Convergence of Padé approximants of Stieltjes-type meromorphic functions and relative asymptotics for orthogonal polynomials on the real line

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Abstract

We obtain results on convergence of Padé approximants of Stieltjes-type meromorphic functions and relative asymptotics for orthogonal polynomials on unbounded intervals. These theorems extend other results of Guillermo López changing the Carleman condition in his theorems by the determination of the corresponding moment problem. Our technique allows us to stretch other results obtained by López.

Key words: Padé approximants, moment problem, orthogonal polynomials, varying measures, asymptotics
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1. Introduction and notations

Two of the most striking papers of Guillermo López have been [11] and [12]. In the first, he solved a conjecture posed by A. A. Gonchar 10 years earlier about the convergence of Padé approximants of Stieltjes-type meromorphic functions. Gonchar [8] proved the convergence of Padé approximants to Markov-type meromorphic function whose measure α is supported on a bounded interval of the real line, and $\alpha' > 0$ a. e. in this interval. Later on, Rakhmanov [16] showed that the convergence does not hold for arbitrary positive measure on $\mathbb{R}$. In [11] López gave a very general sufficient condition to get convergence of Padé approximants for Stieltjes-type meromorphic functions (the measure can have unbounded support on $\mathbb{R}$). The main idea of López is to reduce the problem to study orthogonal polynomial on the unit circle with respect to varying measures.

In [12] López showed that orthogonal polynomials with respect to varying measures are an effective tool not only for solving problems on rational approximation but also for studying questions on orthogonal polynomials involving fixed measures and observed that orthogonal polynomials with respect to varying measures on the unit circle provide a unified approach to the study of orthogonal polynomials on bounded and unbounded intervals. There he obtains relative asymptotics for orthogonal polynomials on unbounded intervals.

In this paper we extend the results of López in [11] and [12]; here we change the Carleman condition on the moments of the measure by the corresponding moment problem is determinate. This hypothesis carries a thorough analysis of the method developed by López in [11] and [12]. Our main ideas are the use of rational approximation in the unit circle and the relation between determination of moment problem and one side approximation.

Let $\widehat{\alpha}$ denote the Cauchy-Stieltjes transform of α

$$\widehat{\alpha}(z) = \int \frac{1}{z-x}\, d\alpha(x), \quad z \in \mathcal{D} = \mathbb{C} \setminus [0, +\infty),$$

where α is a positive Borel measure on $[0, \infty)$ with finite moments, $\int x^k \, d\alpha(x) < \infty$, $k = 0, 1, \ldots$ By $\mathcal{M}_0$ we denote the class of positive Borel measure on $[0, \infty)$ with finite moments such that the Stieltjes moment is determinate. Let r be a rational function whose poles lie on $\mathcal{D}$ and $r(\infty) = 0$. Let

$$f(z) = \widehat{\alpha}(z) + r(z), \quad z \in \mathcal{D}. \tag{1}$$

Given $n \in \mathbb{Z}_+$, the Padé approximant, $\pi_n(z) = \frac{p_n(z)}{q_n(z)}$, of order n at infinite of f

satisfies:

- p_n and q_n are polynomials with $\deg(p_n) \leq n$, $\deg(q_n) \leq n$, $q_n \not\equiv 0$.
- $q_n(z)f(z) - p_n(z) = \sum_{j=n+1}^{\infty} A_{n,j}/z^j$.

The difficulty of the study of convergence of Padé approximant for Stieltjes meromorphic function can be valued by the fact that the Stieltjes moment problem for the measure α can be determinate, so the corresponding Padé approximants of $\widehat{\alpha}$ converge to the Stieltjes transform $\widehat{\alpha}$; nevertheless after a mass ϵ has been added at the origin, the new measure generates an indeterminate Stieltjes problem and its Padé approximants can not converge to the corresponding Stieltjes transform, $\widehat{\alpha}(z) + \frac{\epsilon}{z}$ (see [10]); another interesting example can be founded in [16]. We obtain the following result:

Theorem 1. *If $\alpha \in \mathcal{M}_0$ and $\alpha' > 0$ almost everywhere on $(0, \infty)$, then $\lim_n \pi_n = f$ uniformly on each compact subset of $\mathcal{D} \setminus \{z : r(z) = \infty\}$.*

Under more restrictive assumption on the measure α this theorem was proved by López in [11]. He assumes that the moments of the measure α satisfy the Carleman condition. This is a very well known sufficient conditions for the determinacy of the Stieltjes moment problem. Our technique allows us to extend another results obtained by López changing the Carleman condition by the determination of the corresponding moment problem. The another most important extension is the following theorem on relative asymptotics for orthogonal polynomials on $\mathbb{R}$. A simple example does not cover by López's condition but cover by our assumptions is the measure on $\mathbb{R}$ with image measure on $[0, \infty)$ by $b(x) = x^2$ equal to $\frac{e^{-x^\lambda}}{\Gamma(1/\lambda)} dx$, $1/2 \leq \lambda < 1$, $x \in (0, \infty)$ (see Section 2 for further details).

Theorem 2. *Let $\mathcal{M}$ denote the class of positive Borel measure on $\mathbb{R}$ with finite moments whose Hamburger moment problem is determinate. Let $\nu \in \mathcal{M}$ be* *such that $\nu' > 0$ almost everywhere in $\mathbb{R}$ and let $g \in L^1(\mathbb{R})$ be such that $g \geq 0$, $g\,d\nu \in \mathcal{M}$, and there exist a polynomial Q and $p \in \mathbb{N}$ such that $\frac{Q(x)g(x)}{(1+x^2)^p} \in L^\infty(\nu)$. Then*

$$\lim_{n\to\infty} \frac{\mathcal{H}_n(gd\nu, z)}{\mathcal{H}_n(\nu, z)} = \frac{S(g, \Omega, z)}{S(g, \Omega, i)},$$

uniformly on each compact subset of $\Omega = \{z \in \mathbb{C} : \Im z > 0\}$, where $\mathcal{H}_n(g\,d\nu, z)$, $\mathcal{H}_n(\nu, z)$ are the orthogonal polynomials of degree n with respect to $g(x)\,d\nu(x)$ and ν, respectively, normalized by the condition that both are equal to 1 at i, and

$$S(g, \Omega, z) = \exp\left(\frac{1}{2\pi i}\int_{\mathbb{R}} \log g(x) \frac{xz+1}{z-x}\frac{dx}{x^2+1}\right), \quad z \in \Omega,$$

is the Szegő function for g with respect to the region Ω.

Theorem 1 is proved in Section 4, the proof of Theorem 2 is included in Section 5, the auxiliary results on moment problem appear in Section 2, and Section 3 contains the study of orthogonal polynomials with respect to varying measures.

2. Moment problem and one side approximation

Example. First, we observe that there exist measures satisfying conditions in the Theorem 2 whose moments do not satisfy the Carleman condition. Let $\frac{1}{2} \leq \lambda < 1$, if

$$d\sigma_\lambda(x) = \frac{e^{-x^\lambda}}{\Gamma(1/\lambda)}\,dx, \quad x \in (0, \infty),$$

then $s_n = \int_0^\infty x^n d\sigma_\lambda(x) = \frac{\Gamma((n+1)\lambda)}{\lambda\,\Gamma(1/\lambda)}$, so using Stirling's asymptotic formula we have that $s_n^{1/n} \sim k\,n^{1/\lambda}$, where k is a constant, and $\sum_{n=1}^\infty \frac{1}{s_n^{1/2n}} = \infty$ for $\frac{1}{2} \leq \lambda < 1$; so by the Carleman condition (for the Stieltjes case) the Stieltjes moment problem is determinate. The symmetric measures σ_λ^b on $\mathbb{R}$, whose image measure in $(0, \infty)$ by $b(x) = x^2$ is σ_λ, has Hamburger moment determinate (see [4] or [10]) and their moments $\tilde{s}_{2n} = s_n$, $\tilde{s}_{2n+1} = 0$ does not satisfy the Carleman condition for Hamburger moment problem ($\sum_{n=1}^{\infty} \frac{1}{\tilde{s}_{2n}^{1/2n}} < \infty$ since $\tilde{s}_{2n}^{1/(2n)} \sim \tilde{k}\, n^{1/(2\lambda)}$ and $\frac{1}{2} \leq \lambda < 1$). Sometime we use the notation $\sigma_\lambda(x^2) = \sigma_\lambda^b(x)$, $x \in \mathbb{R}$.

Denote $\Gamma = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$. For $\beta \in \mathcal{M}$, let μ^β be the image measure of β in the unit circle by $\psi_1(z) = \left(i\frac{z+1}{z-1}\right)$. Observe that the function $x = i\frac{z+1}{z-1}$, $z \in \Gamma \setminus \{1\}$, $x \in \mathbb{R}$, has inverse $z = \frac{x+i}{x-i}$. Let M_Γ be the class of measure μ on Γ such that the image measure $\beta^\mu \in \mathcal{M}$. The above change of variables establishes an one to one correspondence between $\mathcal{M}$ and $\mathcal{M}_\Gamma$.

We will use the following Riesz's lemmas (see [5], page 73, or [18] for the proof of Lemma 1; and [3], Corollary 3.4. there, or [19] for Lemma 2).

Lemma 1. *Suppose that $\beta \in \mathcal{M}$ and f is a continuous functions on $\mathbb{R}$ such that there exist constants $A > 0$, $B > 0$ and $j \in \mathbb{Z}_+$ such that*

$$|f(x)| \leq A + B\, x^{2j}, \quad x \in \mathbb{R}.$$

Then for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for any $n \geq N$ there are algebraic polynomials u_n and v_n such that $\deg(u_n) \leq n$, $\deg(v_n) \leq n$ *and*

$$u_n(x) \leq f(x) \leq v_n(x), \forall x \in \mathbb{R}, \qquad \int (v_n(x) - u_n(x))\, d\beta(x) < \epsilon.$$

Lemma 2. *Suppose that $\beta \in \mathcal{M}$ and β is non-discrete. Then for every $z_0 \in \mathbb{C}$ and for every $j \in \mathbb{N}$, $|x - z_0|^{2j} d\beta \in \mathcal{M}$.*

We are also interesting in the case when $j < 0$ and $z_0 = i$ in the Lemma above ($|x - i|^{2j} = \frac{1}{(1+x^2)^{-j}}$). The same conclusion of the lemma above is obtained for $j < 0$ in the following two lemmas.

Lemma 3. *(see [1], p. 43 or [18]) If $\beta \in \mathcal{M}$, the polynomials are dense in $L^2(\beta)$.*

Lemma 4. *(see [18]) Let β be a positive Borel measure on $\mathbb{R}$ with finite moments. Then $\mu \in \mathcal{M}$ if and only if the polynomials are dense in $L^2((1+x^2)d\mu)$.*

Let g be a real continuous function on $\Gamma\setminus\{1\}$ such that there exist constants $C>0$, $D>0$ and $j\geq 0$ such that

$$|g(z)|\leq C+D\frac{1}{|z-1|^{2j}},\quad z\in\Gamma\setminus\{1\}.$$

Lemma 5. *Let $k\in\mathbb{Z}$. Under assumption above on g, given $\mu\in\mathcal{M}_\Gamma$, $\epsilon>0$, and $k\in\mathbb{Z}$, there exist two polynomials $u_{n+k}=u_{n+k}(z,z^{-1})$, $v_{n+k}=v_{n+k}(z,z^{-1})$ such that $\deg(u_{n+k})\leq n+k$, $\deg(v_{n+k})\leq n+k$ in each variables z and z^{-1} and*

$$\frac{u_{n+k}(z,z^{-1})}{|z-1|^{2n}}\leq g(z)\leq\frac{v_{n+k}(z,z^{-1})}{|z-1|^{2n}},\ z\in\Gamma\setminus\{1\},$$
$$\int\frac{v_{n+k}(z,z^{-1})-u_{n+k}(z,z^{-1})}{|z-1|^{2n}}d\mu(z)<\epsilon.$$

Proof. Applying the Lemmas 1–4 to

$$f(x)=\frac{g\left(\frac{x+i}{x-i}\right)}{\left|\left(\frac{x+i}{x-i}\right)-1\right|^{2k}}=g\left(\frac{x+i}{x-i}\right)\frac{|x-i|^{2k}}{2^{2k}},$$

and

$$d\beta(x)=\frac{2^{2k}d\beta^{\mu}(x)}{|x-i|^{2k}},\ x\in\mathbb{R},$$

given $\epsilon>0$, we obtain polynomials u_{n+k},v_{n+k} of degree at most $n+k$ such that

$$u_{n+k}(x)\leq f(x)\leq v_{n+k}(x),\ x\in\mathbb{R},\qquad\int\left(v_{n+k}(x)-u_{n+k}(x)\right)\frac{2^{2k}d\beta^{\mu}(x)}{|x-i|^{2k}}<\epsilon.$$

Changing variables, $x=i\frac{z+1}{z-1}$, $z\in\Gamma$, the above relations are transformed into

$$u_{n+k}\left(i\frac{z+1}{z-1}\right)\leq\frac{g(z)}{|z-1|^{2k}}\leq v_{n+k}\left(i\frac{z+1}{z-1}\right),\quad z\in\Gamma\setminus\{1\},$$
$$\int\left(v_{n+k}\left(i\frac{z+1}{z-1}\right)-u_{n+k}\left(i\frac{z+1}{z-1}\right)\right)|z-1|^{2k}d\mu(z)<\epsilon.$$

Since

$$\left(i\frac{z+1}{z-1}\right)^{j}=\frac{(i(z+1)(\frac{1}{z}-1))^{j}|z-1|^{2n+2k-2j}}{|z-1|^{2n+2k}}=$$
$$=(-1)^{j+1}2^{2n+2k-j}\frac{\sin^{j}\theta(1-\cos\theta)^{n+k-j}}{|z-1|^{2n+2k}},\quad z\in\Gamma,\ z=e^{i\theta},$$

the above relations are equivalent to there exist polynomials

$$\widetilde{u}_{n+k}(z,z^{-1}), \quad \widetilde{v}_{n+k}(z,z^{-1})$$

of degree at most $n+k$ in each variables z and z^{-1} such that

$$\frac{\widetilde{u}_{n+k}(z,z^{-1})}{|z-1|^{2n}} \leq g(z) \leq \frac{\widetilde{v}_{n+k}(z,z^{-1})}{|z-1|^{2n}},\ z \in \Gamma \setminus \{1\},$$
$$\int \frac{\widetilde{v}_{n+k}(z,z^{-1}) - \widetilde{u}_{n+k}(z,z^{-1})}{|z-1|^{2n}}\, d\mu(z) < \epsilon.$$

■

If $\rho \in \mathcal{M}_0$ and ρ^b is the measure on $\mathbb{R}$ with the image measure ρ on $[0,\infty)$ by the function $b(x) = x^2$, $x \in \mathbb{R}$, then $\rho^b \in \mathcal{M}$ and if, moreover, ρ is a non-discrete measure, then the Hamburger moment problem is also determinate and for all $j \in \mathbb{Z}$, $(1+x)^j d\rho(x) \in \mathcal{M}_0$ (see [4]); these results are stated in the following lemmas:

Lemma 6. *If $\rho \in \mathcal{M}_0$ and ρ is a non-discrete measure, then $\rho \in \mathcal{M}$ and for all $j \in \mathbb{Z}$, $(1+x)^j d\rho(x) \in \mathcal{M}_0$.*

Lemma 7. *If $\rho \in \mathcal{M}_0$ and ρ^b is the measure on $\mathbb{R}$ with the image measure ρ on $[0,\infty)$ by the function $b(x) = x^2$, $x \in \mathbb{R}$, then $\rho^b \in \mathcal{M}$.*

3. Orthogonal polynomials for varying measure

Let μ be a positive Borel measure on Γ with infinite points in its support and the sequence of measures

$$d\mu_n(z) = \frac{d\mu(z)}{|z-1|^{2n}},\ z \in \Gamma,\ n \in \mathbb{N};$$

we assume that for each $n \in \mathbb{N}$ and $k \in \mathbb{Z}_+$ we have $z^k \in L^1(\mu_n)$. So we can put each pair (n,m) of natural numbers in correspondence with a polynomial

$\varphi_{n,m}(z) = \kappa_{n,m} z^m + \dots$ (with positive leading coefficient $\kappa_{n,m} = \kappa_m(\mu_n)$) of degree m, orthonormal with respect to the measure μ_n:

$$\int_\Gamma \overline{z}^k \,\varphi_{n,m}(z)\, d\mu_n(z) = 0, \quad k = 0, \dots, m-1, \quad \frac{1}{2\pi}\int_\Gamma |\varphi_{n,m}(z)|^2\, d\mu_n(z) = 1.$$

Let $\Phi_{n,m}(z) = \frac{1}{\kappa_{n,m}}\varphi_{n,m}(z)$ denote the monic orthogonal polynomials of degree m. In some case we shall do explicitly reference to the measure writing $\varphi_m(\mu_n, z) = \varphi_{n,m}(z)$. The following relations are well known:

$$\Phi_{n,m+1}(z) = z\Phi_{n,m}(z) + \Phi_{n,m+1}(0)\Phi^*_{n,m}(z), \tag{2}$$

$$\frac{\kappa_{n,m}}{\kappa_{n,m+1}}\varphi_{n,m+1}(z) = z\varphi_{n,m}(z) + \Phi_{n,m+1}(0)\varphi^*_{n,m}(z), \tag{3}$$

$$\frac{\kappa^2_{n,m}}{\kappa^2_{n,m+1}} = 1 - |\Phi_{n,m+1}(0)|^2; \tag{4}$$

moreover, we have $|\Phi_{n,m+1}(0)| < 1$ and the zeros of $\varphi_{n,m}$ lie in the disk $|z| < 1$. Here after, if p is a polynomial of degree m, $p^*(z) = z^m\overline{p(1/\overline{z})}$.

We need also the following well known Geronimus' identity (see [7], or [5], p. 198).

$$\int_\Gamma z^j \frac{|dz|}{|\varphi_{n,m}(z)|^2} = \int_\Gamma z^j\, d\mu_n(z), \quad j = 0, \pm1, \dots, \pm m. \tag{5}$$

Let μ' denote the Radon-Nykodym derivative of μ with respect to the Lebesgue measure ($|dz|$) on Γ. Let $\mu(z) = \mu'(z)|dz| + \mu_s(z)$ be the Lebesgue decomposition of μ; if $\mu' > 0$ almost everywhere on Γ, we can consider that $\mu' = \infty$ ($\Leftrightarrow \frac{1}{\mu'(z)} = 0$) on the support of μ_s which has Lebesgue's measure equal to zero. We use the notations $\|g\|_{L^p(\mu)} = (\frac{1}{2\pi}\int |g|^p d\mu)^{1/p}$ and $L^1 = L^1(|dz|)$. Our main result in this section is the ratio asymptotics $\lim_{n\to\infty}\frac{\varphi_{n,n+k+1}(z)}{\varphi_{n,n+k}(z)}$; for this aim we need the following two lemmas.

Lemma 8. *Let $k \in \mathbb{Z}$. If $\mu' > 0$ a.e. on Γ, then*

$$\left\| \left|\frac{\varphi_{n,n+k}(z)}{(z-1)^n}\right|^2 - \frac{1}{\mu'(z)} \right\|_{L^1(\mu)} \le 2\min\left\{ \left\| \left|\frac{w_{n+k}(z)}{(z-1)^n}\right| - \frac{1}{\sqrt{\mu'(z)}} \right\|_{L^2(\mu)} : w_{n+k} \in \mathcal{P}_{n+k} \right\}, \tag{6}$$

where $\mathcal{P}_{n+k}$ denotes the set of polynomials of degree at most $n+k$. Moreover, if $\beta_\mu(x) = \mu\left(\frac{x+i}{x-i}\right) \in \mathcal{M}$, then

$$\lim_{n\to\infty} \left\| \left| \frac{\varphi_{n,n+k}(z)}{(z-1)^n} \right|^2 - \frac{1}{\mu'(z)} \right\|_{L^1(\mu)} = 0. \tag{7}$$

Proof.

Let $w_{n+k} \in \mathcal{P}_{n+k}$. Using that $(\mu')^{-1/p} \in L^p(\mu)$, and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\left\| \left| \frac{\varphi_{n,n+k}(z)}{(z-1)^n} \right|^2 - \frac{1}{\mu'(z)} \right\|_{L^1(\mu)} \\
&\quad \le \left\| \left| \frac{\varphi_{n,n}(z)}{(z-1)^n} \right|^2 - \frac{1}{\sqrt{\mu'(z)}} \left| \frac{w_{n+k}(z)}{(z-1)^n} \right| \right\|_{L^1(\mu)} + \left\| \frac{1}{\sqrt{\mu'(z)}} \left| \frac{w_{n+k}(z)}{(z-1)^n} \right| - \frac{1}{\mu'(z)} \right\|_{L^1(\mu)} \\
&\quad = \left\| \left| \frac{\varphi_{n,n+k}(z)}{(z-1)^n} \right| \left(\left| \frac{\varphi_{n,n+k}(z)}{(z-1)^n} \right| - \frac{1}{\sqrt{\mu'(z)}} \left| \frac{w_{n+k}(z)}{\varphi_{n,n+k}(z)} \right| \right) \right\|_{L^1(\mu)} \\
&\qquad\qquad + \left\| \frac{1}{\sqrt{\mu'(z)}} \left(\left| \frac{w_{n+k}(z)}{(z-1)^n} \right| - \frac{1}{\sqrt{\mu'(z)}} \right) \right\|_{L^1(\mu)} \\
&\quad \le \left\| \left| \frac{\varphi_{n,n+k}(z)}{(z-1)^n} \right| - \frac{1}{\sqrt{\mu'(z)}} \left| \frac{w_{n+k}(z)}{\varphi_{n,n+k}(z)} \right| \right\|_{L^2(\mu)} + \left\| \left| \frac{w_{n+k}(z)}{(z-1)^n} \right| - \frac{1}{\sqrt{\mu'(z)}} \right\|_{L^2(\mu)}.
\end{aligned}$$

But taking (5) into account, we obtain

$$\begin{aligned}
&\left\| \left| \frac{\varphi_{n,n+k}(z)}{(z-1)^n} \right| - \frac{1}{\sqrt{\mu'(z)}} \left| \frac{w_{n+k}(z)}{\varphi_{n,n+k}(z)} \right| \right\|_{L^2(\mu)}^2 = \\
&\quad = 1 - \frac{2}{2\pi} \int_0^{2\pi} \left| \frac{w_{n+k}(z)}{(z-1)^n} \right| \sqrt{\mu'(z)}\, d\theta + \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{w_{n+k}(z)}{(z-1)^n} \right|^2 d\mu(z) = \\
&\qquad = \left\| \left| \frac{w_{n+k}(z)}{(z-1)^n} \right| - \frac{1}{\sqrt{\mu'(z)}} \right\|_{L^2(\mu)}^2.
\end{aligned}$$

Hence,

$$\left\| \left| \frac{\varphi_{n,n+k}(z)}{(z-1)^n} \right|^2 - \frac{1}{\mu'(z)} \right\|_{L^1(\mu)} \le 2 \left\| \left| \frac{w_{n+k}(z)}{(z-1)^n} \right| - \frac{1}{\sqrt{\mu'(z)}} \right\|_{L^2(\mu)}.$$

This proves (6).

Now, let us show (7). The set of continuous functions is dense in $L^2(\mu)$. The function $1/\sqrt{\mu'}$ belongs to $L^2(\mu)$ and is nonnegative, hence it can be approximated in the metric of this space by positive continuous functions. In turn, using that a positive trigonometric polynomial $v(z, z^{-1})$ of degree $n+k$ can be written as $v(z, z^{-1}) = |w_{n+k}(z)|^2$ with $w_{n+k} \in \mathcal{P}_{n+k}$ (see [5], p. 211), and by Lemma 5 every positive continuous function on Γ can be approximated by functions of the form $\left|\frac{w_{n+k}(z)}{(z-1)^n}\right|$

$$\lim_{n\to\infty} \min \left\{ \left\| \left| \frac{w_{n+k}(z)}{(z-1)^n} \right| - \frac{1}{\sqrt{\mu'(z)}} \right\|_{L^2(\mu)} : w_{n+k} \in \mathcal{P}_{n+k} \right\} = 0.$$

and by (6) the proof is concluded.

■

The following lemma for fixed measure can be founded in [17].

Lemma 9.

$$|\Phi_{n,n+k}(0)| \leq \left\| \left| \frac{\varphi_{n,n+k}(z)}{(z-1)^n} \right|^2 - \frac{1}{\mu'(z)} \right\|_{L^1(\mu)}$$

Proof. Set $a_{n+k} = -\overline{\Phi_{n,n+k}(0)}$ and $S_n(z) = \Re\left(a_{n+k} z \varphi_{n,n+k}(z)/\varphi^*_{n,n+k}(z)\right)$, comparing the squares of the modulus of the left-hand and right-hand sides of (3) on Γ, we obtain

$$\begin{aligned}
&\frac{\kappa^2_{n,n+k}}{\kappa^2_{n,n+k+1}} |\varphi_{n,n+k+1}(z)|^2 = \\
&= |\varphi_{n,n+k}(z)|^2 - 2\Re\left(a_{n+k} z \varphi_{n,n+k}(z) \overline{\varphi^*_{n,n+k}(z)}\right) + |a_{n+k}|^2 |\varphi_{n,n+k}(z)|^2 = \\
&\qquad = \left(1 + |a_{n+k}|^2\right) |\varphi_{n,n+k}(z)|^2 - 2S_n(z) |\varphi_{n,n+k}(z)|^2, \quad z \in \Gamma.
\end{aligned}$$

Integrating with respect to $\frac{d\mu(z)}{2\pi|z-1|^{2n}}$ and using (4), we obtain the representation

$$|a_{n+k}|^2 = \frac{1}{2\pi} \int_\Gamma S_n(z) \frac{|\varphi_{n+k}(z)|^2}{|z-1|^{2n}} d\mu(z).$$

Since $\int_\Gamma S_n(z)|dz| = 0$ and $|S_n(z)| \le |a_{n+k}|$, $z \in \Gamma$, it follows that

$$
\begin{aligned}
|a_{n+k}|^2 &= \frac{1}{2\pi}\int_\Gamma S_n(z)\left(\frac{|\varphi_{n+k}(z)|^2}{|z-1|^{2n}}\mu'(z) - 1\right)|dz| + \frac{1}{2\pi}\int_\Gamma S_n(z)\frac{|\varphi_{n+k}(z)|^2}{|z-1|^{2n}}d\mu_s(z) \\
&\le |a_{n+k}|\left(\left\|\left|\frac{\varphi_{n,n+k}(z)}{(z-1)^n}\right|^2 \mu'(z) - 1\right\|_{L^1} + \frac{1}{2\pi}\int_\Gamma \frac{|\varphi_{n+k}(z)|^2}{|z-1|^{2n}}d\mu_s(z)\right) \\
&= |a_{n+k}|\left\|\left|\frac{\varphi_{n,n+k}(z)}{(z-1)^n}\right|^2 - \frac{1}{\mu'(z)}\right\|_{L^1(\mu)}
\end{aligned}
$$

This proves the lemma. ■

Combining Lemmas 8 and 9, and the relations (2)–(4) we obtain:

Theorem 3. *If $\mu \in \mathcal{M}_\Gamma$ and $\mu' > 0$ almost everywhere on Γ, then for each $k \in \mathbb{Z}$, we have*

$$\lim_{n\to\infty}\frac{\Phi_{n,n+k+1}(z)}{\Phi_{n,n+k}(z)} = \lim_{n\to\infty}\frac{\varphi_{n,n+k+1}(z)}{\varphi_{n,n+k}(z)} = z,$$

uniformly on each compact subset of $\{z \in \mathbb{C} : 1 \le |z|\}$;

$$\lim_{n\to\infty}\frac{\Phi_{n,n}(z)}{\Phi^*_{n,n}(z)} = \lim_{n\to\infty}\frac{\varphi_{n,n}(z)}{\varphi^*_{n,n}(z)} = 0, \tag{8}$$

uniformly on each compact subset of $\{z : |z| < 1\}$; and

$$\lim_{n\to\infty}\frac{\kappa_{n,n+k+1}}{\kappa_{n,n+k}} = 1, \qquad \lim_{n\to\infty}\Phi_{n,n+k}(0) = 0.$$

Remark 1. *Using quantitative results on polynomial approximation (for results on quantitative one side polynomial approximation on $\mathbb{R}$ see, for example, [6]), and Lemmas 8 and 9 we can estimate the rate of convergence of the $\Phi_{n,n+k}(0)$ to 0.*

Remark 2. *In [2] (Lemma 2) it is proved that condition (8) implies that for all continuous function A on Γ there exist two sequences of polynomials $\{u_{n+k}(z)\}_{n=1}^\infty$, $\{v_{n+k}(z)\}_{n=1}^\infty$ with $\deg u_{n+k}(z) \le n+k$, $\deg v_{n+k}(z) \le n+k$, such that*

$$\lim_{n\to\infty}\max\left\{\left|A(z) - \frac{u_{n+k}(z) + v_{n+k}(\frac{1}{z})}{|\varphi_{n,n+k}(z)|^2}\right| : z \in \Gamma\right\} = 0. \tag{9}$$

Moreover, if f is nonnegative on Γ we can find polynomials $u_{n+k}(z)$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \max\left\{\left|A(z) - \left|\frac{u_{n+k}(z)}{\varphi_{n+k}(z)}\right|^2\right| : z \in \Gamma\right\} = 0. \tag{10}$$

Because of Lemma 8 and

$$\left\|\left|\frac{\varphi_{n,n+k}(z)}{(z-1)^n}\right|^2 - \frac{1}{\mu'(z)}\right\|_{L^1(\mu)} =$$

$$= \frac{1}{2\pi}\int \left|\left|\frac{\varphi_{n,n+k}(z)}{(z-1)^n}\right|^2 - \frac{1}{\mu'(z)}\right| \mu'(z)|dz| + \frac{1}{2\pi}\int \left|\left|\frac{\varphi_{n,n+k}(z)}{(z-1)^n}\right|^2 - \frac{1}{\mu'(z)}\right| d\mu_s(z) =$$

$$= \frac{1}{2\pi}\int \left|\left|\frac{\varphi_{n,n+k}(z)}{(z-1)^n}\right|^2 \mu'(z) - 1\right| |dz| + \frac{1}{2\pi}\int \left|\frac{\varphi_{n,n+k}(z)}{(z-1)^n}\right|^2 d\mu_s(z)$$

we obtain:

Lemma 10. *If $\mu \in \mathcal{M}_\Gamma$ and $\mu' > 0$ almost everywhere on Γ, we have*

$$\lim_{n\to\infty} \int_\Gamma \left|\left|\frac{\varphi_{n,n+k}(z)}{(z-1)^n}\right|^2 \mu'(z) - 1\right| |dz| = 0,$$

$$\lim_{n\to\infty} \int_\Gamma \left|\left|\frac{\varphi_{n,n+k}(z)}{(z-1)^n}\right| \sqrt{\mu'(z)} - 1\right|^2 |dz| = 0. \tag{11}$$

Therefore, for any $A \in L^\infty(\mu)$

$$\lim_{n\to\infty} \int_\Gamma A(z) \left|\frac{\varphi_{n,n+k}(z)}{(z-1)^n}\right|^2 \mu'(z)\,|dz| = \int A(z)\,|dz|,$$

$$\lim_{n\to\infty} \int_\Gamma f(z) \left|\frac{\varphi_{n,n+k}(z)}{(z-1)^n}\right|^2 d\mu(z) = \int f(z)\,|dz|.$$

The proof of (11) can be seen in Lemma 2 of [12]. The above lemma for fixed measures appears in [15] (see [15], Theorem 2.1 and Corollaries 2.2 and 5.1).

Lemma 11. *Let μ be a positive Borel measure on Γ with $\mu' > 0$ a. e. on Γ, and let $h \geq 0$, $h \in L^1(\mu)$.*

(a) *If, in addition, $h\,d\mu \in \mathcal{M}_\Gamma$ and there exists a polynomial Q such that $|Q|h^{-1} \in L^\infty(\mu)$, then, for each $k \in \mathbb{Z}$ and any continuous function A on Γ,*

$$\lim_n \int_\Gamma A(z)|Q(z)|^2 \left|\frac{\varphi_{n+k}(hd\mu_n, z)}{\varphi_{n+k}(\mu_n, z)}\right|^2 |dz| = \int_\Gamma A(z)|Q(z)|^2 h^{-1}(z)\,|dz|.$$

(b) *If, instead, $\mu \in \mathcal{M}_\Gamma$ and there exists a polynomial Q such that $|Q|h \in L^\infty(\mu)$, then, for each $k \in \mathbb{Z}$ and any continuous function A on Γ,*

$$\lim_n \int_\Gamma A(z)|Q(z)|^2 \left|\frac{\varphi_{n+k}(\mu_n, z)}{\varphi_{n+k}(hd\mu_n, z)}\right|^2 |dz| = \int_\Gamma A(z)|Q(z)|^2 h(z)\,|dz|.$$

Proof: Assertions (a) and (b) are proved using the same arguments; we will carry out the proof of (a). Note that from condition (a) (see Remark 2) it follows that there exists a rational sequence $\{\frac{u_{n+k}(z,1/z)}{|\varphi_{n+k}(h\,d\mu_n,z)|^2}\}$ which converges to $A|Q|^2$ uniformly on Γ, where $u_{n+k}(z,1/z)$ is a polynomial of degree at most $n+k$ in both variables z and $1/z$, so using Geronimus' identity (5) and Lemma 10, we have

$$\begin{aligned}
&\lim_n \int_\Gamma A(z)|Q(z)|^2 \left|\frac{\varphi_{n+k}(hd\mu_n, z)}{\varphi_{n+k}(\mu_n, z)}\right|^2 |dz| = \\
&\qquad = \lim_n \int_\Gamma \frac{u_{n+k}(z,1/z)}{|\varphi_n(h\,d\mu_n,z)|^2} \left|\frac{\varphi_{n+k}(hd\mu_n, z)}{\varphi_{n+k}(\mu_n, z)}\right|^2 |dz| = \\
&\qquad = \lim_n \int_\Gamma \frac{u_{n+k}(z,1/z)}{|\varphi_{n+k}(\mu_n,z)|^2}|dz| = \lim_n \int_\Gamma u_{n+k}(z,1/z)\,d\mu_n(z) = \\
&\qquad = \lim_n \int_\Gamma h^{-1}(z)\frac{u_{n+k}(z,1/z)}{|\varphi_n(h\,d\mu_n,z)|^2}|\varphi_n(h\,d\mu_n,z)|^2 h(z)\,d\mu_n(z) = \\
&\qquad = \int_\Gamma A(z)|Q(z)|^2 h^{-1}(z)\,|dz|.
\end{aligned}$$

■

Remark 3. *Following the same method of López in [12] we can obtain the Lemma 11 when A is any Riemann integrable function Γ.*

Another result of independent interest is the weak star limit of $\frac{|z-1|^{2n}}{|\varphi_{n,n+k}(z)|^2}$.

Theorem 4. *If $\mu \in \mathcal{M}_\Gamma$ and $\mu' > 0$ almost everywhere on Γ, then for each $k \in \mathbb{Z}$ we have*

$$\lim_{n\to\infty} \int_\Gamma A(z) \frac{|z-1|^{2n}}{|\varphi_{n,n+k}(z)|^2} |dz| = \lim_{n\to\infty} \int_\Gamma A(z)\, d\mu(z), \tag{12}$$

for all continuous function A on Γ; it means that the weak star limit of $\frac{|z-1|^{2n}}{|\varphi_{n,n+k}(z)|^2}$ is μ.

Proof. Taking real and imaginary part we can assume that A is a real function. Actually we proof a more general result: "for all real continuous function A in $\Gamma \setminus \{1\}$ such that there exists constants $\tilde{A} > 0, \tilde{B} > 0$ and $j \in \mathbb{Z}$ such that

$$|A(z)| \leq \tilde{A} + \frac{\tilde{B}}{|z-1|^{2j}}, \quad z \in \Gamma \setminus \{1\},$$

the relation (12) holds."

Let k and f fixed. Using Lemma 5 given $\epsilon > 0$ we can found polynomials $u_{n+k} = u_{n+k}(z, z^{-1})$ and $v_{n+k} = v_{n+k}(z, z^{-1})$ of degree at most $n+k$ such that

$$\frac{u_{n+k}(z, z^{-1})}{|z-1|^{2n}} \leq A(z) \leq \frac{v_{n+k}(z, z^{-1})}{|z-1|^{2n}}, \; z \in \Gamma \setminus \{1\},$$
$$\int_\Gamma (v_{n+k}(z, z^{-1}) - u_{n+k}(z, z^{-1})) d\mu_n(z) < \epsilon$$

Moreover, using Geronimus' identity (5) we obtain

$$\begin{aligned}\int_\Gamma u_{n+k}(z, z^{-1})\, d\mu_n(z) &= \int_\Gamma \frac{u_{n+k}(z, z^{-1})}{|z-1|^{2n}} \frac{|z-1|^{2n}}{|\varphi_{n,n+k}(z)|^2} |dz| \leq \\ &\leq \int_\Gamma A(z) \frac{|z-1|^{2n}}{|\varphi_{n,n+k}(z)|^2} |dz| \leq \int_\Gamma \frac{v_{n+k}(z, z^{-1})}{|z-1|^{2n}} \frac{|z-1|^{2n}}{|\varphi_{n,n+k}(z)|^2} |dz| = \\ &= \int_\Gamma v_{n+k}(z, z^{-1})\, d\mu_n(z)\end{aligned}$$

and

$$\int_\Gamma u_{n+k}(z, z^{-1})\, d\mu_n(z) \leq \int_\Gamma A(z)\, d\mu(z) \leq \int_\Gamma v_{n+k}(z, z^{-1})\, d\mu_n(z)$$

Therefore

$$\left| \int_\Gamma A(z) \frac{|z-1|^{2n}}{|\varphi_{n,n+k}(z)|^2} |dz| - \int_\Gamma A(z)\, d\mu(z) \right| < \epsilon.$$

$\blacksquare$

Now, we can obtain the relative asymptotics of orthogonal polynomials. The following result under more restrictive assumptions on the measure μ and on the function h was proved by López in [12].

Theorem 5. *Let $\mu \in \mathcal{M}_\Gamma$ such that $\mu' > 0$ a. e. Let h be such that $h\,d\mu \in \mathcal{M}_\Gamma$ and there exists a polynomial Q such that $|Q|h^{\pm 1} \in L^\infty(\mu)$. Then for each $k \in \mathbb{Z}$ we have*

$$\lim_n \frac{\varphi_{n+k}(hd\mu_n, z)}{\varphi_{n+k}(\mu_n, z)} = S(h, \{|\zeta| > 1\}, z),$$

uniformly in each compact subset of $\{z \in \mathbb{C} : |z| > 1\}$, where

$$S(h, \{|\zeta| > 1\}, z) = \exp\left(\frac{1}{4\pi}\int_\Gamma \log h(\zeta)\frac{\zeta + z}{\zeta - z}d\zeta\right)$$

is the Szegő function of h in $\{z \in \mathbb{C} : |z| > 1\}$.

Proof: It will be more convenient for us to prove the equivalent relation

$$\lim_n \frac{\varphi^*_{n+k}(hd\mu_n, z)}{\varphi^*_{n+k}(\mu_n, z)} = S^*(z), \quad |z| < 1.$$

where $S^*(z) = \overline{S(h, \{|\zeta| > 1\}, 1/\overline{z})}$. Using Theorem 3 it is sufficient to prove the above relation for $k = 0$. Without loss of generality, we can consider that the polynomial Q in the assumptions of the theorem has no zeros inside the disk $\{|z| < 1\}$ and, therefore, $\frac{Q(z)\varphi^*_{n+k}(hd\mu_n,z)}{\varphi^*_{n+k}(\mu_n,z)}$ is an analytic function in $\{|z| < 1\}$ and $\frac{Q(z)\varphi^*_{n+k}(hd\mu_n,z)}{\varphi^*_{n+k}(\mu_n,z)} \neq 0$, $|z| < 1$. Then, according to Poisson's formula,

$$\log\left|\frac{Q(z)\varphi^*_{n+k}(hd\mu_n, z)}{\varphi^*_{n+k}(\mu_n, z)}\right|^2 = \frac{1}{2\pi}\int_\Gamma \log\left|\frac{Q(\zeta)\varphi^*_{n+k}(hd\mu_n, \zeta)}{\varphi^*_{n+k}(\mu_n, \zeta)}\right|^2 P(z,\zeta)\,|d\zeta|,$$

where $P(z,\zeta) = \frac{1-|z|^2}{|\zeta - z|^2}$ is the Poisson kernel. Using Jensen's inequality, we obtain

$$\left|\frac{Q(z)\varphi^*_{n+k}(hd\mu_n, z)}{\varphi^*_{n+k}(\mu_n, z)}\right|^2 \leq \frac{1}{2\pi}\int_\Gamma \left|\frac{Q(\zeta)\varphi^*_{n+k}(hd\mu_n, \zeta)}{\varphi^*_{n+k}(\mu_n, \zeta)}\right|^2 P(z,\zeta)\,|d\zeta|,$$

Since $|\varphi^*_{n+k}(\mu_n,\zeta)| = |\varphi_{n+k}(\mu_n,\zeta)|$, $|\varphi^*_{n+k}(hd\mu_n,\zeta)| = |\varphi_{n+k}(hd\mu_n,\zeta)|$, $|\zeta| = 1$ and using Lemma 11 we obtain

$$\limsup_n \left|\frac{Q(z)\varphi^*_{n+k}(hd\mu_n,z)}{\varphi^*_{n+k}(\mu_n,z)}\right|^2 \le \frac{1}{2\pi}\int_\Gamma h^{-1}(\zeta)\,|Q(\zeta)|^2\, P(z,\zeta)\,|d\zeta|,\ |z|<1, \tag{13}$$

which in turn yields that the sequence $\{\frac{\varphi^*_{n+k}(hd\mu_n,z)}{\varphi^*_{n+k}(\mu_n,z)}\}$ is uniformly bounded inside (on each compact subset) of the disk $\{|z|<1\}$ (we recall that Q has no zeros in $\{|z|<1\}$). Let us consider an arbitrary subsequence $\Lambda\subset\mathbb{N}$ such that $\{\frac{\varphi^*_{n+k}(hd\mu_n,z)}{\varphi^*_{n+k}(\mu_n,z)} : n\in\Lambda\}$ converges and denote its limit by S_Λ. In virtue of what was said above, it is sufficient for us to prove that for any such sequence Λ we have $S^* = S_\Lambda$. Let $r\in(0,1)$ be arbitrary. Using Lemma 11 once more, we obtain

$$\begin{aligned}\frac{1}{2\pi}\int_\Gamma |Q(r\zeta)S_\Lambda(r\zeta)|^2\,|d\zeta| &= \lim_{n\in\Lambda}\frac{1}{2\pi}\int_\Gamma \left|\frac{Q(r\zeta)\varphi^*_{n+k}(hd\mu_n,r\zeta)}{\varphi^*_{n+k}(\mu_n,r\zeta)}\right|^2 |d\zeta| \\ &\le \lim_{n\in\Lambda}\frac{1}{2\pi}\int_\Gamma \left|\frac{Q(\zeta)\varphi^*_{n+k}(hd\mu_n,\zeta)}{\varphi^*_{n+k}(\mu_n,\zeta)}\right|^2 |d\zeta| = \lim_{n\in\Lambda}\frac{1}{2\pi}\int_\Gamma h^{-1}(\zeta)\,|Q(\zeta)|^2\,|d\zeta|.\end{aligned}$$

Thus, $QS_\Lambda\in H^2(\{|z|<1\})$, and therefore the limit $\lim_{r\to1} Q(r\zeta)S^*_\Lambda(r\zeta)$ exists almost everywhere for $\zeta\in\Gamma$. On the other hand, according to (13), for each fixed $z\in\Gamma$, we have

$$|Q(rz)S_\Lambda(rz)|^2 \le \frac{1}{2\pi}\int_\Gamma h^{-1}(\zeta)\,|Q(\zeta)|^2\, P(rz,\zeta)\,|d\zeta|,$$

It is well known (see, for example, [20], Section 9.5) that the limit as $r\to1$ of the righthand side of this inequality exists for almost all $z\in\Gamma$ and it is equal a.e. to $h^{-1}(\zeta)|Q(\zeta)|^2$. Therefore, $|S_\Lambda(z)|^2\le h^{-1}(z)$ almost everywhere on Γ. Working with $\{\frac{\varphi^*_{n+k}(\mu_n,z)}{\varphi^*_{n+k}(h\,d\mu_n,z)}\}$, we obtain that the inverse inequality is also satisfied. So $|S_\Lambda(z)|^2 = h^{-1}(z)$ a. e. on Γ, which in turn yields

$$\begin{aligned}\log|S_\lambda(z)| &= \frac{1}{2\pi}\int_\Gamma \log|S(\zeta)|\,P(z,\zeta)\,|d\zeta| = \\ &= \frac{1}{4\pi}\int_\Gamma \log h^{-1}(\zeta)\,P(z,\zeta)\,|d\zeta| = \log|S^*(z)|\end{aligned}$$

Since

$$S_\Lambda(0) = \lim_{n\in\Lambda} \frac{\kappa_{n+k}(h\,d\mu_n)}{\kappa_{n+k}(\mu_n)} > 0$$

and $S^*(0) > 0$, it follows that $\log S_\Lambda(z) = \log S^*(z), |z| < 1$, and thus $S_\Lambda(z) = S^*(z)$. The theorem is established.

■

Asymptotic formulas can be obtained from Theorem 5 for Szego's kernel

$$K_{n+k}(\mu_n, z, \zeta) = \sum_{j=0}^{n+k} \varphi_j(\mu_n, z)\overline{\varphi_j(\mu_n, \zeta)}$$

and the Christofel functions

$$\omega_{n+k}(\mu_n) = \inf_{p\in\mathcal{P}_{n+k}} \int_\Gamma \left|\frac{p(\zeta)}{p(z)}\right|^2 \, d\mu_n(\zeta).$$

where $\mathcal{P}_n$ is the set of all polynomials of degree $\leq n$. We recall other expressions for these functions (see, for example, [20], Chapter XI):

$$\begin{aligned} K_{n+k}(\mu_n, z, \zeta) &= \frac{\varphi^*_{n+k}(\mu_n, z)\overline{\varphi^*_{n+k}(\mu_n, \zeta)} - z\overline{\zeta}\varphi_{n+k}(\mu_n, z)\overline{\varphi_{n+k}(\mu_n, \zeta)}}{1 - z\,\overline{\zeta}} \\ &= \frac{\varphi^*_{n+k+1}(\mu_n, z)\overline{\varphi^*_{n+k+1}(\mu_n, \zeta)} - \varphi_{n+k+1}(\mu_n, z)\overline{\varphi_{n+k+1}(\mu_n, \zeta)}}{1 - z\,\overline{\zeta}} \end{aligned}$$

and

$$\omega_{n+k}(z) = K_{n+k}(\mu_n, z, z)^{-1}.$$

Corollary 1. *Under the assumption of Theorem 5 we have*

$$\lim_n \frac{K_{n+k}(h\,d\mu_n, z, \zeta)}{K_{n+k}(\mu_n, z, \zeta)} = S(h, z)\overline{S(h, \zeta)}, \quad |z| > 1,\ |\zeta| > 1,$$

and, in particular,

$$\lim_n \frac{\omega_{n+k}(h\,d\mu_n, z)}{\omega_{n+k}(\mu_n, z)} = |S(h, z)|^{-2}\,.$$

Let ρ be a positive Borel measure in $\Delta = [-1, 1]$; set $d\rho_n(u) = \frac{d\rho(u)}{(1-u)^n}$ and assume that $u^k \in L^1(\rho_n)$ for each $k \geq 0$. Let $l_{n,m}(u) = \tau_{n,m}u^m + \ldots$ be the orthogonal polynomial of degree m with respect to the measure $d\rho_n(u)$ whose leading coefficient, $\tau_{n,m}$, is supposed to be positive, and $L_{n,m}(u) = l_{n,m}(u)/\tau_{n,m}$.

Lemma 12. *If $\rho' > 0$ a.e. in $(-1,1)$ and its image measure $\rho\left(\frac{x-1}{x+1}\right) \in \mathcal{M}_0$, then for each $j \in \mathbb{Z}$ we have*

$$\lim_{n\to\infty} \frac{\tau_{n+j+1}}{\tau_{n+j}} = 2,$$
$$\lim_{n\to\infty} \frac{l_{n,n+j+1}(u)}{l_{n,n+j}(u)} = u + \sqrt{u^2-1} \stackrel{\text{def}}{=} \varphi(u) = 2 \lim_{n\to\infty} \frac{L_{n,n+j+1}(u)}{L_{n,n+j}(u)},$$

uniformly on each compact subset of $\mathbb{C} \setminus \Delta$.

Proof. The proof is carried out as usual, reducing it to the case of the circle. Let μ be the measure on the unit circle Γ defined by

$$\mu(e^{i\theta}) = \rho(\cos\theta) = \rho\left(\frac{1}{2}\left(z + \frac{1}{z}\right)\right), \quad z = e^{i\theta},\, \theta \in [0, 2\pi).$$

Let $d\mu_n(z) = \frac{d\mu(z)}{|z-1|^{4n}}$, $z \in \Gamma$, and let $\varphi_{2n,m}(z) = \kappa_{n,m} z^m + \ldots$ and $\Phi_{2n,m}(z) = \frac{\varphi_{2n,m}(z)}{\kappa_{n,m}}$ the corresponding orthogonal polynomials on Γ:

$$\frac{1}{2\pi} \int_\Gamma \varphi_{2n,j}(z) \overline{\varphi_{2n,k}(z)}\, d\mu_n(z) = \delta_{j,k}, \quad j,k = 0,1,\ldots$$

these polynomials are connected with the polynomials $l_{n,m}$ and $L_{n,m}$ by the well known relations

$$l_{n,m}(x) = \frac{\varphi_{2n,2m}(z) + \varphi^*_{2n,2m}(z)}{z^m \sqrt{2\pi(1 + \Phi_{2n,2m}(0))}}, \quad L_{n,m}(x) = \frac{\varphi_{2n,2m}(z) + \varphi^*_{2n,2m}(z)}{(2z)^m \sqrt{(1 + \Phi_{2n,2m}(0))}}, \tag{14}$$

with $x = \frac{1}{2}(z + \frac{1}{z})$.

We have

$$\mu\left(\frac{x+i}{x-i}\right) = \rho\left(\frac{1}{2}\left(\frac{x+i}{x-i} + \frac{x-i}{x+i}\right)\right) = \rho\left(\frac{x^2-1}{x^2+1}\right) = \rho\left(\frac{t-1}{t+1}\right),$$

where $t = x^2$, $t \in [0,\infty)$, $x \in \mathbb{R}$. By Lemma 7, the measure μ satisfies the assumptions of the Theorem 3; using this theorem and the relations (14), we immediately complete the proof of the lemma.

■

4. Padé approximants of Stieltjes-type meromorphic functions

In this section we prove Theorem 1; more precisely, let α be as in Theorem 1, let Q_n be the denominator of the Padé approximant of f normalized by $Q_n(-1) = (-1)^n$, and let $\mathcal{L}_n$ be the orthogonal polynomials with respect to α normalized also by $\mathcal{L}_n(-1) = (-1)^n$.

Theorem 6. *If $\alpha' > 0$ a.e. on $(0, \infty)$ and the Stieltjes moment problem for α is determinate, then the following statements hold:*

1. $$\lim_{n\to\infty} \frac{Q_n(z)}{\mathcal{L}_n(z)} = \prod_{j=1}^{d} \left(\frac{1+z)(1+a_j)(\Phi(z)-\Phi(a_j))}{4\Phi(z)(z-a_j)} \right), \quad z \in \mathcal{D},$$
   where $a_1, \ldots, a_d$ are the poles of r (counting their multiplicity) and $\Phi(z) = (\sqrt{z}+i)/(\sqrt{z}-i)$ is the conformal mapping of $\mathcal{D}$ onto the exterior of the unit circle ($\Phi(-1) = \infty$)
2. $\lim_n \pi_n = f$ *uniformly on each compact subset of $\mathcal{D} \setminus \{z : r(z) = \infty\}$.*

Under more restrictive assumption on the measure α this theorem was proved by López in [11]. The general scheme of the proof of the above theorem follows the technique developed in [11] (which at the same time in some steps use ideas of Gonchar [8]). The results in Section 3 allow us to extend the corresponding results in [11] under the more general conditions studied here. For an easy reading we include some details. The schedule of the proof is the following: Carrying out a bilinear transformation we pass to the problem of the convergence of Padé approximants $\Pi_n = g_n/h_n$ for functions of type $F(\zeta) = \widehat{\rho}(\zeta) + R(\zeta)$, where ρ is a measure on $\Delta = [-1, 1]$; moreover, F has asymptotic expansion in powers of $(\zeta - 1)$ and the Padé approximants correspond to this expansion. For the new convergence problem, it is possible to apply a known method of Gonchar, based on the fact that the denominators h_n of the new approximants satisfy incomplete orthogonality relations with respect to a certain (varying) measure with compact support. This allows us to reduce the study of the asymptotic behavior of q_n to the question of the existence of the asymptotics of the ratio of orthogonal polynomials with respect to this same measure.

Proof of Theorem 6.

Step 1. Let us make the change of variables $x = (1+u)/(1-u)$, $x \in (0, \infty)$, $u \in (-1, 1)$, in the integral (1) and take $z = (1+\zeta)/(l-\zeta)$ in the argument of f. It can be checked directly that

$$f\left(\frac{1+\zeta}{l-\zeta}\right) = (1-\zeta)\left(\widehat{\rho}(\zeta) + R(\zeta)\right), \tag{15}$$

where

$$d\rho(u) = \frac{1}{2}(1-u)d\alpha\left(\frac{1+u}{1-u}\right) \quad \text{and} \quad (1-\zeta)R(\zeta) = r\left(\frac{1+\zeta}{1-\zeta}\right).$$

Put

$$F(\zeta) = \widehat{\rho}(\zeta) + R(\zeta) = \int_\Delta \frac{d\rho(t)}{\zeta - t} + R(\zeta), \quad \zeta \in \mathbb{C} \setminus \Delta,$$

let $\Pi_n = g_n/h_n$ be the Padé approximant of orden n of the function F corresponding to the point $\zeta = -1$ (this point corresponds to $z = \infty$). We have

$$h_n(\zeta) = (1-\zeta)^n Q_n\left(\frac{1+\zeta}{1-\zeta}\right), \quad g_n(\zeta) = (1-\zeta)^{n-1} P_n\left(\frac{1+\zeta}{1-\zeta}\right), \tag{16}$$

and

$$\Pi_n(\zeta) = \frac{1}{\zeta - 1}\pi_n\left(\frac{1+\zeta}{1-\zeta}\right). \tag{17}$$

Moreover, if $d\rho_n(u) = \frac{d\rho(u)}{(1-u)^n}$, $u \in (-1,1)$, and $R(\zeta) = \frac{l_{d-1}(\zeta)}{t_d(\zeta)}$, and $t_d(\zeta) = \prod_{j=1}^d (\zeta - b_j)$, then

$$\int_\Delta u^j h_n(u)\, t_d(u)\, d\rho_n(u) = 0, \quad j = 0, 1, \ldots, n-d-1, \tag{18}$$

$$F(\zeta) - \Pi_n(\zeta) = \frac{(1-\zeta)^{2n}}{s(\zeta)h_n(\zeta)t_d(\zeta)} \int_\Delta \frac{s(u)h_n(u)t_d(u)}{\zeta - u} d\rho_n(u), \tag{19}$$

where $s(u)$ is an arbitrary polynomial of degree $\leq n-d$.

Combining (15) and (17) the convergence of $\{\pi_n\}$ to f uniformly in each compact subset of $\mathbb{C}\setminus\{[0,\infty)\cup\{r=\infty\}\}$ is equivalent to the convergence of $\{\Pi_n\}$ to F uniformly in each compact subset of $\mathbb{C}\setminus\{\Delta\cup\{R=\infty\}\}$.

If $t_d=1$ ($\Leftrightarrow r\equiv 0$), then using Stieltjes' theorem we know $\lim_n \pi_n(z)=f(z)$ uniformly in each compact subset of $\mathbb{C}\setminus\{[0,\infty)\cup\{r=\infty\}\}$ or equivalent $\lim_n \Pi_n(z)=F(z)$ uniformly in each compact subset of $\mathbb{C}\setminus\{[-1,1]\cup\{R=\infty\}\}$ and by formula (19) with $s=1$ we obtain

$$\lim_{n\to\infty}\frac{(1-\zeta)^{2n}}{L_{n,n}(\zeta)}\int\frac{L_{n,n}(u)}{\zeta-u}d\rho_n(u)=0, \tag{20}$$

where $l_{n,m}(\zeta)=\tau_{n,m}\zeta^m+\dots$ is the orthogonal polynomial of degree m with respect to the measure $d\rho_n$ whose leading coefficient, $\tau_{n,m}$, is supposed to be positive, and $L_{n,m}(\zeta)=l_{n,m}(\zeta)/\tau_{n,m}$.

Step 2. By Lemma 12 for each $j\in\mathbb{Z}$ we have

$$\lim_{n\to\infty}\frac{l_{n,n+j+1}(\zeta)}{l_{n,n+j}(\zeta)}=\zeta+\sqrt{\zeta^2-1}\overset{\text{def}}{=}\varphi(\zeta)=2\lim_{n\to\infty}\frac{L_{n,n+j+1}(\zeta)}{L_{n,n+j}(\zeta)},\quad \zeta\in\mathbb{C}\setminus\Delta.$$

In view of the orthogonality relations (18), the polynomial $h_n(\zeta)t_d(\zeta)$ can be represented in the form of a finite linear combination of the orthogonal polynomials $L_{n,m}$

$$h_n(\zeta)t_d(\zeta)=\lambda^*_{n,0}L_{n,n+d}(\zeta)+\lambda^*_{n,1}L_{n,n+d-1}(\zeta)+\ldots+\lambda^*_{n,2d}L_{n,n-d}(\zeta). \tag{21}$$

Take $\lambda_n=\left(\sum_{j=0}^{2d}|\lambda^*_{n,j}|\right)^{-1}$, $\lambda_{n,j}=\lambda_n\lambda^*_{n,j}$, $j=0,\dots,2d$ and $S_{n+d}(\zeta)=\lambda_n h_n(\zeta)t_d(\zeta)$; since $q_n\not\equiv 0$, λ_n is finite. We have

$$\Psi_n(\zeta)=\frac{S_{n+d}(\zeta)}{L_{n,n+d}(\zeta)}=\sum_{j=1}^{2d}\lambda_{n,j}\frac{L_{n,n+d-j}(\zeta)}{L_{n,n+d}(\zeta)},\quad \sum_{j=1}^{2d}|\lambda_{n,j}|=1.$$

From the condition of the theorem, by Lemma 12 it follows that

$$\lim_{n\to\infty}\frac{L_{n,n+d-j(\zeta)}}{L_{n,n+d}(\zeta)}=\psi(\zeta)^j,\ j=0,1,\dots,2d,$$

where $\psi(\zeta) = 2/\varphi(\zeta)$. The function ψ is a one-to-one representation of $\mathbb{C}\setminus\Delta$ onto the disk of radius 2. Consequently the sequence Ψ_n is uniformly bounded. From those same relations it follows that any limit function of the sequence $\{\Psi_n\}$ is a polynomial of degree $\leq 2d$ of $\psi(\zeta)$. So in any compact subset of $\mathbb{C}\setminus\Delta$, for all sufficiently large n, there lie no more than d zeros of the polynomial h_n.

Further, let $\mathrm{cap}\,(\mathrm{K})$ denote the logarithmic capacity of the compact set K. By $\mathrm{limcap}\,\mathrm{f_n(z)} = \mathrm{f(z)}$, $\mathrm{z} \in \mathrm{G}$, we will denote the convergence in capacity inside G (this notation means that for any $\epsilon > 0$ and any compact set $K \subset G$ we have $\lim_{\epsilon\to 0} \mathrm{cap}\,(\mathrm{K} \cap \{|\mathrm{f_n} - \mathrm{f}| > \epsilon\}) = 0$). Let us show that

$$\mathrm{limcap}\,\Pi_\mathrm{n}(\zeta) = \mathrm{F}(\zeta), \tag{22}$$

in $\mathbb{C}\setminus\Delta$.

We fix a compact $K \subset \mathbb{C}\setminus\Delta$. Let $\delta > 0$ be sufficiently small so that the δ- neighborhood K_δ of K is contained in $\mathbb{C}\setminus\Delta$ together with its closure.

Let $c_n(\zeta) = \zeta^{d'} + \dots$ be the polynomial whose zeros are the zeros of S_{n+d} that lie on $\mathbb{C}\setminus\Delta$. By virtue of what was said above, for all sufficiently large n we have $d' \leq 2d$. Multiplying (19), with $s = 1$, by $c_n(\zeta)$ and using (21), we obtain

$$c_n(\zeta)(\Pi_n(\zeta) - F(\zeta)) = c_n(\zeta)\frac{L_{n,n+d}(\zeta)}{S_{n+d}(\zeta)}\sum_{j=1}^{2d}\lambda_{n,j}\frac{L_{n,n+d-j}(\zeta)}{L_{n,n+d}(\zeta)}I_{n,j}(\zeta),$$

where

$$\begin{aligned} I_{n,j}(\zeta) &= \int_\Delta \frac{L_{n+d-j}(u)}{L_{n+d-j}(\zeta)}(1-\zeta)^{2n}\frac{d\rho_n(u)}{\zeta-u} \\ &= (1-\zeta)^{2(j-d)}\int_\Delta \frac{L_{n+d-j}(u)}{L_{n+d-j}(\zeta)}(1-\zeta)^{2(n+d-j)}\frac{d\rho_n^{(j)}(u)}{\zeta-u}, \end{aligned}$$

and

$$d\rho_n^{(j)}(u) = \frac{d\rho^{(j)(u)}}{(1-u)^{2(n+d-j)}}, \quad d\rho^{(j)}(u) = (1-u)^{2(d-j)}d\rho(u).$$

It is obvious that for each fixed $j = 0, 1, \ldots, 2d$ the measure $\rho^{(j)}$ satisfies the conditions just like the measure ρ (see Lemma 6). Hence, using (20) it follows that $\lim_{n\to\infty} I_{n,j}(\zeta) = 0$, uniformly in each compact $\mathbb{C} \setminus \Delta$, for each $j = 0, 1, \ldots, 2d$. From what was said above, it is also obvious that the sequence of functions $\frac{c_n(\zeta)}{\Psi_n(\zeta)}$, which are analytic on K, is uniformly bounded on K. Therefore,

$$\lim_n c_n(\zeta)(F(\zeta) - \Pi_n(\zeta)) = 0, \quad \zeta \in K.$$

Since by the Fekete's lemma $\mathrm{cap}\,(\{\zeta : |\mathrm{c_n}(\zeta)| < \epsilon\}) = \epsilon^{1/\mathrm{d}'}$ for each $\epsilon > 0$, and $d' \leq 2d$, relation (22) follows.

Suppose that U is a region whose closure is a compact set in $\mathbb{C} \setminus \Delta$ which contains all the poles of $F(\zeta)$ in $\mathbb{C} \setminus \Delta$. As we proved above, the number of poles of Π_n in U, for all sufficiently large n, is not greater than d. The number of poles of F in U is equal to d. Under these conditions it follows from (22), by virtue of Gonchar's lemma ([9], Lemma 1), that for all sufficiently large n the number of poles of Π_n in U is equal to d, and these poles tend to the poles of F as $n \to \infty$ (each pole of F attracts as many poles of Π_n as its order of multiplicity). In turn, this yields that

$$\lim_{n\to\infty} \Pi_n(\zeta) = F(\zeta)$$

uniformly in each compact subset of $\mathbb{C} \setminus \{\Delta \cup \{b_1, \ldots, b_d\}\}$

Step 3. It remains to complete the proof of statement (a). Taking into consideration (16) we have to prove

$$\lim_{n\to\infty} \frac{h_n(\zeta)}{L_{n,n}(\zeta)} = (2\varphi(\zeta))^{-d} \prod_{j=1}^{d} \frac{\varphi(\zeta) - \varphi(b_j)}{\zeta - b_j}, \tag{23}$$

uniformly on each compact subset of $\mathbb{C}\setminus(\Delta\cup\{b_1, \ldots, b_d\})$, where $b_1, \ldots, b_d$ are the poles of r.

The information obtained about the behavior of the zeros of $h_n(z)$ (the poles of Π_n) inside $\mathbb{C} \setminus \Delta$, we can conclude that any limit function of the sequence $\{\Psi_n(\zeta) = \frac{S_{n+d}(\zeta)}{L_{n+d}(\zeta)}\}$ has the form

$$\sum_{j=0}^{2d} \lambda_j \psi^j(z) = C \prod_{j=1}^{d} (\psi(z) - \psi(b_j))^2$$

where $|C| \in (0, +\infty)$. In particular, for any convergent sequence of $\{\Psi_n(\zeta)\}$ we have

$$\Psi_n(\infty) = \lambda^*_{n,0}\lambda_n, \quad \lim_n \Psi_n(\infty) = C \prod_{j=1}^{d} \psi(b_j). \tag{24}$$

Since the leading coefficient of h_n is equal to 1, the quantity $\lambda^*_{n,0}$ can take only the two values 1 (if $\deg(h_n) = n$) or 0 (if $\mathrm{dedeg}(h_n) < n$). By virtue of the compactness of the sequence, from the above relation it follows, first, that $\lambda^*_{n,0} = 1$ ($\deg(h_n) = n$) for all sufficiently large n, and second, that $\liminf_{n\to\infty} \lambda_n > 0$. Hence the sequence of functions

$$\frac{h_n(\zeta)t_d(\zeta)}{L_{n,n+d}(\zeta)} = 1 + \sum_{j=1}^{2d} \lambda^*_{n,j} \frac{L_{n,n+j-d}(\zeta)}{L_{n,n+d}(\zeta)}$$

is uniformly bounded, just like $\{\Psi_n\}$. Using the same arguments as above, based on (16), the behavior of the zeros of h_n in $\mathbb{C}\setminus\Delta$, and the normalizing conditions, we conclude that this sequence converges uniformly inside $\mathbb{C} \setminus \Delta$:

$$\lim_n \frac{h_n(\zeta)t_d(\zeta)}{L_{n,n+d}(\zeta)} = \prod_{j=1}^{2d} \left(1 - \frac{\psi(\zeta)}{\psi(b_j)}\right),$$

uniformly on each compact subset of $\mathbb{C} \setminus \{\Delta \cup \{b_1, \dots, b_d\}\}$. Considering that $\psi(\zeta) = 2/\varphi(\zeta)$ and $\lim_n \frac{L_{n,n}(\zeta)}{L_{n,n+d}(\zeta)} = (\psi(\zeta))^{-d}$ uniformly on each compact subset of $\mathbb{C} \setminus \Delta$, the part (a) of the theorem is proved.

■

5. Relative asymptotics of orthogonal polynomials on the real line

In this section we prove of Theorem 2. Let μ^ν be the image measure of ν by the function $(i\frac{z+1}{z-1})$, $z \in \Gamma$, then the orthogonal polynomial $\mathcal{H}_n(\nu, z)$ with respect to ν normalized by $\mathcal{H}_n(\nu, i) = 1$ are related to the orthogonal polynomial with respect to $d\mu_n(z) = \frac{d\mu^\nu(z)}{|z-1|^{2n}}$ by the relation

$$\begin{aligned}(z-1)^n \mathcal{H}_n(\nu,\omega) &= \frac{\varphi_n^*(\mu_n, z)\varphi_n^*(\mu_n, 1) - z\varphi_n(\mu_n, z)\varphi_n(\mu_n, 1)}{\kappa_n(\mu_n)\varphi_n^*(\mu_n, 1)(1-z)} \\ &= \frac{K_n(\mu_n, z, 1)}{\kappa_n(\mu_n)\varphi_n^*(\mu_n, 1)},\end{aligned}$$

where $z = \frac{\omega+i}{\omega-i}$, $\omega \in \Omega$, and $|z| > 1$. Writing the above formula for $\mathcal{H}_n(g\,d\nu, \omega)$, we obtain

$$\frac{\mathcal{H}_n(g\,d\nu,\omega)}{\mathcal{H}_n(\nu,\omega)} = \frac{\varphi_n(g\,d\mu_n, z)}{\varphi_n(\mu_n, z)}\,\frac{\kappa_n(g\,d\mu_n)}{\kappa_n(\mu_n)}\,\frac{\frac{\varphi_n^*(g\,d\mu_n,z)}{\varphi_n(g\,d\mu_n,z)}\overline{\left(\frac{\varphi_n^*(g\,d\mu_n,1)}{\varphi_n(g\,d\mu_n,1)}\right)} - z}{\frac{\varphi_n^*(\mu_n,z)}{\varphi_n(\mu_n,z)}\overline{\left(\frac{\varphi_n^*(\mu_n,1)}{\varphi_n(\mu_n,1)}\right)} - z},$$

then combining Theorem 3 and Theorem 5 the proof is concluded.

Remark 4. *The previous results pass over easily to the case of orthogonality on* $[0, +\infty)$. *A measure* α, $\mathrm{supp}(\alpha) \subset [0, +\infty)$ *can be put in correspondence with a measure* ν *on* $\mathbb{R}$ *symmetrical with respect to* 0, $d\nu(x) = |x|d\alpha(x^2)$ *(see [12], Theorem 4).*

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