Fundamental groups and finite sheeted coverings☆

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ABSTRACT

It is well known that for a connected locally path-connected semi-locally 1-connected space X, there exists a bi-unique correspondence between the pointed d-fold connected coverings and the transitive representations of the fundamental group of X in the symmetric group Σ_d of degree d.

The classification problem becomes more difficult if X is a more general space, particularly if X is not locally connected. In attempt to solve the problem for general spaces, several notions of coverings have been introduced, for example, those given by Lubkin or by Fox. On the other hand, different notions of 'fundamental group' have appeared in the mathematical literature, for instance, the Brown–Grossman–Quigley fundamental group, the Čech–Borsuk fundamental group, the Steenrod–Quigley fundamental group, the fundamental profinite group or the fundamental localic group.

The main result of this paper determines different 'fundamental groups' that can be used to classify pointed finite sheeted connected coverings of a given space X depending on topological properties of X.

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0. Introduction

If X is a connected locally path-connected and semi-locally 1-connected space and $\pi_1(X, x^0)$, $x^0 \in X$, its fundamental group, then the category **Cov** (X, x^0) of coverings and continuous maps over (X, x^0) is equivalent to the category of $\pi_1(X, x^0)$-sets, i.e. the functor category **Sets**$^{\pi_1(X, x^0)}$, where $\pi_1(X, x^0)$ is considered as a groupoid with one object and the morphisms are given by the elements of $\pi_1(X, x^0)$. If F is a finite set, a covering with fibre F has an associated $\pi_1(X, x^0)$-set structure on the set F given by a representation $\eta \colon \pi_1(X, x^0) \to \operatorname{Aut}(F)$. It is easy to check that two coverings with fibre F are isomorphic if and only if the corresponding representations are conjugated. In the case of a connected pointed covering the conjugation relation is trivial and accordingly, under this correspondence, the connected coverings correspond to transitive representations.

In 1962, Lubkin [1] published a very nice paper titled "The theory of covering spaces". To avoid the difficulties in the non-locally connected case he introduced a new concept of space which generalizes notions of topological space and uniform space. Using Lubkin's notion of covering space the theory of covering spaces becomes analogous to the Galois theory of extensions. The finite Galoisian extensions are similar to regular covering spaces and "Poincare filtered group" plays the role of the Galois group. One of the main results establishes an order-reversing isomorphism from the ordered set of pointed coverings smaller than a given regular pointed covering $(R, r^0) \to (X, x^0)$ and the filtered subgroups of the "Poincare filtered

group"$P(R, X)$ which is given by the group of automorphisms of the covering $R \to X$ together with a suitable filtration of subgroups. Then, if one considers all regular coverings and takes the corresponding "Poincare filtered groups", it is possible to classify all "connected" coverings of a space X.

In 1967, Artin and Mazur proved [2, Corollary 10.6] that if **C** is a locally connected site, closed under arbitrary coproducts, and K is a hypercovering (see Section 8 of [2]), then the category of locally trivial morphisms over the final object of **C** which are trivial over K_0, is equivalent to the category of simplicial covering maps over the simplicial set $c_0(K)$, where $(c_0(K))_q = c_0(K_q)$ and c_0 is the set of connected components that for a locally connected site is a well-defined functor.

Given a locally connected topological space X, we can consider the locally connected distributive category (site) **C** = **Sh**(X) of the sheaves over X. An open cover $\mathcal{U}$ determines a hypercovering K having the given open cover ($\sqcup_{U \in \mathcal{U}} U$) as K_0. In this case the hypercovering pro-simplicial set is isomorphic (in the pro-homotopy category) to the Čech nerve associated with the family of open covers of the space. Now, by Corollary 10.6, we get the classification of the covering mappings of X which are trivial over the given open cover. The Artin–Mazur fundamental progroup is obtained by applying the standard fundamental group functor to the hypercovering pro-simplicial set and, as a consequence of Corollary 10.6, it can be used to classify all the covering mappings of X. However, Artin–Mazur techniques cannot be applied to non-locally connected spaces.

In 1972–73, Fox [3,4] introduced the notion of overlay of a metrizable space. Fox's fundamental theorem of overlays theory establishes the existence of a bi-unique correspondence between the d-fold overlayings of a connected metrizable space X and the representations up to conjugation of the fundamental trope of X in the symmetric group Σ_d of degree d. In the case of overlays with finite number of sheets, the notion of overlay agrees with the notion of covering with a finite fibre F where F is a set with finite cardinal d. Note that Fox's result refers to the class of metrizable spaces, but no local connectivity condition is required.

If G is a profinite group, i.e. the inverse limit of finite groups, we can consider the category **C**(G) of continuous finite G-sets. A category, which is equivalent to a category of the form $\mathcal{C}(G)$, is said to be a Galois category. Grothendieck [5] gave an axiomatic description of such category and proved that the associated profinite group is unique up to isomorphism. In many cases the full subcategory of locally constant objects of a given topos is a Galois category and then, using Grothendieck's result, one can define the profinite fundamental group of the given topos as the profinite group determined by the Galois category of locally constant objects.

If X is a variety over an algebraically closed field and $p: Y \to X$ is finite and etale, then each fibre of p has exactly the same number of points. Thus, a finite etale map is a natural analogue of a finite covering space. We define **FEt**$/X$ to be the category whose objects are the finite etale maps $p: Y \to X$ (sometimes referred to as finite etale coverings of X) and whose arrows are the X-morphisms. An important fact is that **FEt**$/X$ is a Galois category. The profinite group associated to this Galois category is the fundamental group of the variety X. We recall that the category of finite sheeted coverings of a connected locally path-connected and semi-locally 1-connected space Y is equivalent to the category of finite $\pi_1(Y)$-sets. In this case, taking the finite completion $\hat{\pi}_1(Y)$ of $\pi_1(Y)$, we get that the category of continuous finite $\hat{\pi}_1(Y)$-sets is equivalent to the category of finite $\pi_1(Y)$-sets. If X is a non-singular variety over the field $\mathbb{C}$ of complex numbers, one has that the etale fundamental group of the variety X is the finite completion of the fundamental group of the associated topological space $X(\mathbb{C})$.

In 1989, Moerdijk [6] gave a characterization of the topos of the form **B**(G) for a prodiscrete localic group G. He proved that the category of prodiscrete localic groups is equivalent to the category of progroups with epimorphic bonding homomorphisms. In the case of a locally connected space that equivalence carries the Artin–Mazur fundamental progroup to the fundamental localic group. In this way the classification of coverings can be given in terms of 'continuous' representations of the fundamental localic group. Nevertheless this localic group cannot be used to classify coverings over non-locally connected spaces.

In 1998, a notion of covering projection was introduced by Hernández [7]. In the case of finite fibres or locally connected spaces the notion of covering projection is equivalent to the standard covering notion. Consequently, in that case the category of covering projections for an arbitrary space is equivalent to the category of representations of the fundamental progroupoid. It is important to remark that there exist two non-isomorphic progroupoids π, π' such that the category of π-sets is equivalent to the category of π'-sets. This means that the fundamental progroupoid is not unique up to isomorphism and is determined only up to some equivalent notion of progroupoids. A generalization of this notion for fibre bundle with locally constant cocycles is given in [8].

Using a different approach, in 2001, Mardešić–Matijević [9] gave a classification of overlays over a connected topological space. In this paper, for a given resolution $\{X_i | i \in I\}$ of a space X, the relation between the coverings of the spaces X_i and the overlays of X is analyzed and, as a consequence of this study, an extension of Fox's classification theorem to the class of connected spaces is obtained. An interesting study of overlays over spaces which are the inverse limit of torus can be seen in Eda, Mandić and Matijević [10].

New results on atomic connected categories have been obtained by Dubuc [11]. He proved that an atomic connected pointed topos is the classifying topos of the localic group of the automorphism of the point. In a posterior work Dubuc [12] characterized a Galois topos as the classifying topos of a connected localic groupoid having discrete object space and prodiscrete localic hom-sets. Recently, Dubuc [13] has introduced the localic progroupoid of an arbitrary topos.

Previously, Joyal-Tierney had proved that an arbitrary topos is the classifying topos of a localic groupoid. Further results on classifying topos of localic groupoids have been obtained by M. Bunge and I. Moerdijk. These results have interesting applications to the theory of coverings and overlays. Under some topological conditions on the base space, the category of arbitrary coproducts of coverings or overlays can be considered as the classifying topos of a localic groupoid.

In the preceding paragraphs we have considered some notions of coverings in different contexts. We could give here a description of different notions of fundamental groups which have appeared in the mathematical literature. However, we prefer to devote Sections 1.5 and 1.6 and Section 2 for explaining different types of fundamental groups, progroups, topological groups, etc. Furthermore, we analyze some important relations between these invariants.

Our main result is given in Theorem 3.5. This theorem gives the classification of n-sheeted connected coverings of a connected compact metric space in terms of transitive representations of various types of fundamental 'groups' of the space in appropriate symmetric groups. We write 'group' to indicate that we can work with a progroup, a topological group, a localic group, a near-module or a group. We restrict our considerations to the class of connected compact metric spaces (although in some cases this restriction is not necessary), because in this class of spaces the different notions of finite coverings agree which allow us to compare the different approaches. In addition, we analyze the different fundamental 'groups' that can be used in dealing with S^1-movable spaces, locally connected spaces, locally path-connected spaces or, as in the classical case, locally path-connected and semi-locally 1-connected spaces.

1. Preliminaries

Firstly, we recall the notion of covering and connected covering.

1.1. Coverings

An *open cover* of a space Y is a family $\mathcal{V}$ of open subsets of Y such that $\bigcup_{V \in \mathcal{V}} V = Y$.

Definition 1.1. A continuous map $f : X \to Y$ is said to be a *covering* if there is an open cover $\mathcal{V}$ of the space Y such that for each $V \in \mathcal{V}$ there exist an index set $I(V)$ and a family of open subsets U_i of X, $i \in I(V)$ such that

(i) if $i, j \in I(V)$, $i \neq j$, $U_i \cap U_j = \emptyset$,
(ii) $\bigcup_{i \in I(V)} U_i = f^{-1}(V)$ and
(iii) the restriction $f|_{U_i} : U_i \to V$ is a homeomorphism.

Given two coverings $f : X \to Y$, $f' : X' \to Y$, a covering morphism is a continuous map $\phi : X \to X'$ over Y; that is, ϕ satisfies $f'\phi = f$. The category of covering and covering morphisms is denoted by **Cov**(Y) and the category of connected coverings and covering morphisms is denoted by **Cov**$_0(Y)$.

Definition 1.2. Given a pointed space (Y, y^0), a continuous pointed map $f : (X, x^0) \to (Y, y^0)$ is said to be a *pointed covering* if $f : X \to Y$ is a covering and $f(x^0) = y^0$.

Given two pointed coverings $f : (X, x^0) \to (Y, y^0)$, $f_1 : (X_1, x_1^0) \to (Y, y^0)$, a *pointed covering morphism* is a covering morphism $\phi : (X, x^0) \to (X_1, x_1^0)$ over (Y, y^0). The category of pointed coverings and pointed covering morphisms is denoted by **Cov**(Y, y^0) and the category of pointed connected coverings and pointed covering morphism is denoted by **Cov**$_0(Y, y^0)$.

Given a point $y^0 \in Y$, one can consider the fibre $f^{-1}(y^0)$. A covering $f : X \to Y$ is said to be finite sheeted if $f^{-1}(y)$ is finite for every $y \in Y$ and if $f^{-1}(y)$ has n elements for all $y \in Y$, the covering f is said to be finite n-sheeted. Note that if a space Y is connected and locally connected, then two fibres $f^{-1}(y^0)$, $f^{-1}(y^1)$ have the same cardinal.

The corresponding full subcategories of finite n-sheeted coverings are denoted by n-**Cov**(Y), n-**Cov**$_0(Y)$, n-**Cov**(Y, y^0), n-**Cov**$_0(Y, y^0)$ and the sets of isomorphisms classes by n-**Cov**$(Y)/ \cong$, n-**Cov**$_0(Y)/ \cong$, n-**Cov**$(Y, y^0)/ \cong$, n-**Cov**$_0(Y, y^0)/ \cong$. The same notation is used if n is an infinite cardinal.

1.2. Categories and pro-categories

In this paper the following categories and notations will be used:
Top the category of spaces and continuous maps.
Top * the category of pointed spaces (X, x^0) and continuous maps preserving the base point.
Grp the category of groups and homomorphisms.
TGrp the category of topological groups and continuous homomorphisms.
Grp $_R$ the category of right near-modules over a near-ring R, see [14,15].
Given a category **C**, the category **pro C** has as objects functors $X : I \to$ **C** where I is a left filtering small category and given two objects $X : I \to$ **C**, $Y : J \to$ **C** the hom-set is given by

$$\textbf{pro C}(X, Y) = \text{Lim}_j \text{Colim}_i \{ \textbf{C}(X_i, Y_j) \}.$$

For more properties related to pro-categories, we refer the reader to [16–18]. We denote by $\mathbb{N}$ the left filtering small category which has as objects the non-negative numbers $\{0, 1, \ldots\}$ and if $p, q \in \mathbb{N}$ there is a unique morphism $p \to q$ if and only if $p \geq q$.

A pro-object $X : I \to C$ in **pro C** is said to be a tower if $I = \mathbb{N}$ the left filtering small category of non-negative numbers. In some cases, if a pro-object is isomorphic to a tower, we also say that it is a tower. The full subcategory of **pro C** determined by towers is denoted by **tow C**.

A partially ordered set $(L, \leqslant)$ can be considered as a category having morphisms $x \to y$ associated with the relations $x \leqslant y$. A lattice L is a partially ordered set L which, considered as a category, has finite limits and colimits. Denote by $x \vee y$ the

coproduct, by $x \wedge y$ the product, by 0 the initial object and by 1 the final object. A distributive lattice L is a lattice in which the identity $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ holds for all x, y and z. A lattice A, which has infinite coproducts $\bigvee_{i \in I} y_i$ and infinite distributive identities $x \wedge \bigvee_{i \in I} y_i = \bigvee_{i \in I} x \wedge y_i$, is said to be a frame. A morphism of frames $\phi : B \to A$ is a map of partially ordered sets which preserves finite products and arbitrary coproducts.

　　Frames is the category of frames and frame morphisms.

　　Locales is the opposite category **Frames**^{op}.

　　loc Grp is the category of localic groups; that is, group objects in the category **Locales**.

　　Sh (X) is the category of sheaves over a topological space X.

　　Sh (X, x^0) is the pointed category of sheaves over a pointed topological space (X, x^0).

1.3. Closed model categories

　　The category **Top** of spaces and continuous maps admits a closed model structure in the sense of Quillen [19] induced by week equivalences and Serre fibrations. There are also induced closed model structures in the category of pointed spaces **Top*** and in the corresponding categories of prospaces and pro-pointed spaces **pro Top**, **pro Top***, see Edwards–Hastings monography [16]. The categories obtained by formal inversion of week equivalences will be denoted by **Ho(Top)**, **Ho**(**Top***), **Ho(pro Top)**, **Ho**(**pro Top***). We also consider the categories **pro Ho(Top)**, **pro Ho**(**Top***).

　　We refer to [16] for a proof of the following comparison theorem for towers between the hom-set of the category **Ho**(**pro Top***) and the hom-set in the category **pro Ho**(**Top***).

Theorem 1.3. *Let $X = \{X_i\}$, $Y = \{Y_j\}$ be towers of pointed spaces. Then the sequence*

$$0 \to \mathrm{Lim}_j^1 \mathrm{Colim}_i \{ \mathbf{Ho}(\mathbf{Top}^*)(\Sigma X_i, Y_j) \}$$

$$\to \mathbf{Ho}(\mathbf{pro\ Top}^*)(\{X_i\}, \{X_j\})$$

$$\to \mathrm{Lim}_j \mathrm{Colim}_i \{ \mathbf{Ho}(\mathbf{Top}^*)(X_i, Y_j) \} \to 0$$

is exact, where Σ is the suspension functor (A construction of the functors Lim, Lim^1 *can be seen in Section* 1.6.4.)

1.4. Expansions and resolutions

　　Given a pointed space (X, x^0), a pointed open cover $(\mathcal{U}, U^0)$ consists of an open cover $\mathcal{U}$ of X and an open subset U^0 such that $x^0 \in U^0 \in \mathcal{U}$. Given $(\mathcal{U}, U^0)$, $(\mathcal{V}, V^0)$ pointed open covers, it is said that $(\mathcal{U}, U^0) \geqslant (\mathcal{V}, V^0)$ if $U^0 \subseteq V^0$ and given any $U \in \mathcal{U}$ there is $V \in \mathcal{V}$ such that $U \subseteq V$. The set of pointed open covers of (X, x^0), directed by refinement $\geqslant$ will be denoted by COV(X, x^0). Similarly, one has the set COV(X) of open covers of X directed by refinement $\geqslant$ and a canonical cofinal map COV$(X, x^0) \to COV(X)$. Associated with an open cover $\mathcal{U}$ Porter [20] and Edwards–Hastings [16] consider the pointed Vietoris simplicial set

$$V_q'(X, x^0)(\mathcal{U}) = \{(x_0, \dots, x_q) | \text{ there is } U \in \mathcal{U}, \text{ such that } x_0, \dots, x_q \in U\}$$

for $q \geq 0$. Then we can consider the geometric realization functor to obtain $V_q(X, x^0)(\mathcal{U}) = |V_q'(X, x^0)(\mathcal{U})|$. The induced pro-pointed space

$$V(X, x^0): \mathrm{COV}(X, x^0) \to \mathbf{Top}^*$$

is the "expansion" used by Edwards–Hastings [16] to study strong shape theory of a general topological space. The composite

$$\mathrm{COV}(X, x^0) \to \mathbf{Top}^* \to \mathbf{Ho}(\mathbf{Top}^*)$$

will be denoted by $\bar{V}(X, x^0)$.

　　The pointed simplicial Čech nerve of a pointed open cover $(\mathcal{U}, U^0)$ is given by the simplicial set

$$C_q'(X, x^0)(\mathcal{U}, U^0) = \{(U_0, \dots, U_q) | U_0, \dots, U_q \in \mathcal{U}, \text{ and } U_0 \cap \cdots \cap U_q \neq \emptyset\}$$

and its realization is denoted by $\bar{C}_q(X, x^0)(\mathcal{U}, U^0) = |C_q'(X, x^0)(\mathcal{U}, U^0)|$. In this case, if $(\mathcal{U}, U^0) \geqslant (\mathcal{V}, V^0)$, we can choose for each $U \in \mathcal{U}$ an associated $\psi(U) \in \mathcal{V}$ such that $U \subset \psi(U)$ and $\psi(U^0) = V^0$. The map ψ induces a map $\bar{C}(X, x^0)(\mathcal{U}, U^0) \to \bar{C}(X, x^0)(\mathcal{V}, V^0)$ and two different elections induce homotopic maps. Therefore, we get a well-defined map $\bar{C}(X, x^0)(\mathcal{U}, U^0) \to \bar{C}(X, x^0)(\mathcal{V}, V^0)$ in the homotopy category **Ho**(**Top***). Then, as a consequence of the mentioned properties, a well-defined pro-object is obtained:

$$\bar{C}(X, x^0): \mathrm{COV}(X, x^0) \to \mathbf{Ho}(\mathbf{Top}^*).$$

　　By Dowker theorem, see [21], one has that the pro-objects $\bar{V}(X, x^0)$, $\bar{C}(X, x^0)$ are isomorphic in **proHo**(**Top***). If the space X is paracompact, these pro-objects are HPol-expansion in the sense of Mardešić–Segal, [17]. In the case of a compact metrizable space X, the directed sets COV(X, x^0), COV(X) have cofinal sequences $\{(\mathcal{U}_i, U_i^0)\}$, $\{\mathcal{U}_i\}$. If we denote

$$C_q(X, x^0)(\mathcal{U}_i, U_i^0) = \bar{C}_q(X, x^0)(\mathcal{U}_i, U_i^0),$$

we can choose in the bonding homotopy class a representative

$$C_q(X, x^0)(\mathcal{U}_{i+1}, U_{i+1}^0) \rightarrow C_q(X, x^0)(\mathcal{U}_i, U_i^0)$$

to obtain a tower of spaces that will be denoted by $C(X, x^0)$. Note that this construction is not functorial. However, if X is a metrizable compact space, $V(X, x^0)$ (which is functorial) and $C(X, x^0)$ are equivalent in the category **Ho**(**pro Top***).

Given a pointed topological space (X, x^0) and a pro-pointed space $\{(X_i, x_i^0)\}$ such that each X_i is an ANR (absolute neighbourhood retract), a pro-map $f : (X, x^0) \rightarrow \{(X_i, x_i^0)\}$ is an ANR-resolution if it satisfies the pointed version of the properties R1 and R2 given in [17, page 74]. If (X, x^0) is a paracompact space, then an ANR-resolution is equivalent to the Vietoris pro-pointed space $V(X, x^0)$ in the category **Ho**(**pro Top***). It is interesting to note that for a compact metric space (X, x^0) one can take an ANR-resolution $\{(X_i, x_i^0)|i \in \mathbb{N}\}$ indexed by the directed set of natural numbers, which is also equivalent to $C(X, x^0)$ in **Ho(pro Top*****)**.

If (X, x^0) is a pointed locally connected space, then the category **Sh**(X, x^0) of sheaves on X is a pointed locally connected site and one can consider the category of hypercoverings HC(**Sh**(X, x^0)). Taking the connected component set functor c_0: **Sh**$(X, x^0) \rightarrow$ **Sets** one has the Verdier pro-simplicial set $\prod(\mathbf{Sh}(X, x^0)) = \{c_0(K)|K \in \mathrm{HC}(\mathbf{Sh}(X, x^0))\}$. The realization, $|\prod(\mathbf{Sh}(X, x^0))|$, see [2], is isomorphic to $\bar{C}(X, x^0)$ in the category **pro Ho**(**Top***).

Definition 1.4. Let $Z : J \rightarrow$ **Top*** be a pro-pointed space, where the bonding morphism are denoted by $Z_i^j : Z_j \rightarrow Z_i$. The pro-pointed space Z is said to be S^n-movable, if for any $i \in J$ there is $j \rightarrow i$ in J such that for any $k \rightarrow j$ in J and for any pointed map $f : S^n \rightarrow Z_j$ there is a pointed map $g : S^n \rightarrow Z_k$ such that $Z_i^k g$ is homotopic relative to the base point to $Z_i^j f$. A pointed space (X, x_0) is said to be S^n-movable if the pro-pointed space $V(X, x_0)$ is S^n-movable.

Remark 1.5. We have a similar notion for a pro-object in **pro Ho**(**Top***). As a consequence of Dowker theorem, it is easy to check that $V(X, x_0)$ is S^n-movable if and only if $\bar{C}(X, x_0)$ is S^n-movable.

1.5. Fundamental groups and progroups

In this section, we recall different fundamental groups and progroups and introduce adequate notations in order to distinguish the diverse invariants.

1.5.1. The fundamental group of a pointed space
Let $(S^1, *)$ be the pointed 1-sphere. If (X, x^0) is a pointed space, then

$$\pi_1(X, x^0) = \mathbf{Ho}(\mathbf{Top}^*)((S^1, *), (X, x^0))$$

has a group structure and that group is called the *fundamental group* of (X, x^0).

1.5.2. The Spanier normal subgroup of an open cover
Given an open cover $\mathcal{U}$ of the space X, one can consider the normal subgroup $\pi_1(\mathcal{U}, x^0)$ of $\pi_1(X, x^0)$ generated by the elements a that can be represented as $a = [\alpha u \bar{\alpha}]$, where α is a path of X such that $\alpha(0) = x^0$, u is a loop of X contained in some $U \in \mathcal{U}$ and $\bar{\alpha}$ is the inverse path of α. The definition of this normal subgroup can be found in Spanier book [22]. Next we recall the Spanier classification of connected coverings of a connected locally path-connected space.

Definition 1.6. A subgroup H of $\pi_1(X, x^0)$ is said to be *S-open* (S from Spanier) if there is an open cover $\mathcal{U}$ such that $H \supseteq \pi_1(\mathcal{U}, x^0)$.

Using this definition and taking pointed connected coverings we reformulate the Spanier results on coverings in the following classification theorem:

Theorem 1.7. *The class of connected pointed coverings $(Y, y^0) \rightarrow (X, x^0)$ of a connected locally path-connected space (X, x^0) is bijective up to covering isomorphisms with the set of S-open subgroups of $\pi_1(X, x^0)$.*

1.5.3. Fundamental progroups
The fundamental group functor π_1: **Top**$^* \rightarrow$ **Grp** induces canonical functors

pro Top$^* \rightarrow$ **pro Grp**,

Ho(**pro Top***) $\rightarrow$ **pro Grp**,

pro Ho(**Top***) $\rightarrow$ **pro Grp**

denoted by proπ_1 in all cases. If a pro-pointed space X is a tower, then the progroup pro$\pi_1(X)$ will be also denoted by tow$\pi_1(X)$.

Given a pointed space (X, x^0) its fundamental progroup is given by

$$\text{pro}\pi_1(X, x^0) = \text{pro}\pi_1(\bar{V}(X, x^0)).$$

Since $\bar{V}(X, x^0)$ is isomorphic to $\bar{C}(X, x^0)$ in **pro Ho**(**Top***), it follows that $\text{pro}\pi_1(\bar{V}(X, x^0))$ is isomorphic to $\text{pro}\pi_1(\bar{C}(X, x^0))$. If (X, x^0) is a pointed locally connected space $\text{pro}\pi_1(\bar{C}(X, x^0))$ is also isomorphic to the Artin–Mazur progroup, $\text{pro}\pi_1(|\prod(\textbf{Sh}(X, x^0))|)$, see Section 1.4 and [2].

1.5.4. The Brown–Grossman–Quigley fundamental group

This kind of group has been given in different contexts, in proper homotopy theory by Brown [23], for towers of simplicial sets by Grossman [24], in (strong) shape theory by Quigley[25] and for exterior spaces by Calcines–Hernández-Pinillos [26].

In this paper we use the following notation:

Associated with the pointed 1-sphere S^1, we consider the following pro-pointed space

$$cS^1 = \{\vee_0^\infty S^1 \supseteq \vee_1^\infty S^1 \supseteq \vee_2^\infty S^1 \supseteq \cdots\}.$$

Given a pro-pointed space Z in **pro Top***, its Brown–Grossman–Quigley fundamental group is given by

$$\pi_1^{BGQ}(Z) = \textbf{Ho}(\textbf{pro Top}*)(cS^1, Z).$$

For a pointed space (X, x^0), we can consider the corresponding Vietoris nerve $V(X, x^0)$. The Brown–Grossman–Quigley fundamental group of (X, x^0) is given by

$$\pi_1^{BGQ}(X, x^0) = \textbf{Ho}(\textbf{pro Top}^*)(cS^1, V(X, x^0)).$$

Taking into account the remarks on pointed compact metrizable space (X, x^0) in Section 1.4, it follows that

$$\pi_1^{BGQ}(X, x^0) \cong \textbf{Ho}(\textbf{pro Top}^*)(cS^1, C(X, x^0)).$$

1.5.5. The Steenrod–Quigley fundamental group

Several analogues of this type of fundamental group have been given in different contexts, for example, in proper homotopy theory by Porter [27] and by Hughes and Ranicki [40], in homotopy theory of prospaces by Edwards–Hastings [16], in (strong) shape theory by Quigley [25], Cathey [41] and for exterior spaces by Calcines–Hernández-Pinillos [28,29]. The reason to include in the name 'Steenrod' is the existence of a Theorem of Hurewicz type between these groups and the Steenrod homology groups.

Consider the constant pro-pointed space $^{SQ}S^1 = \{(S^1, *)\}$. For a pointed space (X, x^0) its Steenrod–Quigley fundamental group is given by

$$\pi_1^{SQ}(X, x^0) = \textbf{Ho}(\textbf{pro Top}^*)(^{SQ}S^1, V(X, x^0)).$$

If (X, x^0) is a pointed compact metrizable space, then, by the same arguments as above, we get

$$\pi_1^{SQ}(X, x^0) \cong \textbf{Ho}(\textbf{pro Top}^*)(^{SQ}S^1, C(X, x^0)).$$

1.5.6. The Borsuk–Čech fundamental group

This group was introduced by Borsuk [30] and it can be defined (similarly to the case of the Čech homology) by taking an inverse limit. Given a pointed space (X, x^0) its Borsuk–Čech fundamental group is given by

$$\pi_1^{BC}(X, x^0) = \text{Lim}(\text{pro}\pi_1(X, x^0)).$$

Note that this group can be obtained also as the hom-set

$$\pi_1^{BC}(X, x^0) = \textbf{pro Ho}(\textbf{Top}^*)(^{SQ}S^1, \bar{V}(X, x^0))$$

or equivalently

$$\pi_1^{BC}(X, x^0) = \textbf{pro Ho}(\textbf{Top}^*)(^{SQ}S^1, \bar{C}(X, x^0)).$$

1.5.7. The fundamental localic group

We consider the following constructions in order to define the fundamental localic group of a locally connected pointed space (X, x^0).

The reader can find the basic facts about locals in the book [31] and in Moerdijk's paper [6].

Each topological space $(X, \mathcal{O}(X))$ has an associated frame $\mathcal{O}(X)$ in **Frames** and a locale loc$X$ in the opposite category **Locales**. For a locale L, $\mathcal{O}(L)$ also denotes the corresponding frame. A continuous map $f : X \to Y$ has an associated map loc$f$: loc$X \to$ loc$Y$ which is dual to the map $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$.

The restriction of functor loc: **Top** $\to$ **Locales** to sober spaces gives a full embedding. Another interesting property is that loc preserves products of locally compact spaces.

A *localic group* is a group object in the category of locales; a locale G equipped with maps $m : G \times G \to G$, $r : G \to G$ and $e : 1 \to G$ satisfying the usual identities, where 1 denotes the terminal object in the category of locales. The category of localic groups and localic morphisms will be denoted by **loc Grp**.

Since a discrete group is a locally compact group, one has that $\text{loc}(G \times G) \cong \text{loc}(G) \times \text{loc}(G)$. Using this property we can show that, for a discrete group G, loc$G$ has an induced localic group structure.

We also denote by loc the following functor

$$\text{loc} : \textbf{pro Grp} \to \textbf{loc Grp}$$

which carries a progroup $H = \{H_i | i \in I\}$ to $\text{loc}(H) = \text{Lim}\{\text{loc}(H_i)\}$, where H_i is considered as a discrete group and Lim is the inverse limit in the category **loc Grp**.

A localic group G is said to be *prodiscrete* if it is isomorphic to some $\text{loc}(H)$ for an inverse system $H = \{H_i | i \in I\}$ of discrete groups.

Given a pointed space (X, x^0), we can consider the fundamental progroup $\text{pro}\pi_1(X, x^0)$. The *fundamental localic group* of (X, x^0), denoted by $\text{loc}\pi_1(X, x^0)$, is given by

$$\text{loc}\pi_1(X, x^0) = \text{loc}(\text{pro}\pi_1(X, x^0)).$$

For a locally connected pointed space (X, x^0), one has that, in the category **pro Ho**(**Top***), the pro-pointed space $\bar{C}(X, x^0)$ is isomorphic to the realization of the pro-simplicial pointed set given by the Verdier functor $|\prod(\textbf{Sh}(X, x^0))|$. In this case we get

$$\text{loc}\pi_1(X, x^0) \cong \text{loc}\left(\text{pro}\pi_1 \left|\prod(\textbf{Sh}(X, x^0))\right|\right).$$

1.6. Well-known transformations between fundamental groups

1.6.1. Brown–Grossman–Quigley and Steenrod–Quigley homotopy groups

The Brown–Grossman–Quigley homotopy groups and the Steenrod–Quigley homotopy groups are related by a long exact sequence given in different contexts by Quigley [25], Porter [27], Hernández [32] and G. Pinillos-Hernández-Rivas [33].

Theorem 1.8. *Let* (X, x^0) *be a pointed space. Then the following long sequence*

$$\ldots \to \pi_{q+1}^{BGQ}(X, x^0) \to \pi_q^{SQ}(X, x^0) \to \pi_q^{BGQ}(X, x^0) \longrightarrow \pi_q^{BGQ}(X, x^0) \to$$

$$\ldots \to \pi_1^{BGQ}(X, x^0) \to \pi_0^{SQ}(X, x^0) \to \pi_0^{BGQ}(X, x^0) \longrightarrow \pi_0^{BGQ}(X, x^0)$$

is exact.

1.6.2. Fundamental progroups and Brown–Grossman–Quigley homotopy groups

In 1975, Brown [23] gave a definition of the proper fundamental group $^B\pi_1^{\infty}(X)$ of a σ–compact space X with a base ray. He also constructed a functor $\mathcal{P}^{\infty}$: **towGrp** $\longrightarrow$ **Grp** which gives the relation between the tower of fundamental groups $\{\pi_1(X_i)\}$, where X_i is a base of "neighbourhoods at infinity", and Brown's proper fundamental group.

There is another relation between global tower of groups and global versions of Brown-Grossman homotopy groups given by a global version $\mathcal{P}$ of Brown's functor. We refer the reader to [34] and [32] for the exact formulation of this global and other versions of this functor.

In the context of shape theory we have the following:

Theorem 1.9. *Let* (X, x^0) *be a pointed compact metrizable space. If* $\{(X_i, *) | i \geq 0\}$ *is a pointed ANR-resolution of* (X, x^0)*, then*

$$\pi_q^{BGQ}(X, x^0) \cong \mathcal{P}^{\infty}\{\pi_q(X_i, *)\}.$$

1.6.3. Fundamental progroup and fundamental localic group

In the next sections, the following result given in Moerdijk's paper [6] will be used.

A progroup $H = \{H_i | i \in I\}$ is said to be *surjective* if the bonding maps $H_i \rightarrow H_j$ are surjective. We denote by **pro Grp**$_{\text{surjective}}$ the full subcategory determined by surjective progroups. Similarly, **loc Grp**$|_{\text{prodiscrete}}$ denotes the full subcategory determined by prodiscrete localic groups.

Theorem 1.10. *The functor* loc: **pro Grp** $\rightarrow$ **loc Grp** *restricts to an equivalence of categories*

loc: **pro Grp**$_{\text{surjective}}$ $\rightarrow$ **loc Grp**$|_{\text{prodiscrete}}$.

1.6.4. Limit and first derived functor
In this paper we shall also use the inverse limit functor and its first derived in the case of tower of groups.

Let $\cdots \rightarrow G_2 \xrightarrow{p_1} G_1 \xrightarrow{p_0} G_0$ be a tower of groups. Consider the map $d : \prod_{i=0}^{\infty} G_i \rightarrow \prod_{i=0}^{\infty} G_i$ given by

$$d(g_0, g_1, g_2, \ldots) = (g_0^{-1} p_0(g_1), g_1^{-1} p_1(g_2), g_2^{-1} p_2(g_3), \ldots).$$

Then the inverse limit is given by $\text{Lim}\{G_i, p_i\} = \text{Ker} d$. We have the right action $\prod_{i=0}^{\infty} G_i \times \prod_{i=0}^{\infty} G_i \rightarrow \prod_{i=0}^{\infty} G_i$ given by $x \cdot g = (x_0, x_1, x_2, \ldots) \cdot (g_0, g_1, g_2, \ldots) = (g_0^{-1} x_0 p_0(g_1), g_1^{-1} x_1 p_1(g_2), g_2^{-1} x_2 p_2(g_3), \ldots)$. The pointed set of orbits of this action is denoted by $\text{Lim}^1\{G_i, p_i\}$ and is called the first derived of the Lim functor. We shall use notations $\text{Lim}\{G_i\}$ and $\text{Lim}^1\{G_i\}$ for short. For more properties of these functors we refer the reader to [18,16].

1.6.5. Stennrod-Quigley and Borsuk–Čech homotopy groups
The exact sequence given in Theorem 1.8 and the Lim and Lim^1 functors are related as follows.

Theorem 1.11. *Let* (X, x^0) *be a pointed compact metrizable space. If* $\{(X_i, *) | i \geq 0\}$ *is a pointed ANR-resolution of* (X, x^0), *then in the exact sequence*

$$\ldots \rightarrow \pi_{q+1}^{BGQ}(X, x^0) \rightarrow \pi_q^{SQ}(X, x^0) \rightarrow \pi_q^{BGQ}(X, x^0) \longrightarrow \pi_q^{BGQ}(X, x^0) \rightarrow \cdots$$

we have that $\text{Ker}(\pi_q^{BGQ}(X, x^0) \longrightarrow \pi_q^{BGQ}(X, x^0)) \cong \text{Lim}\{\pi_q(X_i, *)\}$ *and for* $q > 0$, $\text{Coker}(\pi_q^{BGQ}(X, x^0) \longrightarrow \pi_q^{BGQ}(X, x^0)) \cong \text{Lim}^1\{\pi_q(X_i, *)\}$. *Consequently, the following sequence is exact*

$$1 \rightarrow \text{Lim}^1\{\pi_q(X_i, *)\} \rightarrow \pi_q^{SQ}(X, x^0) \rightarrow \text{Lim}\{\pi_q(X_i, *)\} \rightarrow 1.$$

2. More fundamental (pro)groups and relations

In this section, we give some new versions of fundamental groups and progroups and we establish new relations between different invariants.

2.1. The Spanier fundamental topological group and progroup

In Section 1.5.2, for each open cover $\mathcal{U} \in \text{COV}(X)$ we have considered the normal subgroups $\pi_1(\mathcal{U}, x^0)$ introduced by Spanier [22].

This is the reason because we call Spanier fundamental topological group and Spanier fundamental progroup to the following invariants:

Definition 2.1. Let (X, x^0) be a pointed space. The fundamental group $\pi_1(X, x^0)$ provided with the unique topological group structure on $\pi_1(X, x^0)$ having as neighbourhood base at the identity element the family of normal subgroups

$$\{\pi_1(\mathcal{U}, x^0) | \mathcal{U} \in \text{COV}(X)\}$$

will be called the Spanier fundamental topological group and it will be denoted by $\Pi_1^{\text{Sp}}(X, x^0)$.

Let $\pi_1^{\text{Sp}}(X, \mathcal{U}, x^0)$ denote the quotient group

$$\pi_1^{\text{Sp}}(X, \mathcal{U}, x^0) = \pi_1(X, x^0)/\pi_1(\mathcal{U}, x^0).$$

It is easy to see that if $\mathcal{U} \geqslant \mathcal{V}$ then $\pi_1(\mathcal{U}, x^0) \subseteq \pi_1(\mathcal{V}, x^0)$. This inclusion induces an epimorphism $\pi_1^{\text{Sp}}(X, \mathcal{U}, x^0) \rightarrow \pi_1^{\text{Sp}}(X, \mathcal{V}, x^0)$.

Definition 2.2. Let (X, x^0) be a pointed space. The progroup

$$\{\pi_1^{\text{Sp}}(X, \mathcal{U}, x^0)\}_{\mathcal{U} \in \text{COV}(X)}$$

will be called the Spanier fundamental progroup and denoted by $\text{pro}\pi_1^{\text{Sp}}(X, x^0)$.

Let (X, x^0) be a connected locally path-connected pointed space. Take an open cover $\mathcal{U}$ such that if $U \in \mathcal{U}$ then U is path-connected and take $x^0 \in U^0 \in \mathcal{U}$. It is easy to check that there is an induced epimorphism

$$\pi_1^{Sp}(X, \mathcal{U}, x^0) \to \pi_1 \bar{C} X(\mathcal{U}, U^0).$$

The following open questions arise:

Under which topological conditions is $\pi_1^{Sp}(X, \mathcal{U}, x^0) \to \pi_1 \bar{C} X(\mathcal{U}, U^0)$ a monomorphism?

Are the progroups $\mathrm{pro}\pi_1^{Sp}(X, x^0)$, $\mathrm{pro}\pi_1(\bar{C}(X, x^0))$ isomorphic?

The following lemma and proposition give a partial answer to these questions:

Lemma 2.3. *Let $\mathcal{U}$ be an open cover of a topological space X such that*

(i) *If $U \in \mathcal{U}$ then U is path-connected,*
(ii) *If $U, V \in \mathcal{U}$ then $U \cap V$ is path-connected.*

If $x^0 \in U^0 \in \mathcal{U}$, then the natural epimorphism $\pi_1^{Sp}(X, \mathcal{U}, x^0) \to \pi_1 \bar{C} X(\mathcal{U}, U^0)$ is a monomorphism.

Proof. Consider the natural epimorphism $\theta : \pi_1^{Sp}(X, \mathcal{U}, x^0) \to \pi_1 C X(\mathcal{U}, U^0)$. Take a loop $f : I \to X, f(0) = x^0 = f(1)$ and suppose that $\theta([f])$ is given as the finite sequence of 1-simplexes associated with the vertexes $U_0, \ldots, U_{i-1}, U_i, U_{i+1}, \ldots, U_n = U_0$, $x^0 \in U_0$ and we have a partition $0 = t_0 < t_1 < \cdots < t_i < t_{i+1} < \cdots < t_n < t_{n+1} = 1$ such that $f(t_1) \in U_0 \cap U_1, \ldots, f(t_i) \in U_{i-1} \cap U_i, f(t_{i+1}) \in U_i \cap U_{i+1}, \ldots, f(t_n) \in U_{n-1} \cap U_0$. The elementary relations between chains of 1-simplexes are of the following form: If $U_{i-1} \cap U_i \cap U_{i+1} \neq \emptyset$, then the chain associated with $U_0, \ldots, U_{i-1}, U_i, U_{i+1}, \ldots, U_n = U_0$ is equivalent to the chain associated with $U_0, \ldots, U_{i-1}, U_{i+1}, \ldots, U_n = U_0$. Take $x \in U_{i-1} \cap U_i \cap U_{i+1}$, since $U_{i-1} \cap U_i, U_i \cap U_{i+1}$ are path-connected, there are paths α, β such that $\mathrm{Im}(\alpha) \subset U_{i-1} \cap U_i$ and $\alpha(0) = f(t_i), \alpha(1) = x$ and $\mathrm{Im}(\beta) \subset U_i \cap U_{i+1}$ and $\beta(0) = x, \beta(1) = f(t_{i+1})$. Note that the loop $\alpha\beta(\overline{f|_{[t_i, t_{i+1}]}})$ is contained in $U_i \in \mathcal{U}$. Therefore f and $f|_{[t_0, t_i]} \alpha\beta(\overline{f|_{[t_i, t_{i+1}]}})(\overline{f|_{[t_0, t_i]}})f$ represent the same element in $\pi_1^{Sp}(X, \mathcal{U}, x^0)$. However $f|_{[t_0, t_i]} \alpha\beta(\overline{f|_{[t_i, t_{i+1}]}})(\overline{f|_{[t_0, t_i]}})f$ is homotopic relative to $\{0, 1\}$ to $f_1 = f|_{[t_0, t_i]} \alpha\beta f|_{[t_{i+1}, 1]}$. Now we have that $\theta([f_1])$ is represented by the chain $U_0, \ldots, U_{i-1}, U_{i+1}, \ldots, U_n = U_0$ and that $\mathrm{Im}(f_1) \subset U_0 \cup U_i \cup U_{i+1} \cup U_n$. Now suppose that for $\theta([f])$ after m elementary equivalences, we get the trivial 1-simplexes chain associated to $U_0, U_1 = U_0$. We have f and f_m represent the same element in $\pi_1^{Sp}(X, \mathcal{U}, x^0)$. Since $\mathrm{Im}(f_m) \subset U_0$ one has that f_m and f represent the zero element in $\pi_1^{Sp}(X, \mathcal{U}, x^0)$. This implies that θ is a monomorphism. $\square$

Proposition 2.4. *Let X be a connected topological space having property that for any open cover $\mathcal{V}$ there exists a refinement $\mathcal{U}$ such that*

(i) *If $U \in \mathcal{U}$ then U is path-connected,*
(ii) *If $U, V \in \mathcal{U}$ then $U \cap V$ is path-connected.*

Then the progroups $\mathrm{pro}\pi_1^{Sp}(X, x^0)$, $\mathrm{pro}\pi_1(\bar{C}(X, x^0))$ are isomorphic.

Proof. From the properties (i) and (ii) it follows that the family

$$\{(\mathcal{U}, U^0) | x^0 \in U^0 \in \mathcal{U}, \mathcal{U} \text{ open cover satisfying (i) ,(ii)}\}$$

is cofinal in $\mathrm{COV}(X. x^0)$. Given $(\mathcal{U}, U^0)$ in the cofinal family, by Lemma 2.3 one has that $\pi_1^{Sp}(X, \mathcal{U}, x^0) \to \pi_1 \bar{C} X(\mathcal{U}, U^0)$ is an isomorphism. Then the progroups $\mathrm{pro}\pi_1^{Sp}(X, x^0)$, $\mathrm{pro}\pi_1(\bar{C}(X, x^0))$ are isomorphic. $\square$

As a consequence of the existence of the continuous homomorphisms

$$\Pi_1^{Sp}(X, x^0) \to \pi_1^{Sp}(X, \mathcal{U}, x^0) \to \pi_1 C X(\mathcal{U}, U^0),$$

we obtain an induced continuous homomorphism

$$\Pi_1^{Sp}(X, x^0) \to \pi_1^{BC}(X, x^0),$$

assuming that $\pi_1^{BC}(X, x^0)$ is provided with the inverse limit topology.

In general, this canonical homomorphism is neither an epimorphism nor a monomorphism. In the following example, this map is a monomorphism but not an epimorphism. In Steenrod's Reviews, subject van Kampen's theorem, there is a reference to an article which discusses a one point union of two cones over the earring. This space has no nontrivial covering space, though it is locally connected and has uncountable fundamental group. (See also Theorem 2.6 [35].) One can check that for this space the above map is not a monomorphism.

Example 2.5. The Hawaiian earring E is a compact metric space formed by the union of a sequence of circles $C_1, C_2, C_3, \ldots$ which are all tangent to each other at the same point and such that the sequence of radii converges to zero. In Fig. 1, circles $C_n, n \geq 1$, with center $(0, \frac{1}{n})$ and radius $\frac{1}{n}$ are chosen.
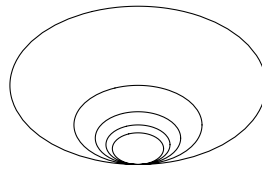
Fig. 1. The Hawaiian earring.

In this example, taking $x^0 = (0, 0)$, one has the following properties:

(i) There is a sequence of open covers $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n, \ldots$ such that $\mathcal{U}_{n+1}$ refines $\mathcal{U}_n$ and $\mathcal{U}_n$ satisfies the properties given in Lemma 2.3. We can also take $U_n^0 \in \mathcal{U}_n$ such that $x^0 \in U_n^0$ and $(\mathcal{U}_n, U_n^0)$ is cofinal in $\mathrm{COV}(E, x^0)$.

(ii) Moreover, one can take $\mathcal{U}_n$ such that $\bar{C}X(\mathcal{U}_n, U^0)$ has the homotopy type of a wedge of $(n-1)$ pointed circles.

(iii) The natural epimorphism $\pi_1^{\mathrm{Sp}}(E, \mathcal{U}_n, x^0) \to \pi_1 \bar{C}E(\mathcal{U}_n, U^0)$ is an isomorphism.

(iv) The progroups $\mathrm{pro}\pi_1^{\mathrm{Sp}}(E, x^0)$, $\mathrm{pro}\pi_1(\bar{C}(E, x^0))$ are isomorphic.

(v) The continuous homomorphism

$$\Pi_1^{\mathrm{Sp}}(E, x^0) \to \pi_1^{\mathrm{BC}}(E, x^0)$$

is not an epimorphism. Denote by x_i the generator of the fundamental group of the circle C_i, $i \geq 1$, Consider the sequence of elements

$$a_{n-1} = [x_1, x_2][x_1, x_3] \cdots [x_1, x_{n-1}] \in \pi_1 \bar{C}E(\mathcal{U}_n, U^0),$$

where $[x, y] = xyx^{-1}y^{-1}$ is the commutator of x, y. Since the map

$$\pi_1 \bar{C}E(\mathcal{U}_{n+1}, U^0) \to \pi_1 \bar{C}E(\mathcal{U}_n, U^0)$$

carries x_i to x_i for $i < n$ and x_n to 1, it follows that a_n is carried to a_{n-1}. Therefore $\{a_n\}_{n \geq 1}$ determines an element in $\pi_1^{\mathrm{BC}}(E, x^0) = \mathrm{Lim}\, \pi_1 \bar{C}E(\mathcal{U}_n, U^0)$. Notice that x_1 is included $(n-1)$ times in the reduced word of a_n for $n \geq 3$. Nevertheless, if $b \in \Pi_1^{\mathrm{Sp}}(E, x^0)$ and $N_n(x_1)$ denotes the number of times that x_1 appears in the reduced image of b by the map $\Pi_1^{\mathrm{Sp}}(E, x^0) \to \pi_1 \bar{C}E(\mathcal{U}_n, U^0)$, one has that there is some N_0 such that $N_n(x_1) \leq N_0$ for all $n \geq 1$. This implies that $\{a_n\}_{n \geq 1}$ is not in the image of the map $\Pi_1^{\mathrm{Sp}}(E, x^0) \to \pi_1^{\mathrm{BC}}(E, x^0)$.

(vi) The continuous homomorphism $\Pi_1^{\mathrm{Sp}}(E, x^0) \to \pi_1^{\mathrm{BC}}(E, x^0)$ is injective. Different proofs have been given by Morgan and Morrison [36], de Smit [37] and Fabel [38].

We have given a reformulation of Spanier [22] results in terms of S-open subgroups in Theorem 1.7 and taking transitive representations, see Definition 3.1, we have the following equivalent version:

Theorem 2.6. *If (X, x^0) is a connected locally path-connected pointed space and (F, x^0) is a discrete pointed space whose cardinal is denoted by $|F|$, then $|F|$-$\mathbf{Cov}_0(X, x^0)/ \cong$ is bijective to the set of transitive continuous representations of the form $\Pi_1^{\mathrm{Sp}}(X, x^0) \to \mathrm{Aut}(F)$.*

Proof. Notice that for an S-open subgroup H of $\Pi = \Pi_1^{\mathrm{Sp}}(X, x^0)$ such that $(\Pi/H, H)$ is pointed bijective to (F, x^0) we have an induced transitive continuous representation $\Pi \to \mathrm{Aut}(F)$ given by $a \cdot (bH) = (ab)H$ for $a, b \in \Pi$. Conversely, for a transitive continuous representation $\Pi \to \mathrm{Aut}(F)$ and a base point x^0 in F we have the S-open subgroup given by the isotropy group at x^0. $\square$

Note that, for a locally path-connected connected space X, there does not exist a connected covering of X having fibres whose cardinality is higher than the cardinal of its fundamental group.

Recall that $\pi_1^{\mathrm{Sp}}(X, \mathcal{U}, x^0) = \pi_1(X, x^0)/\pi_1(\mathcal{U}, x^0)$. Then the family of continuous epimorphisms $\Pi_1^{\mathrm{Sp}}(X, x^0) \to \pi_1^{\mathrm{Sp}}(X, \mathcal{U}, x^0)$ induces a pro-map

$$\phi \colon \Pi_1^{\mathrm{Sp}}(X, x^0) \to \{\pi_1^{\mathrm{Sp}}(X, \mathcal{U}, x^0)\} = \mathrm{pro}\pi_1^{\mathrm{Sp}}(X, x^0).$$

Proposition 2.7. *Let D be a discrete group. Then the map ϕ induces an isomorphism*

$$\phi^* \colon \mathbf{pro\ Grp}(\mathrm{pro}\pi_1^{\mathrm{Sp}}(X, x^0), D) \to \mathbf{TGrp}(\Pi_1^{\mathrm{Sp}}(X, x^0), D).$$

Proof. It follows from the fact that $\Pi_1^{\mathrm{Sp}}(X, x^0) \to \pi_1^{\mathrm{Sp}}(X, \mathcal{U}, x^0)$ is an epimorphism and one has to take into account that $\{\pi_1(\mathcal{U}, x^0)\}$ is a base of neigbourhoods at the identity element. $\square$

2.2. The near-module structure of the fundamental Brown–Grossman–Quigley homotopy group and the $\mathcal{P}$-functor

Associated with the category **C** with coproducts, one has the category, **tow C**, of towers in **C** and the category, **pro C**, of pro-objects in **C**. An object G in **C** induces a pro-object $cG \colon \mathbb{N} \longrightarrow C$ given by

$$(cG)_i = \sum_{j \geq i} G, \quad i \in \mathbb{N}$$

and the bonding maps are given by inclusions

$$(cG)_{i+1} = \sum_{j \geq i+1} G \longrightarrow \sum_{j \geq i} G = (cG)_i.$$

Associated with the infinity cyclic group C_∞ of the category of groups **Grp**, we have the pro-object cC_∞

$$(cC_\infty)_i = \star_{j \geq i} C_\infty, \quad i \in \mathbb{N}.$$

The endomorphism group $\mathcal{P}C_\infty = \textbf{pro Grp}(C_\infty, C_\infty)$ has the structure of a zero-symmetric unitary left near-ring. For any object X of **pro Grp**, we consider the natural action

$$\textbf{pro Grp}(cC_\infty, X) \times \textbf{pro Grp}(cC_\infty, cC_\infty) \longrightarrow \textbf{pro Grp}(cC_\infty, X)$$

which associates (f, φ) to $f\varphi$, for every $f \in \textbf{pro Grp}(cC_\infty, X)$ and every $\varphi \in \textbf{pro Grp}(cC_\infty, cC_\infty)$.

The morphism set **pro Grp**(cC_∞, X) has a group structure and the action satisfies the left distributive law:

$$f(\alpha + \beta) = f\alpha + f\beta, \quad f \in \textbf{pro Grp}(cC_\infty, X), \quad \alpha, \beta \in \mathcal{P}cC_\infty.$$

Notice that the sum $+$ need not be commutative. In this case, $\mathcal{P}cC_\infty$ becomes a zero-symmetric unitary left near-ring and **pro Grp**(cC_∞, X) is a right $\mathcal{P}cC_\infty$-group; that is, a right near-module over the near ring $\mathcal{P}cC_\infty$, see [14,15].

Denote by $\textbf{Grp}_{\mathcal{P}cC_\infty}$ the category of $\mathcal{P}cC_\infty$-groups (near-modules).

Then the enriched version of Brown's functor, denoted by $\mathcal{P}$ instead of $\mathcal{P}^\infty$, is the functor $\mathcal{P} \colon \textbf{pro Grp} \to \textbf{Grp}_{\mathcal{P}cC_\infty}$, given by $\mathcal{P}X = \textbf{pro Grp}(cC_\infty, X)$ for a progroup X and similarly for morphisms. Properties of this version of the functor are studied in [32].

For a group Y, we can consider the reduced product $IY = \prod_{i \geq 0} Y / \sim$ of Y as the quotient of $\prod_{i \geq 0} Y$ given by the relation: $(y_0, y_1, \ldots) \sim (y_0', y_1', \ldots)$ if there is a positive integer k_0 such that for $k \geq k_0$, $y_k = y_k'$. We denote by $[(y_0, y_1, \ldots)]$ the equivalence class of $(y_0, y_1, \ldots)$. The same construction can be given for a set, pointed set or abelian group.

We remark that a group Y can be considered as a constant progroup, and in this case, one has that $\mathcal{P}Y \cong IY$. Moreover, IY has the structure of a right $\mathcal{P}cC_\infty$-group.

If F is a set, then there exists a natural transformation $\eta \colon \mathcal{P}\text{Aut}F \to \text{Aut}\mathcal{P}F$, given by $\eta[(f_0, f_1, \ldots)]([z_0, z_1, \ldots)] = [(f_0(z_0), f_1(z_1), \ldots)]$, where $f_i \in \text{Aut}F$ and $z_i \in F, i \in \mathbb{N}$.

Proposition 2.8. *If F is a finite set, then $\eta \colon \mathcal{P}\text{Aut}F \to \text{Aut}\mathcal{P}F$ is a monomorphism.*

Proof. Suppose that $\eta[(f_0, f_1, \ldots)] = \eta[(f_0', f_1', \ldots)]$, then for every $p = [(z_0, z_1, \ldots)]$, there is i_p such that $f_i(z_i) = f_i'(z_i)$ for all $i \geq i_p$. For each $z \in F$ we have the element $p(y) = [(y, y, \ldots)]$ and the positive integer $i_{p(z)}$ verifies $f_i(z) = f_i'(z)$ for all $i \geq i_{p(z)}$. Then if $i_0 = \max\{i_{p(z)} | z \in F\}$ one has that $f_i(z) = f_i'(z)$ for all $i \geq i_0$ and all $z \in F$. This implies that $f_i = f_i'$ for all $i \geq i_0$. Therefore $[(f_0, f_1, \ldots)] = [(f_0', f_1', \ldots)]$, that is, η is a monomorphism. $\square$

Definition 2.9. Let X be a progroup. A group homomorphism $\theta \colon \mathcal{P}X \to \text{Aut}\mathcal{P}F$ is said to be a $\mathcal{P}$-factorizable representation if there is a $\mathcal{P}cC_\infty$-group homomorphism $\phi \colon \mathcal{P}X \to \mathcal{P}\text{Aut}F$ such that $\eta\phi = \theta$.

Definition 2.10. An object X of **tow Grp** is said to be finitely generated if there is an (effective) epimorphism of the form $\sum_{finite} cC_\infty \longrightarrow X$.

The following theorem has been proved in [34].

Theorem 2.11. *The restriction functor $\mathcal{P} \colon \textbf{tow Grp}/fg \longrightarrow \textbf{Grp}_{\mathcal{P}cC_\infty}$ is a full embedding, where $\textbf{tow Grp}/fg$ denotes the full subcategory of $\textbf{tow Grp}$ determined by finitely generated towers.*

For a given pro-pointed space Z, we can consider the action given by

$$\textbf{Ho}(\textbf{pro Top}^*)(cS^1, Z) \times \textbf{Ho}(\textbf{pro Top}^*)(cS^1, cS^1) \to \textbf{Ho}(\textbf{pro Top}^*)(cS^1, Z).$$

Now we can apply the comparison Theorem 1.3 to obtain that

$$\begin{aligned}
\textbf{Ho}(\textbf{pro Top}^*)(cS^1, cS^1) &\cong \text{Lim}_j\text{Colim}_i\{\textbf{Ho}(\textbf{Top}^*)(cS_i^1, cS_j^1)\} \\
&\cong \text{Lim}_j\text{Colim}_i\{\textbf{Grp}((C_\infty)_i, (C_\infty)_j)\} \\
&\cong \textbf{pro Grp}(cC_\infty, cC_\infty) = \mathcal{P}cC_\infty.
\end{aligned}$$

Therefore, the Brown–Grossman–Quigley group $\pi_1^{BGQ}(Z)$ of a pro-pointed space has an enriched structure of a near-module over the near-ring $\mathcal{P}cC_\infty$. Accordingly, we say that $\pi_1^{BGQ}(Z)$ has the structure of a $\mathcal{P}cC_\infty$-group.

Definition 2.12. Given a pro-pointed space Z, the group $\pi_1^{BGQ}(Z)$ enriched with $\mathcal{P}cC_\infty$-group structure is said to be the Brown–Grossman–Quigley fundamental $\mathcal{P}cC_\infty$-group and it will be denoted by $\Pi_1^{BGQ}(Z)$. If (X, x_0) is a pointed space the Brown–Grossman–Quigley fundamental $\mathcal{P}cC_\infty$-group is given by $\Pi_1^{BGQ}(V(X, x_0))$.

It is easy to check that the following diagram is commutative up to isomorphisms.
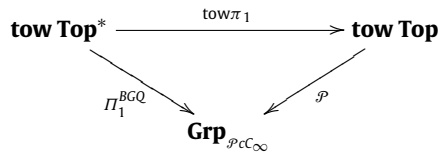
Proposition 2.13. *Let Z be a tower of pointed spaces. Then, there exists a natural $\mathcal{P}cC_\infty$-group isomorphism, $\Pi_1^{BGQ}(Z) \cong \mathcal{P}\text{tow}\pi_1(Z)$.*

2.2.1. The Borsuk–Čech topological fundamental group

The Borsuk–Čech fundamental group of a pointed space (X, x^0) is given by $\pi_1^{BC}(X, x^0) = \text{Lim pro}\pi_1(X, x^0)$. This group can be enriched with the following topological structure: In the progroup $\text{pro}\pi_1(X, x^0) = \{\pi_1(\bar{C}(X, x_0)(\mathcal{U}, U_0))\}$, we take the discrete topology on each $\pi_1(\bar{C}(X, x_0)(\mathcal{U}, U_0))$ and then the inverse limit topology on the inverse limit group $\pi_1^{BC}(X, x^0)$. The corresponding topological group is denoted by $\Pi_1^{BC}(X, x^0)$.

2.2.2. The Steenrod–Quigley topological fundamental group of a connected compact metrizable space

If (X, x^0) is a pointed connected compact metrizable space we have that

$$\pi_1^{SQ}(X, x^0) \cong \mathbf{Ho}(\mathbf{pro\,Top}^*)(^{SQ}S^1, C(X, x^0))$$

and, by Theorem 1.11, there exists a canonical epimorphism

$$\pi_q^{SQ}(X, x^0) \to \Pi_1^{BC}(X, x^0).$$

This canonical map induces the initial topology on $\pi_q^{SQ}(X, x^0)$ satisfying that $\pi_1^{SQ}(X, x^0) \to \Pi_1^{BC}(X, x^0)$ is continuous and any other topology with this property is finer than the initial topology. The group $\pi_1^{SQ}(X, x^0)$ endowed with the above topology is denoted by $\Pi_1^{SQ}(X, x^0)$ and is called the Steenrod–Quigley topological fundamental group of a connected compact metrizable space.

The Kernel of this map $\text{Lim}^1\{\pi_2(X_i, *)\}$ is contained in the intersection of all neighbourhoods of the identity element. Then we have

Proposition 2.14. *Let (X, x^0) be a pointed connected compact metrizable space.*

(i) *If (X, x^0) is S^2-movable, then the map $\Pi_1^{SQ}(X, x^0) \to \Pi_1^{BC}(X, x^0)$ is a homeomorphism,*
(ii) *If D is a discrete group, the induced map*

$$\mathbf{TGpr}(\Pi_1^{BC}(X, x^0), D) \to \mathbf{TGpr}(\Pi_1^{SQ}(X, x^0), D)$$

is an isomorphism.

3. Representations of groups and progroups and finite sheeted coverings

Firstly, we recall the notion of transitive representation for groups.

Definition 3.1. Given a group G, a representation $\eta: G \to \text{Aut}(S)$ into the group of automorphism of a set S is said to be transitive if for every $a, b \in S$ there exists $g \in G$ such that $\eta(g)(a) = b$.

The notion of transitive representation for progroups can be given as follows:

Definition 3.2. Given a progroup $\{G_i\}$ with bonding homomorphisms $G_i^j: G_j \to G_i$ and a set S, a representation $\eta: \{G_i\} \to \text{Aut}(S)$ is said to be transitive if for any homomorphism $f_i: G_i \to \text{Aut}(S)$ representing the pro-map, $\eta = [f_i]$, there exists a map $j \to i$ in I such that for any $k \to j$ the induced representation $f_i G_i^k: G_k \to \text{Aut}(S)$ is transitive.

Note that, for a finite set S, a pro-homomorphism $\eta: \{G_i\} \to \text{Aut}(S)$ represented by $f_{i_0}: G_{i_0} \to \text{Aut}(S)$ has the property that the induced progroup $\{\text{Im}(f_{i_0} G_{i_0}^j)\}$ is isomorphic to a constant progroup. In this case, the action is transitive if the action of this constant progroup (group) is transitive.

Now we can use the functors $\mathcal{P}: \mathbf{pro\,Grp} \to \mathbf{Grp}_{\mathcal{P}cC_\infty}$ and loc: $\mathbf{pro\,Grp} \to \mathbf{loc\,Grp}$ to define the following transitivity analogues:

Definition 3.3. Let S be a finite set. A $\mathcal{P}$-factorizable representation

$$\theta: \pi_1^{BGQ}(X, x^0) \to \operatorname{Aut}(\mathcal{P}S), \ \theta = \eta\phi,$$

is said to be transitive (see Definition 2.9) if there exists a transitive representation $\psi: \operatorname{pro}\pi_1(X, x^0) \to \operatorname{Aut}(S)$ such that $\mathcal{P}(\psi) = \phi$.

Similarly, a localic representation $\theta: \operatorname{loc}\pi_1(X, x^0) \to \operatorname{locAut}(S)$ is said to be transitive if there exists a transitive representation $\eta: \operatorname{pro}\pi_1(X, x^0) \to \operatorname{Aut}(S)$ such that $\operatorname{loc}(\eta) = \theta$.

In the following proposition, we show that a connected locally connected space is S^1-movable.

Proposition 3.4. *Let X be a connected, locally connected space. Then, for any base point $x_0 \in X$, the pointed space (X, x_0) is S^1-movable.*

Proof. By Definition 1.4 and Remark 1.5 one can prove that $\bar{C}(X, x_0) = \{\bar{C}(X, x_0)(\mathcal{U}, U_0)\}$ is S^1-movable. Let $(\mathcal{U}, U_0)$ be a pointed open cover. Since X is locally connected, there exists a pointed open cover $(\mathcal{V}, V_0)$ which refines $(\mathcal{U}, U_0)$ and such that any $V \in \mathcal{V}$ is connected (in particular V_0 is connected). Let $(\mathcal{U}', U_0')$ be a pointed open cover which refines $(\mathcal{V}, V_0)$. Now we take a pointed open cover $(\mathcal{W}, W_0)$ such that any $W \in \mathcal{W}$ is connected, $(\mathcal{W}, W_0)$ refines $(\mathcal{U}', U_0')$ and we can suppose that $(\mathcal{W}, W_0)$ has the following additional property: if $W_1 \subset W$, $W \in \mathcal{W}$, W_1 is a non-empty connected open subset, then $W_1 \in \mathcal{W}$. Take $V_0, V_1, \ldots, V_n = V_0$ a chain such that $V_i \cap V_{i+1} \neq \emptyset$. Note that $x_0 \in V_n \cap V_0$ and take $x_i \in V_{i-1} \cap V_i$. The connected subspace V_i determines the following family $\{W \in \mathcal{W} | W \subset V_i\}$. Since V_i is connected and $\{W \in \mathcal{W} | W \subset V_i\}$ is an open cover of V_i there is $W_0^i, \ldots W_{k_i}^i \in \mathcal{W}$ such that $x_{i-1} \in W_0^i$, $W_{j-1}^i \cap W_j^i \neq \emptyset$, $x_i \in W_{k_i}^i$. Now the canonical map $\pi_1(C(X, x_0)(\mathcal{W}, W_0)) \to \pi_1(C(X, x_0)(\mathcal{U}', U_0')) \to \pi_1(C(X, x_0)(\mathcal{V}, V_0))$ applies the loop represented by $W_0^1, \ldots, W_{k_1}^1, W_0^2, \ldots W_{k_2}^2, \ldots, W_0^n, \ldots W_{k_n}^n$, $W_0^1 = W_0 = W_{k_n}^n$ into a loop which is homotopic to one represented by $V_0, V_1, \ldots, V_n$. Therefore $\bar{C}(X, x_0) = \{\bar{C}(X, x_0)(\mathcal{U}, U_0)\}$ is S^1-movable. Accordingly, (X, x_0) is S^1-movable. □

Next we give the main result of the paper. We remark that, in some cases, the restriction to compact metrizable spaces or to coverings satisfying the finite fibre condition is not necessary. Firstly, we include Table 1 which explains the notation and refers to papers or sections where it is introduced.

Theorem 3.5. *Let (X, x^0) be a connected compact metrizable pointed space. Consider the set $n - \mathbf{Cov}_0(X, x^0)/\cong$ of n-sheeted connected pointed coverings $p: (Y, y^0) \to (X, x^0)$ up to covering isomorphisms. Then*

(i) *$n - \mathbf{Cov}_0(X, x^0)/\cong$ is bijective to the set of transitive representations of the form $\operatorname{pro}\pi_1(X, x^0) \to \Sigma_n$,*

(ii) *Let F be a finite set with n elements. Then $n - \mathbf{Cov}_0(X, x^0)/\cong$ is bijective to the subset of hom-set*

$$\mathbf{Grp}_{\mathcal{P}cC_\infty}(\Pi_1^{BGQ}(X, x^0), \mathcal{P}\operatorname{Aut}(F)),$$

which is in a biunivoque correspondence with $\mathcal{P}$-factorizable transitive representations of the form $\pi_1^{BGQ}(X, x^0) \to \operatorname{Aut}(\mathcal{P}F)$.

(iii) *If (X, x^0) is S^1-movable, then $n - \mathbf{Cov}_0(X, x^0)/\cong$ is bijective to the set of transitive continuous representations of the form $\Pi_1^{CB}(X, x^0) \to \Sigma_n$.*

(iv) *If (X, x^0) is S^1-movable, then $n - \mathbf{Cov}_0(X, x^0)/\cong$ is bijective to the set of transitive continuous representations of the form $\Pi_1^{SQ}(X, x^0) \to \Sigma_n$.*

(v) *If (X, x^0) is locally connected, then $n - \mathbf{Cov}_0(X, x^0)/\cong$ is bijective to the set of transitive representations of the form*

$$\operatorname{pro}\pi_1(X, x^0) \cong \operatorname{pro}\pi_1\left(\left|\prod(\mathbf{Sh}(X, x^0))\right|\right) \to \Sigma_n.$$

(vi) *If (X, x^0) is locally connected, then $n - \mathbf{Cov}_0(X, x^0)/\cong$ is bijective to the set of transitive localic representations of the form*

$$\operatorname{loc}\pi_1(X, x^0) \cong \operatorname{loc}\pi_1\left(\left|\prod(\mathbf{Sh}(X, x^0))\right|\right) \to \operatorname{loc}\Sigma_n.$$

(vii) *If (X, x^0) is locally connected, then $n - \mathbf{Cov}_0(X, x^0)/\cong$ is bijective to the set of transitive continuous representations of the form $\Pi_1^{CB}(X, x^0) \to \Sigma_n$.*

(viii) *If (X, x^0) is locally path-connected, then $n - \mathbf{Cov}_0(X, x^0)/\cong$ is bijective to the set of transitive continuous representations of the form $\Pi_1^{Sp}(X, x^0) \to \Sigma_n$.*

(ix) *If (X, x^0) is locally path-connected, then $n - \mathbf{Cov}_0(X, x^0)/\cong$ is bijective to the set of transitive representations of the form $\operatorname{pro}\pi_1^{Sp}(X, x^0) \to \Sigma_n$.*

(x) *If (X, x^0) is locally path-connected, and semi-locally 1-connected, then $n - \mathbf{Cov}_0/\cong$ is bijective to the set of representations of the form $\pi_1(X, x^0) \to \Sigma_n$.*

Table 1

$\mathrm{pro}\pi_1(X, x^0)$	Fundamental progroup	[16,4,17,18]
Σ_n	Symmetric group or degree n	
C_∞	Infinite cyclic group	
cC_∞	Tower of groups; $(cC_\infty)_i = \star_{j \geq i} C_\infty, \ \ i \in \mathbb{N}$	[34,32]
$\mathcal{P}$	Brown's functor	[23,34]
$\mathcal{P}cC_\infty$	Endomorphism near-ring	[34,14,15]
Grp	Category of groups	
Grp$_{\mathcal{P}cC_\infty}$	Category of near-modules over $\mathcal{P}cC_\infty$	[14,34]
$\Pi_1^{BGQ}(X, x^0)$	Brown–Grossman–Quigley fundamental group enriched with a near-module structure over $\mathcal{P}cC_\infty$	[34,32]
$\pi_1^{BGQ}(X, x^0)$	Brown–Grossman–Quigley fundamental group (without the structure of a near-module)	[24,23,25], [27,39,34]
S^1-movable pro-pointed space	Their bonding morphisms satisfy certain lifting property	Definition 1.4
S^1-movable pointed space	Pointed space which has a S^1-movable resolution	Definition 1.4
$\Pi_1^{BC}(X, x^0)$	Borsuk–Čech fundamental group enriched with the prodiscrete topology	[16,5] Section 2.2.1
$\Pi_1^{SQ}(X, x^0)$	Šteenrod-Quigley fundamental group enriched with a topology	[25,16] Section 2.2.2
Sh(X, x^0)	The category of sheaves **Sh**(X) over a space X with a base point $\mathbf{Sh}(X) \to \mathbf{Sets}$, where **Sets** is the category of sets	[31]
$\| \prod(\mathbf{Sh}(X, x^0)) \|$ X loc connected	Pro-object in Ho(Top*) obtained using the realization and the Verdier $\prod$ functors	[2]
$\mathrm{pro}\pi_1(\| \prod(\mathbf{Sh}(X, x^0)) \|)$ X loc connected	Artin–Mazur fundamental progroup obtained using the Verdier functor	[2]
Frames	Category of complete Heyting algebras	[31,6]
Locales	Dual category of **Frames**	[31,6]
Localic group	Group object in **Locales**	[31,6]
Localic fundamental group $\mathrm{loc}\pi_1(X, x^0)$	Localic inverse limit of the groups of the fundamental progroup	[6]
$\mathrm{loc}\pi_1(\| \prod(\mathbf{Sh}(X, x^0)) \|)$ X loc connected	Localic inverse limit of the groups of the fundamental Artin–Mazur progroup	[2,6]
$\mathrm{locAut}(F)$	Localic version of the discrete group Aut(F)	
$\Pi_1^{Sp}(X, x^0)$	Spanier fundamental topological group	Definition 2.1
$\mathrm{pro}\Pi_1^{Sp}(X, x^0)$	Spanier fundamental progroup	Definition 2.2
$\pi_1(X, x^0)$	Fundamental group	

Proof. We remark that (i) is well known, you can see a proof in [4] using the terminology of tropes of groups in [7] or in [9] and (viii) has been proved in Theorem 2.6.

(i) implies (ii): The functor $\mathcal{P} \colon \mathbf{pro\ Grp} \to \mathbf{Grp}_{\mathcal{P}cC_\infty}$ induces the following hom-set map

$$\mathbf{pro\ Grp}(\mathrm{pro}\pi_1(X, x^0), \mathrm{Aut}(F)) \to \mathbf{Grp}_{\mathcal{P}cC_\infty}(\mathcal{P}(\mathrm{pro}\pi_1(X, x^0)), \mathcal{P}\mathrm{Aut}(F)).$$

In order to apply Theorem 2.11, we have to prove that $\mathrm{pro}\pi_1(X, x^0)$ and $\mathrm{Aut}(F)$ are finitely generated in the sense of Definition 2.10. Since (X, x^0) is a connected compact metrizable pointed space, we can take a cofinal sequence of open covers $(\mathcal{U}_0, U_0) \leq (\mathcal{U}_1, U_1) \leq \cdots \leq (\mathcal{U}_i, U_i)$ such that for every $i \geq 0$ the Čech nerve $C(X, x^0)(\mathcal{U}_i, U_i)$ is a finite CW-complex with finite presented fundamental group and a finite set of generators S_i. Denote by $F[S]$ the free group generated by a set S. We can take the following tower of groups $\cdots \to F[\sqcup_{i=2}^\infty S_i] \to F[\sqcup_{i=1}^\infty S_i] \to F[\sqcup_{i=0}^\infty S_i]$ which is isomorphic to cC_∞. The family of epimorphisms $F[S_i] \to \pi_1(C(X, x^0)(\mathcal{U}_i, U_i))$ induces an epimorphism $F[\sqcup_{i \geq i}^\infty S_i] \to \pi_1(C(X, x^0)(\mathcal{U}_i, U_i))$ for each i and then we have an effective epimorphism $cC_\infty \to \mathrm{pro}\pi_1(X, x^0)$. Therefore $\mathrm{pro}\pi_1(X, x^0)$ is finitely generated. Now since $\mathrm{Aut}(F)$ is a finite group, there is a finite set S of generators and a natural epimorphism $F[S] \to \mathrm{Aut}(F)$. Then we get induced epimorphisms $F[\sqcup_{k \geq i}^\infty S] \to \mathrm{Aut}(F)$ and an effective epimorphism $cC_\infty \to \mathrm{Aut}(F)$. Now, applying Theorem 2.11 we obtain that

$$\mathbf{pro\ Grp}(\mathrm{pro}\pi_1(X, x^0), \mathrm{Aut}(F)) \cong \mathbf{Grp}_{\mathcal{P}cC_\infty}(\mathcal{P}(\mathrm{pro}\pi_1(X, x^0)), \mathcal{P}\mathrm{Aut}(F)).$$

By Proposition 2.13, we have that $\mathcal{P}(\mathrm{pro}\pi_1(X, x^0)) \cong \Pi_1^{BGQ}(X, x^0)$. Then

$$\mathbf{pro\ Grp}(\mathit{pro}\pi_1(X, x^0), \mathrm{Aut}(F)) \cong \mathbf{Grp}_{\mathcal{P}cC_\infty}(\Pi_1^{BGQ}(X, x^0), \mathcal{P}\mathrm{Aut}(F)).$$

By Proposition 2.8, if F is a finite set, then $\eta \colon \mathcal{P}\mathrm{Aut}F \to \mathrm{Aut}\mathcal{P}F$ is a monomorphism. Thus

$$\mathbf{Grp}_{\mathcal{P}cC_\infty}(\Pi_1^{BGQ}(X, x^0), \mathcal{P}\mathrm{Aut}(F)) \cong \{\theta \in \mathbf{Grp}(\pi_1^{BGQ}(X, x^0), \mathrm{Aut}(\mathcal{P}F)) | \theta \text{ is } \mathcal{P}-\text{factorizable}\}.$$

Therefore, we also have

$$\{\psi \in \mathbf{pro\,Grp}(\mathrm{pro}\pi_1(X, x^0), \mathrm{Aut}(F))| \psi \text{ is transitive}\}$$

$$\cong \{\theta \in \mathbf{Grp}(\pi_1^{BGQ}(X, x^0), \mathrm{Aut}(\mathscr{P}F))|\theta \text{ is } \mathscr{P}\text{-factorizable and transitive}\}.$$

(i) implies (iii): Given the natural map $\mathrm{Lim\,pro}\pi_1 C(X, x^0) \rightarrow \mathrm{pro}\pi_1 C(X, x^0)$ in **pro Grp** and the discrete group Σ_n considered as constant progroup, there is an induced map

$$\phi: \mathbf{pro\,Grp}(\mathrm{pro}\pi_1 C(X, x^0), \Sigma_n) \rightarrow \mathbf{pro\,Grp}(\mathrm{Lim\,pro}\pi_1 C(X, x^0), \Sigma_n).$$

The inverse limit, $\mathrm{Lim\,pro}\pi_1 C(X, x^0)$, provided with the inverse limit topology, see Section 2.2.1, is the Borsuk–Čech topological fundamental group $\Pi_1^{BC}(X, x^0)$. A map from the progroup, $\mathrm{pro}\pi_1 C(X, x^0)$, to the discrete finite group, Σ_n, can be represented by a homomorphism of the form

$$\pi_1(C(X, x^0)(\mathcal{U}, U^0)) \rightarrow \Sigma_n$$

which is carried by ϕ to the continuous map

$$\Pi_1^{BC}(X, x^0) = \mathrm{Lim\,pro}\pi_1 C(X, x^0) \rightarrow \pi_1(C(X, x^0)(\mathcal{U}, U^0)) \rightarrow \Sigma_n.$$

Therefore the image of ϕ is contained in $\mathbf{TGrp}(\Pi_1^{BC}(X, x^0), \Sigma_n) \subset \mathbf{Grp}(\Pi_1^{BC}(X, x^0), \Sigma_n)$.

Next, we shall take into account the following properties:

(1) the progroup $\mathrm{pro}\pi_1 C(X, x^0)$ is isomorphic to a tower (this follows since (X, x^0) is a connected compact metrizable pointed space),

(2) the tower $\mathrm{pro}\pi_1 C(X, x^0)$ is isomorphic to a tower $\{G_i | i \in \mathbb{N}\}$ which has surjective bonding homomorphisms $\eta_i^j: G_j \rightarrow G_i$ (this property can be obtained from the fact that (X, x^0) is S^1-movable).

Taking the discrete topology on G_i, from the properties (1) and (2) we obtain that $\Pi_1^{BC}(X, x^0) \cong \mathrm{Lim}\,\{G_i\}$, the canonical map $\eta_i: \Pi_1^{BC}(X, x^0) \rightarrow G_i$ is a continuous epimorphism and the topology on $\Pi_1^{BC}(X, x^0)$ is the initial topology induced by η_i, $i \in \mathbb{N}$ (we remark that for an arbitrary progroup with surjective bonding maps in general the canonical map $\mathrm{Lim}\,\{G_j\} \rightarrow G_j$ need not be an epimorphism.)

Given a continuous homomorphism $f: \Pi_1^{BC}(X, x^0) \rightarrow \Sigma_n$, we have that $\mathrm{Ker}f$ is an open neighbourhood of the identity element. Therefore there exists $\eta_i: \Pi_1^{BC}(X, x^0) \rightarrow G_i$ such that $\mathrm{Ker}\eta_i \subset \mathrm{Ker}f$. This implies that f factors as $\bar{f}\eta_i$ and $\bar{f}: G_i \rightarrow \Sigma_n$ represent a map in **pro Grp**$(\mathrm{pro}\pi_1 C(X, x^0), \Sigma_n)$ which is applied by ϕ into f. Given a homomorphism $h: G_i \rightarrow \Sigma_n$ such that $\eta_i h = 0$, since η_i is an epimorphism it follows that $h = 0$. Therefore

$$\mathbf{pro\,Grp}(\mathrm{pro}\pi_1 C(X, x^0), \Sigma_n) \rightarrow \mathbf{TGrp}(\Pi_1^{BC}(X, x^0), \Sigma_n)$$

is an isomorphism. We also remark that transitivity property is also preserved by this isomorphism.

(iii) implies (iv): Since, by Proposition 2.14, one has that

$$\mathbf{TGpr}(\Pi_1^{BC}(X, x^0), \Sigma_n) \rightarrow \mathbf{TGpr}(\Pi_1^{SQ}(X, x^0), \Sigma_n)$$

is an isomorphism and it is easy to check that the transitivity property is also preserved.

(i) implies (v): If $(X.x^0)$ is a pointed locally connected space, $\mathrm{pro}\pi_1(C(X, x^0))$ is isomorphic to the Artin–Mazur progroup, $\mathrm{pro}\pi_1(|\prod(\mathbf{Sh}(X, x^0))|)$, see Section 1.4 and [2].

(i) implies (vi): Since (X, x^0) is locally connected connected space we have that $\mathrm{pro}\pi_1(X, x^0)$ is a surjective group. The constant progroup Σ_n is also surjective. Then applying Theorem 1.10 we obtain the bijection

$$\mathbf{pro\,Grp}(\mathrm{pro}\pi_1(X, x^0), \Sigma_n) \cong \mathbf{loc\,Grp}(\mathrm{loc}(\mathrm{pro}\pi_1(X, x^0)), \mathrm{loc}\Sigma_n)$$

Taking into account Definition 3.3, we have that the transitivity property is preserved by this bijection.

(iii) implies (vii): Since, by Proposition 3.4, one has that if (X, x_0) is a connected locally connected space, then (X, x_0) is S^1-movable.

(viii) implies (ix): It follows from Proposition 2.7.

(ix) implies (x): In this case, one has that $\mathrm{pro}\pi_1^{Sp}(X, x^0)$ is isomorphic to $\pi_1(X, x^0)$. $\quad\square$

Remark 3.6. The profinite completion $\hat{G}$ of a topological group G satisfies that for any finite discrete group H the canonical map $G \rightarrow \hat{G}$ induces an isomorphism $\mathbf{TGrp}(\hat{G}, H) \rightarrow \mathbf{TGrp}(G, H)$. The prodiscrete completion in general does not verify a similar property for discrete groups, but under the conditions of Theorem 3.5 the above property is also satisfied for discrete groups. Moreover one has the following:

(a) The topological group $\Pi_1^{BC}(X, x^0)$ is prodiscrete and we have

$$\check{\Pi}_1^{SQ}(X, x^0) \cong \Pi_1^{CB}(X, x^0) \cong \check{\Pi}_1^{Sp}(X, x^0).$$

(b) There are induced isomorphisms on the profinite completions,

$$\hat{\Pi}_1^{SQ}(X, x^0) \cong \hat{\Pi}_1^{CB}(X, x^0) \cong \hat{\Pi}_1^{Sp}(X, x^0).$$

(c) The progroups $\mathrm{pro}\pi_1(X, x^0)$ and $\mathrm{pro}\pi_1^{Sp}(X, x^0)$ have the same profinite progroup completion.

Remark 3.7. (a) If Y is connected, locally path-connected and semi-locally simply connected, then fundamental group $\pi_1(Y, y^0)$ satisfies that the category Cov(Y) is equivalent to the category of $\pi_1(Y, y^0)$-sets.

(b) If Y is connected, locally path-connected, then $\Pi_1^{Sp}(Y, y^0)$, pro$\pi_1^{Sp}(Y, y^0)$ satisfy that the category Cov(Y) is equivalent to the category of continuous $\Pi_1^{Sp}(Y, y^0)$-sets and to the category of pro$\pi_1^{Sp}(Y, y^0)$-sets.

(c) If Y is connected, locally connected, then pro$\pi_1(Y, y^0)$, loc$\pi_1(Y, y^0)$ satisfy that the category Cov(Y) is equivalent to the category of continuous $\Pi_1^{Sp}(Y, y^0)$-sets and to the category of pro$\pi_1^{Sp}(Y, y^0)$-sets.

(d) If Y is a space, and CovProj(Y) denotes the category of covering projection given in [7,8] then CovProj(Y) is equivalent to the category of pro$\pi(Y)$-sets, where $\pi(Y)$ is the fundamental progroupoid of Y. If Y is locally connected then CovProj$(Y) =$ Cov(Y).

(e) It is interesting to remark that in general Cov(Y) is not closed by arbitrary coproducts. However we can extend these categories by taking arbitrary coproducts to obtain the classifying topos of the corresponding fundamental 'groups'.

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