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Betti numbers and minimal free resolutions for multi-state system reliability bounds☆

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ABSTRACT

Every coherent system has a monomial ideal associated with it and the knowledge of its multigraded Betti numbers provides reliability bounds for the corresponding system, which are the tightest among a certain class of such bounds. Some alternative methods for computing the multigraded Betti numbers are used in this paper and applied in the study of reliability. We obtain special results for well known examples and show that computational commutative algebra techniques can be used beneficially in the reliability analysis of systems of different types.

0. Introduction

Improved reliability bounds for multi-state coherent systems can be computed using the techniques of (computational) commutative algebra. Every coherent multi-state system has a monomial ideal associated with it and the knowledge of its multigraded Betti numbers provides good reliability bounds for the corresponding system. This use of monomial ideals in system reliability was introduced by Giglio and Wynn (2004) following work on so-called discrete tube theory by Naiman and Wynn (1992, 1997). These methods can be considered as removing redundancy in the classical Bonferroni–Fréchet bounds of probability theory. The latter correspond to the Taylor resolution and the improved bounds are based on minimal free resolutions, whose terms are given by the multigraded Betti numbers. The usual way, then, to obtain these numbers is to compute the minimal free resolution of the ideal. But this is computationally hard in general. Some alternative

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computational methods are used in this paper and applied in the study of reliability, with tractable results in some standard cases. Then, we give the tractable results. This application of monomial resolutions to multi-state coherent systems constitutes a promising new area of application of symbolic computation.

The first two sections of the paper introduce the basic notions on reliability and monomial ideals that are needed in the subsequent sections. The following section presents improved reliability bounds based on the computation of Hilbert series and resolutions of monomial ideals. In the next section we cover the actual computation of multigraded Betti numbers of monomial ideals, in particular using Mayer–Vietoris tree based algorithms. Finally, in the last section we apply our ideas to several relevant coherent systems, in particular (consecutive) k-out-of-n and series–parallel systems.

1. System reliability

A multi-state system is defined here as a system of n components whose states are described by real variables $Y = (Y_1, \dots, Y_n)$, which can be in one of a set of states which we define as the n-dimensional non-negative integer grid $\mathcal{Y} = \mathbb{N}^n$. There is a distinguished subset, $\mathcal{F} \subset \mathcal{Y}$, called the *failure set*, with the interpretation that if $Y \in \mathcal{F}$ the system is said to fail. A member of $\mathcal{F}$ is called a *cut*. Let $\leq$ be the usual multivariate inequality $y \leq z \;\Leftrightarrow y_i \leq z_i,\; i = 1, \dots, n$, and let $y < z$ when $y \leq z$ and $y_i < z_i$ for at least one $i = 1, \dots, n$. Also define $x \vee y = (\max(x_1, y_1), \dots, \max(x_n, y_n))$. Then we call the system *coherent* if

$$y \in \mathcal{F},\; y \leq z \Rightarrow z \in \mathcal{F}. \tag{1}$$

Note that we use y to refer to a particular value (point) in $\mathcal{Y}$ and use Y for the random variable describing the stochastic behavior of the system. Coherency is the principle that if a system has failed and the components move to a worse (higher) state value then the system remains failed.

In reliability, Y is a random variable, which summarises the consequence of internal degradation or external shock to the system liable to increase the values of states, although by repair one can also decrease the value. Indeed, in Markovian systems one can consider Y moving around $\mathcal{Y}$ according to a Markov chain; see, for example, the study of maintenance systems.

A major concern of system reliability is to evaluate or bound the probability of failure $P(\mathcal{F}) = \text{prob}\{Y \in \mathcal{F}\}$. We will be concerned not so much with the dependence of $P(\mathcal{F})$ on the distribution of Y, but rather with the set $\mathcal{F}$ itself. Thus for any set $U \subseteq \mathcal{Y}$ we define the indicator

$$I_U(y) = \begin{cases} 1 & \text{if } y \in U \\ 0 & \text{otherwise.} \end{cases}$$

Then $P(\mathcal{F}) = \mathrm{E}\,(I_{\mathcal{F}}(\mathcal{Y}))$ and identities and bounds on indicator functions give identities and bounds on $P(\mathcal{F})$, whatever the distribution of Y.

2. Monomial ideals

The first step in the algebraization of coherent systems is to encode a point $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathcal{Y}$ by a monomial $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, where $x = (x_1, \dots, x_n)$ is a vector of variables. We see immediately from the 'coherence property', (1), that $\mathcal{F} \subseteq \mathbb{N}^n$ is coded into a set of monomials which defines a monomial ideal $Id_{\mathcal{F}}$ in $\mathbf{k}[x_1, \dots, x_n]$ (where $\mathbf{k}$ is a field of characteristic 0):

$$Id_{\mathcal{F}} = \langle x^\alpha : \alpha \in \mathcal{F} \rangle$$

and (1) is equivalent to the 'ideal property':

$$x^\alpha \in Id_{\mathcal{F}}, \quad \alpha \leq \beta \;\Rightarrow\; x^\beta \in Id_{\mathcal{F}}.$$

Conversely, any monomial ideal gives a failure set, under coherency. The minimal generating sets for the monomial ideal $Id_{\mathcal{F}}$ can be identified with the set, $\mathcal{F}^*$, of *minimal cuts*, in the reliability context. Thus α is a minimal cut if and only if $\alpha \in \mathcal{F},\; \beta < \alpha \Rightarrow \beta \notin \mathcal{F}$ and moreover $Id_{\mathcal{F}} = \langle x^\alpha \mid \alpha \in \mathcal{F}^* \rangle$.

A subset $U \subset \mathcal{Y}$ has a unique generating function:

$$U(x) = \sum_{\alpha \in U} x^{\alpha},$$

and identities and inequalities on their indicator functions, $I_U(y)$, can be translated precisely to those for the corresponding generating functions. In particular we shall be interested in identities and bounds for $\mathcal{F}(x)$, the generating function of the failure set $\mathcal{F}$. The generating function for the whole of $\mathcal{Y} = \mathbb{N}^n$ and for the monomial ideal generated by a single monomial are respectively

$$\mathcal{Y}(x) = \frac{1}{\prod_{i=1}^{n}(1-x_i)},$$
$$\{\beta\}(x) = \frac{x^{\beta}}{\prod_{i=1}^{n}(1-x_i)}.$$

As an example, consider just two minimal cuts, $\mathcal{F}^* = \{\beta, \gamma\}$. Then the failure ideal is $Id_{\mathcal{F}} = \langle x^{\beta}, x^{\gamma} \rangle$, and the generating function of the associated monomial set is

$$\mathcal{F}(x) = \frac{x^{\alpha} + x^{\beta} - \text{lcm}(x^{\alpha}, x^{\beta})}{\prod_{i=1}^{n}(1-x_i)} = \{\alpha\}(x) + \{\beta\}(x) - \{\alpha \vee \beta\}(x). \quad (2)$$

This represents inclusion–exclusion for the failure set of the relevant upper orthants in the original system $\mathcal{Y}$:

$$I_{Q(\alpha)\cup Q(\beta)} = I_{Q(\alpha)}(y) + I_{Q(\beta)}(y) - I_{Q(\alpha)\cap Q(\beta)}(y) = I_{Q(\alpha)}(y) + I_{Q(\beta)}(y) - I_{Q(\alpha)}(y) I_{Q(\beta)}(y),$$

where $Q(\alpha) = \{\gamma | \alpha \leq \gamma\}$, etc, are the orthants. Note that if we omit the last term on the right hand side we obtain an upper bound to the indicator function which gives the elementary Bonferroni bound: $\text{prob}(Q(a) \cup Q(b)) \leq \text{prob}(Q(a)) + \text{prob}(Q(b))$.

A little care is needed with regard to probability statements. For a particular α, $P(\alpha)$ is interpreted as $\text{Prob}\{Y = \alpha\}$. This means that α occurs and no other α', whereas $P(Q(\alpha))$ is the probability that α occurs and all worse events. In terms of cuts this means distinguishing the probability of exactly a particular cut (and nothing else) and the probability of the totality of all outcomes which include that cut in the sense of being at least as bad in terms of the $\leq$ ordering. In the binary cases, discussed below, where individual components may fail, this means that $Q(\alpha)$ is all cuts which simply *include* the components indicated by α. In that case $P(Q(\alpha))$ is the *marginal* probability of the latter components being cut.

3. Improved bound via the multigraded Hilbert series

Consider a multigraded R-module, $\mathcal{M}$, over the ring $R = \mathbf{k}[x_1, \ldots, x_n]$ with the usual multigrading $md(x_i) = (0, \ldots, \overset{i}{1}, \ldots, 0)$ considered as a $\mathbf{k}$ vector space over each of its multigraded components $\mathcal{M}_{\alpha}$. If each of the dimensions is finite we can define the *multigraded Hilbert series* as the formal power series

$$\mathcal{H}(\mathcal{M}; x) = \sum_{\alpha \in \mathbb{N}^n} \dim_{\mathbf{k}}(\mathcal{M}_{\alpha}) x^{\alpha}.$$

For a resolution (P_i, ∂) of the quotient of R by a monomial ideal I we have, from the rank-nullity principle, that

$$\mathcal{H}(R/I; x) = \sum_{i=0}^{d} (-1)^i \mathcal{H}(P_i; x),$$

where the P_i, $i = 0, \ldots, d$, are the modules in the resolution of R/I. If the resolution is multigraded each $P_i = \bigoplus_{\alpha\in\mathbb{N}^n} \gamma_{i,\alpha} P_{i,\alpha}$ for scalars $\gamma_{i,\alpha}$, of which only a finite number are non-zero. Then

$$\mathcal{H}(R/I; x) = \frac{\sum_{i=0}^{d} (-1)^i (\sum_{\alpha\in\mathbb{N}^n} \gamma_{i,\alpha} x^\alpha)}{\prod_{j=1}^{n} (1 - x_j)}.$$

If the resolution is minimal then

$$\mathcal{H}(R/I; x) = \frac{\sum_{i=0}^{d} (-1)^i (\sum_{\alpha\in\mathbb{N}^n} \beta_{i,\alpha} x^\alpha)}{\prod_{j=1}^{n} (1 - x_j)},$$

where $\beta_{i,\alpha}$ are the multigraded Betti numbers and, importantly,

$$\beta_{i,\alpha} \le \gamma_{i,\alpha} \quad \forall \alpha, i. \tag{3}$$

When $I = Id_{\mathcal{F}}$ the Hilbert series of I and R/I are, respectively, the generating functions of $\mathcal{F}$ and $\mathcal{Y} \setminus \mathcal{F}$, the latter being the non-failure set (where the systems works), and

$$\mathcal{H}(I; x) = \frac{\sum_{i=1}^{d} (-1)^{i-1} (\sum_{\alpha\in\mathbb{N}^n} \gamma_{i,\alpha} x^\alpha)}{\prod_{i} (1 - x_i)}.$$

The key idea for system reliability is that if we truncate this (non-simplified) form of the multigraded Hilbert series, using exactness and the optimality (3), (i) we obtain upper and lower bounds for the Hilbert series and (ii) for a minimal resolution these bounds are at least as tight as for any other resolution:

$$\begin{aligned} &\frac{\sum_{i=1}^{k+1} (-1)^{i-1} (\sum_{\alpha\in\mathbb{N}^n} \gamma_{i,\alpha} x^\alpha)}{\prod_{i} (1 - x_i)} \le \frac{\sum_{i=1}^{k+1} (-1)^{i-1} (\sum_{\alpha\in\mathbb{N}^n} \beta_{i,\alpha} x^\alpha)}{\prod_{i} (1 - x_i)} \le \mathcal{H}(I; x) \\ &\quad \le \frac{\sum_{i=1}^{k} (-1)^{i-1} (\sum_{\alpha\in\mathbb{N}^n} \beta_{i,\alpha} x^\alpha)}{\prod_{i} (1 - x_i)} \le \frac{\sum_{i=1}^{k} (-1)^{i-1} (\sum_{\alpha\in\mathbb{N}^n} \gamma_{i,\alpha} x^\alpha)}{\prod_{i} (1 - x_i)} \end{aligned} \tag{4}$$

$k = 1, \ldots, d-1$; k odd.

Now, the inequalities in (4) yield inequalities for indicator functions, by the equivalence mentioned above, and we can, for any probability distribution on $\mathcal{Y} = \mathbb{N}^n$, capture "improved" inclusion–exclusion inequalities, based on the $\beta_{i,\alpha}$. Moreover they will be the best, that is tightest, within the class arising from resolutions in this way.

The proof of (4) is based on elementary properties of exact sequences that can be easily extended to resolutions. Thus, since the Hilbert series of a monomial ideal can be seen as a way to count the monomials that are in the ideal, and from the equality

$$\frac{\sum_{i=1}^{d} (-1)^{i-1} (\sum_{\alpha\in\mathbb{N}^n} \beta_{i,\alpha} x^\alpha)}{\prod_{i} (1 - x_i)} = \mathcal{H}(I; x),$$

we have that the central inequalities in (4) amount to the usual inclusion–exclusion principle applied to the set of monomials in I. To prove the exterior inequalities in (4), observe that in any non-minimal resolution redundant generators of the modules in the resolutions occur in pairs appearing in consecutive dimensions. Since the resolution is multigraded, the generators in each redundant pair have the same multidegree. Those such pairs are clearly canceled in the numerator of the expression $\frac{\sum_{i=1}^{k+1}(-1)^{i-1}(\sum_{\alpha\in\mathbb{N}^n}\gamma_{i,\alpha}x^\alpha)}{\prod_i(1-x_i)}$, except for those in which one of the elements of the pair is in the module of dimension $k+2$. Let the multidegrees of these be $x^{\mu_1},\ldots,x^{\mu_s}$. Then

$$\frac{\sum_{i=1}^{k+1}(-1)^{i-1}(\sum_{\alpha\in\mathbb{N}^n}\gamma_{i,\alpha}x^\alpha)}{\prod_i(1-x_i)} = \frac{\sum_{i=1}^{k+1}(-1)^{i-1}(\sum_{\alpha\in\mathbb{N}^n}\beta_{i,\alpha}x^\alpha)+(-1)^k(x^{\mu_1}+\cdots+x^{\mu_s})}{\prod_i(1-x_i)},$$

whence the inequalities.

3.1. Different resolutions

Let $\mathcal{F}$ be the failure set for a coherent system and label its elements $\mathcal{F}^* = \{\alpha^{(i)}, i = 1,\ldots,r\}$. For an index set $J \subset \{1,\ldots,r\}$ define $m_J = \text{lcm}\{x^{\alpha^{(j)}}, j \in J\}$. Then the classical inclusion–exclusion lemma corresponds to the Taylor resolution and we can write the generating function, equivalently the Hilbert series, as

$$\mathcal{H}(Id_{\mathcal{F}};x) = \frac{\sum_{j=1}^{r}(-1)^{j-1}\sum_{|J|=j}m_J}{\prod_i(1-x_i)}.$$

Since the minimal resolution is a subresolution of the Taylor resolution, from (4) we can claim that truncated inclusion–exclusion bounds based on minimal free resolutions are at least as good as the truncated inclusion–exclusion bounds, sometimes referred to as generalised Bonferroni–Fréchet bounds, or simply Bonferroni bounds.

It may be that we have repetitions of m_J in the Taylor complex. A simplicial complex similar to the Taylor complex in construction but which is restricted to unique labels ($m_I = m_J \Rightarrow I = J$) is the Scarf complex; see Miller and Sturmfels (2004). If in addition the generators x^α, $\alpha \in \mathcal{F}^*$ are in generic position (no variable x_i appears with the same (non-zero) exponent in two distinct generators) then the Scarf complex gives a minimal free resolution of $Id_{\mathcal{F}}$. In general, a perturbation of this complex is needed, which may yield non-minimal resolutions; see Giglio and Wynn (2004). Any resolution of a monomial ideal corresponding to a coherent system gives an expression for the multigraded Hilbert function that can be truncated to obtain bounds for the reliability of the system. The efficient computation of the minimal free resolution of a monomial ideal is in general a complicated task and different approaches to this problem have been given in the literature. Some authors develop non-minimal resolutions which can be more easily obtained, like the ones given in Taylor (1960) or Lyubeznik (1998). Other authors give minimal resolutions for some classes of ideals, like stable ideals in Eliahou and Kervaire (1990) or generic ideals in Miller and Sturmfels (2004). There are a number of resolutions of other types, like cellular resolutions, resolutions obtained using discrete Morse theory; see Orlik and Welker (2007), etc. This is a very active area of research.

4. Computation of multigraded Betti numbers of monomial ideals

Since we are interested in the multigraded Betti numbers of $Id_{\mathcal{F}}$, we can use methods that compute them without necessarily computing the minimal free resolution. These include simplicial Koszul complexes (see Bayer (1996)) and Mayer–Vietoris trees, introduced in Sáenz-de-Cabezón (2006), which, in addition to the general algebraic uses, appear to be effective for certain kinds of problems in reliability. Both methods make use of the equality between the Betti numbers and the dimension of the Koszul homology modules, which comes from the equivalent ways of computing $Tor_\bullet(\mathbf{k}, I)$ for any ideal $I \subseteq \mathbf{k}[x_1,\ldots,x_n]$ either using resolutions of I or resolutions of $\mathbf{k}$, such as the Koszul complex

$\mathbb{K}(I)$; see for example Sáenz-de-Cabezón (2006). The basic subjacent identity is

$$\beta_{i,\alpha}(I) = \dim_{\mathbf{k}}(Tor_{i,\alpha}(I, \mathbf{k})) = \dim_{\mathbf{k}}(H_{i,\alpha}(\mathbb{K}(I))$$

where $\mathbb{K}(I)$ denotes the Koszul complex of I.

4.1. Simplicial Koszul complexes

Definition 4.1. Let I be a monomial ideal, $\alpha = (a_1, \ldots, a_n) \in \mathbb{N}^n$ such that $x^\alpha \in I$. The *Koszul simplicial complex* of I at α is given by

$$\Delta^I_\alpha = \{\text{squarefree vectors } \tau | \mathbf{x}^{\alpha-\tau} \in I\}.$$

The following result relates the reduced simplicial homology of the Koszul simplicial complex to the multidegree α Betti numbers of I (Bayer, 1996; Miller and Sturmfels, 2004):

Theorem 4.2. *Let I be a monomial ideal and Δ^I_α its Koszul simplicial complex at α; then*

$$\beta_{i,\alpha}(I) = \dim_{\mathbf{k}}(H_{i,\alpha}(\mathbb{K}(I))) = \dim_{\mathbf{k}}(\widetilde{H}_{i-1}(\Delta^I_\alpha)) \quad \forall i.$$

If we call L_I the lcm-lattice of $I = \langle m_1, \ldots, m_r \rangle$, i.e. the lattice with elements labeled by the least common multiples of subsets of $\{m_1, \ldots, m_r\}$ ordered by divisibility, we have that

$$\beta_{i,\alpha}(I) = 0 \quad \text{if } x^\alpha \notin L_I. \tag{5}$$

Therefore, to compute the dimensions of the multigraded Koszul homology modules of I, i.e. the multigraded Betti numbers of I, we need only compute the dimensions of the homology of the simplicial Koszul complexes at the points in L_I, which is a finite set.

Remark 4.3. Simplicial homology is computationally expensive in general, but in some applications it may be a good option for the computation of multigraded Betti numbers. Such an example is given later, namely k-out-of-n systems in Section 5.1.

4.2. Mayer–Vietoris trees

Given a monomial ideal I minimally generated by $\{m_1, \ldots, m_r\}$, we can construct an analogue of the well known Mayer–Vietoris sequence from topology, in the following way:

Definition 4.4. For each $1 \leq s \leq r$ define $I_s := \langle m_1, \ldots m_s \rangle$, $\tilde{I}_s := I_{s-1} \cap \langle m_s \rangle = \langle m_{1,s}, \ldots, m_{s-1,s} \rangle$, where $m_{i,j}$ denotes $\mathrm{lcm}(m_i, m_j)$. Then, for each s we have

$$\cdots \longrightarrow H_{i+1}(\mathbb{K}(I_s)) \xrightarrow{\Delta} H_i(\mathbb{K}(\tilde{I}_s)) \longrightarrow H_i(\mathbb{K}(I_{s-1}) \oplus \mathbb{K}(\langle m_s \rangle)) \longrightarrow H_i(\mathbb{K}(I_s)) \xrightarrow{\Delta} \cdots \tag{6}$$

and since the Koszul differential respects multidegrees, we also have a multigraded version of the sequence. The set of these sequences for each s is called the (recursive) *Mayer–Vietoris sequence* of I.

Using these exact sequences recursively for every $\alpha \in \mathbb{N}^n$ we were able to compute the Koszul homology of $I = \langle m_1, \ldots, m_r \rangle$. The ideals involved can be displayed as a tree, the root of which is I, and every node J has as *children* $\tilde{J}$ on the left and J' on the right (if J is generated by r monomials, $\tilde{J}$ denotes $\tilde{J}_r$ and J' denotes J_{r-1}). This is what we call a *Mayer–Vietoris tree* of the monomial ideal I, and we will denote it as $MVT(I)$. Each node in a Mayer–Vietoris tree is given a *position*: the root has position 1 and the left and right children of the node in position p have, respectively, positions $2p$ and $2p+1$. The node of $MVT(I)$ in position p is denoted as $MVT_p(I)$.

Remark 4.5. Strictly speaking, the definition of Mayer–Vietoris sequences of monomial ideals is not fully precise, in the sense that the Mayer–Vietoris sequence associated with a given ideal is not uniquely defined; it depends on how the minimal generators are sorted. The choice of the last generator of the ideal I to be the one which defines the *Mayer–Vietoris sequence* is just a matter of convenience in notation. The important fact is that we select *some* particular generator to define the sequence. With this one, we associate the subindex s which constitutes the *breaking point* for generating the sequence. Several selection strategies can be applied to select the distinguished generator, and they can be changed during the process.

The properties of Mayer–Vietoris trees allow us to perform Koszul homology computations. The following propositions are proved in Sáenz-de-Cabezón (2006, 2008) together with other features of Mayer–Vietoris trees.

Proposition 4.6. *If $H_{i,\alpha}(\mathbb{K}(I)) \neq 0$ for some i, then x^α is a generator of some node J in any Mayer–Vietoris tree MVT(I).*

Thus, all the multidegrees of Koszul generators (equivalently Betti numbers) of I appear in $MVT(I)$. For a sufficient condition, we need the following notation: among the nodes in $MVT(I)$ we call *relevant nodes* those in an even position or in position 1.

Proposition 4.7. *If x^α appears only once as a generator of a relevant node J in MVT(I) then there exists exactly one generator in $H_*(\mathbb{K}(I))$ which has multidegree α.*

The homological degree to which relevant multidegrees contribute can also be read from their position in the tree (in fact it is given by the number of zeros of the binary expression of the position of the corresponding node).

4.2.1. Mayer–Vietoris ideals

Let I be a monomial ideal and $MVT(I)$ a Mayer–Vietoris tree of I. Let $\alpha \in \mathbb{N}^n$; let $\overline{\beta}_{i,\alpha}(I) = 1$ if α is the multidegree of some non-repeated generator in some relevant node of dimension i in $MVT(I)$ and $\overline{\beta}_{i,\alpha}(I) = 0$ in other case. Let $\widehat{\beta}_{i,\alpha}(I)$ be the number of times α appears as the multidegree of some generator of dimension i in some relevant node in $MVT(I)$. Then for all $\alpha \in \mathbb{N}^n$ we have

$$\overline{\beta}_{i,\alpha}(I) \leq \beta_{i,\alpha}(I) \leq \widehat{\beta}_{i,\alpha}(I).$$

Definition 4.8. Let I be a monomial ideal.

- If there exists a Mayer–Vietoris tree of I such that there is no repeated generator in the ideals of the relevant nodes, then we say that I is a *Mayer–Vietoris ideal of type A*. In this case, $\overline{\beta}_{i,\alpha}(I) = \beta_{i,\alpha}(I) = \widehat{\beta}_{i,\alpha}(I)\ \forall i \in \mathbb{N},\ \alpha \in \mathbb{N}^n$.
- If $\overline{\beta}_{i,\alpha}(I) = \beta_{i,\alpha}(I)$ for all $\alpha \in \mathbb{N}^n$ then we say that I is a *Mayer–Vietoris ideal of type B1*.
- If $\widehat{\beta}_{i,\alpha}(I) = \beta_{i,\alpha}(I)$ for all $\alpha \in \mathbb{N}^n$ then we say that I is a *Mayer–Vietoris ideal of type B2*.

Remark 4.9. It is not hard to show (see Sáenz-de-Cabezón (2006, 2008)) that Mayer–Vietoris trees provide resolutions of the corresponding ideals. Therefore, the alternating sums of the upper bounds $\widehat{\beta}_{i,\alpha}(I)$ of the Betti numbers that are given by these trees provide reliability bounds in the sense exposed above. If the corresponding ideal is Mayer–Vietoris of type A or $B2$ then the resolution given by the Mayer–Vietoris tree is minimal. If it is of type $B1$ and not of type $B2$, the minimal resolution is not directly obtained by the tree (we need to perform further computations to minimize it) but the multigraded Betti numbers are immediately read from the tree, so sharp reliability bounds are also provided. Observe that generic ideals are Mayer–Vietoris of type $B1$. In the other cases, the resolutions obtained by the tree are not minimal in general, but their size is relatively small and therefore the reliability bounds provided by Mayer–Vietoris trees are fairly good on average for general ideals.

5. Special examples in reliability

Classical system reliability deals with two-state or binary systems in which $\mathcal{Y} = \{0, 1\}^d$: every component can fail or not fail. Because in general such systems are not generic, the minimal resolution cannot be derived from the Scarf complex and some kind of algorithm for finding the minimal resolution must be used. In Giglio and Wynn (2004) a special perturbation method was used. A starting point for the present collaboration arose when it transpired that some of the examples in that paper were indeed minimal resolutions and some not. This pointed to systematic application of a minimal free resolution method to reliability. We begin with two classical problems, k-out-of-n and consecutive k-out-of-n systems and then address an important class of problems at the heart of reliability theory, namely series and parallel systems. In these problems our aim is always to derive the multigraded Betti numbers which give the optimal bounds in the sense of (4). The results may be purely computational, for example in some complex case, or may lead to a theoretical result in which the Betti numbers can be given a closed form or be related to the structure of the problem in some way.

5.1. k-out-of-n systems

A k-out-of-n system is one in which if at least k out of a total of n components fail then the system is said to fail. There is a considerable literature in the area within reliability but it may first have arisen in the context of occupancy problems and is covered in the classical text (Feller, 1968–1971) the first edition of which was in 1950 and contains a footnote on M. Fréchet. The formula in Feller (1968–1971) Chapter IV, Section 5, is exactly as derived here by our methods.

A k-out-of-n system can be modeled by the ideal

$$I_{k,n} = \langle x^\mu : x^\mu \text{ is a squarefree monomial of degree } k \text{ in } n \text{ variables}\rangle.$$

For example, $I_{3,5} = \langle xyz, xyu, xyv, xzu, xzv, xuv, yzu, yzv, yuv, zuv\rangle$ is the ideal corresponding to the 3-out-of-5 problem. Observe that $I_{k,n}$ has a minimal generating set which consists of $\binom{n}{k}$ monomials. Using the result pointed in Eq. (5), we know that for any ideal I, we have to check the Koszul homology only in the multidegrees that are in the lcm-lattice of I, namely L_I. It is easy to see that $L_{I_{k,n}}$ consists of all squarefree monomials involving a number of variables between k and n. The following lemma characterizes the Koszul simplicial complex at each of these multidegrees:

Lemma 5.1. *If $x^\alpha \in L_{I_{k,n}}$ has $k+i$ non-zero indices, $k < k+i \leq n$, the simplicial Koszul complex $\Delta_\alpha^{I_{k,n}}$ consists of all possible j-faces with $0 \leq j \leq i-1$ and the empty face.*

Proof. Let x^α be a squarefree monomial consisting of the product of $k+i$ variables, $k < k+i \leq n$. If we divide x^α by the product of $j \geq 0$ of these variables then: If $j \leq i$ then the resulting monomial is the product of a set of $k+i-j$ variables, and thus, a $j-1$ face is present in the Koszul simplicial complex. If $j > i$ then the result of the division is the product of $k+i-j$ variables, with $j > i$, $k+i-j < k$, and thus this product is not in $I_{k,n}$, so no $j-1$ face is in the simplicial Koszul complex for $j > i$. □

Thus, the (α, i)-th Betti number at the multidegree given by any combination of $k+i$ variables is $\dim_{\mathbf{k}}(\tilde{H}_{i-1}(C_{k,i}))$, where $C_{k,i}$ is the subcomplex of the $k+i$-dimensional simplex Δ_{k+i} having as facets all the $(i-1)$-faces. And then, $\beta_i(I_{k,n}) = \binom{n}{k+i} \cdot \dim_{\mathbf{k}}(\tilde{H}_{i-1}(C_{k,i}))$, for all $i \in \{0, \dots, n-k\}$.

Our next goal is then to compute the dimension of the reduced homology of the complexes $C_{k,i}$. Since all faces in dimension less than or equal to $i-1$ are present in the complex, we know that $C_{k,i}$ has zero homology at all dimensions except possibly at dimension $i-1$. The chain complex of $C_{k,i}$ has the following form:

$$0 \to C_{i-1} \overset{\delta_{i-1}}{\to} \cdots \to C_1 \overset{\delta_1}{\to} C_0 \to 0.$$

We have $\tilde{H}_j(C_{k,i}) = 0\,\forall j < i-1$; thus $\ker \delta_j / \operatorname{im} \delta_{j+1} = 0$ and $\dim_{\mathbf{k}}(\ker \delta_j) = \dim_{\mathbf{k}}(\operatorname{im} \delta_{j+1})$ for all $j < i-1$. On the other hand, we have the usual equality

$$\dim_{\mathbf{k}}(\ker \delta_j) = \dim_{\mathbf{k}}(C_j) - \dim_{\mathbf{k}}(\operatorname{im} \delta_j).$$

Putting these together we have that

$$\dim_{\mathbf{k}}(\tilde{H}_{i-1}(C_{k,i})) = \dim_{\mathbf{k}}(\ker \delta_{i-1}) = \binom{k+i}{i} - \binom{k+i}{i-1} + \cdots + (-1)^{i-1}\binom{k+i}{1} + (-1)^i.$$

We can use now the following combinatorial identity:

$$\binom{k+i}{i} - \binom{k+i}{i-1} + \cdots + (-1)^{i-1}\binom{k+i}{1} + (-1)^i = \binom{i+k-1}{k-1}$$

and we obtain that for every $\alpha \in L_I$ where α is the product of $k+i$ variables, we have that

$$\beta_{i,\alpha}(I_{k,n}) = \binom{i+k-1}{k-1}.$$

and since we have $\binom{n}{k+i}$ such α's, it follows that

$$\beta_i(I_{k,n}) = \binom{n}{k+i}\binom{i+k-1}{k-1} \qquad \forall 0 \le i \le n-k.$$

These considerations lead us to the following formula for the *multigraded Hilbert series* of I:

$$\mathcal{H}(I_{k,n}; x) = \frac{\sum_{i=0}^{n-k}(-1)^i\binom{i+k-1}{k-1}\left(\sum_{\alpha\in[n,k+i]} x^{\alpha}\right)}{\prod_i (1-x_i)},$$

where $[n, k+i]$ denotes the set of vectors with 1 in the indices of the $(k+i)$-subsets of $\{1, \dots, n\}$ and 0 in the other entries.

Example 5.2. For $I_{3,5}$ we have

$$\mathcal{H}(I_{3,5}; \mathbf{x}) = \frac{xyz + xyu + xyv + xzu + xzv + xuv + yzu + yzv + yuv + zuv}{(1-x)(1-y)(1-z)(1-u)(1-v)} - \frac{3(xyzu + xyzv + xyuv + xzuv + yzuv)}{(1-x)(1-y)(1-z)(1-u)(1-v)} + \frac{6(xyzuv)}{(1-x)(1-y)(1-z)(1-u)(1-v)},$$

and the Betti numbers of $I_{3,5}$ are then: $\beta_0 = 10$, $\beta_1 = 15$ and $\beta_2 = 6$.

Remark 5.3. k-out-of-n systems constitute an example in which the application of simplicial homology is an optimal way to obtain the multigraded Betti numbers of the corresponding ideal. Also, it is not hard to show that a k-out-of-n ideal is Mayer–Vietoris of type $B2$. Therefore, its Mayer–Vietoris tree provides the minimal resolution.

Remark 5.4. Let us consider now systems of n components in which every component can reach a finite number of states $\{0, 1, \dots, i\}$. Assume that such a system fails whenever the sum of the states of its components reaches a level k. We call such a system an *i-multi-state k-out-of-n* system. The reason for this terminology is that a 1-multi-state k-out-of-n system is just the ordinary k-out-of-n system studied above (considering the value 1 indicates failure).

These systems are modelled by ideals of the form $J^k_{[n,i]}$ minimally generated by all monomials in n variables of degree k such that each variable has an exponent less than or equal to i. It is not difficult to see that all ideals $J^k_{[n,i]}$ are Mayer–Vietoris of type $B2$ and therefore their Mayer–Vietoris resolution is minimal. These ideals have even linear resolutions, as can be seen from an inspection of their Mayer–Vietoris trees.

5.2. Consecutive k-out-of-n systems

Consecutive, also called "sequential", k-out-of-n systems fail whenever at least k consecutive components in an ordered list of n components fail. This is also covered by Feller (1968–1971) Chapter XIII. It is of some interest that (Dohmen, 2003) investigates them using a version of the methods in Naiman and Wynn (1992) and Naiman and Wynn (1997). In addition to a significant literature within reliability, the topic has received renewed interest because of its use in the fast detection of fluctuations in data streams using statistics collected from windows of data: so-called "scan statistics"; see Glaz et al. (2001). In the probability literature the emphasis is in computing probabilities under given distributional assumptions, whereas, as pointed out in Section 1, the bounds that we derive are distribution free.

Consecutive k-out-of-n systems can be modelled by the ideals

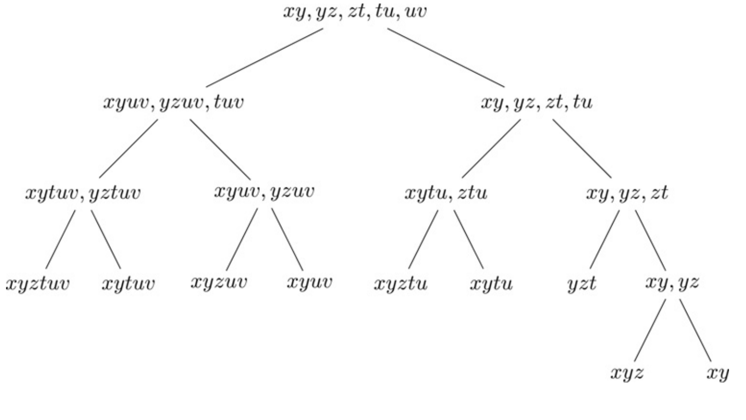
$$\bar{I}_{k,n} = \langle x^{\mu} : \mu \text{ is a squarefree monomial in } n \text{ variables formed by } k \text{ consecutive variables}\rangle.$$

For example, $\bar{I}_{3,5} = \langle xyz, yzu, zuv\rangle$ is the ideal corresponding to the consecutive 3-out-of-5 system. In order to find the multigraded Betti numbers and Hilbert series of $\bar{I}_{k,n}$ we will use its lexicographic Mayer–Vietoris tree. The explicit construction of this tree will give us the results that we need. We will denote the monomials by their subscripts in brackets, e.g. the monomial $x_1x_3x_6$ will be denoted by $[1, 3, 6]$; since we are dealing with squarefree monomials, this notation suffices.

We sort the generators of $\bar{I}_{k,n}$ using the lexicographic order. The construction of $MVT(\bar{I}_{k,n})$ is as follows:

(1) The root node is just $\bar{I}_{k,n}$, which is minimally generated by $n - k + 1$ monomials.
(2) The right child of the root, i.e. $MVT_3(\bar{I}_{k,n})$, is $\bar{I}_{k,n-1}$, so we hang here the corresponding tree.
(3) The left child of the root, $MVT_2(\bar{I}_{k,n})$, consists of the following $n - 2k + 1$ monomials:
  - $[j, \dots (j+k-1), (n-k+1), \dots, n]$ for $1 \le j \le n-2k$ which are the least common multiples of each of the first $n - 2k$ generators of the root with the last one. These generators have $2k$ variables.
  - $[n-k, \dots, n]$ which is the lcm of the last two generators of $MVT_1(\bar{I}_{k,n})$ and divides $[n-k-j, \dots, n]$ for $1 \le j \le (k-1)$ and hence these latter will not appear as minimal generators of this node. This generator has $k + 1$ variables and since we are using lexicographic order, it will appear as the last generator in $MVT_2(\bar{I}_{k,n})$.
(4) The following nodes to consider are $MVT_4(\bar{I}_{k,n})$ and $MVT_5(\bar{I}_{k,n})$, but only if $MVT_2(\bar{I}_{k,n})$ has more than one generator, i.e. if $2k < n$; otherwise they are empty.
  - $MVT_4(\bar{I}_{k,n})$ consists of $n - 2k$ generators, namely the lcms of the first $n - 2k$ generators of $MVT_2(\bar{I}_{k,n})$ with the last one. These have the form $[j, \dots (j+k-1), (n-k), \dots, n]$ for $1 \le j \le n-2k$ and, hence, this node is equivalent to $\bar{I}_{k,n-k-1}$ with each monomial in it multiplied by $[n-k, \dots, n]$. Hence, we hang here a tree 'isomorphic' to $MVT(\bar{I}_{k,n-k-1})$.
  - $MVT_5(\bar{I}_{k,n})$ is completely analogous to $MVT_4(\bar{I}_{k,n})$ and hence equivalent to $\bar{I}_{k,n-k-1}$ but this time each monomial in it is multiplied by $[n-k+1, \dots, n]$. Hence, we also hang here a tree isomorphic to $MVT(\bar{I}_{k,n-k-1})$. The trees that we have hanging from the corresponding nodes are of the same form, except that they have fewer variables; in particular they are of the form $MVT(\bar{I}_{k,j})$ with $j < n$. Eventually, we will have the situation in which $2k \ge n$ and in this case, the left child of the root has only one generator, namely $[j-k, \dots, j]$, and the right node is the consecutive k-out-of-$(j-1)$ tree, so we proceed in this manner until $j = k + 1$.

Example 5.5. Here is the tree corresponding to the consecutive 2-out-of-6 system:



Taking into account the properties of the Mayer–Vietoris trees of these ideals, we see that we can read the multigraded Betti numbers directly from the tree:

Proposition 5.6. *The ideal corresponding to the consecutive k-out-of-n system is Mayer–Vietoris of type A.*

Proof. Assume that we have $\bar{I}_{k,n}$ as the root of our tree, sorted with respect to lexicographic order; then the variable n appears only in the left child of the root, and it will appear in every multidegree of every node in the tree hanging from this node. Thus, no multidegree of the tree hanging from the left child will appear in the tree hanging from the right child, and vice versa. If $2k \geq n$ then we are done, since the left node has just one generator, and the tree hanging from the right node is the one corresponding to the k-out-of-$(n-1)$ system. If the left child of the root has more than one generator, then we look at its children, $MVT_4(\bar{I}_{k,n})$ and $MVT_5(\bar{I}_{k,n})$. The generators of the first one are not present in any node seen so far, and all of them contain the variables $(n-k), \ldots, n$; moreover, every generator of the nodes of the tree hanging from it will have these variables. On the other hand, the variable $n-k$ does not appear in the generators of $MVT_5(\bar{I}_{k,n})$; hence, no multidegree of a generator in the tree hanging from it will appear in the tree hanging from $MVT_4(\bar{I}_{k,n})$ and vice versa. Finally, we see that no multidegree appearing in any relevant node of the tree hanging from $MVT_5(\bar{I}_{k,n})$ is in $MVT_2(\bar{I}_{k,n})$. We know that $MVT_5(\bar{I}_{k,n})$ is generated by the generators of $MVT_2(\bar{I}_{k,n})$ except the last one. Now, every generator of every node in the tree hanging from $MVT_5(\bar{I}_{k,n})$ will have at least $2k+1$ different variables, k of which will be $(n-k+1), \ldots, n$, and on the other hand, the generators in $MVT_2(\bar{I}_{k,n})$ have at most $2k$ different variables. □

With this proposition we have that collecting all the generators of the relevant nodes in $MVT(\bar{I}_{k,n})$ we have the multigraded Betti numbers of $\bar{I}_{k,n}$ in this case; since no generator in the relevant nodes is repeated, we have that the Betti number at each multidegree is 1, and every multidegree appears only once in the minimal resolution of the ideal. The description of the tree and its recursive construction give us also means to count how many multidegrees appear in each dimension (i.e. the Betti numbers) and which multidegrees are present. A thorough description of this process would be tedious, but it is not difficult to obtain a complete list of the multidegrees of the Betti numbers, and hence, of the Hilbert series. However, here we only give an idea of the procedure; an algorithm has been implemented by the authors to generate this list. The main lines of the construction of this list of multidegrees are the following.

(1) In dimension 0 collect all the generators of $\bar{I}_{k,n}$.
(2) In dimension 1 collect all the multidegrees of the form $[j, \ldots, j+k]$ for $1 \leq j \leq (n-k)$.[2] Moreover, for $2k+1 \leq j \leq n$, add the multidegrees $[1, \ldots, k, (j-k+1), \ldots, j], \ldots, [(j-2k), \ldots, (j-k-1), (j-k+1), \ldots, j]$.
(3) For every dimension l add the corresponding multidegrees that appear in $\bar{I}_{k,j-k-1}$ in dimension $(l-2) \geq 0$ multiplied by $[(j-k), \ldots, j]$ and the multidegrees that appear in $\bar{I}_{k,j-k-1}$ in dimension $(l-1) \geq 0$ multiplied by $[(j-k+1), \ldots, j]$ for all $(2k+1) \leq j \leq n$.

Example 5.7. As we can see from the tree of $\bar{I}_{2,6}$, the Betti numbers are $\beta_0 = 5$, $\beta_1 = 7$, $\beta_2 = 4$, $\beta_3 = 1$. The multigraded Hilbert series is

$$\begin{aligned}\mathcal{H}(R/\bar{I}_{2,6}; \mathbf{x}) &= \frac{1-(xy+yz+zt+tu+uv)}{(1-x)(1-y)(1-z)(1-t)(1-u)(1-v)} \\ &\quad + \frac{(xyuv+yzuv+tuv+xytu+ztu+yzt+xyz)}{(1-x)(1-y)(1-z)(1-t)(1-u)(1-v)}\end{aligned}$$

[2] Note that in the case $2k \geq n$ these are the only ones that we have to add, and the corresponding formula is equivalent to the one appearing in Dohmen (2003).

$$- \frac{(xytuv + yztuv + xyzuv + xyztu)}{(1-x)(1-y)(1-z)(1-t)(1-u)(1-v)} + \frac{(xyztuv)}{(1-x)(1-y)(1-z)(1-t)(1-u)(1-v)}.$$

Investigation of the operation of the algorithm leads to recurrence relations for the (standard) Betti numbers. First label $\beta_{i,k,n} = \beta_i(\bar{I}_{k,n})$ and note that $1 \le k \le n$. For $n \le 2k$ we have

$$\begin{aligned} \beta_{0,k,n} &= n - k + 1 \\ \beta_{1,k,n} &= n - k \\ \beta_{i,k,n} &= 0, \quad \text{for } i \ge 2. \end{aligned}$$

For $n \ge 2k + 1$ we have

$$\begin{aligned} \beta_{0,k,n} &= n - k + 1 \\ \beta_{1,k,n} &= n - 2k + 1 + \beta_{1,k,n-1} \\ \beta_{i,k,n} &= \beta_{i-2,k,n-k-1} + \beta_{i-1,k,n-k-1} + \beta_{i,k,n-1}, \quad \text{for } i \ge 2. \end{aligned}$$

Using standard methods we obtain the double generating function

$$G_k(x, y) = \sum_{i=0}^{\infty} \sum_{n=k}^{\infty} \beta_{i,k,n} x^i y^n = \frac{y^k(1 + xy)}{(1 - y)(1 - x^2y^{k+1} - xy^{k+1} - y)}.$$

We first fix k. Then, briefly, the large bracket in the denominator derives from the general form of the generating function. The other terms derive from the boundary conditions which operate for small i and n. We can confirm that $\sum_{i=0}^{\infty}(-1)^i \beta_{i,k,n} = 1$ by considering $G_k(-1, y)$ and we obtain a generating function for $\sum_i \beta_{i,k,n}$:

$$G_k(1, y) = \frac{y^k(1 + y)}{(1 - y)(1 - 2y^{k+1} - y)} \quad 1 \le k \le n.$$

This analysis points to the possibility of obtaining the generating function for the multigraded Betti numbers, or equivalently a closed form for the Hilbert series itself. In further research we will develop probabilistic statements and asymptotic results in particular as $n \to \infty$.

5.3. Series and parallel systems

We turn now to the series–parallel system, a special although very natural type of network. Consider an edge p joining two nodes I and O. We call such a network a *basic* series–parallel network. Consider now two series–parallel networks N_1 and N_2. We can connect them in series or in parallel, and the result is a series–parallel network. This is done in the following way:

- First, we rename the edges in each node so that each edge has a different label. If the edge p_S for some set S of subindices is in network i we can rename it $p_{\{i\}\cup S}$. After this, we can still rename them just by counting them in lexicographic order.
- If the initial (input) node of N_i is labelled I_i and its final (output) node is labelled O_i for $i = 1, 2$, then the parallel union of N_1 and N_2, which we will denote as $N = N_1 \times N_2$, identifies I_1 and I_2 in one node I, which will be the initial node of N, and identifies O_1 and O_2 in one node O, which will be its final node.
- With the same notation as above, the series union of N_1 and N_2, which we will denote as $N = N_1 + N_2$, has as initial node I_1, as final node O_2, and identifies O_1 and I_2 in one intermediate node S.

We just formalize these considerations in the following definition of *series–parallel networks:*

Definition 5.8. We say that a network N is a *parallel–series network* if either N consists of an input node, an output node and a edge joining them, or if $N = N_1 + N_2$ or $N = N_1 \times N_2$ with N_1, N_2 series–parallel networks.

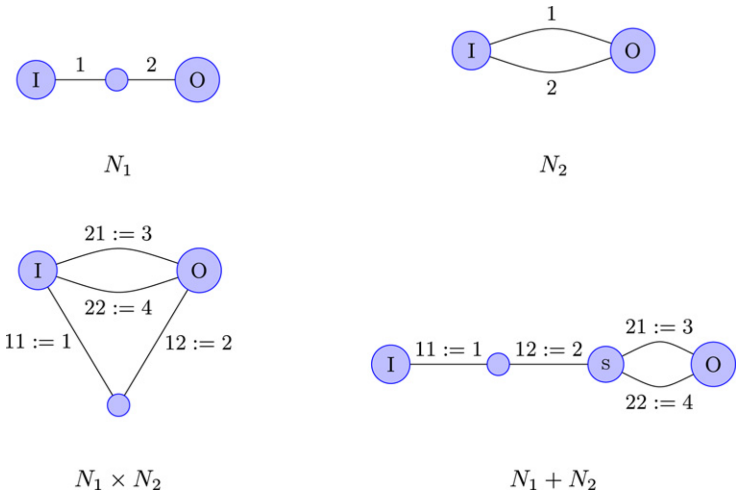


Fig. 1. Example of series–parallel network construction.

These constructions can be seen in Fig. 1, in which the label of the edge p_S is just S.

Given any network N (not necessary a series–parallel one), we associate a monomial ideal with it as follows: Consider one variable x_S for each connection S in N. Then, the monomial ideal I_N associated with N is minimally generated by the monomials $x_{S_1} \cdots x_{S_k}$ where $S_1, \dots, S_k$ is a minimal cut in the network N (Dohmen, 2003; Giglio and Wynn, 2004). Let us consider now the ideals associated with series–parallel networks. It is clear that the ideal I_N of a network N with just one edge p_1 connecting two nodes I and O is just $I_N = \langle x_1 \rangle$. The construction operations $+$ and $\times$ that we have just seen have their counterpart in the ideals of the resulting networks:

Proposition 5.9. *Let N_1 and N_2 be two networks the edges of which are labelled (after renaming as seen above) $p_1, \dots, p_{n_1}$ and $p_{n_1+1}, \dots, p_{n_1+n_2}$. Then,*

$$I_{N_1+N_2} = I_{N_1} + I_{N_2} \qquad I_{N_1 \times N_2} = I_{N_1} \cap I_{N_2}$$

where $I_{N_1+N_2}$ and $I_{N_1 \times N_2}$ are ideals in $\mathbf{k}[x_1, \dots, x_{n_1+n_2}]$.

Proof. We have that for any network N,

$$I_N = \langle x_S | S = \{s_1, \dots, s_{k_s}\} \text{ is a minimal cut in N} \rangle.$$

Any minimal cut in N_1 or N_2 is a minimal cut in $N_1 + N_2$, and there is no mixture among them. Then it is clear that the generating set of $I_{N_1+N_2}$ is just the union of the generating sets of I_{N_1} and I_{N_2}, each being generated in a different set of variables.

Now, the minimal cuts of $N_1 \times N_2$ can be always considered as a combination of one minimal cut in N_1 and one minimal cut in N_2 and there are no other minimal paths. Since there is no intersection between the set of variables of I_{N_1} and I_{N_2} the combination of minimal cuts simply means product of their variables, and hence the result. □

Example 5.10. Consider the networks in Fig. 1, where := expresses relabeling. After relabeling, the edges in N_1 are p_1 and p_2, and the edges in N_2 are p_3 and p_4. We have that

$$I_{N_1} = \langle x_1, x_2 \rangle, \qquad I_{N_2} = \langle x_3 x_4 \rangle, \qquad I_{N_1+N_2} = \langle x_1, x_2, x_3 x_4 \rangle, \qquad I_{N_1 \times N_2} = \langle x_1 x_3 x_4, x_2 x_3 x_4 \rangle.$$

Mayer–Vietoris trees give a good way to compute the multigraded Betti numbers of series–parallel ideals, and hence, the reliability of the corresponding network:

Proposition 5.11. *The ideals associated with series–parallel networks, i.e.* series–parallel ideals, *are Mayer–Vietoris ideals of type A.*

Proof. If N is a basic series–parallel network with unique edge p_1 then $I_N = \langle x_1 \rangle$ which is Mayer–Vietoris of type A. Now consider two series–parallel networks N_1 and N_2 whose ideals are Mayer–Vietoris of type A, i.e. there is some strategy for selecting the pivot monomials when constructing a Mayer–Vietoris tree such that it is of type A. We have to prove that $I_{N_1} + I_{N_2}$ and $I_{N_1} \cap I_{N_2}$ are Mayer–Vietoris of type A:

- The generators of $I_{N_1} + I_{N_2}$ are the union of the generating sets of I_{N_1} and I_{N_2}. We sort them so that the generators of I_{N_2} all appear after the generators of I_{N_1}. We now proceed, taking as pivot monomial always a generator of I_{N_2} following the strategy used to build the minimal Mayer–Vietoris tree of I_{N_2}. Doing so, we have that $MVT_p(I_{N_1} + I_{N_2})$ has as generators the generators of I_{N_1}, each one multiplied by some product of the variables of I_{N_2} and also the generators of $MVT_p(I_{N_2})$. So far, we have no repeated generators in the relevant nodes: Assume that there is some generator repeated in two relevant nodes at positions p and q; then they have the same exponents in the variables of I_{N_1} and the same in the variables of I_{N_2}. If the generator has only variables of the second ideal, being equal would mean that they are equal in $MVT(I_{N_2})$. Since those generators with 'mixed variables' are all of the form $m \cdot m'$ with m a minimal generator of I_{N_1}, no two of these are repeated.

  This procedure takes us to nodes in which no further element only in the variables of I_{N_2} is available. From this moment, on each node we follow the strategy of $MVT(I_{N_1})$, since these nodes in positions p have as generators all the minimal generators of I_{N-1} times some polynomial m'_p in the variables of the second ideal. Since the m'_p are different for different p, we have that all the trees hanging from these nodes are isomorphic to $MVT(I_{N_1})$; therefore, there is no repeated generator in the relevant nodes in each of them. There is also no repetition among the different 'copies' of $MVT(I_{N_1})$ because each $m'{}_p$ is unique.
- $I_{N_1 \times N_2}$: Let us denote by $m_1, \ldots, m_r$ the generators of I_{N_1}, and by $n_1, \ldots, n_s$ the generators of I_{N_2}. Assume without loss of generality that the minimal Mayer–Vietoris trees of I_{N_1} and I_{N_2} were obtained using always the last generator as the pivot monomial. Then $I_{N_1 \times N_2} = I_{N_1} \cap I_{N_2}$ is generated by $\{m_i n_j | i = 1, \ldots, r; j = 1, \ldots, s\}$. To build $MVT(I_{N_1 \times N_2})$ consider as pivot monomial $m_r n_s$; then, $MVT_2(I_{N_1 \times N_2})$ is generated by

  $$m_r n_1 n_s, \ldots, m_r n_{s-1} n_s, m_1 m_r n_s, \ldots, m_{r-1} m_r n_s.$$

  Now, select $m_{r-1} n_s$ as pivot monomial in $MVT_3(I_{N_1 \times N_2})$ and we obtain that $MVT_6(I_{N_1 \times N_2})$ is generated by

  $$m_{r-1} n_1 n_s, \ldots, m_{r-1} n_{s-1} n_s, m_1 m_{r-1} n_s, \ldots, m_{r-2} m_{r-1} n_s.$$

  It is clear that since I_{N_1} and I_{N_2} are Mayer–Vietoris of type A and they are generated in disjoint sets of variables, there is no repeated relevant multidegree inside the subtree hanging from $MVT_2(I_{N_1 \times N_2})$ or inside the subtree hanging from $MVT_6(I_{N_1 \times N_2})$.

  On the other hand we have that no multidegree appears in a relevant node in both trees. The reason is the following: Every multidegree in $MVT_2(I_{N_1 \times N_2})$ is of the form $n_\sigma m_\alpha$ with $s \in \sigma$ and $r \in \alpha$; and every element in $MVT_6(I_{N_1 \times N_2})$ is of the form $n_{\sigma'} m_{\alpha'}$ with $s \in \sigma', \alpha' \subseteq \{1, \ldots, r-1\}$. If $n_\sigma m_\alpha = n_{\sigma'} m_{\alpha'}$ then in particular $m_\alpha = m_{\alpha'}$, but then they would be repeated in $MVT(I_{N_1})$, which is a contradiction.

  Following the same argument while taking elements of the form $m_i n_s$ as pivot monomials in the nodes of dimension 0 we obtain the same contradiction, based on the fact that $MVT(I_{N_1})$ has no repeated relevant multidegrees. Then we turn to taking pivot monomials of the form $m_i n_{s-1}$ ($i \in \{1, \ldots, r\}$) and we can follow a symmetric argument, the contradiction coming now from the fact that $MVT(I_{N_2})$ has no repeated relevant multidegrees. A recursive application of these two arguments yields the result.

So, both $I_{N_1+N_2}$ and $I_{N_1 \times N_2}$ are Mayer–Vietoris of type A. □

6. Conclusions

It has been a long standing challenge to obtain improved bounds of Bonferroni type in system reliability, with many different types of improvement being suggested. We have shown that, among a class of bounds of resolution type, which include the classical case (equivalent to the Taylor resolution), the minimal free resolution is optimal and moreover this resolution is completely described by the multigraded Betti numbers. The computation of these numbers is usually done via minimal free resolutions, but these are in general hard to compute. In certain important classes of systems, alternative methods, such as the one proposed by the first author, can be used to obtain the multigraded Betti numbers in a more efficient way. On one hand, these alternative methods should be used for such situations, and on the other hand, algebraic techniques can be used in many cases to improve the bounds given in the literature on coherent systems.

We have studied three types of system: two rather special and one, the series–parallel systems, which is rather more general. But there are many other systems of which a leading example is given by a general network and gives rise to the area of network reliability. Immediate questions are: What are the multigraded Betti numbers for a general network and are there fast algorithms which use the network structure?

An advantage of the current methods is that they apply naturally to the multi-state coherent system cases which are less thoroughly covered in the reliability literature. Indeed, the key connection is to code a state by the exponent of a monomial ideal. A big challenge from both the viewpoints of algebra and reliability is to generalise the notion of coherency. This would require "geometries" different from that of unions of upper orthants to be included. Other geometries were used in the original work on discrete tubes (Naiman and Wynn (1992), Naiman and Wynn (1997)), and include unions of balls or half-spaces.

The connection of the present work with that in Dohmen (2003) needs to be studied. In addition to his application of discrete tube theory to reliability, that author makes interesting links with other areas of combinatorics such as lace expansions, chromatic numbers and the Whitney broken circuit theorem. It is likely that minimal free resolutions and multigraded Betti numbers will be found to play a role in those theories also.

As pointed out, the bounds given here are distribution free: they are independent of the distribution of the random variable Y defining the (stochastic) system. But where the distribution takes a particular form, e.g. independent failure of components or, say, a Markov chain, it is to be hoped that there is synergy between the minimal bounds given here and the distributions. This may lead to useful formulae for failure probabilities in particular cases. In statistics and probability there is interest in extreme events, for example for testing some kinds of simple null hypotheses, such as independence. We have good prospects in the case of (consecutive) k-out-of-n of finding new results for so-called scan statistics. Our bounds may contribute to an asymptotic theory as the failure set is pushed outwards, so that the first few terms of the bounds give simple formulae. To ask the question bluntly: Do multigraded Betti numbers play a part in certain large deviation theories?

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