The surprising almost everywhere convergence of Fourier-Neumann series*

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Abstract

For most orthogonal systems and their corresponding Fourier series, the study of the almost everywhere convergence for functions in L^p requires very complicated research, harder than in the case of the mean convergence. For instance, for trigonometric series, the almost everywhere convergence for functions in L^2 is the celebrated Carleson theorem, proved in 1966 (and extended to L^p by Hunt in 1967).

In this paper, we take the system

$$j_n^\alpha(x) = \sqrt{2(\alpha+2n+1)}\, J_{\alpha+2n+1}(x) x^{-\alpha-1}, \qquad n = 0, 1, 2, \dots$$

(with J_μ being the Bessel function of the first kind and of the order μ), which is orthonormal in $L^2((0,\infty), x^{2\alpha+1}\,dx)$, and whose Fourier series are the so-called Fourier-Neumann series. We study the almost everywhere convergence of Fourier-Neumann series for functions in $L^p((0,\infty), x^{2\alpha+1}\,dx)$ and we show that, surprisingly, the proof is relatively simple (inasmuch as the mean convergence has already been established).

Keywords: Bessel functions, Fourier-Neumann series, almost everywhere convergence.

1 Introduction and main theorem

Let J_μ denote the Bessel function of the first kind and of the order μ. It is well-known that, for $\alpha \ge -1/2$, we have

$$\int_0^\infty J_{\alpha+2n+1}(x) J_{\alpha+2m+1}(x)\,\frac{dx}{x} = \frac{\delta_{nm}}{2(2n+\alpha+1)}, \quad n, m = 0, 1, 2, \dots$$

(see [1, Ch. XIII, 13.41 (7), p. 404] and [1, Ch. XIII, 13.42 (1), p. 405]). Then, the system

$$j_n^\alpha(x) = \sqrt{2(\alpha+2n+1)}\, J_{\alpha+2n+1}(x) x^{-\alpha-1}, \qquad n = 0, 1, 2, \dots$$

is orthonormal in $L^2((0,\infty), d\mu_\alpha)$ ($L^2(d\mu_\alpha)$ from now on), with $d\mu_\alpha(x) = x^{2\alpha+1}\,dx$. For each suitable function f, take

$$S_n^\alpha f(x) = \sum_{k=0}^n c_k^\alpha(f) j_k^\alpha(x), \quad c_k^\alpha(f) = \int_0^\infty f(y) j_k^\alpha(y)\,d\mu_\alpha(y),$$

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so $S_n^\alpha f$ denotes the n-th partial sum of its Fourier series with respect to the system $\{j_n^\alpha\}_{n=0}^\infty$, which is usually called the Fourier-Neumann series.

Most orthogonal systems (the trigonometric system, Jacobi, Hermite and Laguerre orthogonal polynomials and functions, etc.) are complete in their corresponding L^2 space. However, this does not happen with $\{j_n^\alpha\}_{n=0}^\infty$. To see this, let us consider the so-called modified Hankel transform H_α, that is

$$H_\alpha f(x) = \int_0^\infty \frac{J_\alpha(xy)}{(xy)^\alpha} f(y) y^{2\alpha+1}\, dy, \quad x > 0, \tag{1}$$

(defined for suitable functions). In the usual way, H_α can be extended to functions $f \in L^2(d\mu_\alpha)$: it becomes an isometry on $L^2(d\mu_\alpha)$, where H_α^2 is the identity. Moreover, the Bessel functions and the Jacobi polynomials are related by means of

$$\int_0^\infty J_{\alpha+2n+1}(t) J_\alpha(xt)\, dt = x^\alpha P_n^{(\alpha,0)}(1-2x^2)\chi_{[0,1]}(x); \tag{2}$$

see, for instance, [2, Ch. 8.11, (5), p. 47]. From this formula, $H_\alpha j_n^\alpha$ is supported on $[0,1]$, so consequently $\{j_n^\alpha\}_{n=0}^\infty$ is not complete in $L^2(d\mu_\alpha)$. Furthermore, (2) allows expressing $H_\alpha j_n^\alpha$ in terms of the Jacobi polynomials (which are a complete system). Then, by using that H_α is an isometry on $L^2(d\mu_\alpha)$, the subspace $B_{2,\alpha} = \overline{\operatorname{span}\{j_n^\alpha\}_{n=0}^\infty}$ (closure in $L^2(d\mu_\alpha)$) can be identified with $E_{2,\alpha} = \{f \in L^2(d\mu_\alpha) : M_\alpha f = f\}$, where M_α is the multiplier defined by

$$H_\alpha(M_\alpha f) = \chi_{[0,1]} H_\alpha f.$$

Let us point out that, as $H_\alpha j_n^\alpha$ is supported on $[0,1]$, we have $M_\alpha j_n^\alpha = j_n^\alpha$ so $j_n^\alpha \in E_{2,\alpha}$ indeed.

The $L^p(d\mu_\alpha)$-mean convergence of the Fourier-Neumann series was studied by one of the authors in [3], and later extended in [4]. An important part of these papers is devoted to identifying $B_{p,\alpha} = \overline{\operatorname{span}\{j_n^\alpha\}_{n=0}^\infty}$ (closure in $L^p(d\mu_\alpha)$). When $p \neq 2$, H_α can be defined on $L^p(d\mu_\alpha)$ under some circumstances, but it is not an isometry, and then the relation (2) can no longer be used. However, for a certain range of p's (summed up in (4)), the extension of the multiplier M_α as a bounded operator from $L^p(d\mu_\alpha)$ into itself can be done in the usual way, by using suitable bounds of the Bessel functions. The operator M_α has several interesting properties, such as $M_\alpha^2 f = M_\alpha f$ and

$$\int_0^\infty f(y) M_\alpha g(y)\, d\mu_\alpha(y) = \int_0^\infty M_\alpha f(y) g(y)\, d\mu_\alpha(y), \tag{3}$$

which is valid whenever $f \in L^p(d\mu_\alpha)$ and $g \in L^{p'}(d\mu_\alpha)$ (with $1/p + 1/p' = 1$). Moreover, the space $E_{p,\alpha} = \{f \in L^p(d\mu_\alpha) : M_\alpha f = f\}$ can be defined, and some of its properties proved: $E_{s,\alpha} \subseteq E_{r,\alpha}$ when $s < r$ (the inclusion being continuous and dense), the duality $E'_{p,\alpha} = E_{p',\alpha}$ and, finally, $B_{p,\alpha} = E_{p,\alpha}$. The details can be found in [3, 5].

The goal of this paper is to analyze the almost everywhere convergence of the Fourier-Neumann series $S_n^\alpha f$ for functions $f \in L^p(d\mu_\alpha)$, with $\alpha \geq -1/2$. A partial study is done in [4], but now we are going to extend it, showing a further result.

For most orthogonal systems and their corresponding Fourier series, the study of the almost everywhere convergence for functions in L^p is rather complicated, much more than that of the mean convergence. For instance, for trigonometric series, the almost everywhere convergence for functions in L^2 was conjectured by Lusin in 1915, and proved by Carleson in 1966 (extended by Hunt in 1967 for L^p with $1 < p < \infty$). However, we are going to see that the proofs of the almost everywhere convergence of the Fourier-Neumann series for functions in $L^p(d\mu_\alpha)$ are, surprisingly, relatively simple (supposing that the mean convergence has been previously established).

We will restrict our analysis to the "natural" interval $p \in (p_0(\alpha), p_1(\alpha))$ given by

$$p_0(\alpha) = \frac{4(\alpha+1)}{2\alpha+3} < p < \frac{4(\alpha+1)}{2\alpha+1} = p_1(\alpha). \tag{4}$$

In fact, the first requirement for having the partial sum of the Fourier-Neumann series in $L^p(d\mu_\alpha)$ (with $1 < p < \infty$) is that $j_n^\alpha \in L^p(d\mu_\alpha)$ for every n. By using well-known estimates for the Bessel functions (namely (6) and (7)), this is equivalent to $p > p_0(\alpha)$. And, since the Fourier coefficients $c_n^\alpha(f)$ must exist for every $f \in L^p(d\mu_\alpha)$, we must have $j_n^\alpha \in L^{p'}(d\mu_\alpha)$ for every n, where $1/p + 1/p' = 1$. This is equivalent to $p < p_1(\alpha)$.

As we can see in [3], the range (4) is just the one for which the multiplier $M_\alpha : L^p(d\mu_\alpha) \to L^p(d\mu_\alpha)$ is a bounded operator, which allows defining the subspaces $E_{p,\alpha}$. However, the interval of mean convergence of the Fourier-Neumann series is not all of the range (4), but

$$\max\left\{\frac{4}{3}, p_0(\alpha)\right\} < p < \min\left\{4, p_1(\alpha)\right\}. \tag{5}$$

That is, if p satisfies (5) and $f \in E_{p,\alpha}$, then $S_n^\alpha f \to f$ when $n \to \infty$ in the $L^p(d\mu_\alpha)$-norm. Conversely, (5) is a necessary condition for the mean convergence. When $f \in L^p(d\mu_\alpha)$, the Fourier-Neumann series converges to $M_\alpha f$ (recall that $M_\alpha f = f$ for functions in $E_{p,\alpha}$).

In [4] we prove that, if p satisfies (5), $S_n^\alpha f \to f$ almost everywhere for every $f \in E_{p,\alpha}$ (and the convergence is to $M_\alpha f$ for $f \in L^p(d\mu_\alpha)$). Here, we extend this result by removing the condition $4/3 < p < 4$ (which affects the case $-1/2 \leq \alpha < 0$) for the almost everywhere convergence. Thus, we have

Theorem. *Let us have $\alpha \geq -1/2$ and p satisfying* (4)*. Then,*

$$S_n^\alpha f \to M_\alpha f$$

almost everywhere for every $f \in L^p(d\mu_\alpha)$.

2 Auxiliary results

The Bessel functions satisfy the asymptotic formulas (see, for instance, [1, Ch. III, 3.1 (8), p. 40] and [1, Ch. VII, 7.21 (1), p. 199])

$$J_\nu(x) = \frac{x^\nu}{2^\nu \Gamma(\nu+1)} + O(x^{\nu+2}), \qquad x \to 0^+, \tag{6}$$

$$J_\nu(x) = \left(\frac{2}{\pi x}\right)^{1/2} \left[\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O(x^{-1})\right], \qquad x \to \infty. \tag{7}$$

We will also use bounds with constants independent of the parameter ν of the Bessel function. These bounds are a consequence of the very precise estimates that appear in [6]. To be more precise, we will use the following bound that can be found in [4, 3]:

$$|J_\nu(x)| \leq C x^{-1/4} \left(|x-\nu| + \nu^{1/3}\right)^{-1/4}, \quad x \in (0, \infty), \tag{8}$$

where C is a positive constant independent of ν.

With this information, let us estimate $\|j_n^\alpha\|_{L^p(d\mu_\alpha)}$:

Lemma 1. *Let $\alpha \geq -1/2$ and $p > p_0(\alpha)$. Then, $\{j_n^\alpha\}_{n=0}^\infty \subseteq L^p(d\mu_\alpha)$ and*

$$\|j_n^\alpha\|_{L^p(d\mu_\alpha)} \leq C \begin{cases} n^{-(\alpha+1)+2(\alpha+1)/p}, & \text{if } p < 4, \\ n^{-(\alpha+1)/2}(\log n)^{1/4}, & \text{if } p = 4, \\ n^{-(5/6+\alpha)+(6\alpha+4)/(3p)}, & \text{if } p > 4, \end{cases}$$

with C a positive constant independent of n.

Proof. The assertion that $j_n^\alpha \in L^p(d\mu_\alpha)$ for every $n = 0, 1, 2, \dots$ follows from (6) and (7). Then, estimates (8) show that $\|j_n^\alpha\|_{L^p(d\mu_\alpha)}$ is bounded above by a constant times the right hand side. For a similar expression, see [7]. $\square$

Now, let us note that, for $x \in (0,\infty)$ fixed, the Bessel function $|J_\nu(x)|$ has a huge decay when ν grows to ∞ (and consequently the same happens with $|j_n^\alpha(x)|$ when $n \to \infty$). In fact, according to [1, Ch. III, 3.31 (1), p. 49], we have

$$|J_\nu(x)| \le \frac{2^{-\nu} x^\nu}{\Gamma(\nu+1)}, \qquad \nu \ge -1/2. \tag{9}$$

Then, we have

Lemma 2. *Let $\alpha \ge -1/2$ and p with $1 < p < p_1(\alpha)$. Then, for any $f \in L^p(d\mu_\alpha)$ the Fourier series $\sum_{n=0}^\infty c_n^\alpha(f) j_n^\alpha(x)$ converges absolutely for every $x \in (0,\infty)$. (Note that we do not assert that this convergence is to $f(x)$, not even almost everywhere.)*

Proof. Recall that

$$c_n^\alpha(f) = \int_0^\infty f(y) j_n^\alpha(y) y^{2\alpha+1}\, dy. \tag{10}$$

Since $p < p_1(\alpha)$, it follows that $p' > p_0(\alpha)$ (with $1/p + 1/p' = 1$). Then, from Lemma 1, we have $j_n^\alpha \in L^{p'}(d\mu_\alpha)$ and, moreover, $\|j_n^\alpha\|_{L^{p'}(d\mu_\alpha)} \le C n^\delta$ for some constant $\delta = \delta(p,\alpha)$. Thus, by Hölder's inequality,

$$|c_n^\alpha(f)| \le \|f\|_{L^p(d\mu_\alpha)} \|j_n^\alpha\|_{L^{p'}(d\mu_\alpha)} \le C \|f\|_{L^p(d\mu_\alpha)}\, n^\delta.$$

On the other hand, as a consequence of (9), we have

$$\begin{aligned} |j_n^\alpha(x)| &= \sqrt{2(\alpha+2n+1)}\, |J_{\alpha+2n+1}(x)|\, x^{-\alpha-1} \\ &\le \frac{\sqrt{2(\alpha+2n+1)}\, 2^{-(\alpha+2n+1)} x^{2n}}{\Gamma(\alpha+2n+2)}. \end{aligned}$$

Therefore,

$$|c_n^\alpha(f) j_n^\alpha(x)| \le C \|f\|_{L^p(d\mu_\alpha)} \frac{n^{\delta+1/2} (x/2)^{2n}}{\Gamma(\alpha+2n+2)} \tag{11}$$

and the series $\sum_{n=0}^\infty c_n^\alpha(f) j_n^\alpha(x)$ converges absolutely. $\square$

3 Proof of the theorem

By Lemma 2, $S_n^\alpha f$ converges to some g pointwise when p satisfies (4). We want to prove that, almost everywhere, $g = f$ if $f \in E_{p,\alpha}$, or, more generally, $g = M_\alpha f$ if $f \in L^p(d\mu_\alpha)$.

As established in the introduction, we know that, when p satisfies (5), $S_n^\alpha f$ converges to $M_\alpha f$ in the $L^p(d\mu_\alpha)$-norm; then, $S_n^\alpha f$ has a subsequence that converges to $M_\alpha f$ almost everywhere. Consequently $g = M_\alpha f$ and the convergence $S_n^\alpha f \to M_\alpha f$ almost everywhere is proved under the hypothesis (5).

This is the argument used in [4]. Let us see how to remove the condition $4/3 < p < 4$. For that, we are going to apply the summation process used in [5]. Thus, let us take

$$R_n^\alpha f = \frac{\lambda_0 S_0^\alpha f + \cdots + \lambda_n S_n^\alpha f}{\lambda_0 + \cdots + \lambda_n}$$

with $\lambda_k = 2(\alpha + 2k + 2)$. Actually, as established in that paper, this method is equivalent to the one given by the Cesàro means of order 1, but the kernels that appear with R_n^α are easier to handle, and consequently the use of R_n^α is more convenient for studying the uniform boundedness of the operators involved (and hence the mean convergence).

In [5] it is proved that, when p satisfies (4) (i.e., without $4/3 < p < 4$), $B_{p,\alpha} = E_{p,\alpha}$ and $R_n^\alpha f \to f$ in the $L^p(d\mu_\alpha)$-norm for every $f \in E_{p,\alpha}$. For general $f \in L^p(d\mu_\alpha)$, we always have $M_\alpha f \in E_{p,\alpha}$ (because $M_\alpha^2 f = M_\alpha f$). Moreover, by using (3) and $M_\alpha j_k^\alpha = j_k^\alpha$, it follows that $c_k^\alpha(f) = c_k^\alpha(M_\alpha f)$ for every k, and so $R_n^\alpha f = R_n^\alpha(M_\alpha f)$. As $R_n^\alpha(M_\alpha f)$ converges in mean to $M_\alpha f$, also $R_n^\alpha f$ converges in mean to $M_\alpha f$. Then, there exists a subsequence of $R_n^\alpha f$ that converges almost everywhere to $M_\alpha f$.

On the other hand, R_n^α is a regular summation process. Then, by Lemma 2, given $f \in L^p(d\mu_\alpha)$ with p satisfying (4), we have that $R_n^\alpha f(x)$ converges for every $x \in (0, \infty)$ to the same function $g(x)$ that is the pointwise limit of $S_n^\alpha f(x)$.

Thus, for p satisfying (4), we have: a subsequence of $R_n^\alpha f$ converges almost everywhere to $M_\alpha f$, $S_n^\alpha f$ converges almost everywhere to g, and $R_n^\alpha f$ converges almost everywhere to g. From these facts, $g = M_\alpha f$ and the theorem is proved.

Open question. What happens for $S_n^\alpha f$ with $f \in L^p(d\mu_\alpha)$ and $1 < p \le p_0(\alpha)$? Under these conditions, every $c_k^\alpha(f)$ exists, so the partial sums $S_n^\alpha f$ are well defined. Futhermore, Lemma 2 ensures that, pointwise, $S_n^\alpha f(x)$ converges to some function $g(x)$. But, what is $g(x)$? Let us note that, when $1 < p \le p_0(\alpha)$, the bounded multiplier $M_\alpha : L^p(d\mu_\alpha) \to L^p(d\mu_\alpha)$ no longer exists.

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