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Dynamics of a new family of iterative processes for quadratic polynomials

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ABSTRACT

In this work we show the presence of the well-known Catalan numbers in the study of the convergence and the dynamical behavior of a family of iterative methods for solving nonlinear equations. In fact, we introduce a family of methods, depending on a parameter $m \in \mathbb{N} \cup \{0\}$. These methods reach the order of convergence $m + 2$ when they are applied to quadratic polynomials with different roots. Newton's and Chebyshev's methods appear as particular choices of the family appear for $m = 0$ and $m = 1$, respectively. We make both analytical and graphical studies of these methods, which give rise to rational functions defined in the extended complex plane. Firstly, we prove that the coefficients of the aforementioned family of iterative processes can be written in terms of the Catalan numbers. Secondly, we make an incursion into its dynamical behavior. In fact, we show that the rational maps related to these methods can be written in terms of the entries of the Catalan triangle. Next we analyze its general convergence, by including some computer plots showing the intricate structure of the Universal Julia sets associated with the methods.

1. Introduction

The application of iterative methods for solving nonlinear equations $f(z) = 0$, with $f : \mathbb{C} \to \mathbb{C}$ can give rise to rational functions whose dynamics are not well known. The easiest model (attributed to Cayley, see [1] for more details) is obtained when $f(z)$ is a quadratic polynomial

$$f(z) = (z - a)(z - b), \quad a, b \in \mathbb{C}, \text{ with } a \neq b, \tag{1}$$

and the iterative process is Newton's method [2]:

$$z_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}, \quad n \geq 0. \tag{2}$$

In this situation, the dynamics of Newton's iteration are conjugate to the map $z \to z^2$ via the Möbius transformation

$$M(z) = (z - a)/(z - b). \tag{3}$$

Consequently, if the starting point z_0 is inside the unit circle ($|z_0| < 1$) then Newton's iterates converge to 0, if $|z_0| > 1$, Newton's iterates diverge to ∞ and if $|z_0| = 1$, Newton's iterates have a chaotic behavior (see Fig. 1).

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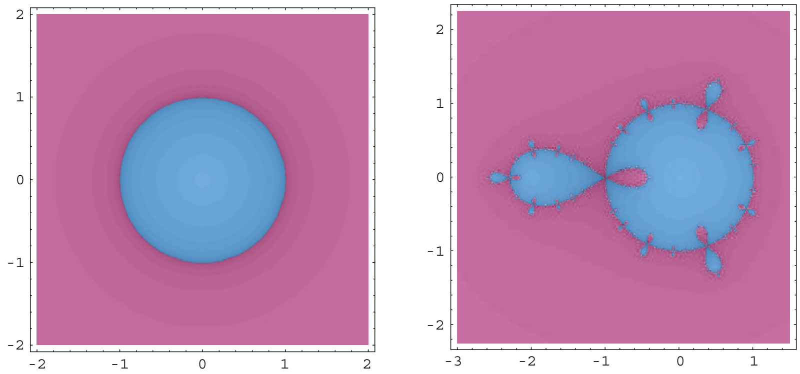

Fig. 1. Universal Julia sets for S_0 (Newton's method) and S_1 (Chebyshev's method).

The situation becomes more complicated if $f(z)$ is a polynomial of degree greater than two. This fact was known yet in [1,3]. But the study of the dynamics of iterative methods applied to quadratic polynomials can be also complicated if other iterative methods, different from Newton's method are considered.

For instance, another well-known example [4] is Chebyshev's method applied to quadratic polynomials. Chebyshev's method is defined by

$$z_{n+1} = z_n - \left(1 + \frac{1}{2} L_f(z_n)\right) \frac{f(z_n)}{f'(z_n)} \quad n \geq 0 \tag{4}$$

where we have denoted

$$L_f(z) = \frac{f(z) f''(z)}{f'(z)^2}. \tag{5}$$

The dynamics of Chebyshev's method are conjugate, via the Möbius transformation (3), to the map $z \to z^3(z+2)/(2z+1)$. Then, its dynamical behavior is more complicated, as shown in Fig. 1.

Following an idea of Gander [5], we consider the following family of iterative methods

$$\begin{cases} z_{n+1} = R_m(z_n) = z_n - H_m(L_f(z_n)) \dfrac{f(z_n)}{f'(z_n)}, \quad n \geq 0, \\ H_m(w) = \displaystyle\sum_{j=0}^{m} A_j w^j, \quad A_j \in \mathbb{R}^+, \ 0 \leq j \leq m \end{cases} \tag{6}$$

and $L_f(z)$ is defined in (5). Notice that for $m = 0$ and $A_0 = 1$ we obtain Newton's method (2) and for $m = 1$ and $A_0 = 1$, $A_1 = 1/2$ we obtain Chebyshev's method (4).

In this paper we present two results for methods (6). Firstly, in Section 2, we find the parameters A_k, $k = 0, \ldots, m$ that allow methods (6) to reach the order of convergence $m + 2$. In fact, these parameters can be written in the following way

$$A_j = \frac{1}{2^j} C_j, \quad 0 \leq j \leq m, \tag{7}$$

where C_j are the well-known Catalan numbers [6,7]:

$$C_j = \frac{1}{j+1} \binom{2j}{j}, \quad j \geq 0, \ j \in \mathbb{N}. \tag{8}$$

Secondly, in Section 3, we study the dynamics of the methods given in (2). In particular we show that these methods are conjugate, via the Möbius transformation (3), to the map

$$S_m(z) = z^{m+2} \frac{P_m(z)}{\hat{P}_m(z)}, \tag{9}$$

where

$$P_m(z) = \sum_{p=0}^{m} B_{m+1,p+1} z^p \quad \text{and} \quad \hat{P}_m(z) = \sum_{p=0}^{m} B_{m+1,m+1-p} z^p \tag{10}$$

Table 1
First entries of the Catalan triangle introduced in [8].

m	p						
	1	2	3	4	5	6	…
1	1						
2	2	1					
3	5	4	1				
4	14	14	6	1			
5	42	48	27	8	1		
6	132	165	110	44	10	1	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

and

$$B_{m,p} := \frac{p}{m}\binom{2m}{m-p}, \quad m, p \in \mathbb{N}, \ p \le m. \tag{11}$$

Notice that he numbers $B_{m,p}$ defined in (11) are the entries of the Catalan triangle introduced in [8]. In Table 1 we can see the first rows of this triangle.

Although the numbers $B_{n,p}$ are not as famous as Catalan numbers, they have also several applications. As a sample, we cite now some of them:

(i) $B_{m,p}$ is the number of leaves at level $p+1$ in all ordered trees with $m+1$ edges.
(ii) $B_{m,p}$ is the number of walks of m steps, each in direction N, S, W or E, starting at the origin, remaining in the upper half plane and ending at height p.
(iii) $B_{m,p}$ denotes the number of pairs of non-intersecting paths of length m and distance p.

Moreover, they satisfy the recurrence relation

$$B_{m,p} = B_{m-1,p-1} + 2B_{m-1,p} + B_{m-1,p+1}, \quad p \ge 2$$

and many other identities as shown in [9,10,8,11].

In addition, in [10] the following new identity

$$\sum_{p=1}^{i} B_{m,p}B_{m,m+p-i}(m+2p-i) = (m+1)C_m\binom{2(m-1)}{i-1}, \quad i \le m, \tag{12}$$

was proved. As we can see in this paper, this identity is key in the understanding of the dynamical behavior of the methods given in (6).

Finally, in Section 3 we study the dynamics of the methods given in (6). In particular, we prove that these root-finding algorithms are generally convergent for quadratic polynomials. In addition, we present some computer graphics showing the intricate dynamical structure of the Universal Julia sets associated with these methods when they are applied to quadratic polynomials.

2. Local convergence for quadratic polynomials

The family of iterative methods we consider in this paper appears as a generalization of a result of Gander [5], where he proves that all iterative method written as

$$\begin{cases} t_{k+1} = t_k - H(L_f(t_k))\dfrac{f(t_k)}{f'(t_k)} \\ H(0) = 1, \qquad H'(0) = 1/2, \qquad |H''(0)| < \infty, \end{cases}$$

has at least cubic order of convergence. Now we consider the methods introduced in (6) with $A_0 = 1$ and $A_1 = 1/2$. Taking into account Gander's result they have at least cubic order of convergence. Then we look for the parameters $A_2, A_3, \ldots, A_m$ that provides us an order of convergence $m+2$, with $m \in \mathbb{N}$. In this paper we find these parameters when methods (6) are applied to a quadratic complex polynomial (1).

Theorem 1. *Let $f : \mathbb{C} \to \mathbb{C}$ be the quadratic polynomial (1). Then, for $m \ge 0$, the methods given in (6) with the coefficients A_j defined in (7), have order of convergence at least $m+2$.*

Proof. Let $R_m(z)$ be the rational function defined in (6), that is,

$$R_m(z) = z - H_m\left(L_f(z)\right)\frac{f(z)}{f'(z)}.$$

It is well known that the method given in (6) converges with order q if

$$R_m(\alpha)=\alpha,\qquad R_m'(\alpha)=R_m''(\alpha)=\cdots=R_m^{(q-1)}(\alpha)=0,\qquad R_m^{(q)}(\alpha)\neq 0,$$

where α is a solution of $f(z)=0$.

Firstly, notice that, as α is a simple root, we can write

$$L_f(z)=(z-\alpha)h(z),$$

with $h(\alpha)\neq 0$.

Now, it is easy to show that $R_m(\alpha)=\alpha$. In addition, we have

$$\begin{aligned}R_m'(z) &= 1-\sum_{j=0}^{m} jA_jL_f(z)^{j-1}(1-2L_f(z))L_f(z)-\sum_{j=0}^{m}A_jL_f(z)^j(1-L_f(z))\\ &= (1-A_0)+(-2A_1+A_0)L_f(z)+\sum_{j=2}^{m}\left((2j-1)A_{j-1}-(j+1)A_j\right)L_f(z)^j+(2m+1)A_mL_f(z)^{m+1}.\end{aligned}$$

As A_j are defined by (7), we deduce that

$$\begin{aligned}R_m'(z) &= \sum_{j=2}^{m}\left((2j-1)A_{j-1}-(j+1)A_j\right)L_f(z)^j+(2m+1)A_mL_f(z)^{m+1}\\ &= (2m+1)A_m(z-\alpha)^{m+1}h(z)^{m+1},\end{aligned}$$

and consequently, $R_m'(\alpha)=0$.

In addition,

$$\begin{aligned}R_m^{(j+1)}(z) &= (2m+1)A_m\sum_{i=0}^{j}\binom{j}{i}((z-\alpha)^{m+1})^{(i)}(h(z)^{m+1})^{(j-i)}\\ &= (2m+1)A_m\sum_{i=0}^{j}\binom{j}{i}(m+1)m\cdots(m+2-i)((z-\alpha)^{m+1-i})(h(z)^{m+1})^{(j-i)}.\end{aligned}$$

Then, $R_m'(\alpha)=R_m''(\alpha)=\cdots=R_m^{(m+1)}(\alpha)=0$ and

$$R_m^{(m+2)}(\alpha)=(2m+1)(m+1)!A_mh(\alpha)^{m+1}\neq 0.$$

This finishes the proof. □

3. Dynamical behavior

In this section, we study the *general convergence* of methods (6) for quadratic polynomials (1). The idea of general convergence of a method for polynomials of a given degree was introduced in [12,13]. To be more precise, a given method is generally convergent if the method converges to a root for almost every starting point and for almost every polynomial of a given degree.

First we give some definitions and properties of rational functions that we will use below. For more information about these concepts, the interested reader can consult the book of Beardon [1].

Let us denote by $\mathbb{C}_\infty=\mathbb{C}\cup\{\infty\}$ the extended complex plane (Riemann Sphere). $R(z)=P(z)/Q(z)$ is a rational map on $\mathbb{C}_\infty$ for polynomials $P(z)$ and $Q(z)$ coprime and not both zero. ζ is a *fixed point* of R if $R(\zeta)=\zeta$.

For $z\in\mathbb{C}_\infty$, we define the *forward orbit of* z as the set

$$\mathrm{Orb}(z)=\{z,R(z),R^2(z),\ldots,R^k(z),\ldots\}$$

where $R^k(z)$ is the kth composition of R.

A rational map R, divides $\mathbb{C}_\infty$ in two subsets, that are known as *Fatou set* and *Julia set*. Fatou set, denoted $\mathcal{F}(R)$ is defined as the set of points $z_0\in\mathbb{C}_\infty$ such that the family of iterates R^n is a normal family in some neighborhood U_{z_0} of z_0. That is, every infinite sequence of R^n contains a subsequence R^{n_k} that converges locally uniformly on U_{z_0} to some continuous function $g\in\mathcal{C}(\mathbb{C}_\infty)$. Recall that $R^{n_k}\to g$ locally uniformly on U_{z_0} if for all $z\in U_{z_0}$, $R^{n_k}\to f$ uniformly on some neighborhood of z. The Julia set, $\mathcal{J}(R)$, is the complement of the Fatou set, $\mathcal{J}(R)=\mathbb{C}_\infty-\mathcal{F}(R)$.

Roughly speaking, the orbits of the points in $\mathcal{F}(R)$ exhibit stable behavior but the orbits of the points in $\mathcal{J}(R)$ exhibit chaotic behavior.

The *basin of attraction* of a fixed point ζ of a rational map R is the set

$$C(\zeta)=\{z\in\mathbb{C}_\infty|R^n(z)\to\zeta,\ \text{for } n\to\infty\}.$$

It is well-known, [1], that $C(\zeta) \subseteq \mathcal{F}(R)$ and $\mathcal{J}(R) = \partial C(\zeta)$.

In addition, if $M(z)$ is a Möbius map

$$M(z) = \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \alpha\delta - \beta\gamma \neq 0,$$

and R, S are conjugate rational maps via M, i.e. $S = MRM^{-1}$, then $\mathcal{F}(S) = M(\mathcal{F}(R))$ and $\mathcal{J}(S) = M(\mathcal{J}(R))$.

In [14] is also introduced the concept of *universal Julia set* for a root-finding algorithm G_f. Thus, a root-finding algorithm G_f has a universal Julia set (for polynomials of degree d) if there exists a rational map R such that for every degree d polynomial f, $\mathcal{J}(G_f)$ is conjugate by a Möbius map to $\mathcal{J}(R)$.

As the rational map arising from Newton's method applied to the quadratic polynomial (1) is conjugate to the map $z \to z^2$ via the Möbius transformation (3), the universal Julia set for Newton's method applied to quadratic polynomials is the unit circle [14,12].

The universal Julia set for Chebyshev's method applied to quadratic polynomials (from now on denoted $\mathcal{J}_1$) is also known (see also [14]). The rational map arising from Chebyshev's method applied to quadratic polynomial (1) is conjugate to the map

$$z \to S_1(z) = z^3 \frac{z+2}{2z+1}$$

via the Möbius transformation (3). The dynamic structure of $\mathcal{J}_1$ is shown in Fig. 1.

As we can see the universal Julia set for Chebyshev's method applied to quadratic polynomials is more complicated than for Newton's method. The map $S_1(z)$ has precisely two forward invariant Fatou components: a superattracting component where iterates converge to ∞ in magenta in Fig. 1 and a superattracting component where iterates converge to 0 (in cyan in Fig. 1). On the other hand, the unit circle is forward invariant and it is contained in $\mathcal{J}_1$ and moreover, $\mathcal{J}_1$ has zero Lebesgue measure on $\mathbb{C}$ (see [14]).

Newton's and Chebyshev's methods are the two first cases of the methods in family (6). Now we look for the universal Julia set for the rest of the methods when they are applied to quadratic polynomials. We will show that the above properties are not exclusive of Newton's and Chebyshev's methods.

Theorem 2. *Let $f(z) = (z-a)(z-b)$ be the quadratic complex polynomial (1). Let $S_m(z)$ be the conjugate map of $R_m(z)$ via the Möbius map (3), that is, $S_m(z) = MR_mM^{-1}(z)$. Then, for $m \geq 0$,*

$$S_m(z) = z^{m+2}\frac{P_m(z)}{\hat{P}_m(z)},$$

where $P_m(z)$ and $\hat{P}_m(z)$ are the polynomials defined in (10) and (11) respectively.

Proof. The inverse of the Möbius map M defined in (3) is $M^{-1}(z) = \frac{bz-a}{z-1}$. If $\omega = M^{-1}(z)$ we have:

$$f(\omega) = \frac{z(b-a)^2}{(z-1)^2}, \qquad f'(\omega) = \frac{(z+1)(b-a)}{(z-1)},$$

and then

$$L_f(\omega) = \frac{f(\omega)f''(\omega)}{f'(\omega)^2} = \frac{2z}{(z+1)^2}.$$

Consequently,

$$S_m(z) = MR_mM^{-1}(z) = MR_m(\omega) = \frac{z\left(z - \sum_{k=1}^{m} C_k \frac{z^k}{(z+1)^{2k}}\right)}{1 - z\sum_{k=1}^{m} C_k \frac{z^k}{(z+1)^{2k}}}$$

$$= \frac{z^2\left((z+1)^{2m} - \sum_{k=1}^{m} C_k z^{k-1}(z+1)^{2(m-k)}\right)}{(z+1)^{2m} - z\sum_{k=1}^{m} C_k z^k (z+1)^{2(m-k)}}.$$

Notice that

$$(z+1)^{2m} = \sum_{j=0}^{2m} \binom{2m}{j} z^j,$$

$$z^{k-1}(z+1)^{2(m-k)} = \sum_{j=0}^{2(m-k)} \binom{2(m-k)}{j} z^{j+k-1} = \sum_{j=k-1}^{2m-(k+1)} \binom{2(m-k)}{j+1-k} z^j.$$

Then,

$$
\begin{aligned}
(z+1)^{2m}-\sum_{k=1}^{m}C_kz^{k-1}(z+1)^{2(m-k)} &= \sum_{j=0}^{2m}\binom{2m}{j}z^j-\sum_{k=1}^{m}C_k\sum_{j=k-1}^{2m-(k+1)}\binom{2(m-k)}{j+1-k}z^j\\
&= \sum_{j=0}^{m-1}\left(\binom{2m}{j}-\sum_{k=1}^{j+1}C_k\binom{2(m-k)}{j+1-k}\right)z^j\\
&\quad+\sum_{j=m}^{2(m-1)}\left(\binom{2m}{j}-\sum_{k=1}^{2m-(j+1)}C_k\binom{2(m-k)}{j+1-k}\right)z^j\\
&\quad+\binom{2m}{2m-1}z^{2m-1}+z^{2m}.
\end{aligned}
$$

Let us consider now Jonah's formula for Catalan numbers [15], that establish the following property:

$$
\binom{n}{j-1}=\sum_{i=1}^{j}C_i\binom{n-2i}{j-i},\quad n\geq 0, j\geq 1. \tag{13}
$$

We can use this formula to deduce the following equalities:

$$
\binom{2m}{j}-\sum_{k=1}^{j+1}C_k\binom{2(m-k)}{j+1-k}=0,
$$

$$
\sum_{j=m}^{2(m-1)}\left(\binom{2m}{j}-\sum_{k=1}^{2m-(j+1)}C_k\binom{2(m-k)}{j+1-k}\right)z^j=z^m\sum_{j=0}^{m-2}\left(\binom{2m}{j+m}-\sum_{k=1}^{m-j-1}C_k\binom{2(m-k)}{j+m+1-k}\right)z^j,
$$

and

$$
\sum_{j=m}^{2(m-1)}\left(\binom{2m}{j}-\sum_{k=1}^{2m-(j+1)}C_k\binom{2(m-k)}{j+1-k}\right)z^j=z^m\sum_{j=0}^{m-2}\left(\binom{2m}{j+m}-\binom{2m}{m-j-2}\right)z^j.
$$

Consequently, the numerator of $S_m(z)$ is:

$$
z^2\left((z+1)^{2m}-\sum_{k=1}^{m}C_kz^{k-1}(z+1)^{2(m-k)}\right)=z^{m+2}\left(\sum_{j=0}^{m-2}\left(\binom{2m}{j+m}-\binom{2m}{m-j-2}\right)z^j+2mz^{m-1}+z^m\right).
$$

In a similar way, the denominator of $S_m(z)$ is

$$
\begin{aligned}
(z+1)^{2m}-z\sum_{k=1}^{m}C_kz^k(z+1)^{2(m-k)} &= \sum_{j=0}^{m}\binom{2m}{j}z^j-\sum_{j=2}^{m}\binom{2m}{j-2}z^j\\
&\quad+z^{m+1}\left(\sum_{j=0}^{m-1}\left(\binom{2m}{j+q-1}-\binom{2m}{m-j-1}\right)z^j\right)\\
&= 1+2mz+\sum_{j=2}^{m}\left(\binom{2m}{j}-\binom{2m}{j-2}\right)z^j.
\end{aligned}
$$

This finishes the proof. □

Theorem 3. *The rational map $S_m(z)$, ($m\geq 0$), defined in (9) satisfies these properties:*

1. *$S_m(z)$ has precisely two forward invariant Fatou components: a superattracting component where iterates converge to ∞ and a superattracting component where iterates converge to 0.*
2. *The unit circle $S^1(z)=\{z\in\mathbb{C};\ |z|=1\}$ is forward invariant and it is contained in $\mathcal{J}(S_m)$.*
3. *$\mathfrak{m}(\mathcal{J}(S_m))=0$, where $\mathfrak{m}$ is the Lebesgue measure on $\mathbb{C}$.*

Proof. Firstly, notice that, from (9), the expression of the first derivative of the conjugate map S_m is given by

$$
S_m'(z)=\frac{z^{m+1}}{\hat{P}_m(z)^2}\left((m+2)P_m(z)\hat{P}_m(z)+z\left(P_m'(z)\hat{P}_m(z)-P_m(z)\hat{P}_m'(z)\right)\right).
$$

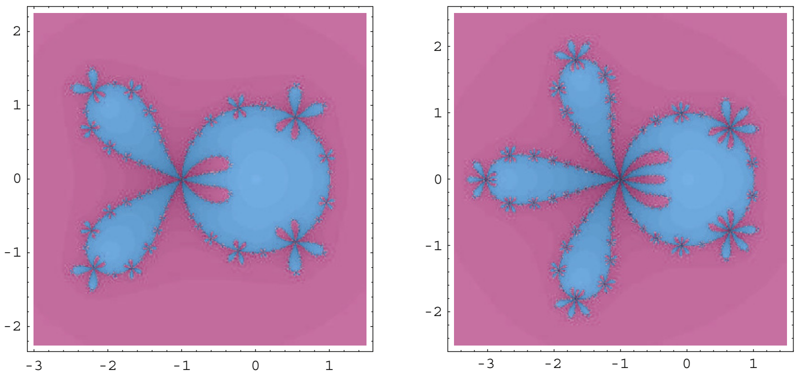

Fig. 2. Universal Julia sets for S_2 and S_3.

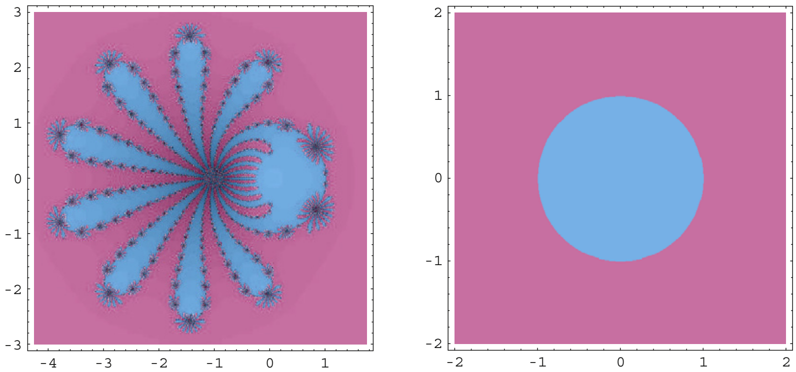

Fig. 3. Universal Julia sets for S_8 and S_∞ (the Euler method).

Taking into account (12), we have

$$S_m'(z) = \frac{(m+2)C_{m+1}z^{m+1}(1+z)^{2m}}{\hat{P}_m(z)^2}. \tag{14}$$

Now, from (14), notice that $z = 0$, $z = \infty$ and $z = -1$ are the only critical points of $S_m(z)$. It is known (see, for example [1]) that there is at least one critical point associated with each forward invariant Fatou component. As $z = 0$ and $z = \infty$ are superattracting fixed points of $S_m(z)$, both of them give rise to a Fatou component.

On the other hand, the critical point $z = -1$ maps to $z = 1$. This is, $S_m(-1) = 1$, and $z = 1$ is a fixed point of $S_m(z)$. As

$$S_m'(1) = \frac{(m+2)C_{m+1}2^{2m}}{\hat{P}_m(1)^2} > 1, \quad m \geq 0,$$

$z = 1$ is a repelling fixed point of $S_m(z)$ and consequently, $z = -1 \in \mathcal{J}(S_m)$. So, S_m has precisely two forward invariant Fatou components.

Secondly, notice that for $z \in S^1$, $|P_m(z)| = |\hat{P}_m(z)|$ for $P_m(z)$ and $\hat{P}_m(z)$ defined in (10). In fact, as $P_m(z) = z^m\hat{P}_m(1/z)$, $\hat{P}_m(1/z) = \hat{P}_m(\bar{z})$ and

$$\hat{P}_m(\bar{z}) = \hat{P}_m(\mathrm{e}^{-\mathrm{i}\theta}) = \sum_{p=0}^{m} B_{m+1,m+1-p}(\cos(\theta p) - \mathrm{i}\sin(\theta p)) = \overline{\hat{P}_m(z)}.$$

Then, for $z \in S^1$, $|\hat{P}_m(z)| = |\hat{P}_m(\bar{z})| = |\hat{P}_m(1/z)| = |P_m(z)|$ and $S^1 \subseteq \mathcal{J}(S_m)$.

Finally, as we have seen the critical points of $S_m(z)$ have finite forward orbits, then it follows from a result of Carleson and Gamelin [16] that $\mathrm{m}(\mathcal{J}(S_q)) = 0$. $\square$

Now, we analyze graphically the dynamical behavior of the rational maps S_m for different values of m. Following [17], we plot the attraction basins associated with the two roots of a quadratic polynomial (1) when we apply S_m. The attraction basins clarify the structures of the universal Julia sets associated with the corresponding iterative methods.

In this way, in Figs. 2 and 3 we have represented the basins for methods (9) with orders of convergence 4 ($m = 2$), 5 ($m = 3$), 10 ($m = 8$) and the special case $m \to \infty$.

Remark 4. The special case obtained when $m \to \infty$ produces an interesting situation. The corresponding method in the family (6) is the well-known Euler method, also called Cauchy's method, [18–21]:

$$\begin{cases} z_{n+1} = z_n - H(L_f(z_n))\dfrac{f(z_n)}{f'(z_n)}, \quad n \geq 0, \\ H(w) = \dfrac{1-\sqrt{1-2w}}{w}. \end{cases}$$

Roughly speaking, this method applied to quadratic polynomial has "order of convergence infinity". This means that we reach one root of polynomial p in only one step, starting from any point in $\mathbb{C}_\infty$.

If we calculate the conjugate map S_∞ associated with the Euler method via the Möbius transformation (3), we obtain:

$$S_\infty(z) = \begin{cases} 0, & |z| < 1 \\ \infty, & |z| > 1. \end{cases}$$

Obviously this map has only two fixed points: $z = 0$ and $z = \infty$. Both of them are superattracting.

For a detailed study about iteration of rational functions theory the works of Beardon [1] and Blanchard [3], can be consulted amongst many others works.

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References

[1] A.F. Beardon, Iteration of Rational Functions, Springer-Verlag, NY, 1991.
[2] I.K. Argyros, Newton Methods, Nova Science Publishers, Inc., Hauppauge, NY, 2005.
[3] P. Blanchard, The dynamics of Newton's method, in: Complex Dynamical Systems (Cincinnati, OH, 1994), in: Proc. Sympos. Appl. Math., vol. 49, Amer. Math. Soc, Providence, RI, 1994, pp. 139–154.
[4] S. Amat, S. Busquier, S. Plaza, A construction of attracting periodic orbits of some classical third order iterative methods, J. Comput. Appl. Math. 189 (1–2) (2006) 22–33.
[5] W. Gander, On Halley's iteration method, Amer. Math. Monthly 92 (1985) 131–134.
[6] R.M. Dickau, Catalan Numbers. http://mathforum.org/advanced/robertd/catalan.html.
[7] E.W. Weisstein, et al. Catalan Number. From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/CatalanNumber.html.
[8] L.W. Shapiro, A Catalan triangle, Discrete Math. 14 (1976) 83–90.
[9] E. Deutsch, L. Shapiro, A survey of the fine numbers, Discrete Math. 241 (2001) 241–265.
[10] J.M. Gutiérrez, M.A. Hernández, P. Miana, N. Romero, New identities in the Catalan triangle, J. Math. Anal. Appl. 341 (2008) 52–61.
[11] N. Sloane, http://www.research.att.com/~njas/sequences/A039598.
[12] S. Smale, On the efficiency of algorithms of analysis, Bull. Allahabad Math. Soc. 13 (1985) 87–121.
[13] C. McMullen, Families of rational maps and iterative root-finding algorithms, Ann. of Math. 125 (1987) 467–493.
[14] K. Kneisl, Julia sets for the super-Newton method, Cauchy's method and Halley's method, Chaos 11 (2) (2001) 359–370.
[15] P. Hilton, J. Pederson, Catalan numbers, their generalization and their uses, Math. Intelligencer 13 (1991) 64–75.
[16] L. Carleson, T. Gamelin, Complex Dynamics, Springer, NY, 1995.
[17] J.L. Varona, Graphic and numerical comparisons between iterative methods, Math. Intelligencer 24 (1) (2002) 37–46.
[18] S. Amat, C. Bermúdez, S. Busquier, S. Plaza, On the dynamics of the Euler iterative function, Appl. Math. Comput. 197 (2) (2008) 725–732.
[19] S. Amat, C. Bermúdez, S. Busquier, J. Carrasco, S. Plaza, Super-attracting perodic orbits for a classical third order method, J. Comput. Appl. Math. 206 (1) (2007) 599–602.
[20] A. Melman, Geometry and convergence of Euler's and Halley's methods, SIAM Rev. 39 (1997) 728–735.
[21] J.F. Traub, Iterative Methods for the Solution of Equations, Prentice-Hall, Englewood Cliffs, NJ, 1964.