar 0} \oplus D_{\bar 1}$, and $C = D_{\bar 0} \oplus D_{\bar 1}u$, where $u \in Z(A)_{\bar 1}$, $u^2 = 1$. Here, as an associative superalgebra ($\mathbb{Z}_2$-graded algebra), $A_{\bar 0}$ is isomorphic to $M_{r,s}(F)$ for some r, s. Identify $A_{\bar 0}$ to $M_{r,s}(F)$, so that $D_{\bar 0} = \left\{\begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix} : a \in M_r(F), b \in M_s(F)\right\}$, and $D_{\bar 1} = \left\{\begin{pmatrix} 0 & u \\ v & 0\end{pmatrix} : u \in M_{r\times s}(F), v \in M_{s\times r}(F)\right\}$. Let us show that C^+ is a maximal subalgebra of A^+. Since $A^+ = C^+ \oplus (D_{\bar 1} \oplus D_{\bar 0}u)$, it is enough to check that for any nonzero element $x \in D_{\bar 0}u \cup D_{\bar 1}$, the subalgebra of A^+ generated by C^+ and x: $\mathrm{Jalg}\langle C^+, x\rangle$, is the whole A^+.

Take $0 \neq x \in D_{\bar 0}u$. Then

$$x = \begin{pmatrix} x_0 & 0 \\ 0 & x_1 \end{pmatrix} u$$

with $x_0 \in M_r(F)$, and $x_1 \in M_s(F)$ not being both zero. Without loss of generality, assume that $x_0 \neq 0$, and take elements

$$\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \in C$$

with $0 \neq b \in M_r(F)$. Then

$$\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \circ x = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \circ \begin{pmatrix} x_0 & 0 \\ 0 & x_1 \end{pmatrix} u = \begin{pmatrix} b \circ x_0 & 0 \\ 0 & 0 \end{pmatrix} u \in \mathrm{Jalg}\langle C^+, x\rangle$$

for any $b \in M_r(F)$. Therefore

$$\begin{pmatrix} \mathrm{Jideal}\langle x_0\rangle & 0 \\ 0 & 0 \end{pmatrix} u \subseteq \mathrm{Jalg}\langle C^+, x\rangle$$

and because of the simplicity of $M_n(F)^+$,

$$\begin{pmatrix} M_r(F) & 0 \\ 0 & 0 \end{pmatrix} u \subseteq \mathrm{Jalg}\langle C^+, x\rangle.$$

Thus

$$\begin{pmatrix} M_r(F) & 0 \\ 0 & 0 \end{pmatrix} u \circ \begin{pmatrix} 0 & M_{r\times s}(F) \\ M_{s\times r}(F) & 0 \end{pmatrix} u = \begin{pmatrix} 0 & M_{r\times s}(F) \\ M_{s\times r}(F) & 0 \end{pmatrix} \subseteq \mathrm{Jalg}\langle C^+, x\rangle,$$

that is, $D_{\bar 1} \subseteq \mathrm{Jalg}\langle C^+, x\rangle$, and so $D_{\bar 1} \circ D_{\bar 1}u = D_{\bar 0}u \subseteq \mathrm{Jalg}\langle C^+, x\rangle$ and $\mathrm{Jalg}\langle C^+, x\rangle = A$.

Take now an element $0 \neq x \in D_{\bar 1}$. Then an element $d_{\bar 1}u \in C^+$ can be found such that $0 \neq x \circ d_{\bar 1}u \in D_{\bar 0}u \cap \mathrm{Jalg}\langle C^+, x\rangle$, so the previous arguments apply.

This concludes the proof of the next result:

Theorem 4.5. *Let A be a finite-dimensional simple associative superalgebra over an algebraically closed field of characteristic zero, and let B be a maximal subalgebra of A^+ such that $B' \neq A$ (where B' denotes the associative subalgebra generated by B in A). Then B is a maximal subalgebra of A^+ if and only if there is a maximal subalgebra C of the superalgebra A such that $B = C^+$.*

4.3. $B' = A$ and B is not semisimple

This situation does not appear in the ungraded case [19]. However, consider the associative superalgebra $A = M_{1,1}(F)$ and the subalgebra B of A^+ spanned by $\{e_{11}, e_{22}, e_{12} + e_{21}\}$, which, by dimension count, is obviously maximal and satisfies that $B' = A$. The radical of B consists of the scalar multiples of $e_{12} + e_{21}$, so it is nonzero.

Question. Is this, up to isomorphism, the only possible example of a maximal subalgebra B of A^+, A being a simple finite-dimensional superalgebra over an algebraically field F of characteristic 0, such that $B' = A$ and B is not semisimple?

5. Maximal subalgebras of $H(A, *)$

Consider now the Jordan superalgebra $J = H(A, *)$, where A is a finite-dimensional simple associative superalgebra over an algebraically closed field F of characteristic zero, and $*$ is a superinvolution of A.

Up to isomorphism [10, Theorem 3.1], it is known that $A = M_{p,q}(F)$ and that $*$ is either the orthosymplectic or the transpose superinvolution, that is, $H(A, *)$ is either $\mathrm{osp}_{n,2m}$ or $p(n)$.

Let B be a maximal subalgebra of $H(A, *)$, then again three possible situations appear:

(i) either $B' = A$ and B is semisimple,
(ii) or $B' \neq A$,
(iii) or $B' = A$ and B is not semisimple.

5.1. $B' = A$ and B semisimple

Let us assume first that B is a maximal subalgebra of the simple superalgebra $H(A, *)$, with $B' = A$ and B semisimple. From Lemma 4.3, we know that either B is isomorphic to D_t ($t \neq 0, \pm 1, -2, -\frac{1}{2}$), or $B = H(A, \diamond)$ with $\diamond$ a superinvolution. In the first case we remark that we have given only necessary conditions in Proposition 3.5 if $B' = A$ and $1_A \in B$. In the second case, one has $B = H(A, \diamond) \subseteq H(A, *)$,, but Theorem 4.4 shows that $H(A, \diamond)$ is maximal in A^+, thus obtaining a contradiction.

Therefore:

Theorem 5.1. *Let J be the Jordan superalgebra $H(A, *)$, where A is a finite-dimensional simple associative superalgebra over an algebraically closed field of characteristic zero, and $*$ a superinvolution in A. If B is a maximal subalgebra of J such that $B' = A$ and B is semisimple, then $B = D_t$ ($t \neq 0, \pm 1, -2, -\frac{1}{2}$) and $(A, *)$ is given by Proposition 3.5.*

Question. Given a natural number m, and with t equal either to $-\frac{m}{m+1}$ or to $-\frac{m+1}{m}$, is D_t isomorphic to a maximal subalgebra of the Jordan superalgebra $H(\mathrm{End}_F(V), *)$ (V and $*$ as in Proposition 3.5)?

For $m = 2$ or $m = 3$, this has been checked to be the case.

It should be noted that if the characteristic of the field is not zero, then Theorem 5.1 is not valid, as the next example shows:

Example 5.2. Let F be an algebraically closed field of characteristic 5, and let V be the superspace: $V_{\bar{0}} = Fv_1 + Fv_2 + Fv_3$, $V_{\bar{1}} = Fw_1 + Fw_2$. Consider the associative superalgebra $A = \mathrm{End}_F(V) \cong M_{3,2}(F)$ and the subalgebra B of A^+ generated by the endomorphisms e, f, x, y such that

$$\begin{aligned}
&e(v_i) = v_i \quad i = 1, 2, 3, \qquad e(w_j) = 0 \quad j = 1, 2,\\
&f(v_i) = 0 \quad i = 1, 2, 3, \quad f(w_j) = w_j \quad j = 1, 2,\\
&x(v_1) = 0, \qquad x(v_2) = 4w_1, \qquad x(v_3) = 3w_2, \qquad x(w_1) = 2v_1, \qquad x(w_2) = 4v_2,\\
&y(v_1) = w_1, \qquad y(v_2) = 4w_2, \qquad y(v_3) = 0, \qquad y(w_1) = v_2, \qquad y(w_2) = v_3.
\end{aligned}$$

We notice that $B \cong D_1$ and also that $B' = A$. Then consider the superinvolution $*$ on A determined by the supersymmetric form, (,), such that

$$(v_1, v_3) = 1, \qquad (v_2, v_2) = 1, \qquad (w_1, w_2) = 1$$

and all the other values for basic elements being zero or obtained by supersymmetry. Under the isomorphism $A \cong M_{3,2}(F)$ induced by the basis above, the superinvolution $*$ is given by:

$$\left(\begin{array}{ccc|cc}
a_{11} & a_{12} & a_{13} & b_{14} & b_{15}\\
a_{21} & a_{22} & a_{23} & b_{24} & b_{25}\\
a_{31} & a_{32} & a_{33} & b_{34} & b_{35}\\
\hline
c_{41} & c_{42} & c_{43} & d_{11} & d_{12}\\
c_{51} & c_{52} & c_{53} & d_{21} & d_{22}
\end{array}\right) \xrightarrow{*} \left(\begin{array}{ccc|cc}
a_{33} & a_{23} & a_{13} & -c_{53} & c_{43}\\
a_{32} & a_{22} & a_{12} & -c_{52} & c_{42}\\
a_{31} & a_{21} & a_{11} & -c_{51} & c_{41}\\
\hline
b_{35} & b_{25} & b_{15} & d_{22} & -d_{12}\\
-b_{34} & -b_{24} & -b_{14} & -d_{21} & d_{11}
\end{array}\right).$$

Straightforward computations show that B is a maximal subalgebra of $H(A, *)$. □

5.2. $B' \neq A$

Assume now that the maximal subalgebra B of $H(A,*)$ satisfies $B' \neq A$. The following result settles this case:

Theorem 5.3. *Let J be the Jordan superalgebra $H(A,*)$, where A is a finite-dimensional simple associative superalgebra over an algebraically closed field of characteristic zero, and $*$ is a superinvolution in A. Let B be a subalgebra of J such that $B' \neq A$ (where as always B' is the subalgebra of A generated by B). Then B is maximal if and only if there are even idempotents $e, f \in A$ with $e + f = 1$ such that $B = H(C,*)$ and one of the following possibilities occurs:*

(i) *either $C = eAe + fAf$, $e^* = e$, $f^* = f$, $H(eAe,*)' = eAe$, and $H(fAf,*)' = fAf$.*
(ii) *or $C = eA + Ae^* + ff^*Aff^*$, with $H(ff^*Aff^*,*)' = ff^*Aff^*$.*

Note [9] that given a finite-dimensional simple associative superalgebra C over F with a superinvolution $*$, the associative subalgebra $H(C,*)'$ is the whole C unless $(C,*)$ is either a quaternion superalgebra with the transpose superinvolution or a quaternion algebra with the standard involution.

Proof. If $B' \neq A$, and since $B \subseteq H(A,*)$, it follows that B' is closed under the superinvolution $*$, and so $B' \subseteq C$ with C a maximal subalgebra of $(A,*)$. But using the maximality of B and that $B \subseteq H(A,*)$, one concludes that $B = H(C,*)$. Recall that $H(A,*)$ is isomorphic either to $p(n)$ or to $\mathrm{osp}_{n,2m}$.

If $B = H(C,*)$ with C a maximal subalgebra of $(A,*)$, then the results in [5] show that either $C = (eAe + eAf + fAf) \cap (e^*Ae^* + f^*Ae^* + f^*Af^*)$ with e, f even orthogonal idempotents, or $C = C_A(u)$ with $u \in A_{\bar{1}}$, $0 \neq u^2 \in F$, $u^* \in Fu$. In this last case, since $u^* \in Fu$ it follows that $u^* = \alpha u$ with $\alpha \in F$. But $(u^*)^* = u$ and so $\alpha^2 = 1$, that is, $\alpha = \pm 1$. Thus $u^2 = (u^2)^* = -(u^*)^2 = -u^2$, a contradiction.

Thus, C is of the first type, and then [5, Proposition 4.6] gives two possible cases.

In the first case there is an idempotent e of A such that $C = eAe + fAf$ and $e^* = e$, $f = 1 - e$. If $H(C,*)' \neq C$ then either $H(eAe,*)' \neq eAe$ or $H(fAf,*)' \neq fAf$. It may be assumed that $H(eAe,*)' \neq eAe$, and then the results in [9] show that either eAe is a quaternion superalgebra with the restriction $*|_{eAe}$ being the transpose superinvolution or is a quaternion algebra contained in $A_{\bar{0}}$, with the standard involution. In both cases $e = e_1 + e_2$ with e_1, e_2 orthogonal idempotents and $e_1^* = e_2$. Consider $D = e_1A + Ae_2 + fAf$ and take $0 \neq e_1af \in e_1Af$, then $e_1af + fa^*e_2 \in H(D,*)$ and $e_1af + fa^*e_2 \notin H(C,*)$. In the same vein, take $c \in A$ with $e_2cf \neq 0$. Then $e_2cf + fc^*e_1 \in H(A,*) \setminus H(D,*)$. Therefore $B = H(C,*) \subsetneq H(D,*) \subsetneq H(A,*)$ and $B = H(C,*)$ is not maximal. So $B' = H(C,*)' = C$ if $B = H(C,*)$ with $C = eAe + fAf$ and $e^* = e$.

In the second case [5, Proposition 4.6], there is an idempotent e in A such that e, e^*, ff^* are mutually orthogonal idempotents with $1 = e + e^* + ff^*$, and $C = eA + Ae^* + ff^*Aff^*$. Hence $H(C,*) = H(ff^*aff^*) + \{ea + a^*e^* : a \in A\}$.

If $H(ff^*Aff^*,*)' \neq ff^*Aff^*$, then ff^*Aff^* is a quaternion superalgebra with superinvolution such that $ff^* = e_1 + e_2$ with e_1, e_2 orthogonal idempotents and $e_1^* = e_2$. Consider the subalgebra $D = eA + Ae^* + e_2A + Ae_1$. As $H(C,*) \subsetneq H(D,*) \subsetneq H(A,*)$, $H(C,*)$ is not maximal. Therefore, if $B = H(C,*)$ with $C = eA + Ae^* + ff^*Aff^*$, and e, e^*, ff^* mutually orthogonal idempotents such that $e + e^* + ff^* = 1$, then $H(ff^*Aff^*,*)' = ff^*Aff^*$.

The proof of the converse will be split according to the different possibilities:

(i.1): The superinvolution $*$ on A is the transpose superinvolution, and the conditions in item (i) of the Theorem hold:

Then $*$ is determined, after identifying A with $\mathrm{End}_F(V)$, by a nondegenerate odd symmetric superform $(\ ,\ )$. That is,, $(V_{\bar{0}}, V_{\bar{0}}) = (V_{\bar{1}}, V_{\bar{1}}) = 0$ and $(a_0, b_1) = (b_1, a_0)$ for any $a_0 \in V_{\bar{0}}$, $b_1 \in V_{\bar{1}}$.

In this situation we claim that a basis $\{x_1, \ldots, x_n, y_1 \ldots, y_n\}$ of V can be chosen such that $\{x_1, \ldots, x_n\}$ is a basis of $V_{\bar{0}}$, $\{y_1, \ldots, y_n\}$ is a basis of $V_{\bar{1}}$, and the coordinate matrices of the superform and of e present the following form, respectively,

$$\begin{pmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This follows from the fact that the eigenspaces of the idempotent transformation e are orthogonal relative to $(\ ,\ )$, as $e^* = e$. Under these circumstances, we may identify $H(A,*)$ to

$$p(n) = \left\{ \begin{pmatrix} a & b \\ c & a^t \end{pmatrix} : b \text{ skewsymmetric}, c \text{ symmetric} \right\}$$

in such a way that the subalgebra $H(eAe + fAf,*)$ becomes the subspace of the matrices (in block form)

$$\begin{pmatrix} a_1 & 0 & c_1 & 0 \\ 0 & a_2 & 0 & c_2 \\ d_1 & 0 & a_1^t & 0 \\ 0 & d_2 & 0 & a_2^t \end{pmatrix},$$

where a_1, c_1, d_1 belong to $M_i(F)$, a_2, c_2, d_2 belong to $M_j(F)$, $i + j = n$, and c_1, c_2 are skewsymmetric matrices, while d_1, d_2 are symmetric.

It must be proved that for any homogeneous element x, $\mathrm{Jalg}\langle H(C,*), x\rangle = H(A,*)$ holds.

Let $x \in H(A,*)_{\bar 0} \setminus H(C,*)_{\bar 0}$, that is,

$$x = \sum_{\substack{1\le k\le i\\ 1\le r\le j}} \lambda_{kr}(e_{k,i+r} + e_{n+i+r,n+k}) + \sum_{\substack{1\le r\le j\\ 1\le k\le i}} \mu_{rk}(e_{i+r,k} + e_{n+k,n+i+r}),$$

where $e_{r,s}$ denotes the matrix with 1 in the (r,s)th entry and 0 in all the other entries. Suppose that there exists $\lambda_{pq} \neq 0$. The same proof works if $\mu_{pq} \neq 0$.

Since $H(C,*)' = C$ and $i > 1$ (as $H(eAe,*)' = eAe$), an index $s \in \{1,\dots,i\}$ can be chosen with $s \neq p$, such that $u = e_{s,p} + e_{n+p,n+s} \in H(C,*)$. Let $v = e_{p,p} + e_{n+p,n+p}$ and $w = e_{i+q,i+q} + e_{n+i+q,n+i+q}$ (note that $v, w \in H(C,*)$). Then

$$((v \circ x) \circ w) \circ u = \frac{1}{8}\lambda_{pq}(e_{s,i+q} + e_{n+i+q,n+s}) \in \mathrm{Jalg}\langle H(C,*), x\rangle.$$

Denote this element by α, and then $0 \neq \alpha \in e_1Af_1 + f_1^*Ae_1^*$. Now

$$((e_1ae_1 + e_1^*a^*e_1^*) \circ \alpha) \circ (f_1bf_1 + f_1^*b^*f_1^*) = e_1ae_1\alpha f_1bf_1 + f_1^*b^*f_1^*\alpha e_1^*a^*e_1^*$$

belongs to $\mathrm{Jalg}\langle H(C,*), x\rangle$. Since $\{ae_1\alpha f_1b : a, b \in A\}$ is an ideal of A, and A is simple, it holds that $\{ae_1\alpha f_1b : a, b \in A\} = A$, and so $e_1af_1 + f_1^*a^*e_1^* \in \mathrm{Jalg}\langle H(C,*), x\rangle$ for any $a \in A$.

Consider now an element $y \in f_1Af_1^* \cap H(C,*)$. Since $j > 1$ (because $H(fAf,*)' = fAf$), we can pick up the element $y = e_{l,k} - e_{l+1,k-1}$, with $l = i+1$ and $k = n+i+2$. Take $z = e_{k-1,p} + e_{1,l} \in H(e_1Af_1 + f_1^*Ae_1^*, *) \subseteq \mathrm{Jalg}\langle H(C,*), x\rangle$ and $v = e_{p,1} \in H(C,*) \cap e_1^*Ae_1$, with $p = n+1$. Then $(y \circ z) \circ v = \frac{1}{4}(-e_{l+1,1} - e_{p,k}) \in (f_1Ae_1 + e_1^*Af_1^*) \cap H(A,*)_{\bar 0}$. As before we obtain that $f_1ae_1 + e_1^*a^*f_1^* \in \mathrm{Jalg}\langle H(C,*), x\rangle$, and $H(A,*)_{\bar 0} \subseteq \mathrm{Jalg}\langle H(C,*), x\rangle$.

Now it will be proved that $H(A,*)_{\bar 1}$ is contained in $\mathrm{Jalg}\langle H(C,*), x\rangle$. Take $y = e_{k,n+i+t} - e_{i+t,n+k} \in H(A,*)_{\bar 1} \cap (e_1Af_1^* + f_1Ae_1^*)$, with $k \in \{1,\dots,i\}$, $t \in \{1,\dots,j\}$ and we claim that $y \in \mathrm{Jalg}\langle H(C,*), x\rangle$. Since $H(fAf,*)' = fAf$, there exists $s \in \{1,\dots,j\}$ with $s \neq t$, and consider then the elements $z = e_{n+i+s,n+k} + e_{k,i+s} \in \mathrm{Jalg}\langle H(C,*), x\rangle$, and $u = e_{i+s,n+i+t} - e_{i+t,n+i+s} \in H(C,*)$. Then it follows that $z \circ u = \frac{1}{2}y \in \mathrm{Jalg}\langle H(C,*), x\rangle$. In the same way we obtain that $(e_1^*Af_1 + f_1^*Ae_1) \cap H(A,*)_{\bar 1} \subseteq \mathrm{Jalg}\langle H(C,*), x\rangle$.

So for any $x \in H(A,*)_{\bar 0} \setminus H(C,*)_{\bar 0}$, $H(A,*) = \mathrm{Jalg}\langle H(C,*), x\rangle$ holds.

Now let $x \in H(A,*)_{\bar 1} \setminus H(C,*)_{\bar 1}$. Then

$$x = \sum_{\substack{1\le k\le i\\ 1\le r\le j}} \lambda_{kr}(e_{k,n+i+r} - e_{i+r,n+k}) + \sum_{\substack{1\le k\le i\\ 1\le r\le j}} \mu_{kr}(e_{n+k,i+r} + e_{n+i+r,k})$$

and assume that for some (p,q), one has $\lambda_{pq} \neq 0$.

Since $u = e_{n+p,p} \in H(C,*)$, $0 \neq 2x \circ u = -\sum_{1\le q\le j} \lambda_{pq}(e_{i+q,p} + e_{n+p,n+i+q}) \in H(A,*)_{\bar 0} \setminus H(C,*)_{\bar 0}$, and the above case applies.

In the same way, if $\mu_{pq} \neq 0$ we obtain that $H(C,*)$ is a maximal subalgebra of $H(A,*)$.

(i.2): The superinvolution $*$ on A is an orthosymplectic superinvolution, and the conditions in item (i) of the Theorem hold:

In this and the following cases, we will content ourselves to establish the setting in which one can apply the same kind of not very illuminating arguments like those used in case **(i.1)**.

Here, after identifying A to $\mathrm{End}_F(V)$, the superinvolution $*$ is determined by a nondegenerate symmetric superform $(\ ,\ )$ on V, that is, $(\ ,\ )|_{V_{\bar 0}\times V_{\bar 0}}$ is symmetric, $(\ ,\ )|_{V_{\bar 1}\times V_{\bar 1}}$ is skewsymmetric and $(V_{\bar 0}, V_{\bar 1}) = (V_{\bar 1}, V_{\bar 0}) = 0$.

Since e is idempotent and self-adjoint, there is a basis of V in which the coordinate matrices of the superform and of e are, respectively,

$$\begin{pmatrix} I & 0 & 0 & 0 & 0 & 0\\ 0 & I & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & I & 0\\ 0 & 0 & 0 & 0 & 0 & I\\ 0 & 0 & -I & 0 & 0 & 0\\ 0 & 0 & 0 & -I & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & I & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & I & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where 0, respectively I, denotes the zero matrix, respectively identity matrix (of possibly different orders). Let n be the dimension of $V_{\bar 0}$, $2m$ the dimension of $V_{\bar 1}$, i the rank of the restriction $e|_{V_{\bar 0}}$, $j = n - i$, $2k$ the rank of $e|_{V_{\bar 1}}$ and $l = m - k$. Hence, identifying by means of this basis $H(A,*)$ to $\mathrm{osp}_{n,2m}$, the idempotent e decomposes as $e = e_1 + e_2 + e_2^*$, with $e_1 = \sum_{s=1}^{i} e_{s,s}$, $e_2 = \sum_{s=1}^{k} e_{n+s,n+s}$ and $e_2^* = \sum_{s=1}^{k} e_{n+m+s,n+m+s}$. Similarly, $f = 1 - e$ decomposes as $f = f_1 + f_2 + f_2^*$.

The elements of $H(C, *)$ are then the matrices (in block form)

$$\left(\begin{array}{cc:cccc} c_{11} & 0 & b_{11} & 0 & b_{13} & 0 \\ 0 & c_{22} & 0 & b_{22} & 0 & b_{24} \\ \hdashline b_{13}^t & 0 & a_{11} & 0 & a_{13} & 0 \\ 0 & b_{24}^t & 0 & a_{22} & 0 & a_{24} \\ -b_{11}^t & 0 & a_{31} & 0 & a_{11}^t & 0 \\ 0 & -b_{22}^t & 0 & a_{42} & 0 & a_{22}^t \end{array}\right)$$

with $c_{11} \in M_i(F)$ and $c_{22} \in M_j(F)$ symmetric matrices, $a_{11} \in M_k(F)$, $a_{22} \in M_l(F)$, $b_{11}, b_{13} \in M_{i\times k}(F)$, $b_{22}, b_{24} \in M_{j\times l}(F)$, $a_{13}, a_{31} \in M_k(F)$ skewsymmetric matrices, and $a_{24}, a_{42} \in M_l(F)$ skewsymmetric too.

Note that it is possible that either e_1 or f_1 may be 0. If, for instance, $f_1 = 0$, then since $H(fAf, *)' = fAf$, it follows that $l > 1$.

In this setting, routine arguments like the ones for **(i.1)** apply.

(ii.1): The superinvolution $*$ on A is the transpose superinvolution, and the conditions in item (ii) of the Theorem hold:

Here a basis $\{x_1, \ldots, x_n, y_1 \ldots, y_n\}$ of V ($\{x_1, \ldots, x_n\}$ being a basis of $V_{\bar 0}$ and $\{y_1, \ldots, y_n\}$ of $V_{\bar 1}$), so that the coordinate matrices of the superform and of the idempotents e, e^* and ff^* are, respectively,

$$\begin{pmatrix} 0&0&0&I&0&0\\ 0&0&0&0&I&0\\ 0&0&0&0&0&I\\ I&0&0&0&0&0\\ 0&I&0&0&0&0\\ 0&0&I&0&0&0 \end{pmatrix}, \quad \begin{pmatrix} I&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&I \end{pmatrix},$$
$$\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&I&0&0&0\\ 0&0&0&I&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix}, \quad \begin{pmatrix} 0&0&0&0&0&0\\ 0&I&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&I&0\\ 0&0&0&0&0&0 \end{pmatrix}.$$

This follows from the fact that e, e^* and ff^* are orthogonal idempotents with $1 = e + e^* + ff^*$, so

$$V_{\bar 0} = S(1, e)_{\bar 0} \oplus S(1, ff^*)_{\bar 0} \oplus S(1, e^*)_{\bar 0},$$
$$V_{\bar 1} = S(1, e^*)_{\bar 1} \oplus S(1, ff^*)_{\bar 1} \oplus S(1, e)_{\bar 1},$$

where $S(1, g)$ denotes the eigenspace of the endomorphism g of eigenvalue 1, and from the fact that ff^* is self-adjoint, so

$$V = (S(1, e)_{\bar 0} \oplus S(1, e^*)_{\bar 1}) \oplus (S(1, ff^*)_{\bar 0} \oplus S(1, ff^*)_{\bar 1}) \oplus (S(1, e^*)_{\bar 0} \oplus S(1, e)_{\bar 1}) .$$

After the natural identifications, the elements of $H(C, *) = H(eA + Ae^* + ff^*Aff^*, *)$ are the matrices (in block form)

$$\left(\begin{array}{ccc:ccc} a_{11} & a_{12} & a_{13} & c_{11} & c_{12} & c_{13} \\ 0 & a_{22} & a_{23} & -c_{12}^t & c_{22} & 0 \\ 0 & 0 & a_{33} & -c_{13}^t & 0 & 0 \\ \hdashline 0 & 0 & d_{13} & a_{11}^t & 0 & 0 \\ 0 & d_{22} & d_{23} & a_{12}^t & a_{22}^t & 0 \\ d_{13}^t & d_{23}^t & d_{33} & a_{13}^t & a_{23}^t & a_{33}^t \end{array}\right),$$

where c_{11}, c_{22} are skewsymmetric matrices and d_{22}, d_{33} symmetric matrices. Since $H(ff^*Aff^*, *)' = ff^*Aff^*$, it follows that ff^*Aff^* is not a quaternion superalgebra and so the order of the blocks in the $(2, 2)$ position is > 1.

This is the setting where routine computations can be applied.

(ii.2): The superinvolution $*$ on A is an orthosymplectic superinvolution, and the conditions in item (ii) of the Theorem hold:

Here, with the same sort of arguments as before, the coordinate matrices in a suitable basis of the orthosymplectic superform, and of the idempotents ff^*, e and e^* are, respectively:

$$\begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & -I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -I & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \end{pmatrix}.$$

Now, the superinvolution $*$, identifying the elements in $H(A, *)$ with their coordinate matrices in the basis above, is given by:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} \\ a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} \end{pmatrix} \to \begin{pmatrix} a_{11}^t & a_{31}^t & a_{21}^t & a_{61}^t & a_{71}^t & -a_{41}^t & -a_{51}^t \\ a_{13}^t & a_{33}^t & a_{23}^t & a_{63}^t & a_{73}^t & -a_{43}^t & -a_{53}^t \\ a_{12}^t & a_{32}^t & a_{22}^t & a_{62}^t & a_{72}^t & -a_{42}^t & -a_{52}^t \\ -a_{16}^t & -a_{36}^t & -a_{26}^t & a_{66}^t & a_{76}^t & -a_{46}^t & -a_{56}^t \\ -a_{17}^t & -a_{37}^t & -a_{27}^t & a_{67}^t & a_{77}^t & -a_{47}^t & -a_{57}^t \\ a_{14}^t & a_{34}^t & a_{24}^t & -a_{64}^t & -a_{74}^t & a_{44}^t & a_{54}^t \\ a_{15}^t & a_{35}^t & a_{25}^t & -a_{65}^t & -a_{75}^t & a_{45}^t & a_{55}^t \end{pmatrix}.$$

Therefore the Jordan superalgebra $H(A, *)$ consists of the following matrices:

$$\left(\begin{array}{ccc|cccc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} \\ a_{13}^t & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} \\ a_{12}^t & a_{32} & a_{22}^t & a_{34} & a_{35} & a_{36} & a_{37} \\ \hline -a_{16}^t & -a_{36}^t & -a_{26}^t & a_{44} & a_{45} & a_{46} & a_{47} \\ -a_{17}^t & -a_{37}^t & -a_{27}^t & a_{54} & a_{55} & -a_{47}^t & a_{57} \\ a_{14}^t & a_{34}^t & a_{24}^t & a_{64} & a_{65} & a_{44}^t & a_{54}^t \\ a_{15}^t & a_{35}^t & a_{25}^t & -a_{65}^t & a_{75} & a_{45}^t & a_{55}^t \end{array}\right),$$

where a_{11}, a_{23}, a_{32} are symmetric matrices, while $a_{46}, a_{57}, a_{64}, a_{75}$ are skewsymmetric matrices. Besides, the elements of $H(C, *) = H(eA + Ae^* + ff^*Aff^*, *)$ are the matrices which, in block form, look like

$$\left(\begin{array}{ccc|cccc} a_{11} & 0 & a_{13} & a_{14} & 0 & a_{16} & a_{17} \\ a_{13}^t & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} \\ 0 & 0 & a_{22}^t & 0 & 0 & 0 & a_{37} \\ \hline -a_{16}^t & 0 & -a_{26}^t & a_{44} & 0 & a_{46} & a_{47} \\ -a_{17}^t & -a_{37}^t & -a_{27}^t & a_{54} & a_{55} & -a_{47}^t & a_{57} \\ a_{14}^t & 0 & a_{24}^t & a_{64} & 0 & a_{44}^t & a_{54}^t \\ 0 & 0 & a_{25}^t & 0 & 0 & 0 & a_{55}^t \end{array}\right).$$

Now again routine arguments with matrices give the result. □

5.3. $B' = A$ and B is not semisimple

As for the maximal subalgebras of the Jordan superalgebras A^+, this situation does not appear in the ungraded case [19]. However, consider the associative superalgebra $A = M_{1,2}(F)$, with the natural orthosymplectic superinvolution. Thus, the Jordan superalgebra $J = H(A, *)$ is

$$J = \mathrm{osp}_{1,2} = \left\{ \begin{pmatrix} a & -c & b \\ b & d & 0 \\ c & 0 & d \end{pmatrix} : a, b, c, d \in F \right\}.$$

The subspace

$$B = \left\{ \begin{pmatrix} a & -b & b \\ b & d & 0 \\ b & 0 & d \end{pmatrix} : a, b, d \in F \right\}$$

is a maximal superalgebra of J, and it satisfies $B' = A$, while it is not semisimple, as its radical coincides with its odd part

Question. Is this, up to isomorphism, the only possible example of a maximal subalgebra B of $H(A, *)$, A being a simple finite-dimensional superalgebra over an algebraically field F of characteristic 0, such that $B' = A$ and B is not semisimple?

It seems that a broader knowledge of nonsemisimple Jordan superalgebras is needed here.

The solution to the above question is also related to the Question after Theorem 5.1. Actually, if this question is answered in the affirmative, then the subalgebra B isomorphic to D_t ($t \neq 0, \pm 1, -2, -\frac{1}{2}$) in Theorem 5.1 would indeed be maximal in $H(A, *)$. Otherwise, any maximal subalgebra S containing B would satisfy $S' = A$ (as $B' = A$ already) and would not be semisimple (because of Theorem 5.1).

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