Convergence of Padé approximants of Stieltjes-type meromorphic functions and the relative asymptotics of orthogonal polynomials on the real line

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Abstract

We obtain results on the convergence of Padé approximants of Stieltjes-type meromorphic functions and the relative asymptotics of orthogonal polynomials on unbounded intervals. These theorems extend some results given by Guillermo López in this direction substituting the Carleman condition in his theorems by the determination of the corresponding moment problem.

Key words: Padé approximants, moment problem, orthogonal polynomials, varying measures, asymptotics
2000 MSC: Primary 42C05; Secondary 33C47.

1. Introduction and notations

Two of the most striking papers of Guillermo López have been [11] and [12]. In the first, he solved a conjecture posed by A. A. Gonchar 10 years earlier about the convergence of Padé approximants of Stieltjes-type meromorphic functions. Gonchar [8] proved the convergence of Padé approximants to Markov-type meromorphic function whose measure α is supported on a bounded interval of the real line, and $\alpha' > 0$ a. e. on this interval. In [16], E. A. Rakhmanov showed that the convergence does not hold for arbitrary positive measures on $\mathbb{R}$. In

[11] López gave a very general sufficient condition to get convergence of Padé approximants for Stieltjes-type meromorphic functions (the measure can have unbounded support on $\mathbb{R}$). The main idea of López was to reduce the problem to the study of orthogonal polynomial on the unit circle with respect to varying measures.

In [12] López showed that orthogonal polynomials with respect to varying measures are an effective tool not only for solving problems on rational approximation but also for studying questions on orthogonal polynomials involving fixed measures and observed that orthogonal polynomials with respect to varying measures on the unit circle provide a unified approach to the study of orthogonal polynomials on bounded and unbounded intervals. There he obtains relative asymptotics for orthogonal polynomials on unbounded intervals.

In this paper we extend the results of López in [11] and [12]; here we substitute the Carleman condition on the moments of the measure which he required by the assumption that the corresponding moment problem be determinate. This new hypothesis requires a careful analysis on each step of the method developed by López in [11] and [12]. Our main ideas involve the use of rational approximation on the unit circle and the relation between determination of the moment problem and one sided approximation.

Let ν be a positive Borel measure on the real line having finite moments of every order, i. e. $x^k \in L^1(\nu)$, and set $c_k = \int_{\mathbb{R}} x^k \, d\nu(x)$, $k = 0, 1, \ldots$ The Hamburger moment problem for ν is determinate if no other measure has the moments of ν. We denote by $\mathcal{M}$ this class of measures. One of the best-known sufficient condition for the determinacy of the Hamburger moment problem is the Carleman condition $\sum_{n=0}^{\infty} c_{2n}^{-1/(2n)} = \infty$. See Section 2 for an example of a measure whose Hamburger moment problem is determinate but its moments do not satisfy the Carleman condition. In the same way, when it is additionally required that the support of the measure is a subset of $[0, \infty)$, the term Stieltjes moment problem is used. In this case the Carleman condition becomes $\sum_{n=0}^{\infty} c_n^{-1/(2n)} = \infty$. By $\mathcal{M}_0$ we denote the class of positive Borel measures on $[0, \infty)$ with finite moments such that the corresponding Stieltjes moment problem is determinate.

Let $\widehat{\alpha}$ denote the Cauchy-Stieltjes transform of α

$$\widehat{\alpha}(z) = \int \frac{1}{z - x} \, d\alpha(x), \quad z \in \mathcal{D} = \mathbb{C} \setminus [0, +\infty),$$

where α is a positive Borel measure on $[0, \infty)$ with finite moments. Let r be a rational function whose poles lie in $\mathcal{D}$ and $r(\infty) = 0$. Let

$$f(z) = \widehat{\alpha}(z) + r(z), \quad z \in \mathcal{D}. \tag{1}$$

Given $n \in \mathbb{Z}_+$, the Padé approximant, $\pi_n(z) = \frac{p_n(z)}{q_n(z)}$, of order n at infinity of f is defined to satisfy:

- p_n and q_n are polynomials with $\deg(p_n) \leq n$, $\deg(q_n) \leq n$, $q_n \not\equiv 0$.
- $q_n(z) f(z) - p_n(z) = \sum_{j=n+1}^{\infty} A_{n,j}/z^j$.

The study of the convergence of Padé approximant for Stieltjes type meromorphic functions is a delicate matter. As Stieltjes himself pointed out (see [20] and also [10]) the Stieltjes moment problem for a measure α can be determinate, so the corresponding Padé approximants of $\widehat{\alpha}$ converge to the Stieltjes transform $\widehat{\alpha}$, while if we add a mass ϵ to α the new measure may have an indeterminate Stieltjes moment problem and, consequently the Padé approximants of the new Stieltjes transform $\widehat{\alpha}(z) + \frac{\epsilon}{z}$ cannot converge; another interesting example can be found in [16]. We prove the following result:

Theorem 1. *If $\alpha \in \mathcal{M}_0$ and $\alpha' > 0$ almost everywhere on $(0, \infty)$, then $\lim_n \pi_n = f$ uniformly on each compact subset of $\mathcal{D} \setminus \{z : r(z) = \infty\}$.*

Under the more restrictive assumption on the measure α that its moments satisfy the Carleman condition this theorem was proved by López in [11]. Our technique allows us to extend other results obtained by López changing the Carleman condition by the determinacy of the corresponding moment problem. Another extension is the following theorem on relative asymptotics of orthogonal polynomials on $\mathbb{R}$.

Theorem 2. *Let $\nu \in \mathcal{M}$ be such that $\nu' > 0$ almost everywhere in $\mathbb{R}$ and let $g \in L^1(\mathbb{R})$ be such that $g \geq 0$, $g\, d\nu \in \mathcal{M}$, and there exist a polynomial Q and $p \in \mathbb{N}$ such that $\frac{Q(x)g^{\pm 1}(x)}{(1+x^2)^p} \in L^\infty(\nu)$. Then*

$$\lim_{n\to\infty} \frac{\mathcal{H}_n(gd\nu, z)}{\mathcal{H}_n(\nu, z)} = \frac{S(g, \Omega, z)}{S(g, \Omega, i)},$$

uniformly on each compact subset of $\Omega = \{z \in \mathbb{C} : \Im z > 0\}$, where $\mathcal{H}_n(g\, d\nu, z)$, $\mathcal{H}_n(\nu, z)$ are the orthogonal polynomials of degree n with respect to $g(x)\, d\nu(x)$ and ν, respectively, normalized by the condition that both are equal to 1 at i, and

$$S(g, \Omega, z) = \exp\left(\frac{1}{2\pi i}\int_{\mathbb{R}} \log g(x) \frac{xz+1}{z-x}\frac{dx}{x^2+1}\right), \quad z \in \Omega,$$

is the Szegő function for g with respect to the region Ω.

Theorem 1 is proved in Section 4, the proof of Theorem 2 is included in Section 5, the auxiliary results on the moment problem appear in Section 2, and Section 3 contains the study of orthogonal polynomials with respect to varying measures.

2. Moment problem and one sided approximation

Next we give an example of a measure such that its Hamburger moment problem is determinate, its Radon-Nikodym derivative with respect to the Lebesgue measure is positive almost everywhere on $\mathbb{R}$ but its moments do not satisfy the Carleman condition.

Example[1]**.** Let

$$d\sigma_{1/2}(x) = \chi_{(0,\infty)}(x)e^{-\sqrt{x}}dx, \quad d\sigma_1^-(x) = \chi_{(-\infty,0)}(x)e^{x}dx,\ x \in \mathbb{R},$$

where χ_E denotes the indicator function of E. Both measures belong to $\mathcal{M}$ because $\sigma_{1/2} \in \mathcal{M}_0$, $\sigma_1^-(-x) \in \mathcal{M}_0$, and they do not have mass at zero (see [10]). Let $\sigma = \sigma_1^- + \sigma_{1/2}$, then $\sigma' > 0$ a. e. on $\mathbb{R}$ and the Carleman condition does not hold. In fact, the $2n$-th moment of σ is $(2n)! + 2(4n+1)! > (4n)!$, so the general term in the Carleman series is majorized by something like n^{-2}. We have $\sigma \in \mathcal{M}$ since $\sigma_{1/2} \in \mathcal{M}$ with infinite index of determinacy because it is absolute continuous with respect to Lebesgue's measure and the determinate measures of finite index are discrete (see [3]); moreover, the measure σ_1^- satisfies the condition $\int e^{1/2|x|}\, d\sigma_1^-(x) < \infty$, so by a Theorem of Yuditskii (see [22]) $\sigma = \sigma_1^- + \sigma_{1/2} \in \mathcal{M}$. Observe that the same conclusions can be drawn if $\sigma_{1/2}$ is changed by $d\sigma_\lambda(x) = \chi_{(0,\infty)}(x)e^{-x^\lambda}dx$ with $\frac{1}{2} \leq \lambda < 1$.

Denote $\Gamma = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$. For $\beta \in \mathcal{M}$, let μ^β be the image measure of β on the unit circle by $\psi_1(z) = \left(i\frac{z+1}{z-1}\right)$. Observe that the function $x = i\frac{z+1}{z-1}$, $z \in \Gamma\setminus\{1\}$, $x \in \mathbb{R}$, has inverse $z = \frac{x+i}{x-i}$. Let M_Γ be the class of measures μ on Γ such that the image measure $\beta^\mu \in \mathcal{M}$. The above change of variables establishes a one to one correspondence between $\mathcal{M}$ and $\mathcal{M}_\Gamma$.

We will use the following Riesz' Lemmas (see [5], page 73, or [18] for the proof of Lemma 1; and [3], Corollary 3.4, or [19] for Lemma 2).

Lemma 1. *Suppose that $\beta \in \mathcal{M}$ and f is a continuous function on $\mathbb{R}$ such that there exist constants $A > 0$, $B > 0$ and $j \in \mathbb{Z}_+$ such that*

$$|f(x)| \leq A + B\,x^{2j}, \quad x \in \mathbb{R}.$$

Then for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for any $n \geq N$ there are

[1] Prof. Christian Berg let me know this example.

algebraic polynomials u_n and v_n such that $\deg(u_n) \le n$, $\deg(v_n) \le n$ *and*

$$u_n(x) \le f(x) \le v_n(x), \forall x \in \mathbb{R}, \qquad \int (v_n(x) - u_n(x))\, d\beta(x) < \epsilon.$$

Lemma 2. *Suppose that $\beta \in \mathcal{M}$ and β is not discrete. Then for every $z_0 \in \mathbb{C}$ and for every $j \in \mathbb{N}$, $|x - z_0|^{2j} d\beta \in \mathcal{M}$.*

We are also interested in the case when $j < 0$ and $z_0 = i$ in the lemma above ($|x-i|^{2j} = \frac{1}{(1+x^2)^{-j}}$). The same conclusion of the lemma above is obtained for $j < 0$ and $z_0 = i$ using the following two lemmas.

Lemma 3. *(see [1], p. 43 or [18]) If $\beta \in \mathcal{M}$, the polynomials are dense in $L^2(\beta)$.*

Lemma 4. *(see [18]) Let β be a positive Borel measure on $\mathbb{R}$ with finite moments. Then $\beta \in \mathcal{M}$ if and only if the polynomials are dense in $L^2((1+x^2)d\beta)$.*

Let g be a real continuous function on $\Gamma \setminus \{1\}$ such that there exist constants $C > 0$, $D > 0$ and $j \ge 0$ for which

$$|g(z)| \le C + D\frac{1}{|z-1|^{2j}}, \quad z \in \Gamma \setminus \{1\}.$$

Lemma 5. *Let $k \in \mathbb{Z}$. Under the previous assumption on g, given $\mu \in \mathcal{M}_\Gamma$, $\epsilon > 0$, and $k \in \mathbb{Z}$, there exist two polynomials $u_{n+k} = u_{n+k}(z, z^{-1})$, $v_{n+k} = v_{n+k}(z, z^{-1})$ such that* $\deg(u_{n+k}) \le n + k$, $\deg(v_{n+k}) \le n + k$ *in each variable z and z^{-1} and*

$$\frac{u_{n+k}(z,z^{-1})}{|z-1|^{2n}} \le g(z) \le \frac{v_{n+k}(z,z^{-1})}{|z-1|^{2n}}, \; z \in \Gamma \setminus \{1\},$$
$$\int \frac{v_{n+k}(z,z^{-1}) - u_{n+k}(z,z^{-1})}{|z-1|^{2n}} d\mu(z) < \epsilon.$$

Proof. Applying Lemmas 1–4 to

$$f(x) = \frac{g\left(\frac{x+i}{x-i}\right)}{\left|\left(\frac{x+i}{x-i}\right) - 1\right|^{2k}} = g\left(\frac{x+i}{x-i}\right)\frac{|x-i|^{2k}}{2^{2k}},$$

and

$$d\beta(x) = \frac{2^{2k} d\beta^{\mu}(x)}{|x-i|^{2k}},\ x \in \mathbb{R},$$

it follows that for each $\epsilon > 0$, there exist polynomials u_{n+k}, v_{n+k} of degree at most $n+k$ such that

$$u_{n+k}(x) \le f(x) \le v_{n+k}(x),\ x \in \mathbb{R}, \qquad \int (v_{n+k}(x) - u_{n+k}(x)) \frac{2^{2k} d\beta^{\mu}(x)}{|x-i|^{2k}} < \epsilon.$$

Changing variables, $x = i\frac{z+1}{z-1}$, $z \in \Gamma$, the above relations are transformed into

$$u_{n+k}\left(i\frac{z+1}{z-1}\right) \le \frac{g(z)}{|z-1|^{2k}} \le v_{n+k}\left(i\frac{z+1}{z-1}\right), \quad z \in \Gamma \setminus \{1\},$$

$$\int \left(v_{n+k}\left(i\frac{z+1}{z-1}\right) - u_{n+k}\left(i\frac{z+1}{z-1}\right)\right) |z-1|^{2k} d\mu(z) < \epsilon.$$

Since

$$\left(i\frac{z+1}{z-1}\right)^j = \frac{(i(z+1)(\frac{1}{z}-1))^j |z-1|^{2n+2k-2j}}{|z-1|^{2n+2k}} =$$

$$= (-1)^{j+1} 2^{2n+2k-j} \frac{\sin^j \theta (1-\cos\theta)^{n+k-j}}{|z-1|^{2n+2k}}, \quad z \in \Gamma,\ z = e^{i\theta},$$

the relations above are equivalent to the existence of polynomials

$$\widetilde{u}_{n+k}(z, z^{-1}), \quad \widetilde{v}_{n+k}(z, z^{-1})$$

of degree at most $n+k$ in each variables z and z^{-1} such that

$$\frac{\widetilde{u}_{n+k}(z, z^{-1})}{|z-1|^{2n}} \le g(z) \le \frac{\widetilde{v}_{n+k}(z, z^{-1})}{|z-1|^{2n}},\ z \in \Gamma \setminus \{1\},$$

$$\int \frac{\widetilde{v}_{n+k}(z, z^{-1}) - \widetilde{u}_{n+k}(z, z^{-1})}{|z-1|^{2n}} d\mu(z) < \epsilon.$$

■

If $\rho \in \mathcal{M}_0$ and ρ^b is the measure on $\mathbb{R}$ with image measure ρ on $[0, \infty)$ by the function $b(x) = x^2$, $x \in \mathbb{R}$, then $\rho^b \in \mathcal{M}$ and if, moreover, ρ is not discrete measure, then the Hamburger moment problem is also determinate and for all $j \in \mathbb{Z}$, $(1+x)^j d\rho(x) \in \mathcal{M}_0$ (see [4]); these results are stated in the following lemmas:

Lemma 6. *If $\rho \in \mathcal{M}_0$ and ρ is not discrete, then $\rho \in \mathcal{M}$ and for all $j \in \mathbb{Z}$, $(1+x)^j d\rho(x) \in \mathcal{M}_0$.*

Lemma 7. *If $\rho \in \mathcal{M}_0$ and ρ^b is the measure on $\mathbb{R}$ with the image measure ρ on $[0,\infty)$ by the function $b(x) = x^2$, $x \in \mathbb{R}$, then $\rho^b \in \mathcal{M}$.*

3. Orthogonal polynomials with respect to varying measures

Let μ be a positive Borel measure on Γ with infinitely many points in its support and consider the sequence of measures

$$d\mu_n(z) = \frac{d\mu(z)}{|z-1|^{2n}},\ z \in \Gamma,\ n \in \mathbb{N}.$$

We assume that for each $n \in \mathbb{N}$ and $k \in \mathbb{Z}_+$ we have $z^k \in L^1(\mu_n)$. Given a pair (n,m) of natural numbers there exists a unique polynomial $\varphi_{n,m}(z) = \kappa_{n,m} z^m + \ldots$ (with positive leading coefficient $\kappa_{n,m} = \kappa_m(\mu_n)$) of degree m, orthonormal with respect to the measure μ_n; that is,

$$\int_\Gamma \overline{z}^k\, \varphi_{n,m}(z)\, d\mu_n(z) = 0, \quad k = 0,\ldots,m-1, \quad \frac{1}{2\pi}\int_\Gamma |\varphi_{n,m}(z)|^2\, d\mu_n(z) = 1.$$

Let $\Phi_{n,m}(z) = \frac{1}{\kappa_{n,m}}\varphi_{n,m}(z)$ denote the monic orthogonal polynomials of degree m. Sometimes we make explicit reference to the measure by writing $\varphi_m(\mu_n, z) = \varphi_{n,m}(z)$. The following relations are well known:

$$\Phi_{n,m+1}(z) = z\Phi_{n,m}(z) + \Phi_{n,m+1}(0)\Phi^*_{n,m}(z), \tag{2}$$

$$\frac{\kappa_{n,m}}{\kappa_{n,m+1}}\varphi_{n,m+1}(z) = z\varphi_{n,m}(z) + \Phi_{n,m+1}(0)\varphi^*_{n,m}(z), \tag{3}$$

$$\frac{\kappa^2_{n,m}}{\kappa^2_{n,m+1}} = 1 - |\Phi_{n,m+1}(0)|^2. \tag{4}$$

Moreover, we have $|\Phi_{n,m+1}(0)| < 1$ and the zeros of $\varphi_{n,m}$ lie in the disk $|z| < 1$. Hereafter, if p is a polynomial of degree m, by $p^*(z) = z^m\overline{p(1/\overline{z})}$ we denote the reversed polynomial.

We also need the following well known identity due to Geronimus (see [7], or [5], p. 198).

$$\int_\Gamma z^j \frac{|dz|}{|\varphi_{n,m}(z)|^2} = \int_\Gamma z^j \, d\mu_n(z), \quad j = 0, \pm 1, \ldots, \pm m. \tag{5}$$

Let μ' denote the Radon-Nykodym derivative of μ with respect to the Lebesgue measure $|dz|$ on Γ. Let $\mu(z) = \mu'(z)|dz| + \mu_s(z)$ be the Lebesgue decomposition of μ; if $\mu' > 0$ almost everywhere on Γ, we can consider that $\mu' = \infty$ ($\Leftrightarrow \frac{1}{\mu'(z)} = 0$) on the support of μ_s which has Lebesgue measure equal to zero. We use the notations $\|g\|_{L^p(\mu)} = (\frac{1}{2\pi}\int_\Gamma |g|^p d\mu)^{1/p}$ and $L^1 = L^1(|dz|)$. Our main result in this section is the ratio asymptotics $\lim_{n\to\infty} \frac{\varphi_{n,n+k+1}(z)}{\varphi_{n,n+k}(z)}$. To this aim we need the following two lemmas.

Lemma 8. *Let $k \in \mathbb{Z}$. If $\mu' > 0$ a.e. on Γ, then*

$$\left\| \left| \frac{\varphi_{n,n+k}(z)}{(z-1)^n} \right|^2 - \frac{1}{\mu'(z)} \right\|_{L^1(\mu)} \leq \tag{6}$$

$$2 \min \left\{ \left\| \left| \frac{w_{n+k}(z)}{(z-1)^n} \right| - \frac{1}{\sqrt{\mu'(z)}} \right\|_{L^2(\mu)} : w_{n+k} \in \mathcal{P}_{n+k} \right\},$$

where $\mathcal{P}_{n+k}$ denotes the set of polynomials of degree at most $n+k$. Moreover, if $\mu \in \mathcal{M}_\Gamma$, then

$$\lim_{n\to\infty} \left\| \left| \frac{\varphi_{n,n+k}(z)}{(z-1)^n} \right|^2 - \frac{1}{\mu'(z)} \right\|_{L^1(\mu)} = 0. \tag{7}$$

Proof.

Let $w_{n+k} \in \mathcal{P}_{n+k}$. Using that $(\mu')^{-1/p} \in L^p(\mu)$, and the Cauchy-Schwarz inequality, we have

$$\left\| \left| \frac{\varphi_{n,n+k}(z)}{(z-1)^n} \right|^2 - \frac{1}{\mu'(z)} \right\|_{L^1(\mu)}$$
$$\leq \left\| \left| \frac{\varphi_{n,n}(z)}{(z-1)^n} \right|^2 - \frac{1}{\sqrt{\mu'(z)}} \left| \frac{w_{n+k}(z)}{(z-1)^n} \right| \right\|_{L^1(\mu)} + \left\| \frac{1}{\sqrt{\mu'(z)}} \left| \frac{w_{n+k}(z)}{(z-1)^n} \right| - \frac{1}{\mu'(z)} \right\|_{L^1(\mu)}$$

$$= \left\| \left| \frac{\varphi_{n,n+k}(z)}{(z-1)^n} \right| \left(\left| \frac{\varphi_{n,n+k}(z)}{(z-1)^n} \right| - \frac{1}{\sqrt{\mu'(z)}} \left| \frac{w_{n+k}(z)}{\varphi_{n,n+k}(z)} \right| \right) \right\|_{L^1(\mu)}$$

$$+ \left\| \frac{1}{\sqrt{\mu'(z)}} \left(\left| \frac{w_{n+k}(z)}{(z-1)^n} \right| - \frac{1}{\sqrt{\mu'(z)}} \right) \right\|_{L^1(\mu)}$$

$$\leq \left\| \left| \frac{\varphi_{n,n+k}(z)}{(z-1)^n} \right| - \frac{1}{\sqrt{\mu'(z)}} \left| \frac{w_{n+k}(z)}{\varphi_{n,n+k}(z)} \right| \right\|_{L^2(\mu)} + \left\| \left| \frac{w_{n+k}(z)}{(z-1)^n} \right| - \frac{1}{\sqrt{\mu'(z)}} \right\|_{L^2(\mu)} .$$

Taking (5) into account, we obtain

$$\left\| \left| \frac{\varphi_{n,n+k}(z)}{(z-1)^n} \right| - \frac{1}{\sqrt{\mu'(z)}} \left| \frac{w_{n+k}(z)}{\varphi_{n,n+k}(z)} \right| \right\|^2_{L^2(\mu)} =$$

$$= 1 - \frac{2}{2\pi} \int_0^{2\pi} \left| \frac{w_{n+k}(z)}{(z-1)^n} \right| \sqrt{\mu'(z)}\, d\theta + \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{w_{n+k}(z)}{(z-1)^n} \right|^2 d\mu(z) =$$

$$= \left\| \left| \frac{w_{n+k}(z)}{(z-1)^n} \right| - \frac{1}{\sqrt{\mu'(z)}} \right\|^2_{L^2(\mu)} .$$

Hence,

$$\left\| \left| \frac{\varphi_{n,n+k}(z)}{(z-1)^n} \right|^2 - \frac{1}{\mu'(z)} \right\|_{L^1(\mu)} \leq 2 \left\| \left| \frac{w_{n+k}(z)}{(z-1)^n} \right| - \frac{1}{\sqrt{\mu'(z)}} \right\|_{L^2(\mu)} .$$

This proves (6).

Now, let us show (7). The set of continuous functions is dense in $L^2(\mu)$. The function $1/\sqrt{\mu'}$ belongs to $L^2(\mu)$ and is nonnegative, hence it can be approximated in the metric of this space by positive continuous functions. In turn, using that a positive trigonometric polynomial $v(z, z^{-1})$ of degree $n+k$ can be written as $v(z, z^{-1}) = |w_{n+k}(z)|^2$ with $w_{n+k} \in \mathcal{P}_{n+k}$ (see [5], p. 211), and by Lemma 5 every positive continuous function on Γ can be approximated by functions of the form $\left| \frac{w_{n+k}(z)}{(z-1)^n} \right|$ in $L^2(\mu)$ it follows that

$$\lim_{n\to\infty} \min \left\{ \left\| \left| \frac{w_{n+k}(z)}{(z-1)^n} \right| - \frac{1}{\sqrt{\mu'(z)}} \right\|_{L^2(\mu)} : w_{n+k} \in \mathcal{P}_{n+k} \right\} = 0.$$

and by (6) the proof is complete.

■

The following lemma for fixed measures may be found in [17].

Lemma 9.

$$|\Phi_{n,n+k}(0)| \leq \left\| \left| \frac{\varphi_{n,n+k}(z)}{(z-1)^n} \right|^2 - \frac{1}{\mu'(z)} \right\|_{L^1(\mu)}$$

Proof. Set $a_{n+k} = -\overline{\Phi_{n,n+k}(0)}$ and $S_n(z) = \Re\left(a_{n+k} z \varphi_{n,n+k}(z)/\varphi^*_{n,n+k}(z)\right)$. Comparing the squares of the modulus of the left-hand and right-hand sides of (3) on Γ, we obtain

$$\begin{aligned}
&\frac{\kappa^2_{n,n+k}}{\kappa^2_{n,n+k+1}} |\varphi_{n,n+k+1}(z)|^2 = \\
&= |\varphi_{n,n+k}(z)|^2 - 2\Re\left(a_{n+k} z \varphi_{n,n+k}(z) \overline{\varphi^*_{n,n+k}(z)}\right) + |a_{n+k}|^2 |\varphi_{n,n+k}(z)|^2 = \\
&\qquad = \left(1 + |a_{n+k}|^2\right) |\varphi_{n,n+k}(z)|^2 - 2S_n(z) |\varphi_{n,n+k}(z)|^2, \quad z \in \Gamma.
\end{aligned}$$

Integrating with respect to $\frac{d\mu(z)}{2\pi|z-1|^{2n}}$ and using (4), we obtain the representation

$$|a_{n+k}|^2 = \frac{1}{2\pi} \int_\Gamma S_n(z) \frac{|\varphi_{n+k}(z)|^2}{|z-1|^{2n}} d\mu(z).$$

Since $\int_\Gamma S_n(z)|dz| = 0$ and $|S_n(z)| \leq |a_{n+k}|$, $z \in \Gamma$, it follows that

$$\begin{aligned}
|a_{n+k}|^2 &= \frac{1}{2\pi} \int_\Gamma S_n(z) \left(\frac{|\varphi_{n+k}(z)|^2}{|z-1|^{2n}} \mu'(z) - 1 \right) |dz| + \frac{1}{2\pi} \int_\Gamma S_n(z) \frac{|\varphi_{n+k}(z)|^2}{|z-1|^{2n}} d\mu_s(z) \\
&\leq |a_{n+k}| \left(\left\| \left| \frac{\varphi_{n,n+k}(z)}{(z-1)^n} \right|^2 \mu'(z) - 1 \right\|_{L^1} + \frac{1}{2\pi} \int_\Gamma \frac{|\varphi_{n+k}(z)|^2}{|z-1|^{2n}} d\mu_s(z) \right) \\
&= |a_{n+k}| \left\| \left| \frac{\varphi_{n,n+k}(z)}{(z-1)^n} \right|^2 - \frac{1}{\mu'(z)} \right\|_{L^1(\mu)}
\end{aligned}$$

This proves the lemma. ∎

Combining Lemmas 8 and 9, and the relations (2)–(4), we obtain

Theorem 3. *If $\mu \in \mathcal{M}_\Gamma$ and $\mu' > 0$ almost everywhere on Γ, then for each $k \in \mathbb{Z}$, we have*

$$\lim_{n\to\infty} \frac{\Phi_{n,n+k+1}(z)}{\Phi_{n,n+k}(z)} = \lim_{n\to\infty} \frac{\varphi_{n,n+k+1}(z)}{\varphi_{n,n+k}(z)} = z,$$

uniformly on each compact subset of $\{z \in \mathbb{C} : 1 \leq |z|\}$*;*

$$\lim_{n\to\infty} \frac{\Phi_{n,n}(z)}{\Phi^*_{n,n}(z)} = \lim_{n\to\infty} \frac{\varphi_{n,n}(z)}{\varphi^*_{n,n}(z)} = 0, \tag{8}$$

uniformly on each compact subset of $\{z : |z| < 1\}$*; and*

$$\lim_{n\to\infty} \frac{\kappa_{n,n+k+1}}{\kappa_{n,n+k}} = 1, \quad \lim_{n\to\infty} \Phi_{n,n+k}(0) = 0.$$

Remark 1. *Using quantitative results of polynomial approximation (for results on quantitative one sided polynomial approximation on* $\mathbb{R}$ *see, for example, [6]), and Lemmas 8 and 9, we can estimate the rate of convergence of the* $\Phi_{n,n+k}(0)$ *to* 0.

Remark 2. *In [2] (Lemma 2) it is proved that condition (8) implies that for every continuous function* A *on* Γ *there exist two sequences of polynomials* $\{u_{n+k}(z)\}_{n=1}^{\infty}$*,* $\{v_{n+k}(z)\}_{n=1}^{\infty}$ *with* $\deg u_{n+k}(z) \leq n+k$*,* $\deg v_{n+k}(z) \leq n+k$*, such that*

$$\lim_{n\to\infty} \max\left\{ \left| A(z) - \frac{u_{n+k}(z) + v_{n+k}(\frac{1}{z})}{|\varphi_{n,n+k}(z)|^2} \right| : z \in \Gamma \right\} = 0. \tag{9}$$

Moreover, if f *is nonnegative on* Γ *we can find polynomials* $u_{n+k}(z)$*,* $n \in \mathbb{N}$*, such that*

$$\lim_{n\to\infty} \max\left\{ \left| A(z) - \left| \frac{u_{n+k}(z)}{\varphi_{n+k}(z)} \right|^2 \right| : z \in \Gamma \right\} = 0. \tag{10}$$

Because of Lemma 8 and

$$\left\| \left| \frac{\varphi_{n,n+k}(z)}{(z-1)^n} \right|^2 - \frac{1}{\mu'(z)} \right\|_{L^1(\mu)} =$$

$$= \frac{1}{2\pi} \int_\Gamma \left| \left| \frac{\varphi_{n,n+k}(z)}{(z-1)^n} \right|^2 - \frac{1}{\mu'(z)} \right| \mu'(z)|dz| + \frac{1}{2\pi} \int_\Gamma \left| \left| \frac{\varphi_{n,n+k}(z)}{(z-1)^n} \right|^2 - \frac{1}{\mu'(z)} \right| d\mu_s(z) =$$

$$= \frac{1}{2\pi} \int_\Gamma \left| \left| \frac{\varphi_{n,n+k}(z)}{(z-1)^n} \right|^2 \mu'(z) - 1 \right| |dz| + \frac{1}{2\pi} \int_\Gamma \left| \frac{\varphi_{n,n+k}(z)}{(z-1)^n} \right|^2 d\mu_s(z),$$

we obtain

Lemma 10. *If $\mu \in \mathcal{M}_\Gamma$ and $\mu' > 0$ almost everywhere on Γ, we have*

$$\lim_{n\to\infty} \int_\Gamma \left| \left| \frac{\varphi_{n,n+k}(z)}{(z-1)^n} \right|^2 \mu'(z) - 1 \right| |dz| = 0,$$

$$\lim_{n\to\infty} \int_\Gamma \left| \left| \frac{\varphi_{n,n+k}(z)}{(z-1)^n} \right| \sqrt{\mu'(z)} - 1 \right|^2 |dz| = 0. \tag{11}$$

Therefore, for any $A \in L^\infty(\mu)$

$$\lim_{n\to\infty} \int_\Gamma A(z) \left| \frac{\varphi_{n,n+k}(z)}{(z-1)^n} \right|^2 \mu'(z)\, |dz| = \int_\Gamma A(z)\, |dz|,$$

$$\lim_{n\to\infty} \int_\Gamma A(z) \left| \frac{\varphi_{n,n+k}(z)}{(z-1)^n} \right|^2 d\mu(z) = \int_\Gamma A(z)\, |dz|.$$

The proof of (11) can be seen in Lemma 2 of [12]. The above lemma for fixed measures appears in [15] (see Theorem 2.1, Corollary 2.2, and Corollary 5.1).

Lemma 11. *Let μ be a positive Borel measure on Γ with $\mu' > 0$ a. e. on Γ, and let $h \geq 0$, $h \in L^1(\mu)$.*

(a) *If, in addition, $h\, d\mu \in \mathcal{M}_\Gamma$ and there exists a polynomial Q such that $|Q|h^{-1} \in L^\infty(\mu)$, then, for each $k \in \mathbb{Z}$ and any continuous function A on Γ,*

$$\lim_n \int_\Gamma A(z)|Q(z)|^2 \left| \frac{\varphi_{n+k}(hd\mu_n, z)}{\varphi_{n+k}(\mu_n, z)} \right|^2 |dz| = \int_\Gamma A(z)|Q(z)|^2 h^{-1}(z)\, |dz|.$$

(b) *If, instead, $\mu \in \mathcal{M}_\Gamma$ and there exists a polynomial Q such that $|Q|h \in L^\infty(\mu)$, then, for each $k \in \mathbb{Z}$ and any continuous function A on Γ,*

$$\lim_n \int_\Gamma A(z)|Q(z)|^2 \left| \frac{\varphi_{n+k}(\mu_n, z)}{\varphi_{n+k}(hd\mu_n, z)} \right|^2 |dz| = \int_\Gamma A(z)|Q(z)|^2 h(z)\, |dz|.$$

Proof: Assertions (a) and (b) are proved using the same arguments; we will carry out the proof of (a). Note that from condition (a) (see Remark 2) it follows that there exists a rational sequence $\{\frac{u_{n+k}(z,1/z)}{|\varphi_{n+k}(h\,d\mu_n,z)|^2}\}$ which converges to $A|Q|^2$

uniformly on Γ, where $u_{n+k}(z,1/z)$ is a polynomial of degree at most $n+k$ in both variables z and $1/z$. Using the Geronimus identity (5) and Lemma 10, we have

$$\lim_n \int_\Gamma A(z)|Q(z)|^2 \left|\frac{\varphi_{n+k}(hd\mu_n,z)}{\varphi_{n+k}(\mu_n,z)}\right|^2 |dz| =$$
$$= \lim_n \int_\Gamma \frac{u_{n+k}(z,1/z)}{|\varphi_n(h\,d\mu_n,z)|^2} \left|\frac{\varphi_{n+k}(hd\mu_n,z)}{\varphi_{n+k}(\mu_n,z)}\right|^2 |dz| =$$
$$= \lim_n \int_\Gamma \frac{u_{n+k}(z,1/z)}{|\varphi_{n+k}(\mu_n,z)|^2}|dz| = \lim_n \int_\Gamma u_{n+k}(z,1/z)\,d\mu_n(z) =$$
$$= \lim_n \int_\Gamma h^{-1}(z)\frac{u_{n+k}(z,1/z)}{|\varphi_n(h\,d\mu_n,z)|^2}|\varphi_n(h\,d\mu_n,z)|^2 h(z)\,d\mu_n(z) =$$
$$= \int_\Gamma A(z)|Q(z)|^2 h^{-1}(z)\,|dz|.$$

■

Remark 3. *Following the same method employed by López in [12], we can obtain Lemma 11 when A is any Riemann integrable function on Γ.*

Another result of independent interest is the weak star limit of $\frac{|z-1|^{2n}}{|\varphi_{n,n+k}(z)|^2}|dz|$.

Theorem 4. *If $\mu \in \mathcal{M}_\Gamma$ and $\mu' > 0$ almost everywhere on Γ, then for each $k \in \mathbb{Z}$ we have*

$$\lim_{n\to\infty} \int_\Gamma A(z)\frac{|z-1|^{2n}}{|\varphi_{n,n+k}(z)|^2}|dz| = \lim_{n\to\infty}\int_\Gamma A(z)\,d\mu(z), \qquad (12)$$

for every continuous function A on Γ. That is, the weak star limit of $\frac{|z-1|^{2n}}{|\varphi_{n,n+k}(z)|^2}|dz|$ is μ.

Proof. Taking real and imaginary parts, we can assume that A is a real function. Actually, we prove the more general statement that for every real continuous function A in $\Gamma \setminus \{1\}$ such that there exists constants $\tilde{A} > 0, \tilde{B} > 0$ and $j \in \mathbb{Z}$ for which

$$|A(z)| \le \tilde{A} + \frac{\tilde{B}}{|z-1|^{2j}}, \quad z \in \Gamma \setminus \{1\},$$

relation (12) holds.

Let k and A be fixed. Using Lemma 5 given $\epsilon > 0$ we can find polynomials $u_{n+k} = u_{n+k}(z, z^{-1})$ and $v_{n+k} = v_{n+k}(z, z^{-1})$ of degree at most $n+k$ such that

$$\frac{u_{n+k}(z, z^{-1})}{|z-1|^{2n}} \leq A(z) \leq \frac{v_{n+k}(z, z^{-1})}{|z-1|^{2n}}, \; z \in \Gamma \setminus \{1\},$$
$$\int_\Gamma (v_{n+k}(z, z^{-1}) - u_{n+k}(z, z^{-1})) d\mu_n(z) < \epsilon.$$

Moreover, using the Geronimus identity (5) we obtain

$$\int_\Gamma u_{n+k}(z, z^{-1})\, d\mu_n(z) = \int_\Gamma \frac{u_{n+k}(z, z^{-1})}{|z-1|^{2n}} \frac{|z-1|^{2n}}{|\varphi_{n,n+k}(z)|^2} |dz| \leq$$
$$\leq \int_\Gamma A(z) \frac{|z-1|^{2n}}{|\varphi_{n,n+k}(z)|^2} |dz| \leq \int_\Gamma \frac{v_{n+k}(z, z^{-1})}{|z-1|^{2n}} \frac{|z-1|^{2n}}{|\varphi_{n,n+k}(z)|^2} |dz| =$$
$$= \int_\Gamma v_{n+k}(z, z^{-1})\, d\mu_n(z)$$

and

$$\int_\Gamma u_{n+k}(z, z^{-1})\, d\mu_n(z) \leq \int_\Gamma A(z)\, d\mu(z) \leq \int_\Gamma v_{n+k}(z, z^{-1})\, d\mu_n(z).$$

Therefore,

$$\left| \int_\Gamma A(z) \frac{|z-1|^{2n}}{|\varphi_{n,n+k}(z)|^2} |dz| - \int_\Gamma A(z)\, d\mu(z) \right| < \epsilon.$$

■

Now, we can obtain the relative asymptotics of orthogonal polynomials. The following result under more restrictive assumptions on the measure μ and on the function h was proved by López in [12].

Theorem 5. *Let $\mu \in \mathcal{M}_\Gamma$ be such that $\mu' > 0$ a. e. Let h be such that $h\, d\mu \in \mathcal{M}_\Gamma$ and suppose there exists a polynomial Q such that $|Q|h^{\pm 1} \in L^\infty(\mu)$. Then, for each $k \in \mathbb{Z}$ we have*

$$\lim_n \frac{\varphi_{n+k}(h d\mu_n, z)}{\varphi_{n+k}(\mu_n, z)} = S(h, \{|\zeta| > 1\}, z),$$

uniformly in each compact subset of $\{z \in \mathbb{C} : |z| > 1\}$*, where*

$$S(h, \{|\zeta| > 1\}, z) = \exp\left(\frac{1}{4\pi}\int_{\Gamma} \log h(\zeta)\frac{\zeta + z}{\zeta - z}|d\zeta|\right)$$

is the Szegő function of h *in* $\{z \in \mathbb{C} : |z| > 1\}$.

Proof: It will be more convenient for us to prove the equivalent relation

$$\lim_{n} \frac{\varphi^*_{n+k}(hd\mu_n, z)}{\varphi^*_{n+k}(\mu_n, z)} = S^*(z), \quad |z| < 1.$$

where $S^*(z) = \overline{S(h, \{|\zeta| > 1\}, 1/\overline{z})}$. Using Theorem 3, it is sufficient to prove the above relation for $k = 0$. Without loss of generality, we can consider that the polynomial Q in the assumptions of the theorem has no zeros inside the disk $\{|z| < 1\}$ and, therefore, $\frac{Q(z)\varphi^*_{n+k}(hd\mu_n,z)}{\varphi^*_{n+k}(\mu_n,z)}$ is an analytic function in $\{|z| < 1\}$ and $\frac{Q(z)\varphi^*_{n+k}(hd\mu_n,z)}{\varphi^*_{n+k}(\mu_n,z)} \neq 0$, $|z| < 1$. Then, according to Poisson's formula,

$$\log\left|\frac{Q(z)\varphi^*_{n+k}(hd\mu_n, z)}{\varphi^*_{n+k}(\mu_n, z)}\right|^2 = \frac{1}{2\pi}\int_{\Gamma} \log\left|\frac{Q(\zeta)\varphi^*_{n+k}(hd\mu_n, \zeta)}{\varphi^*_{n+k}(\mu_n, \zeta)}\right|^2 P(z, \zeta)\,|d\zeta|,$$

where $P(z, \zeta) = \frac{1-|z|^2}{|\zeta - z|^2}$ is the Poisson kernel. Using Jensen's inequality, we obtain

$$\left|\frac{Q(z)\varphi^*_{n+k}(hd\mu_n, z)}{\varphi^*_{n+k}(\mu_n, z)}\right|^2 \leq \frac{1}{2\pi}\int_{\Gamma} \left|\frac{Q(\zeta)\varphi^*_{n+k}(hd\mu_n, \zeta)}{\varphi^*_{n+k}(\mu_n, \zeta)}\right|^2 P(z, \zeta)\,|d\zeta|,$$

Since $|\varphi^*_{n+k}(\mu_n, \zeta)| = |\varphi_{n+k}(\mu_n, \zeta)|$ and $|\varphi^*_{n+k}(hd\mu_n, \zeta)| = |\varphi_{n+k}(hd\mu_n, \zeta)|$, $|\zeta| = 1$, using Lemma 11 we obtain

$$\limsup_{n}\left|\frac{Q(z)\varphi^*_{n+k}(hd\mu_n, z)}{\varphi^*_{n+k}(\mu_n, z)}\right|^2 \leq \frac{1}{2\pi}\int_{\Gamma} h^{-1}(\zeta)\,|Q(\zeta)|^2\, P(z, \zeta)\,|d\zeta|,\ |z| < 1. \tag{13}$$

In turn, this yields that the sequence $\{\frac{\varphi^*_{n+k}(hd\mu_n,z)}{\varphi^*_{n+k}(\mu_n,z)}\}$ is uniformly bounded inside (on each compact subset) of the disk $\{|z| < 1\}$ (we recall that Q has no zeros in $\{|z| < 1\}$). Let us consider an arbitrary subsequence $\Lambda \subset \mathbb{N}$ such that $\{\frac{\varphi^*_{n+k}(hd\mu_n,z)}{\varphi^*_{n+k}(\mu_n,z)} : n \in \Lambda\}$ converges and denote its limit by S_Λ. In virtue of what was said above, it is sufficient for us to prove that for any such sequence Λ we have $S^* = S_\Lambda$. Let $r \in (0,1)$ be arbitrary. Using Lemma 11 once more, we obtain

$$\frac{1}{2\pi}\int_\Gamma |Q(r\zeta)S_\Lambda(r\zeta)|^2\,|d\zeta| = \lim_{n\in\Lambda}\frac{1}{2\pi}\int_\Gamma \left|\frac{Q(r\zeta)\varphi^*_{n+k}(hd\mu_n, r\zeta)}{\varphi^*_{n+k}(\mu_n, r\zeta)}\right|^2 |d\zeta|$$
$$\leq \lim_{n\in\Lambda}\frac{1}{2\pi}\int_\Gamma \left|\frac{Q(\zeta)\varphi^*_{n+k}(hd\mu_n, \zeta)}{\varphi^*_{n+k}(\mu_n, \zeta)}\right|^2 |d\zeta| = \lim_{n\in\Lambda}\frac{1}{2\pi}\int_\Gamma h^{-1}(\zeta)\,|Q(\zeta)|^2\,|d\zeta|.$$

Thus, $QS_\Lambda \in H^2(\{|z|<1\})$, and therefore the limit $\lim_{r\to1} Q(r\zeta)S^*_\Lambda(r\zeta)$ exists almost everywhere for $\zeta \in \Gamma$. On the other hand, according to (13), for each fixed $z \in \Gamma$, we have

$$|Q(rz)S_\Lambda(rz)|^2 \leq \frac{1}{2\pi}\int_\Gamma h^{-1}(\zeta)\,|Q(\zeta)|^2\,P(rz,\zeta)\,|d\zeta|.$$

It is well known (see, for example, [21], Section 9.5) that the limit as $r \to 1$ of the righthand side of this inequality exists for almost all $z \in \Gamma$ and it is equal a.e. to $h^{-1}(\zeta)|Q(\zeta)|^2$. Therefore, $|S_\Lambda(z)|^2 \leq h^{-1}(z)$ almost everywhere on Γ. Working with $\{\frac{\varphi^*_{n+k}(\mu_n,z)}{\varphi^*_{n+k}(h\,d\mu_n,z)}\}$, we obtain that the inverse inequality is also satisfied. So $|S_\Lambda(z)|^2 = h^{-1}(z)$ a. e. on Γ, which implies

$$\log|S_\Lambda(z)| = \frac{1}{2\pi}\int_\Gamma \log|S_\Lambda(\zeta)|\,P(z,\zeta)\,|d\zeta| =$$
$$= \frac{1}{4\pi}\int_\Gamma \log h^{-1}(\zeta)\,P(z,\zeta)\,|d\zeta| = \log|S^*(z)|$$

Since

$$S_\Lambda(0) = \lim_{n\in\Lambda}\frac{\kappa_{n+k}(h\,d\mu_n)}{\kappa_{n+k}(\mu_n)} > 0$$

and $S^*(0) > 0$, it follows that $\log S_\Lambda(z) = \log S^*(z)$, $|z|<1$, and thus $S_\Lambda(z) = S^*(z)$. The theorem has been established.

■

Asymptotic formulas can be obtained from Theorem 5 for the Szego kernel

$$K_{n+k}(\mu_n, z, \zeta) = \sum_{j=0}^{n+k}\varphi_j(\mu_n, z)\overline{\varphi_j(\mu_n,\zeta)}$$

and the Christofel functions

$$\omega_{n+k}(\mu_n,z) = \inf_{p\in\mathcal{P}_{n+k}} \int_\Gamma \left|\frac{p(\zeta)}{p(z)}\right|^2 d\mu_n(\zeta).$$

where $\mathcal{P}_n$ is the set of all polynomials of degree $\leq n$. Other expressions for these functions (see, for example, [21], Chapter XI) are

$$\begin{aligned} K_{n+k}(\mu_n,z,\zeta) &= \frac{\varphi^*_{n+k}(\mu_n,z)\overline{\varphi^*_{n+k}(\mu_n,\zeta)} - z\overline{\zeta}\varphi_{n+k}(\mu_n,z)\overline{\varphi_{n+k}(\mu_n,\zeta)}}{1-z\,\overline{\zeta}} \\ &= \frac{\varphi^*_{n+k+1}(\mu_n,z)\overline{\varphi^*_{n+k+1}(\mu_n,\zeta)} - \varphi_{n+k+1}(\mu_n,z)\overline{\varphi_{n+k+1}(\mu_n,\zeta)}}{1-z\,\overline{\zeta}} \end{aligned}$$

and

$$\omega_{n+k}(z) = K_{n+k}(\mu_n,z,z)^{-1}.$$

Corollary 1. *Under the assumptions of Theorem 5, we have*

$$\lim_n \frac{K_{n+k}(h\,d\mu_n,z,\zeta)}{K_{n+k}(\mu_n,z,\zeta)} = S(h,z)\overline{S(h,\zeta)}, \quad |z|>1,\ |\zeta|>1.$$

In particular,

$$\lim_n \frac{\omega_{n+k}(h\,d\mu_n,z)}{\omega_{n+k}(\mu_n,z)} = |S(h,z)|^{-2}.$$

Let ρ be a positive Borel measure in $\Delta=[-1,1]$. Set $d\rho_n(u) = \frac{d\rho(u)}{(1-u)^n}$ and assume that $u^k \in L^1(\rho_n)$ for each $k\geq 0$. Let $l_{n,m}(u) = \tau_{n,m}u^m + \dots$ be the orthonormal polynomial of degree m with respect to the measure $d\rho_n(u)$ whose leading coefficient $\tau_{n,m}$ is supposed to be positive. Set $L_{n,m}(u) = l_{n,m}(u)/\tau_{n,m}$.

Lemma 12. *If $\rho'>0$ a.e. on $(-1,1)$ and $\rho\left(\frac{x-1}{x+1}\right)\in\mathcal{M}_0$, then for each $j\in\mathbb{Z}$ we have*

$$\lim_{n\to\infty}\frac{\tau_{n+j+1}}{\tau_{n+j}} = 2$$

and

$$\lim_{n\to\infty}\frac{l_{n,n+j+1}(u)}{l_{n,n+j}(u)} = u+\sqrt{u^2-1} \stackrel{\text{def}}{=} \varphi(u) = 2\lim_{n\to\infty}\frac{L_{n,n+j+1}(u)}{L_{n,n+j}(u)},$$

uniformly on each compact subset of $\mathbb{C}\setminus\Delta$.

Proof. The proof is carried out as usual, reducing it to the case of the unit circle. Let μ be the measure on the unit circle Γ defined by

$$\mu(e^{i\theta}) = \rho(\cos\theta) = \rho\left(\frac{1}{2}\left(z + \frac{1}{z}\right)\right), \quad z = e^{i\theta}, \theta \in [0, 2\pi).$$

Let $d\mu_n(z) = \frac{d\mu(z)}{|z-1|^{4n}}$, $z \in \Gamma$. Let $\varphi_{2n,m}(z) = \kappa_{n,m} z^m + \ldots$ and $\Phi_{2n,m}(z) = \frac{\varphi_{2n,m}(z)}{\kappa_{n,m}}$ be the corresponding orthonormal and monic orthogonal polynomials, respectively, on Γ. In particular,

$$\frac{1}{2\pi}\int_\Gamma \varphi_{2n,j}(z)\overline{\varphi_{2n,k}(z)}\, d\mu_n(z) = \delta_{j,k}, \quad j, k = 0, 1, \ldots.$$

These polynomials are connected with the polynomials $l_{n,m}$ and $L_{n,m}$ by the well known relations

$$l_{n,m}(x) = \frac{\varphi_{2n,2m}(z) + \varphi^*_{2n,2m}(z)}{z^m\sqrt{2\pi(1 + \Phi_{2n,2m}(0))}}, \quad L_{n,m}(x) = \frac{\varphi_{2n,2m}(z) + \varphi^*_{2n,2m}(z)}{(2z)^m\sqrt{(1 + \Phi_{2n,2m}(0))}}, \tag{14}$$

with $x = \frac{1}{2}(z + \frac{1}{z})$.

We have

$$\mu\left(\frac{x+i}{x-i}\right) = \rho\left(\frac{1}{2}\left(\frac{x+i}{x-i} + \frac{x-i}{x+i}\right)\right) = \rho\left(\frac{x^2-1}{x^2+1}\right) = \rho\left(\frac{t-1}{t+1}\right),$$

where $t = x^2$, $t \in [0, \infty)$, $x \in \mathbb{R}$. By Lemma 7, the measure μ satisfies the assumptions of the Theorem 3. Using this theorem and the relations (14), we immediately complete the proof of the lemma.

$\blacksquare$

4. Padé approximants of Stieltjes-type meromorphic functions

In this section we prove Theorem 1. More precisely, let α be as in Theorem 1, let Q_n be the denominator of the Padé approximant of f normalized by $Q_n(-1) = (-1)^n$, and let $\mathcal{L}_n$ be the orthogonal polynomials with respect to α normalized also by $\mathcal{L}_n(-1) = (-1)^n$.

Theorem 6. *If $\alpha' > 0$ a.e. on $(0, \infty)$ and $\alpha \in \mathcal{M}_0$, then the following statements hold:*

1. $$\lim_{n\to\infty} \frac{Q_n(z)}{\mathcal{L}_n(z)} = \prod_{j=1}^{d} \left(\frac{(1+z)(1+a_j)(\Phi(z) - \Phi(a_j))}{4\Phi(z)(z-a_j)} \right), \quad z \in \mathcal{D},$$
   where $a_1, \ldots, a_d$ are the poles of r (counting their multiplicity) and $\Phi(z) = (\sqrt{z}+i)/(\sqrt{z}-i)$ is the conformal mapping of $\mathcal{D}$ onto the exterior of the unit circle ($\Phi(-1) = \infty$)

2. $\lim_n \pi_n = f$ *uniformly on each compact subset of $\mathcal{D} \setminus \{z : r(z) = \infty\}$.*

Under more restrictive assumption on the measure α this theorem was proved by López in [11]. The general scheme of the proof of the above theorem follows the technique developed in [11] (which at the same time in some steps uses ideas from Gonchar [8]). For convenience of the reader, we include some details. The scheme of the proof is the following: Carrying out a bilinear transformation we pass to the problem of the convergence of Padé approximants $\Pi_n = g_n/h_n$ for functions of type $F(\zeta) = \widehat{\rho}(\zeta) + R(\zeta)$, where ρ is a measure on $\Delta = [-1, 1]$; moreover, F has asymptotic expansion in powers of $(\zeta - 1)$ and the Padé approximants correspond to this expansion. For the new convergence problem, it is possible to apply a known method of Gonchar, based on the fact that the denominators h_n of the new approximants satisfy incomplete orthogonality relations with respect to a certain (in this case varying) measure with compact support. This allows us to reduce the study of the asymptotic behavior of q_n to the question of the existence of the asymptotics of the ratio of orthogonal polynomials with respect to this same measure.

Proof of Theorem 6.

Step 1. Let us make the change of variables $x = (1+u)/(1-u)$, $x \in (0, \infty)$, $u \in (-1, 1)$, in the integral (1) and take $z = (1+\zeta)/(1-\zeta)$ in the argument of f. It can be checked directly that

$$f\left(\frac{1+\zeta}{1-\zeta}\right) = (1-\zeta)\left(\widehat{\rho}(\zeta) + R(\zeta)\right), \tag{15}$$

where

$$d\rho(u) = \frac{1}{2}(1-u)d\alpha\left(\frac{1+u}{1-u}\right) \quad \text{and} \quad (1-\zeta)R(\zeta) = r\left(\frac{1+\zeta}{1-\zeta}\right).$$

Put

$$F(\zeta) = \widehat{\rho}(\zeta) + R(\zeta) = \int_\Delta \frac{d\rho(t)}{\zeta - t} + R(\zeta), \quad \zeta \in \mathbb{C} \setminus \Delta,$$

let $\Pi_n = g_n/h_n$ be the Padé approximant of orden n of the function F corresponding to the point $\zeta = 1$ (this point corresponds to $z = \infty$). We have

$$h_n(\zeta) = (1-\zeta)^n Q_n\left(\frac{1+\zeta}{1-\zeta}\right), \quad g_n(\zeta) = (1-\zeta)^{n-1} P_n\left(\frac{1+\zeta}{1-\zeta}\right), \tag{16}$$

and

$$\Pi_n(\zeta) = \frac{1}{\zeta - 1}\pi_n\left(\frac{1+\zeta}{1-\zeta}\right). \tag{17}$$

Moreover, if $d\rho_n(u) = \frac{d\rho(u)}{(1-u)^n}$, $u \in (-1,1)$, $R(\zeta) = \frac{l_{d-1}(\zeta)}{t_d(\zeta)}$, and $t_d(\zeta) = \prod_{j=1}^d (\zeta - b_j)$, then

$$\int_\Delta u^j h_n(u)\, t_d(u)\, d\rho_n(u) = 0, \quad j = 0, 1, \ldots, n-d-1, \tag{18}$$

$$F(\zeta) - \Pi_n(\zeta) = \frac{(1-\zeta)^{2n}}{s(\zeta)h_n(\zeta)t_d(\zeta)} \int_\Delta \frac{s(u)h_n(u)t_d(u)}{\zeta - u} d\rho_n(u), \tag{19}$$

where $s(u)$ is an arbitrary polynomial of degree $\leq n - d$.

Combining (15) and (17) the convergence of $\{\pi_n\}$ to f, uniformly on each compact subset of $\mathbb{C} \setminus \{[0,\infty) \cup \{r = \infty\}\}$, is equivalent to the convergence of $\{\Pi_n\}$ to F uniformly on each compact subset of $\mathbb{C} \setminus \{\Delta \cup \{R = \infty\}\}$.

If $t_d = 1$ ($\Leftrightarrow r \equiv 0$), then using Stieltjes' theorem we know $\lim_n \pi_n(z) = f(z)$ uniformly on each compact subset of $\mathbb{C} \setminus \{[0,\infty) \cup \{r = \infty\}\}$ or equivalently $\lim_n \Pi_n(z) = F(z)$ uniformly on each compact subset of $\mathbb{C} \setminus \{[-1,1] \cup \{R = \infty\}\}$. By formula (19) with $s = 1$, we obtain

$$\lim_{n\to\infty} \frac{(1-\zeta)^{2n}}{L_{n,n}(\zeta)} \int_\Delta \frac{L_{n,n}(u)}{\zeta - u} d\rho_n(u) = 0, \tag{20}$$

where $l_{n,m}(\zeta) = \tau_{n,m}\zeta^m + \dots$ is the orthogonal polynomial of degree m with respect to the measure $d\rho_n$ whose leading coefficient $\tau_{n,m}$ is supposed to be positive, and $L_{n,m}(\zeta) = l_{n,m}(\zeta)/\tau_{n,m}$.

Step 2. By Lemma 12 for each $j \in \mathbb{Z}$ we have

$$\lim_{n\to\infty} \frac{l_{n,n+j+1}(\zeta)}{l_{n,n+j}(\zeta)} = \zeta + \sqrt{\zeta^2 - 1} \stackrel{\text{def}}{=} \varphi(\zeta) = 2 \lim_{n\to\infty} \frac{L_{n,n+j+1}(\zeta)}{L_{n,n+j}(\zeta)}, \quad \zeta \in \mathbb{C}\setminus\Delta.$$

In view of the orthogonality relations (18), the polynomial $h_n(\zeta)t_d(\zeta)$ can be represented in the form of a finite linear combination of the orthogonal polynomials $L_{n,m}$

$$h_n(\zeta)t_d(\zeta) = \lambda^*_{n,0}L_{n,n+d}(\zeta) + \lambda^*_{n,1}L_{n,n+d-1}(\zeta) + \ldots + \lambda^*_{n,2d}L_{n,n-d}(\zeta). \tag{21}$$

Take $\lambda_n = \left(\sum_{j=0}^{2d} |\lambda^*_{n,j}|\right)^{-1}$, $\lambda_{n,j} = \lambda_n \lambda^*_{n,j}$, $j = 0, \dots, 2d$ and $S_{n+d}(\zeta) = \lambda_n h_n(\zeta)t_d(\zeta)$. Since $q_n \not\equiv 0$, λ_n is finite. We have

$$\Psi_n(\zeta) = \frac{S_{n+d}(\zeta)}{L_{n,n+d}(\zeta)} = \sum_{j=1}^{2d} \lambda_{n,j} \frac{L_{n,n+d-j}(\zeta)}{L_{n,n+d}(\zeta)}, \quad \sum_{j=1}^{2d} |\lambda_{n,j}| = 1.$$

¿From the condition of the theorem, by Lemma 12 it follows that

$$\lim_{n\to\infty} \frac{L_{n,n+d-j(\zeta)}}{L_{n,n+d}(\zeta)} = \psi(\zeta)^j, \; j = 0, 1, \dots, 2d,$$

where $\psi(\zeta) = 2/\varphi(\zeta)$. The function ψ is a one-to-one representation of $\mathbb{C}\setminus\Delta$ onto the disk of radius 2. Consequently, the sequence Ψ_n is uniformly bounded. From those same relations it follows that any limit function of the sequence $\{\Psi_n\}$ is a polynomial of degree $\leq 2d$ of $\psi(\zeta)$. So in any compact subset of $\mathbb{C}\setminus\Delta$, for all sufficiently large n, there lie no more than d zeros of the polynomial h_n.

Further, let $\mathrm{cap\,(K)}$ denote the logarithmic capacity of the compact set K. By $\mathrm{limcap\,f_n(z) = f(z),\ z \in G}$, we will denote the convergence in capacity inside G (this notation means that for any $\epsilon > 0$ and any compact set $K \subset G$ we have $\lim_{\epsilon\to 0} \mathrm{cap\,(K \cap \{|f_n - f| > \epsilon\}) = 0}$). Let us show that

$$\mathrm{limcap\,\Pi_n(\zeta) = F(\zeta)}, \tag{22}$$

in $\mathbb{C}\setminus\Delta$.

We fix a compact $K \subset \mathbb{C}\setminus\Delta$. Let $\delta > 0$ be sufficiently small so that the δ- neighborhood K_δ of K is contained in $\mathbb{C}\setminus\Delta$ together with its closure. Let $c_n(\zeta) = \zeta^{d'} + \dots$ be the polynomial whose zeros are the zeros of S_{n+d} that lie on $\mathbb{C}\setminus\Delta$. By virtue of what was said above, for all sufficiently large n we have $d' \le 2d$. Multiplying (19), with $s = 1$, by $c_n(\zeta)$ and using (21), we obtain

$$c_n(\zeta)(\Pi_n(\zeta) - F(\zeta)) = c_n(\zeta)\frac{L_{n,n+d}(\zeta)}{S_{n+d}(\zeta)}\sum_{j=1}^{2d}\lambda_{n,j}\frac{L_{n,n+d-j}(\zeta)}{L_{n,n+d}(\zeta)}I_{n,j}(\zeta),$$

where

$$\begin{aligned} I_{n,j}(\zeta) &= \int_\Delta \frac{L_{n+d-j}(u)}{L_{n+d-j}(\zeta)}(1-\zeta)^{2n}\frac{d\rho_n(u)}{\zeta - u} \\ &= (1-\zeta)^{2(j-d)}\int_\Delta \frac{L_{n+d-j}(u)}{L_{n+d-j}(\zeta)}(1-\zeta)^{2(n+d-j)}\frac{d\rho_n^{(j)}(u)}{\zeta - u}, \end{aligned}$$

and

$$d\rho_n^{(j)}(u) = \frac{d\rho^{(j)(u)}}{(1-u)^{2(n+d-j)}}, \quad d\rho^{(j)}(u) = (1-u)^{2(d-j)}d\rho(u).$$

It is obvious that for each fixed $j = 0, 1, \dots, 2d$ the measure $\rho^{(j)}$ satisfies the same conditions as the measure ρ (see Lemma 6). Hence, using (20) it follows that $\lim_{n\to\infty} I_{n,j}(\zeta) = 0$, uniformly on each compact subset K of $\mathbb{C} \setminus \Delta$, for each $j = 0, 1, \ldots, 2d$. From what was said above, it is also obvious that the sequence of functions $\frac{c_n(\zeta)}{\Psi_n(\zeta)}$, which are analytic on K, is uniformly bounded on K. Therefore,

$$\lim_{n} c_n(\zeta)(F(\zeta) - \Pi_n(\zeta)) = 0, \quad \zeta \in K.$$

Since by the Fekete's lemma $\mathrm{cap}\,(\{\zeta : |\mathrm{c_n}(\zeta)| < \epsilon\}) = \epsilon^{1/\mathrm{d}'}$ for each $\epsilon > 0$, and $d' \leq 2d$, relation (22) follows.

Suppose that U is a region whose closure is a compact set in $\mathbb{C} \setminus \Delta$ which contains all the poles of $F(\zeta)$ in $\mathbb{C} \setminus \Delta$. As we proved above, the number of poles of Π_n in U, for all sufficiently large n, is not greater than d. The number of poles of F in U is equal to d. Under these conditions it follows from (22), by virtue of Gonchar's lemma ([9], Lemma 1), that for all sufficiently large n the number of poles of Π_n in U is equal to d, and these poles tend to the poles of F as $n \to \infty$ (each pole of F attracts as many poles of Π_n as its order). In turn, this yields that

$$\lim_{n\to\infty} \Pi_n(\zeta) = F(\zeta)$$

uniformly in each compact subset of $\mathbb{C} \setminus \{\Delta \cup \{b_1, \ldots, b_d\}\}$

Step 3. It remains to complete the proof of statement (a). Taking into consideration (16), we have to prove

$$\lim_{n\to\infty} \frac{h_n(\zeta)}{L_{n,n}(\zeta)} = (2\varphi(\zeta))^{-d} \prod_{j=1}^{d} \frac{\varphi(\zeta) - \varphi(b_j)}{\zeta - b_j}, \tag{23}$$

uniformly on each compact subset of $\mathbb{C}\setminus(\Delta\cup\{b_1, \ldots, b_d\})$, where $b_1, \ldots, b_d$ are the poles of r.

The information obtained about the behavior of the zeros of $h_n(z)$ (the poles of Π_n) inside $\mathbb{C} \setminus \Delta$, we can conclude that any limit function of the sequence $\{\Psi_n(\zeta) = \frac{S_{n+d}(\zeta)}{L_{n+d}(\zeta)}\}$ has the form

$$\sum_{j=0}^{2d} \lambda_j \psi^j(z) = C \prod_{j=1}^{d} (\psi(z) - \psi(b_j))^2$$

where $|C| \in (0, +\infty)$. In particular, for any convergent subsequence of $\{\Psi_n(\zeta)\}$ we have

$$\Psi_n(\infty) = \lambda^*_{n,0}\lambda_n, \quad \lim_n \Psi_n(\infty) = C \prod_{j=1}^{d} \psi(b_j). \tag{24}$$

Since the leading coefficient of h_n is equal to 1, the quantity $\lambda^*_{n,0}$ can take only the two values 1 (if $\deg(h_n) = n$) or 0 (if dedeg$(h_n) < n$). By virtue of the compactness of the sequence, from the above relation it follows, first, that $\lambda^*_{n,0} = 1$ ($\deg(h_n) = n$) for all sufficiently large n, and second, that $\liminf_{n\to\infty} \lambda_n > 0$. Hence the sequence of functions

$$\frac{h_n(\zeta)t_d(\zeta)}{L_{n,n+d}(\zeta)} = 1 + \sum_{j=1}^{2d} \lambda^*_{n,j} \frac{L_{n,n+j-d}(\zeta)}{L_{n,n+d}(\zeta)}$$

is uniformly bounded, just like $\{\Psi_n\}$. Using the same arguments as above, based on (16), the behavior of the zeros of h_n in $\mathbb{C}\setminus\Delta$, and the normalizing conditions, we conclude that

$$\lim_n \frac{h_n(\zeta)t_d(\zeta)}{L_{n,n+d}(\zeta)} = \prod_{j=1}^{2d} \left(1 - \frac{\psi(\zeta)}{\psi(b_j)}\right),$$

uniformly on each compact subset of $\mathbb{C} \setminus (\Delta \cup \{b_1, \ldots, b_d\})$. Considering that $\psi(\zeta) = 2/\varphi(\zeta)$ and $\lim_n \frac{L_{n,n}(\zeta)}{L_{n,n+d}(\zeta)} = (\psi(\zeta))^{-d}$ uniformly on each compact subset of $\mathbb{C} \setminus \Delta$, part (a) of the theorem is proved.

■

Remark 4. *Part 1 of Theorem 6 gives an interesting example of relative asymptotics. We will look at a more general result in the next section.*

5. Relative asymptotics of orthogonal polynomials on the real line

In this section we prove Theorem 2. Let μ^ν be the image measure of ν by the function $(i\frac{z+1}{z-1})$, $z \in \Gamma$. Then the orthogonal polynomials $\mathcal{H}_n(\nu, z)$ with respect to ν normalized by $\mathcal{H}_n(\nu, i) = 1$ are related to the orthogonal polynomial with respect to $d\mu_n(z) = \frac{d\mu^\nu(z)}{|z-1|^{2n}}$ by

$$
\begin{aligned}
(z-1)^n \mathcal{H}_n(\nu, \omega) &= \frac{\varphi_n^*(\mu_n, z)\varphi_n^*(\mu_n, 1) - z\varphi_n(\mu_n, z)\varphi_n(\mu_n, 1)}{\kappa_n(\mu_n)\varphi_n^*(\mu_n, 1)(1-z)} \\
&= \frac{K_n(\mu_n, z, 1)}{\kappa_n(\mu_n)\varphi_n^*(\mu_n, 1)},
\end{aligned}
$$

where $z = \frac{\omega+i}{\omega-i}$, $\omega \in \Omega$, and $|z| > 1$. Writing the above formula for $\mathcal{H}_n(g\, d\nu, \omega)$, we obtain

$$
\frac{\mathcal{H}_n(g\, d\nu, \omega)}{\mathcal{H}_n(\nu, \omega)} = \frac{\varphi_n(\widetilde{g}\, d\mu_n, z)}{\varphi_n(\mu_n, z)} \frac{\kappa_n(\widetilde{g}\, d\mu_n)}{\kappa_n(\mu_n)} \frac{\frac{\varphi_n^*(\widetilde{g}\, d\mu_n, z)}{\varphi_n(\widetilde{g}\, d\mu_n, z)} \overline{\left(\frac{\varphi_n^*(\widetilde{g}\, d\mu_n, 1)}{\varphi_n(\widetilde{g}\, d\mu_n, 1)}\right)} - z}{\frac{\varphi_n^*(\mu_n, z)}{\varphi_n(\mu_n, z)} \overline{\left(\frac{\varphi_n^*(\mu_n, 1)}{\varphi_n(\mu_n, 1)}\right)} - z}.
$$

where $\widetilde{g}(z) = g(i\frac{z+1}{z-1})$. Combining Theorem 3 and Theorem 5 the proof follows.

Remark 5. *The previous results passes over easily to the case of orthogonality on* $[0, +\infty)$. *A measure* α, $\mathrm{supp}(\alpha) \subset [0, +\infty)$ *can be put in correspondence with a measure* ν *on* $\mathbb{R}$ *symmetric with respect to* 0, $d\nu(x) = |x| d\alpha(x^2)$ *(see [12], Theorem 4).*

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References

[1] **N. I. Akhiezer,** The Classical Moment Problem, *Oliver and Boyd, Edinburgh,* 1965.

[2] **M. P. Alfaro, M. Bello-Hernández, J. M. Montaner, J. L. Varona**, Some asymptotic properties for orthogonal polynomials with respect to varying measures, *J. Approx. Th.* **135** (2005), 22–34.

[3] **C. Berg, A. Duran**, The index of determinacy for measures and the l^2-norm of orthogonal polynomials, *Trans. AMS* **347(8)** (1995), 2795–2811.

[4] **C. Berg, M. Thill**, Rotation invariant moment problems, *Acta Math.* **167** (1991), 207–227.

[5] **G. Freud**, Orthogonal polynomials, *Oxford, Pergamon Press.*, 1971.

[6] **G. Freud, P. Nevai**, Weighted L_1 and one-sided weighted L_1 polynomial approximation on the real axis. (Hungarian) *Magyar Tud. Akad. Mat. Fiz. Oszt. Kőzl.* **21** (1973), 485-502.

[7] **Ya. L. Geronimus**, Polynomials orthogonal on the unit circle and on an interval, *Fizmatgiz*, Moscow, 1958; English transl., Orthogonal polynomials, *Consultants Bureau*, New York, 1961.

[8] **A. A. Gonchar**, On convergence of Padé approximants for some classes of meromorphic functions, *Mat. Sb.* **97(139)** (1975), 607–629; English transl. in *Math. USSR Sb.* **26** (1975), 555–575.

[9] **A. A. Gonchar**, On the convergence of generalized Padé approximants of meromorphic functions, *Mat. Sb.* **98(140)** (1975), 564-577; English transl. in *Math. USSR Sb.* **27** (1975), 503–514.

[10] **C. C. Heyde**, Some remarks on the moment problem I and II. *Quart. J. Math. Oxford Ser.* **14 (2)** (1963) 91–96, 97–105.

[11] **G. López Lagomasino**, Convergence of Padé approximants of Stieltjes type meromorphic functions and comparative asymptotics of orthogonal

polynomials, *Mat. Sb.* **136(178)** (1988), 206–226; English transl. in *Math. USSR Sb.* **64** (1989), 207–227.

[12] **G. López Lagomasino** Relative asymptotics for polynomials orthogonal on the real axis, *Mat. Sb.* **137(179)** (1988), 500–527; English transl. in *Math. USSR Sb.*, **65** (1990), 505–529.

[13] **A. Mate, P. Nevai, and V. Totik**, Extensions of Szegős theory of orthogonal polynomials. II, *Const. Approx.* **3** (1987), 51–72.

[14] **A. Mate, P. Nevai, and V. Totik**, Extensions of Szego's theory of orthogonal polynomials. III, *Const. Approx.* **3** (1987), 73–96.

[15] **A. Mate, P. Nevai, and V. Totik**, Strong and weak convergence of orthogonal polynomials, Amer. J. Math. **109** (1987), 239–281.

[16] **E. A. Rakhmanov**, On the convergence of Padé approximants in classes of holomorphic functions, *Mat. Sb.* **112(154)** (1980), 162–169; English transl. in *Math. USSR Sb.* **40** (1981), 149–155.

[17] **E. A. Rakhmanov**, On the asymptotics of polynomials orthogonal on the circle with weights not satisfying Szego's condition, Mat. Sb. **130(172)** (1986), 151–169; English transl. in Math. USSR Sb. **58** (1987), 149–167.

[18] **M. Riesz**, Sur le problème des moments III. *Ark. Mat. Astr. Fys.* **16** (1923), 1–52.

[19] **M. Riesz**, Sur le problème des moments et le théoreme de Parseval correspondant. *Acta Litt. Ac. Sci. Szeged* **1** (1923), 209–225.

[20] **T. J. Stieltjes**, Recherches sur les fractions continues. *Ann. Fac. Sci. Toulouse* **8** (1894), J1-122; **9** (1895), A1–47.

[21] **G. Szego**, Orthogonal polynomials, rev. ed., *Amer. Math. Soc.*, Providence, R. I., 1959.

[22] **P. Yuditskii**, Analytic perturbation preserves determinacy of infinite index, *Math. Scand.* **86** (2000), 288-292.