Journal of Pure and Applied Algebra 215 (2011) 108–130

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

Irreducible Lie–Yamaguti algebras of generic type

Pilar Benito [a], Alberto Elduque [b,*], Fabián Martín-Herce [c]

[a] Departamento de Matemáticas y Computación y Centro de Investigación de Informática, Matemáticas y Estadística, Universidad de La Rioja, 26004 Logroño, Spain

[b] Departamento de Matemáticas e Instituto Universitario de Matemáticas y Aplicaciones, Universidad de Zaragoza, 50009 Zaragoza, Spain

[c] Departamento de Matemáticas y Computación, Universidad de La Rioja, 26004 Logroño, Spain

ARTICLE INFO

Article history:
Received 20 July 2009
Received in revised form 22 February 2010
Available online 10 May 2010
Communicated by C.A. Weibel

MSC:
Primary: 17A30
Secondary: 17B60

ABSTRACT

Lie–Yamaguti algebras (or generalized Lie triple systems) are binary–ternary algebras intimately related to reductive homogeneous spaces. The Lie–Yamaguti algebras which are irreducible as modules over their inner derivation algebras are the algebraic counterparts of the isotropy irreducible homogeneous spaces.

These systems splits into three disjoint types: adjoint type, non-simple type and generic type. The systems of the first two types were classified in a previous paper through a generalized Tits Construction of Lie algebras. In this paper, the Lie–Yamaguti algebras of generic type are classified by relating them to several other nonassociative algebraic systems: Lie and Jordan algebras and triple systems, Jordan pairs or Freudenthal triple systems.

1. Introduction

Given a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ of a Lie algebra $\mathfrak{g}$, so that $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$, $\mathfrak{m}$ a subspace, and $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$, there exist natural binary and ternary products defined in $\mathfrak{m}$, given by

$$\begin{aligned} x \cdot y &= \pi_{\mathfrak{m}}\big([x, y]\big),\\ [x, y, z] &= \big[\pi_{\mathfrak{h}}([x, y]), z\big], \end{aligned} \tag{1.1}$$

for any $x, y, z \in \mathfrak{m}$, where $\pi_{\mathfrak{h}}$ and $\pi_{\mathfrak{m}}$ denote the projections on $\mathfrak{h}$ and $\mathfrak{m}$ respectively, relative to the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.

The vector spaces endowed with such binary and ternary products were introduced by Yamaguti in [30] under the name of *general Lie triple systems*, and were later renamed as *Lie triple algebras* in [18]. Here and in the previous paper [3] we follow the notation in [19, Definition 5.1], and call these systems *Lie–Yamaguti algebras*:

Definition 1.1. A *Lie–Yamaguti algebra* $(\mathfrak{m}, x \cdot y, [x, y, z])$ (*LY-algebra* for short) is a vector space $\mathfrak{m}$ equipped with a bilinear operation $\cdot : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ and a trilinear operation $[\ ,\ ,\ ] : \mathfrak{m} \times \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ such that, for all $x, y, z, u, v, w \in \mathfrak{m}$:

(LY1) $x \cdot x = 0$,
(LY2) $[x, x, y] = 0$,
(LY3) $\sum_{(x,y,z)}\Big([x, y, z] + (x \cdot y) \cdot z\Big) = 0$,
(LY4) $\sum_{(x,y,z)}[x \cdot y, z, t] = 0$,

* Corresponding author.
E-mail addresses: pilar.benito@unirioja.es (P. Benito), elduque@unizar.es (A. Elduque), fabian.martin@dmc.unirioja.es (F. Martín-Herce).

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doi:10.1016/j.jpaa.2010.04.003

(LY5) $[x, y, u \cdot v] = [x, y, u] \cdot v + u \cdot [x, y, v]$,
(LY6) $[x, y, [u, v, w]] = [[x, y, u], v, w] + [u, [x, y, v], w] + [u, v, [x, y, w]]$.

Here $\sum_{(x,y,z)}$ means the cyclic sum on x, y, z.

The LY-algebras with $x \cdot y = 0$ for any x, y are exactly the Lie triple systems, closely related with symmetric spaces, while the LY-algebras with $[x, y, z] = 0$ are the Lie algebras. Less known examples can be found in [2] where a detailed analysis on the algebraic structure of LY-algebras arising from homogeneous spaces which are quotients of the compact Lie group G_2 is given.

For background and motivation on these systems one may consult [3].

Given a Lie–Yamaguti algebra $(\mathfrak{m}, x \cdot y, [x, y, z])$ and any two elements $x, y \in \mathfrak{m}$, the linear map $D(x, y) : \mathfrak{m} \to \mathfrak{m}$, $z \mapsto D(x, y)(z) = [x, y, z]$ is, due to (LY5) and (LY6), a derivation of both the binary and ternary products. These derivations will be called *inner derivations*. Moreover, let $D(\mathfrak{m}, \mathfrak{m})$ denote the linear span of the inner derivations. Then $D(\mathfrak{m}, \mathfrak{m})$ is closed under commutation thanks to (LY6). Consider the vector space $\mathfrak{g}(\mathfrak{m}) = D(\mathfrak{m}, \mathfrak{m}) \oplus \mathfrak{m}$, and endow it with the anticommutative multiplication given, for any $x, y, z, t \in \mathfrak{m}$, by:

$$\begin{aligned}
&[D(x, y), D(z, t)] = D([x, y, z], t) + D(z, [x, y, t]),\\
&[D(x, y), z] = D(x, y)(z) = [x, y, z],\\
&[z, t] = D(z, t) + z \cdot t.
\end{aligned} \tag{1.2}$$

Note that the Lie algebra $D(\mathfrak{m}, \mathfrak{m})$ becomes a subalgebra of $\mathfrak{g}(\mathfrak{m})$.

Then it is straightforward [30] to check that $\mathfrak{g}(\mathfrak{m})$ is a Lie algebra, called the *standard enveloping Lie algebra* of the Lie–Yamaguti algebra $\mathfrak{m}$. The binary and ternary products in $\mathfrak{m}$ coincide with those given by (1.1), where $\mathfrak{h} = D(\mathfrak{m}, \mathfrak{m})$.

Given a Lie algebra $\mathfrak{g}$ and a subalgebra $\mathfrak{h}$, the pair $(\mathfrak{g}, \mathfrak{h})$ will be said to be a *reductive pair* (see [27]) if there is a complementary subspace $\mathfrak{m}$ of $\mathfrak{h}$ with $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$. The decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ will then be called a *reductive decomposition* of the Lie algebra $\mathfrak{g}$ and *symmetric decomposition* if the additional condition $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}$ holds. In the latter case, we shall refer to the pair $(\mathfrak{g}, \mathfrak{h})$ as a *symmetric pair*. In particular, given a LY-algebra $(\mathfrak{m}, x \cdot y, [x, y, z])$, the pair $\big(\mathfrak{g}(\mathfrak{m}), D(\mathfrak{m}, \mathfrak{m})\big)$ is a reductive pair and the pair is symmetric in case $x \cdot y = 0$.

Definition 1.2. A Lie–Yamaguti algebra $(\mathfrak{m}, x \cdot y, [x, y, z])$ is said to be *irreducible* if $\mathfrak{m}$ is an irreducible module for its Lie algebra of inner derivations $D(\mathfrak{m}, \mathfrak{m})$.

The irreducible Lie–Yamaguti algebras constitute the algebraic counterpart to the isotropy irreducible homogeneous spaces considered in [29]. Concerning these irreducible LY-algebras over algebraically closed fields of characteristic zero, it is not difficult to prove (see [3, Proposition 1.3, Theorem 2.1]) the following basic structure results:

Theorem 1.3. *Let $(\mathfrak{m}, x \cdot y, [x, y, z])$ be an irreducible LY-algebra. Then $D(\mathfrak{m}, \mathfrak{m})$ is a semisimple and maximal subalgebra of the standard enveloping Lie algebra $\mathfrak{g}(\mathfrak{m})$. Moreover, $\mathfrak{g}(\mathfrak{m})$ is simple in case $\mathfrak{m}$ and $D(\mathfrak{m}, \mathfrak{m})$ are not isomorphic as $D(\mathfrak{m}, \mathfrak{m})$-modules.* □

Proposition 1.4. *Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be a reductive decomposition of a simple Lie algebra $\mathfrak{g}$, with $\mathfrak{m} \neq 0$. Then $\mathfrak{g}$ and $\mathfrak{h}$ are isomorphic, respectively, to the standard enveloping Lie algebra and the inner derivation algebra of the Lie–Yamaguti algebra $(\mathfrak{m}, x \cdot y, [x, y, z])$ given by (1.1). Moreover, in case $\mathfrak{h}$ is semisimple and $\mathfrak{m}$ is irreducible as a module for $\mathfrak{h}$, either $\mathfrak{h}$ and $\mathfrak{m}$ are isomorphic as $\mathrm{ad}\,\mathfrak{h}$-modules or $\mathfrak{m} = \mathfrak{h}^{\perp}$, the orthogonal complement of $\mathfrak{h}$ relative to the Killing form of $\mathfrak{g}$.* □

From Theorem 1.3, in [3, Section 2] it is proved that the classification of irreducible LY-algebras splits into three non-overlapping types:

$$\begin{array}{ll}
\text{ADJOINT TYPE:} & \mathfrak{m} \text{ is the adjoint module for } D(\mathfrak{m}, \mathfrak{m}),\\
\text{NON-SIMPLE TYPE:} & D(\mathfrak{m}, \mathfrak{m}) \text{ is not simple},\\
\text{GENERIC TYPE:} & \text{Both } \mathfrak{g}(\mathfrak{m}) \text{ and } D(\mathfrak{m}, \mathfrak{m}) \text{ are simple.}
\end{array} \tag{1.3}$$

The LY-algebras of Adjoint Type are just the simple Lie algebras (see [3, Theorem 2.4]) and those of Non-Simple Type can be described through reductive decompositions modeled by a Generalized Tits Construction from [4] using quaternions, octonions and simple Jordan algebras as basic ingredients (see [3, Theorems 4.1, 4.4]).

In the Generic Type, $\mathfrak{m}$ and $D(\mathfrak{m}, \mathfrak{m})$ are not isomorphic as $\mathrm{ad}_{\mathfrak{g}(\mathfrak{m})} D(\mathfrak{m}, \mathfrak{m})$-modules, so following Proposition 1.4, the classification of the irreducible LY-algebras of this type is equivalent to the determination of the reductive decompositions $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ satisfying the following conditions:

$$\begin{array}{ll}
\text{(a)} & \mathfrak{g} \text{ is a simple Lie algebra},\\
\text{(b)} & \mathfrak{h} \text{ is a simple subalgebra of } \mathfrak{g},\\
\text{(c)} & \mathfrak{m} \text{ is an irreducible ad}\,\mathfrak{h}\text{-module (in particular } \mathfrak{m} \neq 0).
\end{array} \tag{1.4}$$

Note that the previous conditions imply

$$\begin{array}{ll}
\text{(d)} & \mathfrak{m} \text{ is } \mathfrak{h}^{\perp} \text{ (orthogonal with respect to the Killing form of } \mathfrak{g}),\\
\text{(e)} & \mathfrak{h} \text{ is a maximal subalgebra of } \mathfrak{g},\\
\text{(f)} & \mathfrak{m} \text{ is not the adjoint module.}
\end{array} \tag{1.5}$$

The purpose in this paper is the classification of the LY-algebras of Generic Type while, at the same time, their close connections to some well-known nonassociative algebraic systems will be highlighted. It will be shown that most irreducible LY-algebras of this type appear inside simple Lie algebras as orthogonal complements of subalgebras of derivations of Lie and Jordan algebras and triple systems, Freudenthal and orthogonal triple systems or Jordan and anti-Jordan pairs.

The paper is structured as follows. Section 2 is devoted to determine the irreducible LY-algebras inside reductive decompositions of simple special linear Lie algebras (classical Cartan type A_n). The classification of these LY-algebras flows parallel to the classification of the simple Jordan linear pairs and the so called anti-Jordan pairs. Following a similar pattern, Sections 3 and 4 provide LY-algebras appearing inside orthogonal and symplectic simple Lie algebras (Cartan types B_n, D_n and C_n) through the classification of simple Lie triple systems, and orthogonal and symplectic triple systems. Irreducible LY-algebras inside exceptional Lie algebras of types G_2, F_4, E_6, E_7 and E_8 are the goal of Section 5. In this case, the classification can be transferred from the complex field. Section 6 is an Appendix section where definitions and classifications of the different pairs and triple systems related to irreducible LY-algebras are included. The paper ends with an epilogue section that summarizes the classification results obtained in this paper and in the previous one [3].

Throughout this paper, *all the algebraic systems considered will be assumed to be finite dimensional over an algebraically closed ground field k of characteristic zero*. The symbol $\oplus$ denotes the direct sum of subspaces and $\otimes$ the tensor product of k-subspaces unless otherwise stated. Basic notation and terminology on representation theory of Lie algebras follows [14].

The authors are much indebted to the referee for his/her very careful analysis and comments.

2. Special linear case

As mentioned in the Introduction, the irreducible LY-algebras of Generic Type appear as orthogonal complements of maximal simple subalgebras of simple Lie algebras. In this section we classify these systems in case their standard enveloping algebras are (simple) special linear Lie algebras $\mathfrak{sl}(V)$ (or $\mathfrak{sl}_{\dim V}(k)$ if a basis for V is fixed).

Any reductive decomposition $\mathfrak{sl}(V) = \mathfrak{h} \oplus \mathfrak{m}$ satisfying (a), (b) and (c) in (1.4) presents two elementary restrictions:

- $\dim V \geq 3$ (the smallest simple Lie algebra is the three-dimensional algebra $\mathfrak{sl}_2(k)$).
- V is irreducible as a module for $\mathfrak{h}$. Otherwise we would have $V = V_1 \oplus V_2$ and $\mathfrak{h} \subseteq \{f \in \mathfrak{sl}(V) : f(V_i) \subseteq V_i, \operatorname{tr} f\ |_{V_i} = 0\}$ (since $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}]$), but this subspace is properly contained in the subalgebra $\{f \in \mathfrak{sl}(V) : f(V_i) \subseteq V_i\}$. This is not possible by the maximality of the subalgebra $\mathfrak{h}$, following condition (e) from (1.5).

The previous restrictions allow us to introduce trilinear products involving V and its dual $\mathfrak{h}$-module V^*, in such a way that the pair (V, V^*) is endowed with a (linear) Jordan or anti-Jordan pair structure (see [21] and [13] for definitions or Section 6.3 of this paper) and $\mathfrak{h}$ can be viewed as the derived subalgebra of the inner derivation algebra of the induced pair.

Since $\mathfrak{sl}(V)$ is embedded in $\mathfrak{gl}(V)$, and this is a module for its subalgebra $\mathfrak{h}$, we can consider the standard isomorphism of $\mathfrak{h}$-modules

$$V \otimes V^* \cong \mathfrak{gl}(V) = \mathfrak{h} \oplus \mathfrak{m} \oplus kI_V \tag{2.1}$$

given by identifying the element $x \otimes \varphi$ with the linear map $y \mapsto x\varphi(y)$. (I_V denotes the identity map of the vector space V.)

Now let $d_{x,\varphi}$ be the projection of $x \otimes \varphi$ onto $\mathfrak{h}$ and, for a fixed $\xi \in k$, let us define the $\mathfrak{h}$-invariant triple products

$$\begin{array}{rcl} V \otimes V^* \otimes V & \to & V \\ x \otimes \varphi \otimes y & \mapsto & \{x\varphi y\}_\xi := d_{x,\varphi}(y) - \xi\varphi(x)y, \end{array} \tag{2.2}$$

$$\begin{array}{rcl} V^* \otimes V \otimes V^* & \to & V^* \\ \varphi \otimes x \otimes \psi & \mapsto & \{\varphi x\psi\}_\xi := \psi \circ d_{x,\varphi} - \xi\varphi(x)\psi. \end{array} \tag{2.3}$$

Products (2.2) and (2.3) are related by

$$\{\varphi x\psi\}_\xi(y) = \psi \circ \{x\varphi y\}_\xi, \tag{2.4}$$

for all $\varphi, \psi \in V^*$, $x, y \in V$, and the subalgebra $\mathfrak{h}$ can be described as

$$\mathfrak{h} = \operatorname{span}\langle d_{x,\varphi} : x \in V, \varphi \in V^*\rangle. \tag{2.5}$$

Then, we have the following result:

Lemma 2.1. *For a given reductive decomposition $\mathfrak{sl}(V) = \mathfrak{h} \oplus \mathfrak{m}$ which satisfies* (a), (b), (c) *in* (1.4)*, consider the vector spaces $U^+ = V$ and $U^- = V^*$. There exists a unique nonzero scalar $\xi \in k$ such that the pair $\mathcal{U} = (U^+, U^-)$ is either a simple Jordan pair under the triple products $\{x_\sigma y_{-\sigma} z_\sigma\}_\xi$ defined in* (2.2) *and* (2.3) *for $\sigma = \pm$, or a contragredient simple anti-Jordan pair with $\langle x_\sigma y_{-\sigma} z_\sigma\rangle := \sigma\{x_\sigma y_{-\sigma} z_\sigma\}_\xi$ as triple products. Moreover $\mathfrak{h}$ is the linear subalgebra*

$$\mathfrak{h} = \operatorname{span}\left\langle \{x\varphi.\}_\xi - \frac{\operatorname{tr}(\{x\varphi.\}_\xi)}{\dim V} I_V : x \in V, \varphi \in V^* \right\rangle, \tag{2.6}$$

which, up to isomorphism, turns out to be the derived subalgebra of the inner derivation Lie algebra of the corresponding pair and $\mathfrak{m}$ is $\mathfrak{h}^\perp$, the orthogonal complement of $\mathfrak{h}$ with respect to the Killing form of $\mathfrak{sl}(V)$.

Proof. First we shall check that, for an arbitrary ξ, the ξ-products in (2.2) and (2.3) satisfy the identity

$$\{x_\sigma y_{-\sigma}\{u_\sigma v_{-\sigma} w_\sigma\}_\xi\}_\xi = \{\{x_\sigma y_{-\sigma} u_\sigma\}_\xi v_{-\sigma} w_\sigma\}_\xi - \{u_\sigma\{y_{-\sigma} x_\sigma v_{-\sigma}\}_\xi w_\sigma\}_\xi + \{u_\sigma v_{-\sigma}\{x_\sigma y_{-\sigma} w_\sigma\}_\xi\}_\xi. \quad (2.7)$$

For $x = x_+, u = u_+ \in U^+ = V$ and $\varphi = y_-, \psi = v_- \in U^- = V^*$, the map

$$L : V \otimes V^* \to \mathrm{End}(V), \quad (2.8)$$

defined by $x \otimes \varphi \mapsto L_{x,\varphi} = \{x\varphi.\}_\xi$, is an $(\mathfrak{h} \oplus kI_V)$-module homomorphism. From (2.4) we have $\{\varphi x\psi\}_\xi = \psi \circ L_{x,\varphi}$ which easily yields

$$[L_{x,\varphi}, L_{u,\psi}] = L_{L_{x,\varphi}(u),\psi} - L_{u,\psi\circ L_{x,\varphi}}. \quad (2.9)$$

But (2.9) is equivalent to (2.7) for $\sigma = +$.

In case $\varphi = x_-, \psi = u_- \in U^- = V^*$ and $x = y_+, y = v_+ \in U^* = V$, identity (2.7) follows from the $(\mathfrak{h} \oplus kI_V)$-module homomorphism

$$\hat{L} : V^* \otimes V \to \mathrm{End}(V^*) \quad (2.10)$$

given by $\varphi \otimes x \mapsto \hat{L}_{\varphi,x} = \{\varphi x.\}_\xi$.

On the other hand, any product defined as in (2.2) is in $\mathrm{Hom}_\mathfrak{h}(V \otimes V^* \otimes V, V)$, so we must look at the previous subspace in order to get our result. Since $\mathfrak{h}$, $\mathfrak{m}$ and kI_V are irreducible and non-isomorphic $\mathfrak{h}$-modules,

$$\begin{aligned}\mathrm{Hom}_\mathfrak{h}(V \otimes V^* \otimes V, V) &\cong \mathrm{Hom}_\mathfrak{h}(V \otimes V^*, V \otimes V^*)\\ &\cong \mathrm{Hom}_\mathfrak{h}(\mathfrak{h}, \mathfrak{h}) \oplus \mathrm{Hom}_\mathfrak{h}(\mathfrak{m}, \mathfrak{m}) \oplus \mathrm{Hom}_\mathfrak{h}(kI_V, kI_V).\end{aligned}$$

Then $\mathrm{Hom}_\mathfrak{h}(V \otimes V^* \otimes V, V)$ is a three-dimensional vector space from Schur's Lemma. Now, using the alternative decomposition

$$\begin{aligned}\mathrm{Hom}_\mathfrak{h}(V \otimes V^* \otimes V, V) &\cong \mathrm{Hom}_\mathfrak{h}(V \otimes V, V \otimes V)\\ &\cong \mathrm{Hom}_\mathfrak{h}(S^2V, V \otimes V) \oplus \mathrm{Hom}_\mathfrak{h}\left(\bigwedge^2 V, V \otimes V\right),\end{aligned}$$

we get that either $\mathrm{Hom}_\mathfrak{h}(\bigwedge^2 V, V \otimes V)$ or $\mathrm{Hom}_\mathfrak{h}(S^2V, V \otimes V)$ is a one-dimensional subspace (S^2V and $\bigwedge^2 V$ stand for the second symmetric and alternating power of V respectively). Hence, two different situations appear:

(a) $\mathrm{Hom}_\mathfrak{h}(\bigwedge^2 V, V \otimes V)$ *is one dimensional.*

In this case, the vector space $\mathrm{Hom}_\mathfrak{h}((\bigwedge^2 V) \otimes V^*, V)$ is also one dimensional and, since $\dim V \geq 3$, we can take the nonzero map

$$x \otimes y \otimes \varphi - y \otimes x \otimes \varphi \mapsto \varphi(x)y - \varphi(y)x$$

as generator. So

$$d_{x,\varphi}(y) - d_{y,\varphi}(x) = \xi(\varphi(x)y - \varphi(y)x),$$

for some unique $\xi \in F$ and therefore, for $x, y \in V, \varphi \in V^*$ we have the identity

$$\{x\varphi y\}_\xi = \{y\varphi x\}_\xi. \quad (2.11)$$

On the other hand, since $\mathrm{Hom}_\mathfrak{h}(\bigwedge^2 V, V \otimes V)$ is assumed to be one dimensional and $V \otimes V = S^2V \oplus \bigwedge^2 V$, we have that $\mathrm{Hom}_\mathfrak{h}(\bigwedge^2 V, S^2V) = \mathrm{Hom}_\mathfrak{h}(S^2V, \bigwedge^2 V) = \mathrm{Hom}_\mathfrak{h}(S^2V \otimes \bigwedge^2 V^*, k) = 0$. This latter condition implies that the restriction of the map $x \otimes y \otimes \varphi \otimes \psi \mapsto \psi(\{x\varphi y\}_\xi)$ on $S^2V \otimes \bigwedge^2 V^*$ is zero. So, using (2.4) and taking into account that the base field is of characteristic zero, we get

$$0 = \psi(\{x\varphi y\}_\xi) - \varphi(\{x\psi y\}_\xi) = (\{\varphi x\psi\}_\xi - \{\psi x\varphi\}_\xi)(y),$$

which implies

$$\{\varphi x\psi\}_\xi = \{\psi x\varphi\}_\xi. \quad (2.12)$$

Then, from (2.7), (2.11) and (2.12), we obtain that $(\mathcal{U}, \{x_\sigma y_{-\sigma} z_\sigma\}_\xi)$ is a Jordan pair.

(b) $\mathrm{Hom}_\mathfrak{h}(S^2V, V \otimes V)$ *is one dimensional.*

In this case, analogous arguments but for symmetric powers, give us

$$d_{x,\varphi}(y) + d_{y,\varphi}(x) = \xi(\varphi(x)y + \varphi(y)x),$$

for some unique $\xi \in k$, which yields

$$\{x\varphi y\}_\xi + \{y\varphi x\}_\xi = 0, \quad (2.13)$$

and

$$\{\varphi x\psi\}_\xi + \{\psi x\varphi\}_\xi = 0, \tag{2.14}$$

for $x, y \in V, \varphi, \psi \in V^*$. Now for $\sigma = \pm$, the products $\langle x_\sigma y_{-\sigma} z_\sigma\rangle = \sigma\{x_\sigma y_{-\sigma} z_\sigma\}_\xi$ satisfy

$$\langle x_\sigma y_{-\sigma} z_\sigma\rangle = -\langle z_\sigma y_{-\sigma} x_\sigma\rangle, \tag{2.15}$$

and using (2.7):

$$\begin{aligned}\langle x_\sigma y_{-\sigma}\langle u_\sigma v_{-\sigma} z_\sigma\rangle\rangle &= \sigma^2\{x_\sigma y_{-\sigma}\{u_\sigma v_{-\sigma} z_\sigma\}_\xi\}_\xi\\ &= \sigma^2\{\{x_\sigma y_{-\sigma} u_\sigma\}_\xi v_{-\sigma} z_\sigma\}_\xi - \sigma^2\{u_\sigma\{y_{-\sigma} x_\sigma v_{-\sigma}\}_\xi z_\sigma\}_\xi + \sigma^2\{u_\sigma v_{-\sigma}\{x_\sigma y_{-\sigma} z_\sigma\}_\xi\}_\xi\\ &= \langle\langle x_\sigma y_{-\sigma} u_\sigma\rangle v_{-\sigma} z_\sigma\rangle + \langle u_\sigma\langle y_{-\sigma} x_\sigma v_{-\sigma}\rangle z_\sigma\rangle + \langle u_\sigma v_{-\sigma}\langle x_\sigma y_{-\sigma} z_\sigma\rangle\rangle.\end{aligned} \tag{2.16}$$

Now identities (2.15) and (2.16) prove that $(\mathcal{U}, \langle x_\sigma y_{-\sigma} z_\sigma\rangle)$ is an anti-Jordan pair.

Following [13, Section 1], for a given Jordan or anti-Jordan pair with triple products $a_\sigma b_{-\sigma} c_\sigma = D_\sigma(a_\sigma, b_{-\sigma})(c_\sigma)$, the so called inner derivation algebra is the Lie algebra spanned by the (inner) derivations $D(a_+, b_-) = (D_+(a_+, b_-), -D_-(b_-, a_+))$ in the Jordan pair case or $D(a_+, b_-) = (D_+(a_+, b_-), D_-(b_-, a_+))$ in the anti-Jordan pair case. In this way, comparing inner derivation maps for the pair and anti-pair obtained from (a) or (b), we arrive at the relationship

$$(\langle x\varphi.\rangle, \langle\varphi x.\rangle) = (\{x\varphi.\}_\xi, -\{\varphi x.\}_\xi).$$

Thus in both cases,

$$\text{Inder}\,\mathcal{U} = \text{span}\langle(\{x\varphi.\}_\xi, -\{\varphi x.\}_\xi) : x \in V, \varphi \in V^*\rangle.$$

Now, the Lie algebra Inder $\mathcal{U}$ is isomorphic to $\mathfrak{h}$ in case $\xi = 0$ and to $\mathfrak{h} \oplus kI_V$ otherwise. The previous assertion follows from the map $\mathfrak{h} \oplus kI_V \to \text{Inder}\,\mathcal{U}$, given by $d \mapsto (d, -\tilde{d})$, where $\tilde{d}(\varphi) = \varphi \circ d$. Moreover, since V and V^* are $\mathfrak{h}$-irreducible, according to [13, Proposition 1.2], $\mathcal{U}$ is a simple Jordan pair or anti-Jordan pair. Attached to the anti-Jordan pair structure, we have the nondegenerate bilinear map $V \otimes V^* \to k$ defined by $x \otimes \varphi \mapsto \varphi(x)$ which, because of (2.4) satisfies

$$\psi(\langle x\varphi y\rangle) + \langle\varphi x\psi\rangle(y) = \psi(\{x\varphi y\}_\xi) - \{\varphi x\psi\}_\xi(y) = 0,$$

for $x, y \in V, \varphi, \psi \in V^*$. Then the anti-Jordan pair is contragredient following [13, Section 2]. Now, from [25, Theorem 2] and [13, Sections 2 and 3], the inner derivation Lie algebra of either a simple Jordan pair or a simple contragredient anti-Jordan pair is never simple (see Tables 5 and 6 in the Appendix section for a complete description of these algebras). So, up to isomorphisms, the Lie algebra Inder $\mathcal{U}$ is $\mathfrak{h} \oplus kI_V$ which proves that $\xi \neq 0$. So the derived subalgebra $\text{Inder}_0\,\mathcal{U} = [\text{Inder}\,\mathcal{U}, \text{Inder}\,\mathcal{U}]$ is isomorphic to $\mathfrak{h}$. Moreover, using (2.5), the algebra $\mathfrak{h}$ is spanned by the zero trace maps $d_{x,\varphi} = L_{x,\varphi} + \xi\varphi(x)I_V$, therefore $\xi\varphi(x) = -\frac{\text{tr}(L_{x,\varphi})}{\dim V}$ (see (2.8)). The final assertion on $\mathfrak{m}$ follows from condition (d) in (1.5). □

Now, we can establish the main result for the generic $\mathfrak{sl}$-case:

Theorem 2.2. *Let $(\mathfrak{m}, a \cdot b, [a, b, c])$ be an irreducible LY-algebra of generic type and standard enveloping Lie algebra of type $\mathfrak{sl}$. Then precisely one of the following two cases occurs:*

(i) *There is a vector space V and an involution on the associative algebra $\text{End}(V)$ such that $\mathfrak{m}$ is, up to isomorphism, the simple Lie triple system consisting of the zero trace symmetric elements in $\text{End}(V)$, with the natural triple product $[a, b, c] = [[a, b], c]$ (inside $\mathfrak{sl}(V)$). Moreover, $\dim V \geq 5$ if the involution is orthogonal, and $\dim V \geq 4$ if it is symplectic. In particular, the binary product $a \cdot b$ is trivial.*

(ii) *There is a simple Jordan triple system J of one of the following types:*
   (1) *the subspace of $n \times n$ symmetric matrices for $n \geq 2$ with the triple product $\{xyz\} = xy^tz + zy^tx$,*
   (2) *the subspace of $n \times n$ skew symmetric matrices for $n \geq 5$ again with the triple product $\{xyz\} = xy^tz + zy^tx$,*
   (3) *the subspace of 1×2-matrices over the algebra of octonions $\mathcal{O}$ with the triple product $\{xyz\} = x(\bar{y}^tz) + z(\bar{y}^tx)$,*
   (4) *the exceptional Jordan algebra $\mathcal{H}_3(\mathcal{O})$ (multiplication denoted by juxtaposition) with its triple product $\{xyz\} = x(zy) + z(xy) - (zx)y$,*

   such that, up to isomorphism, $\mathfrak{g}(\mathfrak{m}) = \mathfrak{sl}(J)$, $\mathfrak{h} = D(\mathfrak{m}, \mathfrak{m}) = \mathcal{L}_0(J) = [\mathcal{L}(J), \mathcal{L}(J)]$, where $\mathcal{L}(J) = \text{span}\langle\{xy.\}; x, y \in J\rangle$. Here the LY-algebra $\mathfrak{m}$ appears as the orthogonal complement to $\mathfrak{h}$ in $\mathfrak{g}(\mathfrak{m})$ relative to the Killing form, with the binary and ternary products in (1.1).

There are no isomorphisms among the LY-algebras in the different items above.

Conversely, the LY-algebras in items (i) and (ii) are indeed irreducible of generic type with standard enveloping algebra of type $\mathfrak{sl}$.

Proof. According to Lemma 2.1, $\mathfrak{m} = \mathfrak{h}^\perp$ where $\mathfrak{h}$ is as described in (2.6) and $V = \mathcal{U}^+$, the (+)-component of either a suitable simple Jordan pair or a simple contragredient anti-Jordan pair $\mathcal{U} = (\mathcal{U}^+, \mathcal{U}^-)$ with triple products $\{x_\sigma y_\sigma z_\sigma\}$ for $\sigma = \pm$. We also note that the subalgebra $\mathfrak{h}$ is described by means of the (+)-product operators $D_+(x_+, y_-) = \{x_+ y_- .\}$, $x_+ \in \mathcal{U}^+$ and $y_- \in \mathcal{U}^-$. Moreover, for contragredient anti-Jordan pairs we must have the isomorphism $(\mathcal{U}^+)^* \cong \mathcal{U}^-$, as modules for Inder $\mathcal{U}$. (From [13, Corollary 2.2] it follows that any simple Jordan pair is contragredient, that is, U^+ and U^-

are dually paired by the nondegenerate bilinear form $m(x_+, y_-) = \operatorname{tr} D_+(x_+, y_-)$; the assertion is not true for anti-Jordan pairs as shown in [13, Example 2.7].)

Conditions (b) and (c) in (1.4) and the previous initial restrictions on irreducibility and dimension of V imply that we must look only for simple Jordan or simple contragredient anti-Jordan pairs $\mathcal{U} = (\mathcal{U}^+, \mathcal{U}^-)$ such that:

(1) the derived subalgebra $\operatorname{Inder}_0 \mathcal{U} = [\operatorname{Inder} \mathcal{U}, \operatorname{Inder} \mathcal{U}]$ of the inner derivation algebra is simple,
(2) $\mathcal{U}^+ = V(\lambda)$ is an irreducible module of dominant weight λ (recall that the notation in [14] is used throughout) for the $(+)$-component of $\operatorname{Inder}_0 \mathcal{U}$ and $\dim \mathcal{U}^+ \geq 3$,
(3) $\mathcal{U}^+ \otimes \mathcal{U}^- = V(\lambda) \otimes V(\lambda^*)$ has just two nontrivial irreducible components which correspond to the module decomposition for $\mathfrak{sl}(\mathcal{U}^+)$.

Isomorphisms between either Jordan pairs or anti-Jordan pairs provide isomorphic LY-algebras. A look at the classification of simple Jordan pairs in [21, Theorem 17.12] and of simple anti-Jordan pairs in [13, Section 3 and 4] (both classifications are outlined in the Appendix – Section 6.3 and Table 5 –) show that the possibilities for Jordan pairs $\mathcal{U} = (\mathcal{U}^+, \mathcal{U}^-)$ satisfying conditions (1) and (2) above are the following:
(Condition (3) is checked case by case from the $\mathfrak{h}$-module decomposition $\mathfrak{gl}(\mathcal{U}^+) = \mathcal{U}^+ \otimes \mathcal{U}^- = \mathfrak{sl}(\mathcal{U}^+) \oplus V(0)$.)

$(\mathcal{M}_{p,1}(k), \mathcal{M}_{p,1}(k))_{p\geq 3}$:
In this case, the pair $(\operatorname{Inder}_0 \mathcal{U}, \mathcal{U}^+)$ is, up to isomorphism, the pair $\big(\mathfrak{sl}_p(k), V(\lambda_1)\big)$ (recall that we follow the notations of [14]). Then, as modules for $\mathfrak{sl}_p(k)$, $\mathcal{U}^+ \otimes \mathcal{U}^- = V(\lambda_1) \otimes V(\lambda_{p-1}) = V(\lambda_1 + \lambda_{p-1}) \oplus V(0)$.
Therefore, this case must be discarded (see (2.1)).

$(\mathcal{A}_n(k), \mathcal{A}_n(k))_{n\geq 5}$:
Here the pair $(\operatorname{Inder}_0 \mathcal{U}, \mathcal{U}^+)$ is, up to isomorphism, $(\mathfrak{sl}_n(k), V(\lambda_2))$ and, as modules for $\mathfrak{sl}_n(k)$, $\mathcal{U}^+ \otimes \mathcal{U}^- = V(\lambda_2) \otimes V(\lambda_{n-2}) = V(\lambda_1 + \lambda_{n-1}) \oplus V(\lambda_2 + \lambda_{n-2}) \oplus V(0)$.

$(\mathcal{H}_n(k), \mathcal{H}_n(k))_{n\geq 2}$:
Here the pair $(\operatorname{Inder}_0 \mathcal{U}, \mathcal{U}^+)$ is, up to isomorphism, $(\mathfrak{sl}_n(k), V(2\lambda_1))$ and, as modules for $\mathfrak{sl}_n(k)$, $\mathcal{U}^+ \otimes \mathcal{U}^- = V(2\lambda_1) \otimes V(2\lambda_{n-1}) = V(2\lambda_1 + 2\lambda_{n-1}) \oplus V(\lambda_1 + \lambda_{n-1}) \oplus V(0)$.

$(k^n, k^n)_{n\geq 5}$:
Here the pair $(\operatorname{Inder}_0 \mathcal{U}, \mathcal{U}^+)$ is, up to isomorphism, $(\mathfrak{so}_n(k), V(\lambda_1))$ and, as modules for $\mathfrak{so}_n(k)$, $\mathcal{U}^+ \otimes \mathcal{U}^- = V(\lambda_1) \otimes V(\lambda_1)$ is equal to $V(2\lambda_1) \oplus V(\lambda_2) \oplus V(0)$ for $n \geq 7$ and for $n = 5, 6$ it decomposes as $V(2\lambda_1) \oplus V(2\lambda_2) \oplus V(0)$ and $V(2\lambda_1) \oplus V(\lambda_2 + \lambda_3) \oplus V(0)$ respectively.

$(\mathcal{M}_{1,2}(\mathcal{O}), \mathcal{M}_{1,2}(\mathcal{O}))$:
Here the pair $(\operatorname{Inder}_0 \mathcal{U}, \mathcal{U}^+)$ is, up to isomorphism, $(\mathfrak{so}_{10}(k), V(\lambda_4))$ and, as modules for $\mathfrak{so}_{10}(k)$, $\mathcal{U}^+ \otimes \mathcal{U}^- = V(\lambda_4) \otimes V(\lambda_5) = V(\lambda_4 + \lambda_5) \oplus V(\lambda_2) \oplus V(0)$.

$(\mathcal{H}_3(\mathcal{O}), \mathcal{H}_3(\mathcal{O}))$:
Here the pair $(\operatorname{Inder}_0 \mathcal{U}, \mathcal{U}^+)$ is, up to isomorphism, $(E_6, V(\lambda_1))$ and, as modules for E_6, $\mathcal{U}^+ \otimes \mathcal{U}^- = V(\lambda_1) \otimes V(\lambda_6) = V(\lambda_1 + \lambda_6) \oplus V(\lambda_2) \oplus V(0)$.

For anti-Jordan pairs, the results in Section 6.3 and Table 6 show that the series $\mathcal{U} = (\mathcal{M}_{p,1}(k), \mathcal{M}_{p,1}(k))$ for $p \geq 3$ with $(\mathfrak{sl}_p(k), V(\lambda_1))$ and $\mathcal{U} = (k^{2n}, k^{2n})$ with $(\mathfrak{sp}_{2n}(k), V(\lambda_1))$ for $n \geq 2$ are the unique possibilities. The decomposition $\mathcal{U}^+ \otimes \mathcal{U}^-$ as $\mathfrak{sl}_p(k)$-module in the first case is analogous to the corresponding series of Jordan pairs, and hence this case must be discarded. For the anti-Jordan pairs $\mathcal{U} = (k^{2n}, k^{2n})$, the decomposition as modules for $\mathfrak{sp}_{2n}(k)$, is given by

$(k^{2n}, k^{2n})_{n\geq 2}$:
$V(\lambda_1) \otimes V(\lambda_1) = V(2\lambda_1) \oplus V(\lambda_2) \oplus V(0)$.

The Jordan pairs and anti-Jordan pairs of type $\mathcal{U} = (k^n, k^n)$ present a special common feature. Any of these structures can be described as a pair (V, V) where V is a vector space endowed with a nondegenerate ϵ-symmetric form b and with triple product $\{xyz\} = b(x, y)z + b(y, z)x - \epsilon b(x, z)y$, where $\epsilon = 1$ for Jordan pairs and $\epsilon = -1$ for anti-Jordan pairs (so $\dim V$ is even in the latter case). Moreover, the operators appearing in (2.6) are of the form

$$d_{x,y} = \{xy.\} - \frac{\operatorname{tr}(\{xy.\})}{\dim V} = b(y, .)x - \epsilon b(x, .)y,$$

and hence the subalgebra $\mathfrak{h} = \operatorname{span}\langle d_{x,y} = b(y, .)x - \epsilon b(x, .)y : x, y \in V\rangle$ is the Lie algebra $\mathfrak{so}(V)$ in case $\epsilon = 1$ and $\mathfrak{sp}(V)$ for $\epsilon = -1$. On the other hand, the map $f \mapsto f^*$, where f^* is the adjoint map relative to the form b, induces an involution on the associative algebra $\operatorname{End}(V)$ for which $\mathfrak{h} = \{f \in \operatorname{End}(V) : f^* = -f\} = \mathcal{S}(V, *)$ is just the Lie algebra $\mathfrak{so}(V)$ or $\mathfrak{sp}(V)$ and the set $\mathcal{J} = \mathcal{H}(V, *) = \{f \in \operatorname{End}(V) : f^* = f\}$, under the symmetrized product $f \cdot g = fg + gf$, is a central simple Jordan algebra. In this case, the decomposition $\mathfrak{gl}(V) = \mathcal{S}(V, *) \oplus \mathcal{H}(V, *)$ is symmetric and its restriction to $\mathfrak{sl}(V)$ provides the symmetric decomposition $\mathfrak{sl}(V) = \mathfrak{h} \oplus \mathcal{H}(V, *)_0$, where $\mathcal{H}(V, *)_0$ consist of the zero trace elements in $\mathcal{J}$. This symmetric decomposition satisfies (1.4), so $\mathfrak{m} = \mathfrak{h}^\perp = \mathcal{H}(V, *)_0$ and therefore the LY-algebra $\mathfrak{m}$ has trivial binary product and the ternary one is given by $[f, g, h] = [[f, g], h] = -(f, h, g) = -(f \cdot h) \cdot g + f \cdot (h \cdot g) = (g, f, h)$. This provides item (i) in the Theorem.

Finally, the remaining admissible Jordan pairs above are all of the form (J, J) for a Jordan triple system J and the assertion (ii) in the Theorem follows.

In all the cases considered, it has been proved that $\mathfrak{m}$ is an irreducible module for the simple Lie algebra $\mathfrak{h}$ and it is not isomorphic to the adjoint module. Hence the converse is clear. □

3. Orthogonal case

In this section we classify LY-algebras of Generic Type whose standard enveloping algebra is a (simple) orthogonal Lie algebra $\mathfrak{so}(V, b)$, so V is a vector space of dimension ≥ 5 (note that $\mathfrak{so}_4(k)$ is not simple and that $\mathfrak{so}_3(k)$ is isomorphic to $\mathfrak{sl}_2(k)$), endowed with a nondegenerate symmetric form b.

As in the previous section, we are looking for decompositions $\mathfrak{so}(V, b) = \mathfrak{h} \oplus \mathfrak{m}$ in which conditions (a), (b) and (c) in (1.4) hold. Our discussion in the $\mathfrak{so}$-case will be based on the following elementary facts:

- Considering both $\mathfrak{so}(V)$ and V as modules for $\mathfrak{h}$, the linear map $x \wedge y \mapsto \sigma_{x,y} = b(x, .)y - b(y, .)x$ defines an isomorphism of $\mathfrak{h}$-modules:

$$\bigwedge^2 V \cong \mathfrak{so}(V, b), \tag{3.1}$$

  from the second alternating power of V onto the Lie algebra $\mathfrak{so}(V)$.
- Any tensor product of irreducible modules $V(\lambda) \otimes V(\mu)$ contains a (unique) copy of the irreducible module $V(\lambda + \mu)$. This copy is generated by $v = v_\lambda \otimes v_\mu$, the only vector (up to scalars) of (highest) weight $\lambda + \mu$. (Here v_λ denotes a nonzero vector of weight λ.) Moreover, in case $\lambda = \mu$ this copy is located inside the second symmetric power of $V(\lambda)$, that is:

$$V(2\lambda) \subseteq S^2 V(\lambda). \tag{3.2}$$

- For a given dominant weight λ and any simple root α not orthogonal to λ ($\langle \lambda, \alpha \rangle \neq 0$), the second alternating power $\bigwedge^2 V(\lambda)$ contains a (unique) copy of the irreducible module $V(2\lambda - \alpha)$. This copy is generated by $v = v_\lambda \otimes v_{\lambda-\alpha} - v_{\lambda-\alpha} \otimes v_\lambda$, the only vector (up to scalars) of (highest) weight $2\lambda - \alpha$. Hence,

$$V(2\lambda - \alpha) \subseteq \bigwedge^2 V(\lambda), \quad \text{in case } \langle \lambda, \alpha \rangle \neq 0. \tag{3.3}$$

Lemma 3.1. *For a given reductive decomposition $\mathfrak{so}(V, b) = \mathfrak{h} \oplus \mathfrak{m}$ satisfying* (a), (b) *and* (c) *in* (1.4) *with* $\dim V \geq 5$, *one has that, as a module for $\mathfrak{h}$, either:*

(i) *V decomposes as $V = kv \oplus W$, an orthogonal sum of a trivial module kv and an irreducible module W with* $\dim W \geq 5$. *In this case, the subalgebra $\mathfrak{h}$ is $\mathfrak{h} = \sigma_{W,W} = \operatorname{span}\langle \sigma_{x,y} : x, y \in W \rangle$, ($\sigma_{x,y}$ as in (3.1)), so it is isomorphic to $\mathfrak{so}(W, b)$ and for the subspace $\mathfrak{m}$ we have that $\mathfrak{m} = \sigma_{v,W}$. Moreover, the reductive decomposition*

$$\mathfrak{so}(kv \oplus W, b) = \sigma_{W,W} \oplus \sigma_{v,W} \tag{3.4}$$

is symmetric; or

(ii) *$V = V(m\lambda_i)$ is an irreducible module for $\mathfrak{h}$ whose dominant weight is a multiple of the fundamental weight λ_i relative to some system of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_n\}$, and one of the following holds:*
  (ii-a) *$\mathfrak{m} = V(2m\lambda_i - \alpha_i)$ for some i, $1 \leq i \leq n$.*
  (ii-b) *$\mathfrak{h}$ is a simple Lie algebra of type B_3, $V = V(\lambda_3)$ and $\mathfrak{m} = V(\lambda_1)$*
  (ii-c) *$\mathfrak{h}$ is a simple Lie algebra of type G_2 and $V = \mathfrak{m} = V(\lambda_1)$.*

Proof. If V is not irreducible as a module for $\mathfrak{h}$, let W be a proper and irreducible $\mathfrak{h}$-submodule. Assume first $b(W, W) \neq 0$, thus the restriction of b to W is nondegenerate by irreducibility of W, so V decomposes as the orthogonal sum $V = W \oplus W^\perp$. Since $\mathfrak{h} \subset \mathfrak{so}(W) \oplus \mathfrak{so}(W^\perp) \subset \mathfrak{so}(V, b)$, the maximality of $\mathfrak{h}$ (condition (e) in (1.5)) forces

$$\mathfrak{h} = \mathfrak{so}(W, b) = \mathfrak{so}(W) \oplus \mathfrak{so}(W^\perp).$$

But $\mathfrak{h}$ is a simple Lie algebra, so $W^\perp$ must be one dimensional. So we have the orthogonal decomposition $V = kv \oplus W$ and we get the natural $\mathbb{Z}_2$-graduation in (3.4) with $\mathfrak{h} = \sigma_{W,W} = \operatorname{span}\langle \sigma_{x,y} : x, y \in W \rangle$ (the maps $\sigma_{x,y}$ as in (3.1)), and $\sigma_{v,W} = \operatorname{span}\langle \sigma_{v,x} : x \in W \rangle$, which is an irreducible module for $\mathfrak{h}$ isomorphic to W. Thus from (1.5), we have $\mathfrak{m} = \mathfrak{h}^\perp = \sigma_{v,W}$ which provides item (i) in the Lemma.

On the other hand, if V is not irreducible as a module for $\mathfrak{h}$, but the restriction of b to any irreducible $\mathfrak{h}$-submodule is trivial, by Weyl's theorem on complete reducibility, given an irreducible submodule W_1 there is another irreducible submodule W_2 with $b(W_1, W_2) \neq 0$. So, W_1 and W_2 are isotropic, that is $b(W_i, W_i) = 0$, and contragredient modules, and $V = W_1 \oplus W_2 \oplus (W_1 \oplus W_2)^\perp$. Arguing as before, we may assume that $V = W_1 \oplus W_2$. Then our subalgebra $\mathfrak{h}$ lies inside the subalgebra $\{f \in \mathfrak{so}(V,b) : f(W_i) \subseteq W_i\} = \sigma_{W_1,W_2}$ and this contradicts the maximality of $\mathfrak{h}$, since σ_{W_1,W_2} is contained properly in the subalgebra $\sigma_{W_1,W_2} \oplus \sigma_{W_1,W_1}$.

Now, in case $V = V(\lambda)$ remains irreducible as a module for $\mathfrak{h}$, its dominant weight λ relative to a Cartan subalgebra of $\mathfrak{h}$ and a choice of a system $\Delta = \{\alpha_1, \dots, \alpha_n\}$ of simple roots, decomposes as $\lambda = \sum_{i=1}^n m_i\lambda_i$, where as in [14], $\lambda_1, \dots, \lambda_n$ denote the fundamental weights. Note that $m_i = \langle \lambda, \alpha_i\rangle \geq 0$ is a non-negative integer for any i. Let α_i be a simple root which is not orthogonal to λ, that is $m_i \neq 0$. From (3.3), a copy of the irreducible module $V(2\lambda - \alpha_i)$ appears in $\bigwedge^2 V(\lambda) \cong \mathfrak{so}(V) = \mathfrak{h} \oplus \mathfrak{m}$. In case $\mathfrak{h} \cong V(2\lambda - \alpha_i)$, we have that $2\lambda - \alpha_i$ is the highest root ω of $\mathfrak{h}$, and hence $\omega + \alpha_i$ is twice a dominant weight λ (while $\mathfrak{h}$ being a proper subspace of $\bigwedge^2 V(\lambda)$). A quick look at the Dynkin diagrams (see [14]) shows that the only possibilities are the ones that appear in items (ii-b) and (ii-c).

Otherwise we must assume that the highest root of $\mathfrak{h}$ is not of the form $2\lambda - \alpha_i$ for some simple root α_i such that $\langle \lambda, \alpha_i\rangle \neq 0$. As $\mathfrak{so}(V)$ has exactly two irreducible components as a module for $\mathfrak{h}$, there exists exactly one simple root α_i not orthogonal to λ. Hence $\lambda = m_i\lambda_i$ with $m_i \geq 1$ and $\mathfrak{m} = V(2m_i\lambda_i - \alpha_i)$ which provides item (ii-a). □

Following Lemma 3.1, for any reductive and nonsymmetric decomposition $\mathfrak{so}(V,b) = \mathfrak{h} \oplus \mathfrak{m}$ satisfying (a), (b) and (c) in (1.4), the vector space V, considered as a module for $\mathfrak{h}$ must be (nontrivial) irreducible with dominant weight of the form $m\lambda_i$, λ_i being a fundamental weight relative to some system of simple roots Δ of $\mathfrak{h}$. The irreducibility of V allows us to endow this space with a structure of either a Lie triple system or an orthogonal triple system (see [26, Section V], [7, Definition 4.1] or Section 6.2 of the Appendix in this paper for the definition of the latter systems), in such a way that the subalgebra $\mathfrak{h}$ becomes its inner derivation Lie algebra. In this way, the classification in the $\mathfrak{so}$-case will follow from known results on these triple systems.

For an arbitrary reductive decomposition $\mathfrak{so}(V,b) = \mathfrak{h} \oplus \mathfrak{m}$, by using the isomorphism as modules for $\mathfrak{h}$ in (3.1), we can define the map

$$\begin{array}{ccccc} V \otimes V & \to & \mathfrak{so}(V,b) & \to & \mathfrak{h} \\ x \otimes y & \mapsto & \sigma_{x,y} & \mapsto & d_{x,y}, \end{array} \tag{3.5}$$

where $d_{x,y}$ denotes the projection of the operator $\sigma_{x,y}$ onto $\mathfrak{h}$, so the subalgebra $\mathfrak{h}$ can be written as $\mathfrak{h} = \mathrm{span}\langle d_{x,y} : x, y \in V\rangle$. Now, let us define the triple product on V given by

$$xyz := d_{x,y}z. \tag{3.6}$$

This product satisfies the identities:

$$xxz = 0, \tag{3.7}$$

$$xy(uvw) = (xyu)vw + u(xyv)w + uv(xyw), \tag{3.8}$$

$$b(xyu, v) + b(u, xyv) = 0, \tag{3.9}$$

for any $x, y, z, u, v \in V$.

Note that (3.7) is equivalent to $d_{x,y}$ being skew symmetric as a function of x and y. Identity (3.8) tells us that the map given in (3.5) is a homomorphism of modules for $\mathfrak{h}$ and (3.9) follows from $\mathfrak{h}$ being a subalgebra of $\mathfrak{so}(V,b)$. Since $d_{x,y}z = xyz$, the subalgebra $\mathfrak{h}$ is the inner derivation Lie algebra of the triple V:

$$\mathfrak{h} = \mathrm{span}\langle d_{x,y} : x, y \in V\rangle = \mathrm{Inder}\, V, \tag{3.10}$$

and we get the following result:

Lemma 3.2. *Given a reductive decomposition $\mathfrak{so}(V,b) = \mathfrak{h} \oplus \mathfrak{m}$ satisfying* (a), (b), (c) *in* (1.4)*, such that the vector space V is irreducible as a module for $\mathfrak{h}$, the vector space V endowed with the triple product $xyz = d_{x,y}z$ defined in* (3.6) *is either a simple Lie triple system or a simple orthogonal triple system with associated bilinear form ξb for some unique nonzero scalar ξ. Moreover, the subalgebra $\mathfrak{h}$ satisfies the equation*

$$\mathfrak{h} = \mathrm{span}\langle d_{x,y} : x, y \in V\rangle, \tag{3.11}$$

and therefore coincides with the inner derivation Lie algebra of the corresponding triple system, and the subspace $\mathfrak{m}$ is the orthogonal complement $\mathfrak{h}^\perp$ to $\mathfrak{h}$ relative to the Killing form of $\mathfrak{so}(V,b)$.

Proof. First we shall check that the vector space $\mathrm{Hom}_{\mathfrak{h}}((\bigwedge^2 V) \otimes V, V)$ is two dimensional. Since $V \cong V^*$ and $\bigwedge^2 V \cong \mathfrak{so}(V)$,

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{h}}\left(\left(\bigwedge^2 V\right) \otimes V, V\right) &\cong \mathrm{Hom}_{\mathfrak{h}}\left(\bigwedge^2 V, V \otimes V\right) \\ &\cong \mathrm{Hom}_{\mathfrak{h}}(\mathfrak{h}, V \otimes V) \oplus \mathrm{Hom}_{\mathfrak{h}}(\mathfrak{m}, V \otimes V). \end{aligned} \tag{3.12}$$

Moreover, the irreducibility of $\mathfrak{h}$ as a module for itself gives

$$\dim \mathrm{Hom}_{\mathfrak{h}}(\mathfrak{h}, V \otimes V) = \dim \mathrm{Hom}_{\mathfrak{h}}(V \otimes V^*, \mathfrak{h}). \tag{3.13}$$

Lemma 3.1 shows that $V = V(m\lambda_i)$ (the irreducible module of dominant weight $m\lambda_i$) as a module for $\mathfrak{h}$, so [11, Theorem 1] proves that $\mathrm{Hom}_{\mathfrak{h}}(\mathfrak{h}, V \otimes V)$ is a one-dimensional vector space. From the different possibilities for $\mathfrak{m}$ described in (ii-a), (ii-b) and (ii-c) of Lemma 3.1, the same assertion holds for $\mathrm{Hom}_{\mathfrak{h}}(\mathfrak{m}, V \otimes V)$:

(ii-a) $\mathfrak{m} = V(m\lambda_i - \alpha_i)$.
The assertion follows from (3.3) and comments therein.

(ii-b) $\mathfrak{h} \cong B_3$, $V = V(\lambda_3)$ and $\mathfrak{m} = V(\lambda_1)$.
The assertion follows since the tensor product decomposition

$$V(\lambda_3) \otimes V(\lambda_3) \cong V(2\lambda_3) \oplus V(\lambda_2) \oplus V(\lambda_1) \oplus V(0), \tag{3.14}$$

contains only one copy of $\mathfrak{m}$.

(ii-c) $\mathfrak{h} \cong G_2$, $V = \mathfrak{m} = V(\lambda_1)$.
Again the tensor product decomposition

$$V(\lambda_1) \otimes V(\lambda_1) \cong V(2\lambda_1) \oplus V(\lambda_2) \oplus V(\lambda_1) \oplus V(0), \tag{3.15}$$

contains only one copy of $\mathfrak{m}$.

On the other hand, in a easy way we can get the following $\mathfrak{h}$-module decomposition for the tensor product $(\bigwedge^2 V) \otimes V$:

$$\left(\bigwedge^2 V\right) \otimes V = \bigwedge^3 V \oplus S, \tag{3.16}$$

where $\bigwedge^3 V$ embeds in $(\bigwedge^2 V) \otimes V$ by means of $x \wedge y \wedge z \mapsto (x \wedge y) \otimes z + (y \wedge z) \otimes x + (z \wedge x) \otimes y$, and $S = \operatorname{span}\langle (x \wedge y) \otimes z + (z \wedge y) \otimes x : x, y, z \in V\rangle$. The nonzero $\mathfrak{h}$-homomorphism $\varphi : S \to V$ given by:

$$\varphi((x \wedge y) \otimes z + (z \wedge y) \otimes x) = \sigma_{x,y}(z) + \sigma_{z,y}(x) = 2b(x, z)y - b(y, z)x - b(y, x)z,$$

with $\sigma_{x,y}$ as in (3.1), provides the alternative decomposition

$$\left(\bigwedge^2 V\right) \otimes V = \bigwedge^3 V \oplus \operatorname{Ker}\varphi \oplus V, \tag{3.17}$$

and therefore

$$\operatorname{Hom}_{\mathfrak{h}}\left(\left(\bigwedge^2 V\right) \otimes V, V\right) = \operatorname{Hom}_{\mathfrak{h}}\left(\bigwedge^3 V, V\right) \oplus \operatorname{Hom}_{\mathfrak{h}}(\operatorname{Ker}\varphi, V) \oplus \operatorname{Hom}_{\mathfrak{h}}(V, V). \tag{3.18}$$

Since the dimension of the vector spaces $\operatorname{Hom}_{\mathfrak{h}}((\bigwedge^2 V) \otimes V, V)$ and $\operatorname{Hom}_{\mathfrak{h}}(V, V)$ is 2 and 1 respectively, either $\operatorname{Hom}_{\mathfrak{h}}(\bigwedge^3 V, V) = 0$ or $\operatorname{Hom}_{\mathfrak{h}}(S, V) = k\varphi$. In the first case, we have that the triple product xyz defined in (3.6) must be trivial when restricted to $\bigwedge^3 V$. Therefore this triple product satisfies the additional identity

$$xyz + yzx + zxy = 0, \tag{3.19}$$

for any $x, y, z \in V$. Hence, from (3.7), (3.8) and (3.19) we get that (V, xyz) is a Lie triple system (see Section 6.1 in the Appendix for the definition). Moreover, as $\mathfrak{h} = \operatorname{Inder} V$ and $V = V(m\lambda_i)$ is $\mathfrak{h}$-irreducible, this triple system is simple.

Otherwise $\operatorname{Hom}_{\mathfrak{h}}(S, V) = k\varphi$ holds, so the restriction of the triple product xyz to S give us the relationship

$$xyz + zyx = \xi\varphi(x \wedge y \otimes z + z \wedge y \otimes x) = 2\xi b(x, z)y - \xi b(y, z)x - \xi b(y, x)z, \tag{3.20}$$

for some $\xi \in k$ and any $x, y, z \in V$. Moreover, let us show that ξ must be nonzero. Assume on the contrary that $\xi = 0$, from (3.20) we get

$$xyz + zyx = 0, \tag{3.21}$$

for all $x, y, z \in V$ and the triple product is totally antisymmetric. Then the triple products $\langle x_\sigma y_{-\sigma} z_\sigma\rangle = \sigma x_\sigma y_{-\sigma} z_\sigma$ defined on the vector space pair $\mathcal{U} = (U^+, U^-)$ with $U^\sigma = V$ and $\sigma = \pm$ satisfy:

$$\langle x_\sigma y_{-\sigma} z_\sigma\rangle = \sigma x_\sigma y_{-\sigma} z_\sigma = -\sigma z_\sigma y_{-\sigma} x_\sigma = -\langle z_\sigma y_{-\sigma} x_\sigma\rangle,$$

and using (3.7) and (3.8),

$$\begin{aligned}\langle x_\sigma y_{-\sigma}\langle u_\sigma v_{-\sigma} w_\sigma\rangle\rangle &= \sigma^2 x_\sigma y_{-\sigma}(u_\sigma v_{-\sigma} w_\sigma)\\ &= \sigma^2((x_\sigma y_{-\sigma} u_\sigma)v_{-\sigma} w_\sigma) - u_\sigma(y_{-\sigma} x_\sigma v_{-\sigma})w_\sigma + u_\sigma v_{-\sigma}(x_\sigma y_{-\sigma} w_\sigma))\\ &= \langle\langle x_\sigma y_{-\sigma} u_\sigma\rangle v_{-\sigma} w_\sigma\rangle + \langle u_\sigma\langle y_{-\sigma} x_\sigma v_{-\sigma}\rangle w_\sigma\rangle + \langle u_\sigma v_{-\sigma}\langle x_\sigma y_{-\sigma} w_\sigma\rangle\rangle.\end{aligned}$$

Hence, $\mathcal{U}$ is an anti-Jordan pair for which the inner derivation operators are of the form

$$\begin{aligned}(D_+(x_+, y_-), D_-(y_-, x_+)) &= (\langle x_+ y_- .\rangle, \langle y_- x_+ .\rangle)\\ &= (x_+ y_- ., -y_- x_+ .).\end{aligned} \tag{3.22}$$

We note that the linearization $xyz = -yxz$ of (3.7) is equivalent to $D_{x,y} = -D_{y,x}$, thus $(D_+(x_+, y_-), D_-(y_-, x_+)) = (D_{x_+,y_-}, D_{x_+,y_-})$. This shows that the Lie algebra $\operatorname{Inder}\mathcal{U}$ is isomorphic to $\mathfrak{h}$ and therefore it is a simple Lie algebra. Moreover, since V is an irreducible module for $\mathfrak{h}$, $\mathcal{U}$ is a simple anti-Jordan pair [13, Proposition 1.2]. But according to Table 6 in the Appendix , the inner derivation algebras of simple anti-Jordan pairs such that $\mathcal{U}^+ = \mathcal{U}^-$ are not simple. Hence $\xi \neq 0$.

Now Eq. (3.20) with $z = y$ and (3.7) give the identity

$$xyy = \xi b(x, y)y - \xi b(y, y)x, \tag{3.23}$$

for any $x, y \in V$, which together with (3.7)–(3.9) (see Section 6.2 in the Appendix) prove that the vector space V is an orthogonal triple system under the triple product xyz and the symmetric bilinear form $\xi b(x, y)$. Since the form is nondegenerate, V is a simple orthogonal triple system ([7, Proposition 4.4]) and the subalgebra $\mathfrak{h}$ is its inner derivation Lie algebra. The last assertion ($\mathfrak{m} = \mathfrak{h}^\perp$) follows from condition (d) in (1.5). □

Now we can formulate the main result for the generic $\mathfrak{so}$-case:

Theorem 3.3. *Let $(\mathfrak{m}, a \cdot b, [a, b, c])$ be an irreducible LY-algebra of generic type and standard enveloping Lie algebra of type $\mathfrak{so}$. Then precisely one of the following cases occurs:*

(i) *There is a vector space V of dimension ≥ 5, endowed with a nondegenerate symmetric bilinear form b such that $\mathfrak{m}$ is, up to isomorphism, the simple Lie triple system defined on V with triple product $[u, v, w] = b(u, w)v - b(v, w)u$. In particular, the binary product $u \cdot v$ is trivial.*

(ii) *Up to isomorphism, $\mathfrak{m}$ coincides with the space $\mathcal{O}_0$ of trace zero octonions with binary and ternary products $a \cdot b = ab - ba = [a, b]$ and $[a, b, c] = 2\big([[a, b], c] - 3\big((ac)b - a(cb)\big)\big)$ for any $a, b, c \in \mathcal{O}_0$, where ab denotes the multiplication in $\mathcal{O}$.*

(iii) *There is a simple Lie triple system T endowed with a nondegenerate symmetric bilinear form b of one of the following types:*

(iii.a) *a simple Lie algebra of type different from A with its natural triple product $d_{x,y}z = [xyz] = [[x, y], z]$ endowed with its Killing form,*

(iii.b) *the subspace of trace zero elements of a simple Jordan algebra of degree ≥ 3, not isomorphic either to $\mathrm{Mat}_n(k)^+$ $(n \geq 3)$ or to $\mathcal{H}_4(k)$, with its triple product $d_{x,y}z = [xyz] = (x, z, y) = (x \circ z) \circ y - x \circ (y \circ z)$ (where $x \circ y$ denotes the multiplication in the Jordan algebra), endowed with the nondegenerate bilinear form given by its generic trace,*

(iii.c) *the Lie triple systems attached to the exceptional symmetric pairs (F_4, B_4), (E_6, C_4), (E_7, A_7) or (E_8, D_8), endowed with the nondegenerate bilinear form given by the restriction of the Killing form of the ambient Lie algebra,*

such that, up to isomorphism, $\mathfrak{g}(\mathfrak{m}) = \mathfrak{so}(T, b)$, $\mathfrak{h} = D(\mathfrak{m}, \mathfrak{m}) = \mathrm{Inder}\, T = d_{T,T} = \mathrm{span}\langle d_{x,y} : x, y \in T\rangle$. Here the LY-algebra $\mathfrak{m}$ appears as the orthogonal complement to $\mathfrak{h}$ in $\mathfrak{g}(\mathfrak{m})$ relative to the Killing form, with the binary and ternary products in (1.1)*.*

There are no isomorphisms among the LY-algebras in the different items above.

Conversely, the LY-algebras in items (i)–(iii) *are irreducible of generic type with standard enveloping algebra of type $\mathfrak{so}$.*

Proof. From Lemmas 3.1 and 3.2 we know that either $\mathfrak{m} = \sigma_{v,W} = \mathrm{span}\langle \sigma_{v,x} : x \in W\rangle$, where $\sigma_{v,x} = b(v, .)x - b(x, .)v$, inside the symmetric decomposition $\mathfrak{so}(kv \oplus W, b) = \sigma_{W,W} \oplus \sigma_{v,W}$ where b is a nondegenerate symmetric form with $b(v, W) = 0$ or $\mathfrak{m} = \mathfrak{h}^\perp$ where $\mathfrak{h}$ is the linear Lie algebra of inner derivations related to either a simple Lie triple system or a simple orthogonal triple system which is irreducible as a module for its inner derivation algebra.

In the first case, since $[\sigma_{x,y}, \sigma_{a,b}] = \sigma_{\sigma_{x,y}(a),b} + \sigma_{a,\sigma_{x,y}(b)}$ for any $x, y, a, b \in V$, it follows that

$$[[\sigma_{v,x}, \sigma_{v,y}], \sigma_{v,z}] = b(v, v)\sigma_{v,\sigma_{x,y}(z)}, \tag{3.24}$$

for any $x, y, z \in W$. Besides, since the ground field k is assumed to be algebraically closed, one may take v with $b(v, v) = 1$. Hence $\mathfrak{m} = \sigma_{v,W}$ can be identified to W with trivial binary product and triple product given by $[xyz] = \sigma_{x,y}(z)$, thus obtaining the situation in item (i).

Otherwise, we must look for either simple Lie triple systems or simple orthogonal triple systems (V, xyz) such that:

(1) the inner derivation algebra $\mathrm{Inder}\, V$ is simple,
(2) $V = V(m\lambda_i)$ is an irreducible module for $\mathrm{Inder}\, V$ with dominant weight m-times a fundamental weight λ_i, and $\dim V \geq 5$,
(3) $\bigwedge^2 V$ decomposes as a sum of two irreducible modules.

Since isomorphic irreducible orthogonal or Lie triple systems provide isomorphic LY-algebras, we need to check the previous conditions in the classifications, up to isomorphisms, of such systems given in [7, Theorem 4.7], [10, Table I] and [11, Table III], which are outlined in the Appendix of this paper: Tables 1 and 4. Then, using Table 4 in the Appendix , and restrictions (1)–(2)–(3), we get the following possibilities for the triple $(V, \mathrm{Inder}\, V, V(m\lambda_i))$ for orthogonal triple systems: the G-type triple system defined on the seven-dimensional space $\mathcal{O}_0$ of trace zero octonions with triple $(\mathcal{O}_0, G_2, V(\lambda_1))$ and the F-type triple systems defined on an eight-dimensional vector space V having a 3-fold vector cross product with ternary description $(V, \mathfrak{so}_7(k), V(\lambda_3))$. The respective decompositions of $\bigwedge^2 V$ as a module for $\mathfrak{h} = \mathrm{Inder}\, V$ are the following:

(In both cases condition (3) holds.)

$(\mathcal{O}_0, G_2, V(\lambda_1))$:
$\bigwedge^2 V(\lambda_1) = V(\lambda_1) \oplus V(\lambda_2)$ as a module for Inder $\mathcal{O}_0 = G_2$.

$(V, \mathfrak{so}_7(k), V(\lambda_3))$:
$\bigwedge^2 V(\lambda_3) = V(\lambda_1) \oplus V(\lambda_2)$ as a module for Inder $V \simeq \mathfrak{so}_7(k)$.

Following [7], the orthogonal triple system of G-type satisfies that $\mathfrak{h} = \text{Inder}\, V$ is the simple Lie algebra of type G_2 given by the Lie algebra of derivations of $\mathcal{O}$, considered as a subalgebra of $\mathfrak{so}(\mathcal{O}_0, n)$, where n denotes the norm of the octonion algebra. Then $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{h}$ relative to the Killing form of $\mathfrak{g} = \mathfrak{so}(\mathcal{O}_0, n)$. But $\mathfrak{so}(\mathcal{O}_0, n)$ decomposes as

$$\mathfrak{so}(\mathcal{O}_0, n) = \text{Der}(\mathcal{O}) \oplus \text{ad}_{\mathcal{O}_0},$$

(see [28, Chapter III, §8] or [9]), where $\text{ad}_x(y) = [x, y] = xy - yx$. Since $\text{ad}_{\mathcal{O}_0}$ is irreducible as a module for $\text{Der}(\mathcal{O})$ it turns out that $\text{ad}_{\mathcal{O}_0}$ is necessarily the orthogonal complement to $\mathfrak{h} = \text{Der}(\mathcal{O})$ relative to the Killing form of $\mathfrak{so}(\mathcal{O}_0, n)$. Besides for any $x, y \in \mathcal{O}_0$ we have:

$$\begin{aligned}[\text{ad}_x, \text{ad}_y] &= [L_x - R_x, L_y - R_y] \\ &= [L_x, L_y] + [R_x, R_y] - [R_x, L_y] - [L_x, R_y] \\ &= D_{x,y} - 3[L_x, R_y],\end{aligned}$$

where L_x and R_x denote the left and right multiplications by x in $\mathcal{O}$, and where $D_{x,y} = [L_x, L_y] + [L_x, R_y] + [R_x, R_y] = \text{ad}_{[x,y]} - 3[L_x, R_y]$ is the inner derivation of $\mathcal{O}$ generated by the elements x and y. Therefore

$$[\text{ad}_x, \text{ad}_y] = -\text{ad}_{[x,y]} + 2D_{x,y},$$

and hence, if we identify $\mathfrak{m} = \text{ad}_{\mathcal{O}_0}$ with $\mathcal{O}_0$ by means of $x \mapsto -\text{ad}_x$, the binary and ternary multiplications in (1.1) are given by:

$$x \cdot y = [x, y],$$
$$[xyz] = 2D_{x,y}(z),$$

for any $x, y, z \in \mathcal{O}_0$ and we obtain item (ii).

For F-type orthogonal triple systems (see [7] and [8]), we have that $V = \mathcal{O}$ with bilinear form $b(x, y) = \alpha n(x, y)$ where $n(x, y)$ is as in the previous paragraph, the triple product is given by $d_{x,y}z = xyz = (x\bar{y})z + 4b(x, z)y - 4b(y, z)x - b(x, y)z$ and Inder $V = \text{span}\langle d_{x,y}\rangle$. In this case Inder V is a Lie algebra of type B_3 that can be described as Inder $V = \text{span}\langle D_{x,y}, L_x + 2R_x : x, y \in \mathcal{O}_0\rangle$. Moreover, the automorphism θ of $\mathfrak{so}(\mathcal{O}, n) \cong \mathfrak{so}_8(k)$ given by $L_x \mapsto L_x + R_x$ and $R_x \mapsto -R_x$ (see [28, Chapter III, Section 8] makes $\theta(\text{Inder}\, V) = \text{span}\langle D_{x,y}, L_x - R_x = \text{ad}_x : x, y \in \mathcal{O}_0\rangle = \{f \in \mathfrak{so}(V) : f(1) = 0\} = \mathfrak{so}(\mathcal{O}_0, n)$, a Lie algebra for which $\mathcal{O}_0$ is irreducible and orthogonal to $k1$ which is a one dimensional and $\theta(\text{Inder}\, V)$-invariant subspace. Hence, the LY-algebra $\mathfrak{m} = (\text{Inder}\, V)^{\perp}$ is isomorphic to $\sigma_{1,\mathcal{O}_0} = \text{span}\langle \sigma_{1,x} = n(1, .)x - n(x, .)1 : x \in \mathcal{O}_0\rangle$ obtained as in item (i). Hence nothing new appears here.

For simple Lie triple systems, [10, Table I] presents the complete classification of such triple systems encoded through affine Dynkin diagrams. Using this classification, in [11, Table III] a complete list of all simple Lie triple systems which are irreducible for Inder V is given. Table III also provides the inner derivation algebra Inder V, and the structure for each irreducible Lie triple V as a module for Inder V. The results in [10] and [11] are displayed on Table 1 in the Appendix. Table 1 provides the dominant weights for the different irreducible Lie triple systems V as well as the $\bigwedge^2 V$-decomposition of those triples with simple inner derivation Lie algebra. So using Table 1 under the restrictions (1)–(2)–(3), we arrive at the possibilities described below. We also note that in all the cases, V is a contragredient and irreducible module for Inder V, and hence there exists a unique Inder V-invariant form b on V up to scalars, which is either symmetric or skew symmetric. The $\bigwedge^2 V$-decomposition in Table 1 proves that the form b is always symmetric for irreducible triple systems with simple inner derivation algebra.

$(\mathcal{L} \times \mathcal{L}, \mathcal{L})$:
Here $\mathcal{L}$ represents a Lie algebra of type different from A_n, $n \geq 1$, but considered as a triple system by means of the product $d_{x,y}z = [xyz] = [[x, y], z]$, endowed with the Killing form. These triple systems are called adjoint in [3].

$(A_{2n}, B_n)_{n\geq 1}$, $(A_{2n-1}, C_n)_{n\geq 3}$, $(A_{2n-1}, D_n)_{n\geq 3}$:
Following Theorem 2.2, these triple systems correspond to the space of trace zero elements in a simple Jordan algebra $\mathcal{J}$ of type B or C and degree $n \geq 3$ with triple product given by the associator. Then, since $d_{x,y}z = [xyz] = (y, z, x) = [L_x, L_y](z)$ (L_x denotes the multiplication by x in the Jordan algebra), we have that Inder $\mathcal{J}_0 = \text{span}\langle [L_x, L_y] : x, y \in \mathcal{J}_0\rangle = \text{Der}\, \mathcal{J}$ and b is the generic trace.

(F_4, B_4), (E_6, F_4), (E_6, C_4), (E_7, A_7), (E_8, D_8):
Table 1 shows that all the simple Lie triple systems with exceptional simple standard enveloping Lie algebra and simple inner derivation Lie algebra work. The triple system related to the pair (E_6, F_4) consists of the trace zero elements of the exceptional simple Jordan algebra (Albert algebra) with associator as triple product (see [15]).

On the other hand, Table 1 shows that the only symmetric decompositions with standard enveloping Lie algebra of type $\mathfrak{so}$ are given by symmetric pairs of type $(\mathfrak{so}_n(k), \mathfrak{so}_{n-1}(k))$, up to isomorphisms. It is easy to check that none of the reductive pairs related with the LY-algebras described in items (ii) and (iii) are of this form. So the binary and ternary products of the corresponding LY-algebras are not trivial.

Finally, the restriction on item (iii.b) on the Jordan algebra not being isomorphic to $\mathcal{H}_4(k)$ is due to the fact that for this Jordan algebra, the associated subalgebra $\mathfrak{h}$ is $\mathfrak{so}_4(k)$ which is not simple.

The converse is clear from the arguments above. □

Remark 3.4. The case in item (iii.b) of the previous Theorem corresponding to the Jordan algebra $\mathcal{H}_3(k)$ satisfies that its enveloping algebra is $\mathfrak{so}_5(k)$, which is isomorphic to $\mathfrak{sp}_4(k)$. Hence this case will appear too in the next section (Item (i) of Theorem 4.4). Therefore, we may assume that the Jordan algebra in item (iii.b) above is not isomorphic to $\mathcal{H}_3(k)$. This will be done in our final Table 9.

4. Symplectic case

For LY-algebras of Generic Type and standard enveloping Lie algebra a (simple) symplectic Lie algebra $\mathfrak{sp}(V, b)$, we will follow a similar procedure to that used in the special and orthogonal cases. In the symplectic case, V is an even-dimensional vector space endowed with a nondegenerate skew symmetric form b.

Given a suitable reductive decomposition $\mathfrak{sp}(V) = \mathfrak{h} \oplus \mathfrak{m}$, we may view V and $\mathfrak{sp}(V)$ as modules for $\mathfrak{h}$. The map $(x, y) \mapsto \gamma_{x,y} = b(x, .)y + b(y, .)x$ provides an isomorphism of $\mathfrak{h}$-modules:

$$S^2V \cong \mathfrak{sp}(V), \tag{4.1}$$

where S^2V is the second symmetric power of V. This isomorphism and the following easy Lemma on representations of Lie algebras are used in an essential way throughout this section:

Lemma 4.1. *Let μ_1 and μ_2 be two dominant weights of a simple Lie algebra (relative to a fixed system of simple roots). Then the modules $\bigwedge^2 V(\mu_1)$ and $S^2V(\mu_2)$ are isomorphic if and only if one of the following holds:*

(i) *$\mathfrak{h}$ is a simple Lie algebra of type A_1, $\mu_1 = k\lambda_1$ and $\mu_2 = (k-1)\lambda_1$ with $k \geq 1$,*
(ii) *$\mathfrak{h}$ is a simple Lie algebra of type B_2, $\mu_1 = \lambda_1$ and $\mu_2 = \lambda_2$.*

Proof. For a given simple root α_i non-orthogonal to μ_1, the weight $2\mu_1 - \alpha_i$ is maximal in the set of weights of the module $\bigwedge^2 V(\mu_1)$ relative to the usual partial order where $\lambda > \mu$ if $\lambda - \mu$ is a sum of positive roots. Since $2\mu_2$ is the only maximal weight for $S^2V(\mu_2)$, we have $2\mu_1 - \alpha_i = 2\mu_2$ and there is a unique simple root α_i not orthogonal to μ_1. The last assertion implies $\mu_1 = k\lambda_i$, $k \geq 1$ and $\alpha_i = 2(\mu_1 - \mu_2)$. Now it is easy to check that the only possibilities are the following (see [14, Chapter III, Section 11]):

- A_1 with $\alpha_i = \alpha_1 = 2\lambda_1$, which implies item (i) in the Lemma.
- B_2 and $\alpha_i = \alpha_1 = 2(\lambda_1 - \lambda_2)$:
  In this case, $\mu_1 = k\lambda_1$ and $\mu_2 = (k-1)\lambda_1 + \lambda_2$. But computing the dimension of the corresponding irreducible modules $V(k\lambda_1)$ and $V((k-1)\lambda_1 + \lambda_2)$, we get $k = 1$ as the only possibility, thus item (ii) in Lemma follows.
- $\mathfrak{h} = C_n$, $n \geq 3$ and $\alpha_i = \alpha_n = 2(-\lambda_{n-1} + \lambda_n)$:
  Then $\mu_1 = k\lambda_n$ and $\mu_2 = \lambda_{n-1} + (k-1)\lambda_n$. But the formula
  $$\frac{\dim V(k\lambda_n)}{\dim V((k-1)\lambda_n + \lambda_{n-1})} = \frac{2k+n+1}{2kn},$$
  implies that $\dim V(k\lambda_n) < \dim V((k-1)\lambda_n + \lambda_{n-1})$ except for $n = 3$ and $k = 1$. So, the only possibility for both modules to be isomorphic is $\mu_1 = \lambda_3$ and $\mu_2 = \lambda_2$. But then $\dim \bigwedge^2 V(\lambda_3) = 91 < \dim S^2V(\lambda_2) = 105$ and therefore this situation does not hold.

The converse is easily checked by using the Clebsch–Gordan formula and the isomorphism between the B_2-type Lie algebra $\mathfrak{so}_5(k)$ and the C_2-type $\mathfrak{sp}_4(k)$. □

Recall that given an irreducible module $V(\lambda)$ for a dominant weight λ of a simple Lie algebra, the dual module $V(\lambda)^*$ is isomorphic to $V(-\sigma\lambda)$, where σ is the element of the Weyl group sending the given system of simple roots to its opposite (see [14, Section 21, Exercise 6]). We will write $-\sigma\lambda = \lambda^*$ and will say that the dominant weight λ is self-dual in case $\lambda = \lambda^*$, that is, in case $V(\lambda)$ is a self-dual module.

Lemma 4.2. *Let $\mathfrak{sp}(V) = \mathfrak{h} \oplus \mathfrak{m}$ be a reductive decomposition satisfying* (a), (b) *and* (c) *in* (1.4). *Then* $\dim V \geq 4$ *and as a module for $\mathfrak{h}$, $V = V(\lambda)$ is irreducible and either its dominant weight λ is a fundamental and self-dual weight or $\mathfrak{h} = A_1$,* $\dim V = 4$ *and $\lambda = 3\lambda_1$. In any case, $\mathfrak{m}$ is an irreducible module for $\mathfrak{h}$ whose dominant weight is 2λ.*

Proof. In case V is reducible as a module for $\mathfrak{h}$, the arguments in the proof of Lemma 3.1 show that the vector space V can be decomposed as an orthogonal sum, $V = W \oplus W^{\perp}$ with W irreducible and nontrivial and $\mathfrak{h} = \mathfrak{sp}(W) \oplus \mathfrak{sp}(W^{\perp})$ which is not a simple Lie algebra. Hence V must be irreducible and the assertion on its dimension is clear. In what follows let λ be the dominant weight of the irreducible self-dual module V.

On the other hand, (3.2) and (4.1) show that $V(2\lambda)$ appears as a submodule in $S^2V \cong \mathfrak{sp}(V) = \mathfrak{h} \oplus \mathfrak{m}$, thus as modules over $\mathfrak{h}$ either $\mathfrak{h}$ or $\mathfrak{m}$ is isomorphic to $V(2\lambda)$. In the first case the only possibility is that $2\lambda = 2\lambda_1$ for the simple Lie algebra of type C_n. This implies $\mathfrak{sp}(V) = \mathfrak{sp}(V(\lambda_1)) = \mathfrak{h}$, which is not possible. Hence $\mathfrak{m}$ is irreducible with 2λ as dominant weight.

Now let $\lambda = \sum m_i\lambda_i$ be the decomposition of λ as a sum of fundamental weights λ_i, and assume that λ is not fundamental. Then λ can be decomposed in two different ways:

$$\begin{array}{ll} \text{(a)} & \lambda = \lambda_i + \lambda_i^*, \\ & \text{with } \lambda_i \text{ fundamental and nonself-dual.} \\ \text{(b)} & \lambda = \lambda' + \lambda'', \\ & \text{with } \lambda', \lambda'' \text{ nonzero and self-dual dominant weights.} \end{array} \tag{4.2}$$

Suppose $\lambda = \lambda_i + \lambda_i^*$ and note that $\prod(\lambda_i^*) = \{-\mu : \mu \in \prod(\lambda_i)\}$ is the set of weights for the module $V(\lambda_i)^* = V(\lambda_i^*)$. (As in [14], $\prod(\lambda)$ denotes the set of weights of $V(\lambda)$.) Since $f_\mu(v_{-\mu}) \neq 0$ in case $v_{-\mu} \in V(\lambda_i)$ and $f_\mu \in V(\lambda_i^*)$ are $(\pm\mu)$-weight vectors, the symmetric $\mathfrak{h}$-invariant form

$$\begin{aligned} &b : (V(\lambda_i) \otimes V(\lambda_i)^*) \otimes (V(\lambda_i) \otimes V(\lambda_i)^*) \to F \\ &(v_1 \otimes f_1) \otimes (v_2 \otimes f_2) \mapsto f_1(v_2)f_2(v_1), \end{aligned} \tag{4.3}$$

satisfies $b(v_{\lambda_i} \otimes f_{\lambda_i^*}, v_{-\lambda_i^*} \otimes f_{-\lambda_i}) \neq 0$. As the copy of $V(\lambda)$ generated by $v_{\lambda_i} \otimes f_{\lambda_i^*}$ that appears in $V(\lambda_i) \otimes V(\lambda_i)^*$ contains too the element $v_{-\lambda_i^*} \otimes f_{-\lambda_i}$, the symmetric form b induces a symmetric and nonzero $\mathfrak{h}$-invariant form on $V(\lambda)$, but this is not possible: because of the irreducibility of V, up to scalars there is exactly one $\mathfrak{h}$-invariant form on V, which must be skew symmetric.

Hence from (4.2) λ decomposes as $\lambda = \lambda' + \lambda''$. Then, the self-dual modules $V(\lambda')$ and $V(\lambda'')$ are endowed with nondegenerate and $\mathfrak{h}$-invariant forms $b_1(x', y')$ and $b_2(x'', y'')$. From b_1 and b_2, we can define on the tensor product $V(\lambda') \otimes V(\lambda'')$ the $\mathfrak{h}$-invariant form

$$\begin{aligned} &\hat{b} : V(\lambda') \otimes V(\lambda'') \otimes V(\lambda') \otimes V(\lambda'') \to F \\ &v_1' \otimes v_1'' \otimes v_2' \otimes v_2'' \mapsto b_1(v_1', v_2')b_2(v_1'', v_2''), \end{aligned} \tag{4.4}$$

which satisfies $\hat{b}(v_{\lambda'} \otimes v_{\lambda''}, v_{-\lambda'} \otimes v_{-\lambda''}) \neq 0$. Now, a copy of $V(\lambda)^{\otimes^2}$ appears in $(V(\lambda') \otimes V(\lambda''))^{\otimes^2}$ so $\hat{b}$ defines a nonzero and $\mathfrak{h}$-invariant form on $V(\lambda)^{\otimes^2}$ which must be skew symmetric. Consequently and without loss of generality, we can assume that b_1 is symmetric and b_2 is skew symmetric. Now let c' and c'' be the module homomorphisms given by

$$\begin{aligned} &c' : V(\lambda') \otimes V(\lambda'') \otimes V(\lambda') \otimes V(\lambda'') \to V(\lambda'') \otimes V(\lambda'') \\ &v_1' \otimes v_1'' \otimes v_2' \otimes v_2'' \mapsto b_1(v_1', v_2')v_1'' \otimes v_2'', \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} &c'' : V(\lambda') \otimes V(\lambda'') \otimes V(\lambda') \otimes V(\lambda'') \to V(\lambda') \otimes V(\lambda') \\ &v_1' \otimes v_1'' \otimes v_2' \otimes v_2'' \mapsto b_2(v_1'', v_2'')v_1' \otimes v_2'. \end{aligned} \tag{4.6}$$

Since b_1 is symmetric, we have

$$c'(v_{\lambda'} \otimes v_{\lambda''} \otimes v_{-\lambda'} \otimes v_{-\lambda''} + v_{-\lambda'} \otimes v_{-\lambda''} \otimes v_{\lambda'} \otimes v_{\lambda''}) = b_1(v_{\lambda'}, v_{-\lambda'})(v_{\lambda''} \otimes v_{-\lambda''} + v_{-\lambda''} \otimes v_{\lambda''}) \neq 0.$$

Hence, as the symmetric modules $S^2V(\lambda)$ and $S^2V(\lambda'')$ are generated by the vectors $v_{\lambda'} \otimes v_{\lambda''} \otimes v_{-\lambda'} \otimes v_{-\lambda''} + v_{-\lambda'} \otimes v_{-\lambda''} \otimes v_{\lambda'} \otimes v_{\lambda''}$ and $v_{\lambda''} \otimes v_{-\lambda''} + v_{-\lambda''} \otimes v_{\lambda''}$ respectively, we get a copy of the second symmetric power $S^2V(\lambda'')$ inside $S^2V(\lambda)$. Moreover, since $\dim V(\lambda) > \dim V(\lambda'')$, we have $S^2V(\lambda'') \neq S^2V(\lambda)$. In this way using (4.1), $S^2V(\lambda'')$ appears as a proper submodule inside $\mathfrak{sp}(V) = \mathfrak{h} \oplus \mathfrak{m}$. Similar arguments for c'' and the skew symmetric b_2 yield

$$c''(v_{\lambda'} \otimes v_{\lambda''} \otimes v_{-\lambda'} \otimes v_{-\lambda''} + v_{-\lambda'} \otimes v_{-\lambda''} \otimes v_{\lambda'} \otimes v_{\lambda''}) = b_2(v_{\lambda''}, v_{-\lambda''})(v_{\lambda'} \otimes v_{-\lambda'} - v_{-\lambda'} \otimes v_{\lambda'}) \neq 0,$$

and therefore the second alternating power $\bigwedge^2 V(\lambda')$ also appears properly in the module decomposition of $S^2V(\lambda) \cong \mathfrak{h} \oplus \mathfrak{m}$. Since $\mathfrak{h}$ is contained in both $\mathfrak{so}(V(\lambda'), b_1) \simeq \bigwedge^2 V(\lambda')$ and in $\mathfrak{sp}(V(\lambda''), b_2) \simeq S^2V(\lambda'')$, and $\mathfrak{m}$ is irreducible, we get

$$\mathfrak{so}(V(\lambda')) \cong \bigwedge^2 V(\lambda') \cong \mathfrak{h} \cong S^2V(\lambda'') \cong \mathfrak{sp}(V(\lambda'')). \tag{4.7}$$

Then Lemma 4.1 shows that either $\mathfrak{h}$ is a simple Lie algebra of type A_1, $\lambda' = 2\lambda_1$ and $\lambda'' = \lambda_1$; or $\mathfrak{h}$ is simple of type B_2 and therefore $\lambda' = \lambda_1$ and $\lambda'' = \lambda_2$. The latter possibility does not work from the dimensionality of the different modules involved: $\dim V = \dim V(\lambda_1 + \lambda_2) = 16$, $\dim \mathfrak{h} = \dim B_2 = 10$ and $\dim \mathfrak{m} = \dim V(2\lambda_1 + 2\lambda_2) = 81$, but $\dim \mathfrak{sp}(V) = 136 > \dim \mathfrak{h} + \dim \mathfrak{m} = 91$. Hence, in case λ is not fundamental, $\mathfrak{h}$ is of type A_1 with $\lambda = \lambda' + \lambda'' = 3\lambda_1$ and this completes the proof. □

Lemma 4.2 shows that the irreducible LY-algebras which appear inside reductive decompositions $\mathfrak{sp}(V) = \mathfrak{h} \oplus \mathfrak{m}$ satisfying (a), (b) and (c) in (1.4) are given by simple and maximal linear subalgebras $\mathfrak{h}$ with natural action on V given by a fundamental and self-dual dominant weight except for $\mathfrak{sp}_4(k) \cong \mathfrak{so}_5(k)$. This allows us to endow V with a structure of either a symplectic triple system (see [31] and [7, Definition 2.1] for a definition) or an anti-Lie triple system (see [13]), such that $\mathfrak{h}$ becomes its inner derivation algebra. In this way, the classification in the $\mathfrak{sp}$-case will follow from known results on these triple systems.

For an arbitrary reductive decomposition $\mathfrak{sp}(V, b) = \mathfrak{h} \oplus \mathfrak{m}$, the $\mathfrak{h}$-module isomorphism in (4.1) allows us to define the map

$$\begin{aligned} V \otimes V &\to \mathfrak{sp}(V, b) \to \mathfrak{h} \\ x \otimes y &\mapsto \gamma_{x,y} \mapsto d_{x,y}, \end{aligned} \tag{4.8}$$

where $d_{x,y}$ denotes the projection of $\gamma_{x,y} = b(x, .)y + b(y, .)x$ onto $\mathfrak{h}$, so the subalgebra $\mathfrak{h}$ appears as $\mathfrak{h} = \text{span}\langle d_{x,y} : x, y \in V\rangle$. Using these projections $d_{x,y}$, we define the triple product on V

$$xyz := d_{x,y}z, \tag{4.9}$$

which satisfies the following identities:

$$xyz = yxz, \tag{4.10}$$

$$xy(uvw) = (xyu)vw + u(xyv)w + uv(xyw), \tag{4.11}$$

$$b(xyu, v) + b(u, xyv) = 0, \tag{4.12}$$

for $x, y, z \in V$. Identity (4.10) is equivalent to the operator $d_{x,y}$ being symmetric as a function of x and y. Identity (4.11) states that (4.8) is an $\mathfrak{h}$-module homomorphism and (4.12) follows because $\mathfrak{h}$ is a subalgebra of $\mathfrak{sp}(V, b)$. Moreover, since $d_{x,y}z = xyz$, the subalgebra $\mathfrak{h}$ becomes the inner derivation algebra of the triple (V, xyz), so that

$$\mathfrak{h} = \text{span}\langle d_{x,y} : x, y \in V\rangle = \text{Inder}\, V. \tag{4.13}$$

Then, we have the following result, which is parallel to Lemma 3.2:

Lemma 4.3. *Given a reductive decomposition $\mathfrak{sp}(V, b) = \mathfrak{h} \oplus \mathfrak{m}$ satisfying (a), (b), (c) in (1.4), the vector space V endowed with the triple product $d_{x,y}z = xyz$ defined in (4.9) is either a simple anti-Lie triple system of classical type or a simple symplectic triple system with associated symmetric form ξb for some nonzero scalar ξ. Moreover, the subalgebra $\mathfrak{h}$ satisfies the equation*

$$\mathfrak{h} = \text{span}\langle d_{x,y} : x, y \in V\rangle, \tag{4.14}$$

and therefore coincides with the inner derivation algebra of the corresponding triple system, and the subspace $\mathfrak{m}$ is the orthogonal complement $\mathfrak{h}^\perp$ to $\mathfrak{h}$ relative to the Killing form of $\mathfrak{sp}(V, b)$.

Proof. Since the triple product (4.9) belongs to $\text{Hom}_{\mathfrak{h}}(S^2V \otimes V, V)$, we will describe the previous vector space in order to get the different possible products. Following Lemma 4.2, V is an irreducible and self-dual module for $\mathfrak{h}$. Also (4.1) gives $S^2V \cong \mathfrak{sp}(V) = \mathfrak{h} \oplus \mathfrak{m}$, so

$$\begin{aligned} \text{Hom}_{\mathfrak{h}}(S^2V \otimes V, V) &\cong \text{Hom}_{\mathfrak{h}}(S^2V, V \otimes V) \\ &\cong \text{Hom}_{\mathfrak{h}}(\mathfrak{h}, V \otimes V) \oplus \text{Hom}_{\mathfrak{h}}(\mathfrak{m}, V \otimes V). \end{aligned} \tag{4.15}$$

Then, using [10, Theorem 1] and the dimension equality

$$\dim \text{Hom}_{\mathfrak{h}}(\mathfrak{h}, V \otimes V) = \dim \text{Hom}_{\mathfrak{h}}(V \otimes V, \mathfrak{h}),$$

the first summand in (4.15) is one dimensional. The same is deduced from the Clebsch–Gordan formula for the second summand in case $\mathfrak{h}$ is of type A_1 and $V = V(3\lambda_1)$, as then we have $V(3\lambda_1) \otimes V(3\lambda_1) \cong V(6\lambda_1) \oplus V(4\lambda_1) \oplus V(2\lambda_1) \oplus V(0)$. Otherwise by Lemma 4.2, $V = V(\lambda_i)$ with λ_i fundamental and $\mathfrak{m} = V(2\lambda_i)$, so the result follows from (3.2). Hence, $\text{Hom}_{\mathfrak{h}}$ $(S^2V \otimes V, V)$ is always a two-dimensional vector space.

On the other hand, $S^2V \otimes V$ can be decomposed as the module sum:

$$S^2V \otimes V = S^3V \oplus \text{span}\langle (x \otimes y + y \otimes x) \otimes z - (z \otimes y + y \otimes z) \otimes x\rangle. \tag{4.16}$$

Write $S = \text{span}\langle (x \otimes y + y \otimes x) \otimes z - (z \otimes y + y \otimes z) \otimes x : x, y, z, \in V\rangle$, then we can consider the nonzero $\mathfrak{h}$-module homomorphism $\varphi : S \to V$ given by

$$\varphi((x \otimes y + y \otimes x) \otimes z - (z \otimes y + y \otimes z) \otimes x) = \gamma_{x,y}(z) - \gamma_{z,y}(x) = 2b(x, z)y + b(y, z)x - b(y, x)z,$$

where $\gamma_{x,y}$ is given in (4.1). We have the alternative decomposition

$$S^2V \otimes V = S^3V \oplus \text{Ker}\,\varphi \oplus V, \tag{4.17}$$

and therefore we can display $\text{Hom}_{\mathfrak{h}}(S^2V \otimes V, V)$ as

$$\text{Hom}_{\mathfrak{h}}(S^2V \otimes V, V) = \text{Hom}_{\mathfrak{h}}(S^3V, V) \oplus \text{Hom}_{\mathfrak{h}}(\text{Ker}\,\varphi, V) \oplus \text{Hom}_{\mathfrak{h}}(V, V). \tag{4.18}$$

Since $\mathrm{Hom}_{\mathfrak{h}}(S^2V \otimes V, V)$ is two dimensional and V is irreducible as an $\mathfrak{h}$-module, Eq. (4.18) shows that either $\mathrm{Hom}_{\mathfrak{h}}(S^3V, V)$ is a trivial vector space or $\mathrm{Hom}_{\mathfrak{h}}(S, V)$ is a one-dimensional vector space spanned by φ. In case $\mathrm{Hom}_{\mathfrak{h}}(S^3V, V) = 0$, the triple product xyz defined in (4.9) restricted to S^3V must be trivial. Then this product satisfies the additional identity

$$xyz + zxy + yzx = 0, \tag{4.19}$$

for all $x, y, z \in V$. Hence, using (4.10), (4.11) and (4.19) we have that (V, xyz) is an anti-Lie triple system with $\mathfrak{h}$ as inner derivation algebra. Moreover, the triple system is simple and of classical type by the $\mathfrak{h}$-irreducibility of V.

Otherwise, $\mathrm{Hom}_{\mathfrak{h}}(S, V) = k\varphi$, and the restriction of the triple product to S gives us the relationship

$$xyz - zyx = \xi(2b(x, z)y + b(y, z)x - b(y, x)z), \tag{4.20}$$

for some $\xi \in k$. Moreover ξ must be nonzero: otherwise, for all $x, y, z \in V$ we have $xyz = zyx$ and the triple products $\langle x_\sigma y_{-\sigma} z_\sigma\rangle = \sigma x_\sigma y_{-\sigma} z_\sigma$ defined on the vector space pair $\mathcal{U} = (V^+, V^-)$ with $V^\sigma = V$ and $\sigma = \pm$, satisfy

$$\langle x_\sigma y_{-\sigma} z_\sigma\rangle = \sigma x_\sigma y_{-\sigma} z_\sigma = \sigma z_\sigma y_{-\sigma} x_\sigma = \langle z_\sigma y_{-\sigma} x_\sigma\rangle,$$

and from (4.10) and (4.11)

$$\begin{aligned}\langle x_\sigma y_{-\sigma}\langle u_\sigma v_{-\sigma} w_\sigma\rangle\rangle &= \sigma^2 x_\sigma y_{-\sigma}(u_\sigma v_{-\sigma} w_\sigma)\\ &= \sigma^2((x_\sigma y_{-\sigma} u_\sigma)v_{-\sigma} w_\sigma) + u_\sigma(y_{-\sigma} x_\sigma v_{-\sigma})w_\sigma + u_\sigma v_{-\sigma}(x_\sigma y_{-\sigma} w_\sigma))\\ &= \langle\langle x_\sigma y_{-\sigma} u_\sigma\rangle v_{-\sigma} w_\sigma\rangle - \langle u_\sigma\langle y_{-\sigma} x_\sigma v_{-\sigma}\rangle w_\sigma\rangle + \langle u_\sigma v_{-\sigma}\langle x_\sigma y_{-\sigma} w_\sigma\rangle\rangle.\end{aligned}$$

Therefore $\mathcal{U}$ is a Jordan pair for which the inner derivation operators are of the form

$$\begin{aligned}(D_+(x_+, y_-), D_-(y_-, x_+)) &= (\langle x_+ y_- ., \rangle, -\langle y_- x_+ .\rangle)\\ &= (x_+ y_- ., y_- x_+ .).\end{aligned} \tag{4.21}$$

Now from (4.10) we have $d_{x,y}z = xyz = d_{y,x}z$, thus $(D_+(x_+, y_-), D_-(y_-, x_+)) = (D_{x_+,y_-}, D_{x_+,y_-})$, which shows that the Lie algebra Inder $\mathcal{U}$ is isomorphic to $\mathfrak{h}$. Since V is $\mathfrak{h}$-irreducible, $\mathcal{U}$ is a simple Jordan pair ([13, Proposition 1.2]). But from Table 5 we deduce that the inner derivation Lie algebras of the simple Jordan pairs are not simple. Hence $\xi \neq 0$. Now, from (4.20) we get $xyz - zyx = \xi(2b(x, z)y + b(y, z)x - b(y, x)z)$ for any x, y, z and using (4.10) we obtain

$$yxz - yzx = xyz - zyx = \xi b(y, z)x - \xi b(y, x)z + 2\xi b(x, z)y.$$

The previous identity, together with (4.10)–(4.12), shows that (V, xyz) is a symplectic triple system with associated skew symmetric bilinear form $\xi b(x, y)$. Moreover, as $b(x, y)$ is nondegenerate, V is a simple triple system ([7, Proposition 2.4]) with $\mathfrak{h}$ as its inner derivation algebra. □

Now we have all the ingredients in order to state the main result of this section:

Theorem 4.4. *Let $(\mathfrak{m}, a \cdot b, [a, b, c])$ be an irreducible LY-algebra of generic type and standard enveloping Lie algebra of type $\mathfrak{sp}$. Then there is a simple symplectic triple system $(T, [\ldots], b)$ of one of the following forms:*

(i) *$\mathcal{T}_k$, the symplectic triple system associated to the Jordan algebra $J = k$ with cubic form $n(\alpha) = \alpha^3$,*
(ii) *$\mathcal{T}_{\mathcal{H}_3(\mathcal{C})}$, the symplectic triple system associated to the Jordan algebra $J = \mathcal{H}_3(\mathcal{C})$, where $\mathcal{C}$ is either k, $k \times k$, $\mathrm{Mat}_2(k)$ or the algebra of octonions $\mathcal{O}$,*

such that, up to isomorphism, $\mathfrak{g}(\mathfrak{m}) = \mathfrak{sp}(T, b)$ and $\mathfrak{h} = \mathrm{Inder}\, T$. The LY-algebra $\mathfrak{m}$ appears as the orthogonal complement to $\mathfrak{h}$ in $\mathfrak{g}(\mathfrak{m})$ relative to the Killing form, with the binary and ternary products in (1.1).

Conversely, these LY-algebras are irreducible of generic type and standard enveloping algebra of type $\mathfrak{sp}$.

Proof. Lemmas 4.2 and 4.3 show that $\mathfrak{m} = \mathfrak{h}^\perp$, the orthogonal complement of the inner derivation algebra $\mathfrak{h}$ of a simple anti-Lie or symplectic triple system (V, xyz) with the following extra features:

(1) the inner derivation algebra Inder V is a simple Lie algebra,
(2) $V = V(m\lambda_i)$ is the Inder V-irreducible module with dominant weight m-times a fundamental weight λ_i and $\dim V \geq 4$ (actually, either $\dim V = 4$, $\mathfrak{h} = A_1$ and $\lambda = 3\lambda_1$ or λ is a fundamental dominant weight),
(3) S^2V decomposes as a sum of two irreducible modules.

Since isomorphic irreducible anti-Lie or symplectic triple systems provide isomorphic LY-algebras, we just have to check which triple systems satisfy these extra conditions. Following [13], the simple anti-Lie triple systems are the odd parts of the simple Lie superalgebras and therefore the classification of such triple systems is reduced to that of the simple Lie superalgebras in [16]. For simple symplectic triple systems we will follow the classification and comments given in [7, Section 2]. Both classifications are outlined in the Appendix of the paper. Table 2 shows that there are no simple anti-Lie triple systems satisfying (1)–(3) simultaneously. For simple symplectic triple systems, following Table 3 and applying the restrictions (1)–(2)–(3), we get that the only possibilities are given by the simple symplectic triple systems $\mathcal{T}_k$, associated to the one-dimensional Jordan algebra k in item (i), and the simple symplectic triple systems $\mathcal{T}_{\mathcal{J}}$ associated to a simple Jordan algebra $\mathcal{J} = \mathcal{H}_3(\mathcal{C})$ of degree 3 with $\mathcal{C} = k$, $k \times k$, the algebra of quaternions or the algebra of octonions, which proves the theorem. □

5. Exceptional case

In this section we deal with the irreducible LY-algebras of Generic Type and exceptional standard enveloping Lie algebra. These systems appear in reductive decompositions $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ for which (a), (b) and (c) in (1.4) hold, $\mathfrak{g}$ being a simple Lie algebra of type G_2, F_4, E_6, E_7, or E_8. The classification in this case is given in the next result:

Theorem 5.1. *Let* $(\mathfrak{m}, x \cdot y, [x, y, z])$ *be an irreducible LY-algebra of generic type and exceptional standard enveloping algebra. Then one of the following holds:*

(i) $\mathfrak{m}$ *is the Lie triple system associated to one of the symmetric pairs* (F_4, B_4), (E_6, F_4), (E_6, C_4), (E_7, A_7) *or* (E_8, D_8).
(ii) $\mathfrak{m} = \mathfrak{h}^{\perp}$ *is the orthogonal complement with respect to the Killing form in* $\mathfrak{g}$ *associated to one of the reductive pairs* $(\mathfrak{g}, \mathfrak{h}) = (G_2, A_1)$, (E_6, G_2) *or* (E_7, A_2). *Moreover, for the previous pairs, as a module for* $\mathfrak{h}$, $\mathfrak{m}$ *is isomorphic to* $V(10\lambda_1)$, $V(\lambda_1 \oplus \lambda_2)$ *and* $V(4\lambda_1 \oplus 4\lambda_2)$ *respectively.*

Conversely, the listed cases in (i) *and* (ii) *are indeed irreducible LY-algebras of generic type and exceptional standard enveloping algebras.*

Proof. Let $\mathfrak{m}$ be a LY-algebra of generic type and exceptional standard enveloping Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ as in (1.4). Lemma 4.2 in [3] shows that for each such reductive decomposition there exists an analogous decomposition $\tilde{\mathfrak{g}} = \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{m}}$ over the complex numbers. In particular, the highest weight of $\mathfrak{m}$ as a module for $\mathfrak{h}$ coincides with the highest weight of $\tilde{\mathfrak{m}}$ as a module for $\tilde{\mathfrak{h}}$. Then, because of [3, Lemma 4.3], either $\mathfrak{m}$ is a Lie triple system and we obtain (i), or $\mathfrak{h}$ is a simple S-subalgebra of $\mathfrak{g}$ and the different possibilities for the pair $(\mathfrak{g}, \mathfrak{h})$ can be read from [6, Theorem 14.1], where a complete list of the complex maximal and simple S-subalgebras of the exceptional Lie algebras is given: $(\mathfrak{g}, \mathfrak{h}) = (G_2, A_1)$, (F_4, A_1), (E_6, A_1), (E_7, A_1), (E_8, A_1), (E_6, G_2), (E_6, C_4), (E_6, F_4), or (E_7, A_2).

The cases (E_6, C_4) and (E_6, F_4) correspond to symmetric pairs already considered in (i). In case $\mathfrak{h} = A_1$, the irreducibility restriction on $\mathfrak{m}$ forces $\mathfrak{m} = V(n\lambda_1)$ for some $n \geq 1$. Now, given a Cartan subalgebra, spanned by an element h, of A_1 we can pick up a Cartan subalgebra H of $\mathfrak{g}$ with $h \in H$. Then, $[H, h] = 0$, so $H \subseteq C_{\mathfrak{g}}(h) = k \cdot h \oplus V(n\lambda_1)_0$, where $V(n\lambda_1)_0$ is the 0-weight subspace of $V(n\lambda_1)$. Since $\dim V(n\lambda_1)_0$ is 0 or 1, depending on the parity of n, we get that $\mathfrak{g}$ must be a Lie algebra of rank 2. Hence, (G_2, A_1) is the unique possibility that works. A dimension count shows that $n = 10$ in this case.

A dimension count for the other cases (E_6, G_2) and (E_7, A_2) completes the proof. □

Remark 5.2. The reductive pair (G_2, A_1) can be constructed by using transvections. A construction of G_2 from $A_1 = \mathfrak{sl}_2(k)$ and an 11-dimensional module is given in Dixmier [5] (see also [2]). The symmetric pairs (F_4, B_4) and (E_6, F_4) are strongly related to the Albert algebra (the exceptional simple Jordan algebra, see [15]). Nice constructions of the pairs (E_8, D_8), (E_7, A_7) and (E_6, C_4) can be read from constructions in [1].

6. Appendix

6.1. Lie and anti-Lie triple systems

Following [20, Theorem 1.1] Lie triple systems are nothing else but skew symmetric elements relative to involutive automorphisms in Lie algebras. That is, these systems can be viewed as the odd part of $\mathbb{Z}_2$-graded Lie algebras. Anti-Lie triple systems appear in the same vein by using the odd part of Lie superalgebras. Since Lie triple systems are LY-algebras with trivial binary product, because of Definition 1.1 of this paper and [13, Section 5], it is possible to introduce both triple systems in an axiomatically unified way by using a vector space V endowed with a triple product xyz satisfying the identities

$$\begin{aligned} &xyz = \epsilon\, yxz,\\ &xyz + yzx + zxy = 0,\\ &xy(uvw) = (xyu)vw + u(xyv)w + uv(xyw), \end{aligned} \tag{6.1}$$

where $\epsilon = -1$ for Lie triple systems and $\epsilon = 1$ for anti-Lie triple systems.

Given a Lie or anti-Lie triple system (V, xyz), the standard enveloping construction $\mathfrak{g}(V) = D(V, V) \oplus V$ in (1.2), where $D(V, V) = \operatorname{Inder} V = \operatorname{span}\langle d_{x,y} : x, y \in V\rangle$, $d_{x,y}z = xyz$, is the inner derivation Lie algebra of the corresponding triple system, provides either a $\mathbb{Z}_2$-graded Lie algebra or a Lie superalgebra according to V being a Lie or and anti-Lie triple system. Moreover, it is not difficult to prove that the Lie algebra (respectively superalgebra) $\mathfrak{g}(V)$ is graded simple (respectively simple) if and only if the Lie (respectively anti-Lie) triple system V is simple.

Over algebraically closed fields of characteristic zero, simple Lie triple systems were classified in [20] through involutive automorphisms. Table I in [10] presents an alternative classification of these systems by means of (reductive) symmetric pairs $(\mathfrak{g}(V), \operatorname{Inder} V)$ obtained from affine Dynkin diagrams (see [17, Chapter 4]) that encode the Cartan type of the standard enveloping Lie algebra $\mathfrak{g}(V)$ and the inner derivation Lie algebra $\operatorname{Inder} V$ of the Lie triple system V. These diagrams are equipped with some numerical labels which describe the lowest weight of V as a module for $\operatorname{Inder} V$ (the highest weight also is easily checked since simple Lie triple systems are self-dual modules). Using the latter classification, in [11, Table III] all simple and $\operatorname{Inder} V$-irreducible Lie triple systems are listed. Combining the results in [10] and [11] we arrive at Table 1, that displays the irreducible Lie triple systems whose inner derivation algebra is simple.

On the other hand, as simple anti-Lie triple systems are the odd part of simple Lie superalgebras, the classification of these systems can be obtained from that of the simple Lie superalgebras in [16]. Of special interest for our purposes are the

Table 1
Irreducible L.t.s. with simple inner derivation Lie algebra.

$(\mathfrak{g}(V), \operatorname{Inder} V)$	Inder V	$V = V(k\lambda_i)$	$V(2k\lambda_i - \alpha_i)$	$\mathfrak{so}(V) \cong \bigwedge^2 V$
$(A_1 \times A_1, A_1)$	$V(2\lambda_1)$	$V(2\lambda_1)$	$V(2\lambda_1)$	$V(2\lambda_1)$
$(A_n \times A_n, A_n)_{n\geq 2}$	$V(\lambda_1+\lambda_n)$	$V(\lambda_1+\lambda_n)$		
$(B_3 \times B_3, B_3)$	$V(\lambda_2)$	$V(\lambda_2)$	$V(\lambda_1 + 2\lambda_3)$	$V(\lambda_2) \oplus V(\lambda_1 + 2\lambda_3)$
$(B_n \times B_n, B_n)_{n\geq 4}$	$V(\lambda_2)$	$V(\lambda_2)$	$V(\lambda_1 + \lambda_3)$	$V(\lambda_2) \oplus V(\lambda_1 + 2\lambda_3)$
$(C_n \times C_n, C_n)_{n\geq 2}$	$V(2\lambda_1)$	$V(2\lambda_1)$	$V(2\lambda_1 + \lambda_2)$	$V(2\lambda_1) \oplus V(2\lambda_1 + \lambda_2)$
$(D_4 \times D_4, D_4)$	$V(\lambda_2)$	$V(\lambda_2)$	$V(\lambda_1+\lambda_3+\lambda_4)$	$V(\lambda_2) \oplus V(\lambda_1+\lambda_3+\lambda_4)$
$(D_n \times D_n, D_n)_{n\geq 5}$	$V(\lambda_2)$	$V(\lambda_2)$	$V(\lambda_1 + \lambda_3)$	$V(\lambda_2) \oplus V(\lambda_1 + 2\lambda_3)$
$(G_2 \times G_2, G_2)$	$V(\lambda_2)$	$V(\lambda_2)$	$V(3\lambda_1)$	$V(\lambda_2) \oplus V(3\lambda_1)$
$(F_4 \times F_4, F_4)$	$V(\lambda_1)$	$V(\lambda_1)$	$V(\lambda_2)$	$V(\lambda_1) \oplus V(\lambda_2)$
$(E_6 \times E_6, E_6)$	$V(\lambda_2)$	$V(\lambda_2)$	$V(\lambda_4)$	$V(\lambda_2) \oplus V(\lambda_4)$
$(E_7 \times E_7, E_7)$	$V(\lambda_1)$	$V(\lambda_1)$	$V(\lambda_3)$	$V(\lambda_1) \oplus V(\lambda_3)$
$(E_8 \times E_8, E_8)$	$V(\lambda_8)$	$V(\lambda_8)$	$V(\lambda_7)$	$V(\lambda_8) \oplus V(\lambda_7)$
(D_3, B_2)	$V(2\lambda_2)$	$V(\lambda_1)$	$V(2\lambda_2)$	$V(2\lambda_2)$
(B_3, D_3)	$V(\lambda_2+\lambda_3)$	$V(\lambda_1)$	$V(\lambda_2 + \lambda_3)$	$V(\lambda_2 + \lambda_3)$
$(B_n, D_n)_{n\geq 4}$	$V(\lambda_2)$	$V(\lambda_1)$	$V(\lambda_2)$	$V(\lambda_2)$
$(D_{n+1}, B_n)_{n\geq 3}$	$V(\lambda_2)$	$V(\lambda_1)$	$V(\lambda_2)$	$V(\lambda_2)$
(A_2, A_1)	$V(2\lambda_1)$	$V(4\lambda_1)$	$V(6\lambda_1)$	$V(2\lambda_1) \oplus V(6\lambda_1)$
(A_4, B_2)	$V(2\lambda_2)$	$V(2\lambda_1)$	$V(2\lambda_1 + 2\lambda_2)$	$V(2\lambda_2) \oplus V(2\lambda_1 + 2\lambda_2)$
$(A_{2n}, B_n)_{n\geq 3}$	$V(\lambda_2)$	$V(2\lambda_1)$	$V(2\lambda_1 + \lambda_2)$	$V(\lambda_2) \oplus V(2\lambda_1 + \lambda_2)$
$(A_{2n-1}, C_n)_{n\geq 3}$	$V(2\lambda_1)$	$V(\lambda_2)$	$V(\lambda_1 + \lambda_3)$	$V(2\lambda_1) \oplus V(\lambda_1 + \lambda_3)$
$(A_5, D_3)_{n\geq 4}$	$V(\lambda_2+\lambda_3)$	$V(2\lambda_1)$	$V(2\lambda_1+\lambda_2+\lambda_3)$	$V(\lambda_2+\lambda_3) \oplus V(2\lambda_1+\lambda_2+\lambda_3)$
$(A_{2n-1}, D_n)_{n\geq 4}$	$V(\lambda_2)$	$V(2\lambda_1)$	$V(2\lambda_1 + \lambda_2)$	$V(\lambda_2) \oplus V(2\lambda_1 + \lambda_2)$
(E_6, F_4)	$V(\lambda_1)$	$V(\lambda_4)$	$V(\lambda_3)$	$V(\lambda_1) \oplus V(\lambda_3)$
(F_4, B_4)	$V(\lambda_2)$	$V(\lambda_4)$	$V(\lambda_3)$	$V(\lambda_2) \oplus V(\lambda_3)$
(E_6, C_4)	$V(2\lambda_1)$	$V(\lambda_4)$	$V(2\lambda_3)$	$V(2\lambda_1) \oplus V(2\lambda_3)$
(E_7, A_7)	$V(\lambda_1+\lambda_7)$	$V(\lambda_4)$	$V(\lambda_3 + \lambda_5)$	$V(\lambda_1 + \lambda_7) \oplus V(\lambda_3 + \lambda_5)$
(E_8, D_8)	$V(\lambda_2)$	$V(\lambda_8)$	$V(\lambda_6)$	$V(\lambda_2) \oplus V(\lambda_6)$

simple anti-Lie triple systems V which are completely reducible as modules for Inder V. This class of systems appears from the odd parts of the simple classical Lie superalgebras listed in Table 2. We shall refer to them as anti-Lie triple systems of *classical type*. Structural module information on Table 2 follows from [16, Proposition 2.1.2].

6.2. Symplectic and orthogonal triple systems

Symplectic triple systems were introduced in [31] and orthogonal triple systems were defined in [26, Section V]. They are basic ingredients in the construction of some 5-graded Lie algebras and Lie superalgebras respectively (see [7]), and hence they are strongly related to $\mathbb{Z}_2$-graded Lie algebras and to a specific class of Lie superalgebras. These triple systems consist of a vector space V endowed with a trilinear product xyz and a ϵ-bilinear form b, with $\epsilon = -1$ (skew symmetric) for symplectic triple systems and $\epsilon = 1$ (symmetric) for orthogonal ones, satisfying the relations

$$\begin{aligned} &xyz = -\epsilon\, yxz, \\ &xyz + \epsilon xzy = \epsilon\, b(x, y)z + b(x, z)y - \epsilon\, 2b(y, z)x, \\ &xy(uvw) = (xyu)vw + u(xyv)w + uv(xyw). \end{aligned} \tag{6.2}$$

We note that the second relation in the orthogonal case ($\epsilon = 1$) is just the linearization of the identity

$$xyy = b(x, y)y - b(y, y)x, \tag{6.3}$$

and, from the third relation, we can introduce for these systems in the usual way the inner derivation Lie algebra Inder $V = \operatorname{span}\langle d_{x,y} : x, y \in V\rangle$.

Symplectic and orthogonal triple systems are related to the so called $(-\epsilon, -\epsilon)$ balanced Freudenthal–Kantor triple systems introduced in [32]. In [7, Theorems 2.16 and 2.18] it is also shown that symplectic triple systems are closely related to Freudenthal triple systems and a class of ternary algebras defined in [12]: the balanced symplectic Lie algebras. Moreover, following [7, Theorems 2.4 and 4.4], the simplicity of both types of triple systems (fields of characteristic different from 2 and 3) is equivalent to the nondegeneracy of the associated bilinear form b.

The relationship between symplectic triple systems, Freudenthal triple systems and ternary algebras leads to the classification of simple symplectic triple systems over algebraically closed fields of characteristic different from 2 and 3 given in [7, Theorem 2.21] (the classification is based on previous classifications of Freudenthal triple systems in [23] and simple ternary algebras from [12, Theorem 4.1]). For simple orthogonal triple systems, Theorem 4.7 in [7] displays the classification over algebraically closed fields of characteristic zero, by means of a previous classification of the simple $(-1, -1)$ balanced Freudenthal–Kantor triple systems in [8, Theorem 4.3]. The classifications and comments therein [7] provide Tables 3 and 4, where the Cartan type of the Lie algebra Inder V and the Inder V-module structure of V is given for the different types of simple symplectic and orthogonal triple systems.

Table 2
Simple classical Lie superalgebras.

$\mathcal{L}$-Type	$\mathcal{L}_{\bar{0}}$	$\mathcal{L}_{\bar{1}}$ as $\mathcal{L}_{\bar{0}}$-module
$A(m,0)_{m\geq 1}$	$A_m \times Z$[a]	$V(\lambda_1)\oplus V(\lambda_m)$
$A(m,n)_{m>n\geq 1}$	$A_m\times A_n\times Z$	$V(\lambda_1)\otimes V(\lambda_1')\oplus V(\lambda_m)\otimes V(\lambda_n')$
$A(n,n)_{n\geq 1}$	$A_n\times A_n$	$V(\lambda_1)\otimes V(\lambda_1')\oplus V(\lambda_n)\otimes V(\lambda_n')$
$B(0,n)_{n\geq 1}$	C_n	$V(\lambda_1)$
$B(m,n)_{m,n\geq 1}$	$B_m\times C_n$	$V(\lambda_1)\otimes V(\lambda_1')$
$D(2,n)_{n\geq 2}$	$A_1\times A_1\times C_n$	$V(\lambda_1)\otimes V(\lambda_1')\otimes V(\lambda_1'')$
$D(m,n)_{m\geq 3,n\geq 1}$	$D_m\times C_n$	$V(\lambda_1)\otimes V(\lambda_1')$
$C(n)_{n\geq 2}$	$C_{n-1}\times Z$	$V(\lambda_1)\oplus V(\lambda_1)$
$Q(n)_{n\geq 2}$	A_n	$V(\lambda_1+\lambda_n)$
$P(n)_{n\geq 2}$	A_n	$V(\lambda_{n-1})\oplus V(2\lambda_1)$
$D(2,1;\alpha)_{\alpha\neq 0,-1}$	$A_1\times A_1\times A_1$	$V(\lambda_1)\otimes V(\lambda_1')\otimes V(\lambda_1'')$
$F(4)$	$B_3\times A_1$	$V(\lambda_3)\otimes V(\lambda_1')$
$G(3)$	$G_2\times A_1$	$V(\lambda_1)\otimes V(\lambda_1')$

[a] Z stands for a one-dimensional center of $\mathcal{L}_{\bar{0}}$. In case $\mathcal{L}_{\bar{0}}=[\mathcal{L}_{\bar{0}},\mathcal{L}_{\bar{0}}]\times Z$, the highest weight of $\mathcal{L}_{\bar{1}}$ as a module for $[\mathcal{L}_{\bar{0}},\mathcal{L}_{\bar{0}}]$ is considered. For these cases, $\mathcal{L}_{\bar{1}}$ decomposes as sum of two irreducible modules. The elements of the center act as $\alpha\cdot\mathrm{Id}$ in one of the two summands, α being a nonzero scalar, and as $-\alpha\cdot\mathrm{Id}$ on the other summand. The same remark applies to the remaining tables.

Table 3
Simple symplectic triple systems.

V-Type	$\dim V$	Inder V	V as Inder V-module	$\mathfrak{sp}(V)\cong S^2V$
Orthogonal type	8	$A_1\times A_1\times A_1$	$V(\lambda_1)\otimes V(\lambda_1')\otimes V(\lambda_1'')$	
	$4n,\ n\geq 3$	$A_1\times D_n$	$V(\lambda_1)\otimes V(\lambda_1')$	
	6	$A_1\times A_1$	$V(\lambda_1)\otimes V(2\lambda_1')$	
	$4n+2,\ n\geq 2$	$A_1\times B_n$	$V(\lambda_1)\otimes V(\lambda_1')$	
Special type	2	Z	$k\times k$	
	$2n,\ n\geq 2$	$A_{n-1}\times Z$	$V(\lambda_1)\otimes V(\lambda_{n-1})$	
Symplectic type	2	A_1	$V(\lambda_1)$	$V(2\lambda_1)$
	$2n,\ n\geq 2$	C_n	$V(\lambda_1)$	$V(2\lambda_1)$
$\mathcal{T}_k$	4	A_1	$V(3\lambda_1)$	$V(6\lambda_1)\oplus V(2\lambda_1)$
$\mathcal{T}_{\mathcal{H}_3(k)}$	14	C_3	$V(\lambda_3)$	$V(2\lambda_1)\oplus V(2\lambda_3)$
$\mathcal{T}_{\mathcal{H}_3(k\times k)}$	20	A_5	$V(\lambda_3)$	$V(\lambda_1+\lambda_5)\oplus V(2\lambda_3)$
$\mathcal{T}_{\mathcal{H}_3(\mathcal{Q})}$	32	D_6	$V(\lambda_6)$	$V(\lambda_2)\oplus V(2\lambda_6)$
$\mathcal{T}_{\mathcal{H}_3(\mathcal{O})}$	56	E_7	$V(\lambda_7)$	$V(\lambda_1)\oplus V(2\lambda_7)$

Table 4
Simple orthogonal triple systems.

V-Type	$\dim V$	Inder V	V as Inder V-module	$\mathfrak{so}(V)\cong\bigwedge^2 V$
Orthogonal type	3, 5	A_1, B_2	$V(2\lambda_1), V(\lambda_1)$	$V(2\lambda_1), V(2\lambda_2)$
	$2n+1,\ n\geq 3$	B_n	$V(\lambda_1)$	$V(\lambda_2)$
	2, 4	$Z, A_1\times A_1$	$k\times k, V(\lambda_1)\otimes V(\lambda_1')$	
	6	D_3	$V(\lambda_1)$	$V(\lambda_2+\lambda_3)$
	$2n,\ n\geq 4$	D_n	$V(\lambda_1)$	$V(\lambda_2)$
Unitarian type	$2n,\ n\geq 3$	$A_{n-1}\times Z$	$V(\lambda_1)\oplus V(\lambda_{n-1})$	
Symplectic type	$4n,\ n\geq 2$	$A_1\times C_n$	$V(\lambda_1)\otimes V(\lambda_1')$	
D_μ-type	4	$A_1\times A_1$	$V(\lambda_1)\oplus V(\lambda_1')$	
G-type	7	G_2	$V(\lambda_1)$	$V(\lambda_1)\oplus V(\lambda_2)$
F-type	8	B_3	$V(\lambda_3)$	$V(\lambda_1)\oplus V(\lambda_2)$

Among the simple symplectic triple systems a special use of the following ones will be made:

$$\mathcal{T}_{\mathcal{J}}=\left\{\begin{pmatrix}\alpha & a\\ b & \beta\end{pmatrix} : \alpha,\beta\in k, a,b\in\mathcal{J}\right\}, \tag{6.4}$$

where $\mathcal{J}=\mathcal{J}ordan(n,c)$ is the Jordan algebra of a nondegenerate cubic form n with basepoint (see [22, II.4.3] for a definition) of one of the following types: $\mathcal{J}=k$, $n(\alpha)=\alpha^3$ and $t(\alpha,\beta)=3\alpha\beta$ or $\mathcal{J}=H_3(\mathcal{C})$ for a unital composition algebra $\mathcal{C}$. Theorem 2.21 in [7] displays the product and bilinear form for the triple systems $\mathcal{T}_{\mathcal{J}}$ by using the trace form $t(a,b)$ and the cross product $a\times b$ attached to the Jordan algebra $\mathcal{J}$.

Table 5
Simple Jordan pairs.

$\mathcal{U}$-Type	$(\mathcal{U}^+, \mathcal{U}^-)$-description	Inder $\mathcal{U}$	$\mathcal{U}^+$	$\mathcal{U}^-$
$\mathrm{I}_{p,q}$ [a] $p\geq q\geq 1$	$\mathcal{U}^+ = \mathcal{M}_{p,q}(k)$ $\mathcal{U}^- = \mathcal{M}_{p,q}(k)$ $\{xyz\} = xy^tz + zy^tx$	$A_{p-1}\times A_{q-1}\times Z$	$V(\lambda_1)\otimes V(\lambda'_1)$	$V(\lambda_{p-1})\otimes V(\lambda'_{q-1})$
II_n $n\geq 5$	$\mathcal{U}^+ = \mathcal{U}^- = \mathcal{A}_n(k)$ $\{xyz\} = xy^tz + zy^tx$	$A_{n-1}\times Z$	$V(\lambda_2)$	$V(\lambda_{n-2})$
III_n $n\geq 2$	$\mathcal{U}^+ = \mathcal{U}^- = \mathcal{H}_n(k)$ $\{xyz\} = xy^tz + zy^tx$	$A_{n-1}\times Z$	$V(2\lambda_1)$	$V(2\lambda_{n-1})$
IV_{2n} $n\geq 3$	$\mathcal{U}^+ = \mathcal{U}^- = k^{2n}$ $\{xyz\}=b(x,y)z+b_{x,y}(z)$ $b(x,y)=b(y,x)$	$D_n\times Z$	$V(\lambda_1)$	$V(\lambda_1)$
IV_{2n+1} $n\geq 2$	$\mathcal{U}^+ = \mathcal{U}^- = k^{2n+1}$ $\{xyz\}=b(x,y)z+b_{x,y}(z)$ $b(x,y)=b(y,x)$	$B_n\times Z$	$V(\lambda_1)$	$V(\lambda_1)$
V	$\mathcal{U}^+ = \mathcal{U}^- = \mathcal{M}_{1,2}(\mathcal{O})$ $\{xyz\} = x(\bar{y}^tz) + z(\bar{y}^tx)$	$D_5\times Z$	$V(\lambda_4)$	$V(\lambda_5)$
VI	$\mathcal{U}^+ = \mathcal{U}^- = \mathcal{H}_3(\mathcal{O})$ $\{xyz\}=x(zy)+z(xy)-(zx)y$	$E_6\times Z$	$V(\lambda_1)$	$V(\lambda_6)$

[a] The isomorphism $\mathrm{I}_{2,2}\cong \mathrm{IV}_4$ has been omitted in the classification given in [21].

Table 6
Simple anti-Jordan pairs.

$\mathcal{U}$-Type	$(\mathcal{U}^+, \mathcal{U}^-)$-description	Inder $\mathcal{U}$	$\mathcal{U}^+$	$\mathcal{U}^-$
$\mathrm{GL}_{p,q}$ $p\geq q\geq 1$ $pq>1$	$\mathcal{U}^+ = \mathcal{M}_{p,q}(k)$ $\mathcal{U}^- = \mathcal{M}_{p,q}(k)$ $\{xyz\} = xy^tz - zy^tx$	$A_{p-1}\times A_{q-1}\times Z$ $p\neq q$ $A_{p-1}\times A_{q-1}$ $p=q$	$V(\lambda_1)\otimes V(\lambda'_1)$	$V(\lambda_{p-1})\otimes V(\lambda'_{q-1})$
$\mathrm{Sps}(2n)$ $n\geq 1$	$\mathcal{U}^+ = \mathcal{U}^- = k^{2n}$ $\{xyz\} = b(x,y)z+b_{x,y}(z)$ $b(x,y)=-b(y,x)$	$C_n\times Z$	$V(\lambda_1)$	$V(\lambda_1)$
$\mathrm{Sym}(n)$ [a] $n\geq 3$	$\mathcal{U}^+ = \mathcal{H}_n(k)$ $\mathcal{U}^- = \mathcal{A}_n(k)$ $\{xyz\} = xyz - zyx$	A_{n-1}	$V(2\lambda_1)$	$V(\lambda_{n-2})$

[a] In [12, Proposition 2.8] the anti-Jordan pair Sym(2) is erroneously included as simple: $\{\mathcal{A}_2(k)\mathcal{H}_2(k)\mathcal{A}_2(k)\} = 0$, so $(\mathcal{H}_2(k), 0)$ is a proper ideal.

Table 7
Irreducible LY-algebras of Adjoint Type.

$\mathfrak{g}(\mathfrak{m})$	$\mathfrak{h}$	$\mathfrak{m}$-description
$k[t]/(t^2-1)\otimes\mathcal{L}$	$\mathrm{Der}\,\mathcal{L}\cong\mathcal{L}$	$\mathcal{L}$ [a] $a\cdot b = 0$ $[a,b,c] = [[a,b],c]$
$k[t]/(t^2-t-\beta)\otimes\mathcal{L}$ $\beta\neq -1/4$	$\mathrm{Der}\,\mathcal{L}\cong\mathcal{L}$	$\mathcal{L}$ $a\cdot b = [a,b]$ $[a,b,c] = \beta[[a,b],c]$
$k[t]/(t^2)\otimes\mathcal{L}$	$\mathrm{Der}\,\mathcal{L}\cong\mathcal{L}$	$\mathcal{L}$ $a\cdot b = [a,b]$ $[a,b,c] = -1/4[[a,b],c]$

[a] $\mathcal{L}$ stands for a simple Lie algebra with product $[a,b]$, so $\mathcal{L}$ is either a classical linear algebra $\mathfrak{sl}_n(k)$, $n\geq 1$ (Cartan type A_{n-1}), $\mathfrak{so}_n(k)$, $n\geq 5$ (Cartan type B_k or D_k according to $n=2k+1$ or $n=2k$), $\mathfrak{sp}_{2n}$, $n\geq 3$ (Cartan type C_n) or an exceptional algebra of type G_2, F_4, E_6, E_7, E_8.

6.3. Jordan and anti-Jordan pairs

Jordan pairs, axiomatically introduced (for arbitrary fields and dimension) in [21] are basic ingredients in the construction of Lie algebras with 3-gradings. In the context of Lie superalgebras endowed with a consistent 3-grading, the anti-Jordan pairs introduced in [13] constitute the corresponding concept. Following [13] (see also [24, Chapter XI]), over fields of characteristic different from 2 and 3 both types of pairs can be defined by means of a pair of vector spaces $\mathcal{U} = (\mathcal{U}^+, \mathcal{U}^-)$ with trilinear products $\{x_\sigma y_{-\sigma} z_\sigma\}$ for $\sigma = \pm$ satisfying the identities:

$$\begin{aligned}&\{x_\sigma y_{-\sigma} z_\sigma\} = \epsilon\{z_\sigma y_{-\sigma} x_\sigma\}\\ &\{x_\sigma y_{-\sigma}\{u_\sigma v_{-\sigma} w_\sigma\}\} = \{\{x_\sigma y_{-\sigma} u_\sigma\} v_{-\sigma} w_\sigma\} - \epsilon\{u_\sigma\{y_{-\sigma} x_\sigma v_{-\sigma}\} w_\sigma\} + \{u_\sigma v_{-\sigma}\{x_\sigma y_{-\sigma} w_\sigma\}\},\end{aligned} \tag{6.5}$$

with $\epsilon = 1$ for Jordan pairs and $\epsilon = -1$ for anti-Jordan pairs.

Table 8
Irreducible LY-algebras of Non-simple Type.

$\mathfrak{g}(\mathfrak{m})$	$\mathfrak{h}$	$\mathfrak{m}$-description
$\mathfrak{sl}_{pq}(k)$ $2 \le p \le q$ $(p,q) \neq (2,2)$	$\mathfrak{sl}_p(k) \oplus \mathfrak{sl}_q(k)$	$\mathfrak{sl}_p(k) \otimes \mathfrak{sl}_q(k)$ $(a \otimes f) \cdot (b \otimes g) = \frac{1}{2}[a,b] \otimes (fg + gf - \frac{2}{q}\mathrm{tr}(fg)I_q) +$ $\frac{1}{2}(ab + ba - \frac{2}{p}\mathrm{tr}(ab)I_p) \otimes [f,g]$ $[a \otimes f, b \otimes g, c \otimes h] = \frac{1}{q}[[a,b],c] \otimes \mathrm{tr}(fg)h +$ $\frac{1}{p}\mathrm{tr}(ab)c \otimes [[f,g],h]$
$\mathfrak{so}_{p+q}(k)$ $3 \le p \le q$	$\mathfrak{so}_p(k) \oplus \mathfrak{so}_q(k)$	$k^p \otimes k^q$ $(u \otimes x) \cdot (v \otimes y) = 0$ $[u \otimes x, v \otimes y, w \otimes z] = \varphi(x,y)(\psi_{u,v}(w) \otimes z) +$ $\psi(u,v)(w \otimes \varphi_{x,y}(z))$ $b = \varphi, \psi : b(x,y) = b(y,x)$ and $b_{x,y}(z) = b(x,z)y - b(y,z)x$
$\mathfrak{so}_{4q}(k)$ $3 \le q$	$\mathfrak{sp}_2(k) \oplus \mathfrak{sp}_q(k)$	$\mathfrak{sp}_2(k) \otimes \mathcal{H}_q(\mathcal{Q})_0$ $(a \otimes f) \cdot (b \otimes g) = \frac{1}{2}[a,b] \otimes (fg + gf - \frac{2}{q}\mathrm{tr}(fg)I_q)$ $[a \otimes f, b \otimes g, c \otimes h] = \frac{1}{q}[[a,b],c] \otimes \mathrm{tr}(fg)h +$ $\frac{1}{2}\mathrm{tr}(ab)c \otimes [[f,g],h]$
$\mathfrak{sp}_{p+q}(k)$ $2 \le p \le q$	$\mathfrak{sp}_p(k) \oplus \mathfrak{sp}_q(k)$	$k^p \otimes k^q$ $(u \otimes x) \cdot (v \otimes y) = 0$ $[u \otimes x, v \otimes y, w \otimes z] = \varphi(x,y)(\psi_{u,v}(w) \otimes z) +$ $\psi(u,v)(w \otimes \varphi_{x,y}(z))$ $b = \varphi, \psi : b(x,y) = -b(y,x)$ and $b_{x,y}(z) = b(x,z)y + b(y,z)x$
$\mathfrak{sp}_{2q}(k)$ $3 \le q$	$\mathfrak{sp}_2(k) \oplus \mathfrak{so}_q(k)$	$\mathfrak{sp}_2(k) \otimes \mathcal{H}_q(k)_0$ $(a \otimes f) \cdot (b \otimes g) = \frac{1}{2}[a,b] \otimes (fg + gf - \frac{2}{q}\mathrm{tr}(fg)I_q)$ $[a \otimes f, b \otimes g, c \otimes h] = \frac{1}{q}[[a,b],c] \otimes \mathrm{tr}(fg)h +$ $\frac{1}{2}\mathrm{tr}(ab)c \otimes [[f,g],h]$
G_2	$\mathfrak{sp}_2(k) \oplus \mathfrak{sl}_2(k)$	$k^2 \otimes \mathcal{T}_k$ $(u \otimes x) \cdot (v \otimes y) = 0$ $[u \otimes x, v \otimes y, w \otimes z] = \varphi(x,y)(\psi_{u,v}(w) \otimes z) + \psi(u,v)w \otimes xyz$ $b = \varphi, \psi : b(x,y) = -b(y,x)$ and $b_{x,y}(z) = b(x,z)y + b(y,z)x$
F_4	$\mathfrak{sp}_2(k) \oplus \mathfrak{sp}_6(k)$	$k^2 \otimes \mathcal{T}_{\mathcal{H}_3(k)}$ $(u \otimes x) \cdot (v \otimes y) = 0$ $[u \otimes x, v \otimes y, w \otimes z] = \varphi(x,y)(\psi_{u,v}(w) \otimes z) + \psi(u,v)w \otimes xyz$ $b = \varphi, \psi : b(x,y) = -b(y,x)$ and $b_{x,y}(z) = b(x,z)y + b(y,z)x$
F_4	$G_2 \oplus \mathfrak{sl}_2(k)$	$\mathcal{O}_0 \otimes \mathcal{H}_3(k)_0$ $(a \otimes x) \cdot (b \otimes y) = \frac{1}{2}[a,b] \otimes (x \bullet y - t(x \bullet y)1)$ $[a \otimes x, b \otimes y, c \otimes z] = D_{a,b}(c) \otimes t(x \bullet y)z + t(ab)c \otimes d_{x,y}(z)$ $x \bullet y = \frac{1}{2}(xy + yx)$ and $d_{x,y}(z) = x \bullet (y \bullet z) - y \bullet (x \bullet z)$ $D_{a,b}(c) = \frac{1}{4}([[a,b],c] + 3((ac)b - a(cb))$ $t(ab)$ and $t(x \bullet y)$ the normalized traces
E_6	$\mathfrak{sp}_2(k) \oplus \mathfrak{sl}_6(k)$	$k^2 \otimes \mathcal{T}_{\mathcal{H}_3(\mathcal{K})}$ $(u \otimes x) \cdot (v \otimes y) = 0$ $[u \otimes x, v \otimes y, w \otimes z] = \varphi(x,y)(\psi_{u,v}(w) \otimes z) + \psi(u,v)w \otimes xyz$ $b = \varphi, \psi : b(x,y) = -b(y,x)$ and $b_{x,y}(z) = b(x,z)y + b(y,z)x$
E_6	$G_2 \oplus \mathfrak{sl}_3(k)$	$\mathcal{O}_0 \otimes \mathcal{H}_3(k \times k)_0$ $(a \otimes x) \cdot (b \otimes y) = \frac{1}{2}[a,b] \otimes (x \bullet y - t(x \bullet y)1)$ $[a \otimes x, b \otimes y, c \otimes z] = D_{a,b}(c) \otimes t(x \bullet y)z + t(ab)c \otimes d_{x,y}(z)$ $x \bullet y = \frac{1}{2}(xy + yx)$ and $d_{x,y}(z) = x \bullet (y \bullet z) - y \bullet (x \bullet z)$ $D_{a,b}(c) = \frac{1}{4}([[a,b],c] + 3((ac)b - a(cb))$ $t(ab)$ and $t(x \bullet y)$ the normalized traces
E_7	$\mathfrak{sp}_2(k) \oplus \mathfrak{so}_{12}(k)$	$k^2 \otimes \mathcal{T}_{\mathcal{H}_3(\mathcal{Q})}$ $(u \otimes x) \cdot (v \otimes y) = 0$ $[u \otimes x, v \otimes y, w \otimes z] = \varphi(x,y)(\psi_{u,v}(w) \otimes z) + \psi(u,v)w \otimes xyz$ $b = \varphi, \psi : b(x,y) = -b(y,x)$ and $b_{x,y}(z) = b(x,z)y + b(y,z)x$
E_7	$G_2 \oplus \mathfrak{sp}_6(k)$	$\mathcal{O}_0 \otimes \mathcal{H}_3(\mathcal{Q})_0$ $(a \otimes x) \cdot (b \otimes y) = \frac{1}{2}[a,b] \otimes (x \bullet y - t(x \bullet y)1)$ $[a \otimes x, b \otimes y, c \otimes z] = D_{a,b}(c) \otimes t(x \bullet y)z + t(ab)c \otimes d_{x,y}(z)$ $x \bullet y = \frac{1}{2}(xy + yx)$ and $d_{x,y}(z) = x \bullet (y \bullet z) - y \bullet (x \bullet z)$ $D_{a,b}(c) = \frac{1}{4}([[a,b],c] + 3((ac)b - a(cb))$ $t(ab)$ and $t(x \bullet y)$ the normalized traces
E_7	$\mathfrak{sl}_2(k) \oplus F_4$	$\mathcal{Q}_0 \otimes \mathcal{H}_3(\mathcal{Q})_0$ $(a \otimes x) \cdot (b \otimes y) = \frac{1}{2}[a,b] \otimes (x \bullet y - t(x \bullet y)1)$ $[a \otimes x, b \otimes y, c \otimes z] = D_{a,b}(c) \otimes t(x \bullet y)z + t(ab)c \otimes d_{x,y}(z)$ $x \bullet y = \frac{1}{2}(xy + yx)$ and $d_{x,y}(z) = x \bullet (y \bullet z) - y \bullet (x \bullet z)$ $D_{a,b}(c) = \frac{1}{4}[[a,b],c]$ $t(ab)$ and $t(x \bullet y)$ the normalized traces

Continued on next page

Table 8
(Continued.)

$\mathfrak{g}(\mathfrak{m})$	$\mathfrak{h}$	$\mathfrak{m}$-description
E_8	$\mathfrak{sp}_2(k) \oplus E_7$	$k^2 \otimes \mathcal{T}_{\mathcal{H}_3(\mathcal{O})}$ $(u \otimes x) \cdot (v \otimes y) = 0$ $[u \otimes x, v \otimes y, w \otimes z] = \varphi(x, y)(\psi_{u,v}(w) \otimes z) + \psi(u, v)w \otimes xyz$ $b = \varphi, \psi :\ b(x, y) = -b(y, x)$ and $b_{x,y}(z) = b(x, z)y + b(y, z)x$
E_8	$G_2 \oplus F_4$	$\mathcal{O}_0 \otimes \mathcal{H}_3(\mathcal{O})_0$ $(a \otimes x) \cdot (b \otimes y) = \frac{1}{2}[a, b] \otimes (x \bullet y - t(x \bullet y)1)$ $[a \otimes x, b \otimes y, c \otimes z] = D_{a,b}(c) \otimes t(x \bullet y)z + t(ab)c \otimes d_{x,y}(z)$ $x \bullet y = \frac{1}{2}(xy + yx)$ and $d_{x,y}(z) = x \bullet (y \bullet z) - y \bullet (x \bullet z)$ $D_{a,b}(c) = \frac{1}{4}([[a, b], c] + 3((ac)b - a(cb))$ $t(ab)$ and $t(x \bullet y)$ the normalized traces

The simplicity of such a system can be characterized through its inner derivation algebra:

$$\text{Inder}\, \mathcal{U} = \text{span}\langle(\{x_+y_-\cdot\}, -\epsilon\{y_-x_+\cdot\}) : x_+ \in \mathcal{U}^+, y_- \in \mathcal{U}^-\rangle \tag{6.6}$$

which is a Lie subalgebra of $\mathfrak{gl}(\mathcal{U}^+) \times \mathfrak{gl}(\mathcal{U}^-)$. According to [13, Proposition 1.2], $\mathcal{U}$ is a simple pair if and only if $\{\mathcal{U}^\sigma \mathcal{U}^{-\sigma} \mathcal{U}^\sigma\} \neq 0$ and $\mathcal{U}^\sigma$ is an irreducible Inder $\mathcal{U}$-module (via the action of the σ-component). Over algebraically closed fields of characteristic zero, the simple finite-dimensional Jordan and anti-Jordan pairs were classified in [21, Theorem 17.12] and [13, Sections 3 and 4]. In Tables 5 and 6 below a complete description of both classifications is given. The tables include the inner derivation algebra Inder $\mathcal{U}$, the Cartan type of its derived subalgebra $\text{Inder}_0\, \mathcal{U} = [\text{Inder}\, \mathcal{U}, \text{Inder}\, \mathcal{U}]$ and the highest weight of $\mathcal{U}^\sigma$ as a module for $\text{Inder}_0\, \mathcal{U}$ for the different simple Jordan and anti-Jordan pairs $\mathcal{U}$. We follow the matricial description of the original classifications, although alternative descriptions could be displayed. In this way, rectangular $p \times q$ matrices, $n \times n$ symmetric or alternating matrices are represented in the tables by $\mathcal{M}_{p,q}(k)$, $\mathcal{H}_n(k)$ and $\mathcal{A}_n(k)$, while $\mathcal{M}_{1,2}(\mathcal{O})$ represents the space of 1×2 matrices over the octonions $\mathcal{O}$. We use the standard notation $\mathcal{H}_3(\mathcal{O})$ for the 27-dimensional exceptional Jordan algebra (or Albert algebra, see [15] or [28] for a complete description). For a given matrix y, its transpose is denoted by y^t and in case $y \in \mathcal{M}_{1,2}(\mathcal{O})$, $\bar{y}$ represents the standard involution induced in $\mathcal{M}_{1,2}(\mathcal{O})$ by the involution of $\mathcal{O}$. Triple products for Jordan pairs and anti-Jordan pairs of the form $\mathcal{U} = (k^n, k^n)$ are defined by means of the operators

$$b_{x,y} = b(y, \cdot)x - \epsilon b(x, \cdot)y, \tag{6.7}$$

for a nondegenerate ϵ-symmetric form b ($\epsilon = 1$ for b symmetric and $\epsilon = -1$ in case b is skew symmetric).

The structural module information given on these tables can be obtained from a direct computation for the different Jordan and anti-Jordan pairs. Alternatively, the relationship among $\mathbb{Z}_2$-graded simple Lie algebras (respectively superalgebras) having a consistent 3-grading and Jordan pairs (resp. anti-Jordan pairs) allows us to obtain the complete information from the classification of simple and nonirreducible Lie triple systems (using the corresponding affine Dynkin diagrams in [10, Table I]) and simple superalgebras of type $A(m, n)$, $C(n)$ and $P(n)$ (from [16, Proposition 2.1.2]).

7. Epilogue

The aim of this final section is to summarize the complete classification of irreducible LY-algebras while emphasizing their connections to other algebraic systems. Following [3, Theorem 2.4] we arrive at the irreducible LY-algebras of Adjoint Type. They are nothing else but simple Lie algebras with binary and ternary products given by the Lie bracket as Table 7 shows. From [3, Theorems 4.1 and 4.4] we get the information given in Table 8. In this table, the irreducible LY-algebras of non-simple type and exceptional enveloping algebra appear related to the Classical Tits Construction and symplectic (equivalently Freudenthal) triple systems $\mathcal{T}_{\mathcal{J}}$ attached to a simple Jordan algebra $\mathcal{J}$ of degree 3 or equal to the base field k. In the classical enveloping algebra case, the non-simple classification follows from a slight generalization of the Tits Construction due to Benkart and Zelmanov and given in [4]. Along this paper we have seen that in the generic case, apart from the Lie triple systems and the exceptional cases $(G_2, \mathfrak{sl}_2(k))$, (E_7, G_2) and $(E_7, \mathfrak{sl}_3(k))$, the irreducible LY-algebras are related to reductive pairs ($\mathfrak{sl}(V)$ or $\mathfrak{so}(V)$ or $\mathfrak{sp}(V)$, $\text{Der}^\star V$) for a suitable triple system V with $\text{Der}^\star V$ closely related to the (inner) derivation algebra of the system. In this way, either Jordan or anti-Jordan pairs (triple system) appear in the $\mathfrak{sl}$-case, Lie or orthogonal triple systems in the $\mathfrak{so}$-case and symplectic or anti-Lie triple systems in the $\mathfrak{sp}$-case. This yields our final Table 9 according to Theorems 2.2, 3.3 and 4.4. Note that, apart from simple Lie algebras, the basic ingredients in the classification are the composition algebras (k, $k \times k$, $\mathcal{Q}$ and $\mathcal{O}$), and simple Jordan algebras and their trace zero elements ($\mathcal{H}_n(k)$), $\mathcal{H}_n(k \times k)$, $\mathcal{H}_n(\mathcal{Q})$, $\mathcal{H}_n(\mathcal{O})$ and $\mathcal{J}(k^n) = k1 \oplus k^n$ the Jordan algebra of a nondegenerate symmetric bilinear form, so $\mathcal{J}(k^n)_0 = k^n$.

Table 9
Irreducible LY-algebras of Generic Type.

$\mathfrak{g}(\mathfrak{m})$	$\mathfrak{h}$	$(\mathfrak{g}(\mathfrak{m}), \mathfrak{h})$-pair description	$\mathfrak{m}$-description
$\mathfrak{sl}_n(k)$ $5 \le n$	$\mathfrak{so}_n(k)$	$(\mathcal{L}\mathrm{ie}_0(\mathcal{H}_n(k)), \mathrm{Der}\,\mathcal{H}_n(k))^*$	$\mathcal{H}_n(k)_0$ $a \cdot b = 0$ $[a, b, c] = (bc)a - b(ac)$
$\mathfrak{sl}_{2n}(k)$ $2 \le n$	$\mathfrak{sp}_{2n}(k)$	$(\mathcal{L}\mathrm{ie}_0(\mathcal{H}_n(\mathcal{Q})), \mathrm{Der}\,\mathcal{H}_n(\mathcal{Q}))$	$\mathcal{H}_n(\mathcal{Q})_0$ $a \cdot b = 0$ $[a, b, c] = (bc)a - b(ac)$
$\mathfrak{sl}_{\frac{n(n+1)}{2}}(k)$ $2 \le n$	$\mathfrak{sl}_n(k)$	$(\mathfrak{sl}(\mathcal{H}_n(k)), \mathcal{L}_0(\mathcal{H}_n(k)))^*$	$\mathfrak{h}^\perp$
$\mathfrak{sl}_{\frac{n(n-1)}{2}}(k)$ $5 \le n$	$\mathfrak{sl}_n(k)$	$(\mathfrak{sl}(\mathcal{A}_n(k)), \mathcal{L}_0(\mathcal{A}_n(k)))$	$\mathfrak{h}^\perp$
$\mathfrak{sl}_{16}(k)$	$\mathfrak{so}_{10}(k)$	$(\mathfrak{sl}(\mathcal{M}_{1,2}(\mathcal{O})), \mathcal{L}_0(\mathcal{M}_{1,2}(\mathcal{O})))$	$\mathfrak{h}^\perp$
$\mathfrak{sl}_{27}(k)$	E_6	$(\mathfrak{sl}(\mathcal{H}_3(\mathcal{O})), \mathcal{L}_0(\mathcal{H}_3(\mathcal{O})))$	$\mathfrak{h}^\perp$
$\mathfrak{so}_{n+1}(k)$ $5 \le n$	$\mathfrak{so}_n(k)$	$(\mathcal{L}\mathrm{ie}_0(\mathcal{J}(k^n)), \mathrm{Der}\,\mathcal{J}(k^n))$[a]	k^n $x \cdot y = 0$ $[x, y, z] = b(x, z)y - b(y, z)x$ $b(x, y) = b(y, x)$
$\mathfrak{so}_{\dim \mathcal{L}}(k)$	$\mathrm{Der}\,\mathcal{L}$	$(\mathfrak{so}(\mathcal{L}), \mathrm{Der}\,\mathcal{L})$[b]	$\mathfrak{h}^\perp$
$\mathfrak{so}_{\frac{n^2+n-2}{2}}(k)$ $5 \le n$	$\mathfrak{so}_n(k)$	$(\mathfrak{so}(\mathcal{H}_n(k)_0), \mathrm{Der}\,\mathcal{H}_n(k))$	$\mathfrak{h}^\perp$
$\mathfrak{so}_{2n^2-n-1}(k)$ $3 \le n$	$\mathfrak{sp}_{2n}(k)$	$(\mathfrak{so}(\mathcal{H}_n(\mathcal{Q})_0), \mathrm{Der}\,\mathcal{H}_n(\mathcal{Q}))$	$\mathfrak{h}^\perp$
$\mathfrak{so}_{26}(k)$	F_4	$(\mathfrak{so}(\mathcal{H}_3(\mathcal{O})_0), \mathrm{Der}\,\mathcal{H}_3(\mathcal{O}))$	$\mathfrak{h}^\perp$
$\mathfrak{so}_7(k)$	G_2	$(\mathfrak{so}(\mathcal{O}_0), \mathrm{Der}\,\mathcal{O})$	$\mathcal{O}_0$ $a \cdot b = ab - ba$ $[a, b, c] = 2\big([[a, b], c] - 3\big((ac)b - a(cb)\big)\big)$
$\mathfrak{so}_{16}(k)$	$\mathfrak{so}_9(k)$	$(\mathfrak{so}(\mathcal{T}_{(F_4,B_4)}), \mathrm{Inder}\,\mathcal{T}_{(F_4,B_4)})$[c]	$\mathfrak{h}^\perp$
$\mathfrak{so}_{42}(k)$	$\mathfrak{sp}_8(k)$	$(\mathfrak{so}(\mathcal{T}_{(E_6,C_4)}), \mathrm{Inder}\,\mathcal{T}_{(E_6,C_4)})$	$\mathfrak{h}^\perp$
$\mathfrak{so}_{70}(k)$	$\mathfrak{sl}_8(k)$	$(\mathfrak{so}(\mathcal{T}_{(E_7,A_7)}), \mathrm{Inder}\,\mathcal{T}_{(E_7,A_7)})$	$\mathfrak{h}^\perp$
$\mathfrak{so}_{128}(k)$	$\mathfrak{so}_{16}(k)$	$(\mathfrak{so}(\mathcal{T}_{(E_8,D_8)}), \mathrm{Inder}\,\mathcal{T}_{(E_8,D_8)})$	$\mathfrak{h}^\perp$
$\mathfrak{sp}_4(k)$	$\mathfrak{sl}_2(k)$	$(\mathfrak{sp}(\mathcal{T}_k), \mathrm{Inder}\,\mathcal{T}_k)$[d]	$\mathfrak{h}^\perp$
$\mathfrak{sp}_{14}(k)$	$\mathfrak{sp}_6(k)$	$(\mathfrak{sp}(\mathcal{T}_{\mathcal{H}_3(k)})), \mathrm{Inder}\,\mathcal{T}_{\mathcal{H}_3(k)})$	$\mathfrak{h}^\perp$
$\mathfrak{sp}_{20}(k)$	$\mathfrak{sl}_6(k)$	$(\mathfrak{sp}(\mathcal{T}_{\mathcal{H}_3(k\times k)})), \mathrm{Inder}\,\mathcal{T}_{\mathcal{H}_3(k\times k)})$	$\mathfrak{h}^\perp$
$\mathfrak{sp}_{32}(k)$	$\mathfrak{so}_{12}(k)$	$(\mathfrak{sp}(\mathcal{T}_{\mathcal{H}_3(\mathcal{Q})})), \mathrm{Inder}\,\mathcal{T}_{\mathcal{H}_3(\mathcal{Q})})$	$\mathfrak{h}^\perp$
$\mathfrak{sp}_{56}(k)$	E_7	$(\mathfrak{sp}(\mathcal{T}_{\mathcal{H}_3(\mathcal{O})})), \mathrm{Inder}\,\mathcal{T}_{\mathcal{H}_3(\mathcal{O})})$	$\mathfrak{h}^\perp$
G_2	$\mathfrak{sl}_2(k)$		$\mathfrak{h}^\perp$
F_4	$\mathfrak{so}_9(k)$		$\mathcal{T}_{(F_4,B_4)}$
E_6	$\mathfrak{sp}_4(k)$		$\mathcal{T}_{(E_6,C_4)}$
E_6	G_2		$\mathfrak{h}^\perp$
E_6	F_4	$(\mathcal{L}\mathrm{ie}_0(\mathcal{H}_3(\mathcal{O})), \mathrm{Der}\,\mathcal{H}_3(\mathcal{O}))$	$\mathcal{H}_3(\mathcal{O})_0$ $a \cdot b = 0$ $[a, b, c] = (bc)a - b(ac)$
E_7	$\mathfrak{sl}_8(k)$		$\mathcal{T}_{(E_7,A_7)}$
E_7	$\mathfrak{sl}_3(k)$		$\mathfrak{h}^\perp$
E_8	$\mathfrak{so}_{16}(k)$		$\mathcal{T}_{(E_8,D_8)}$

[a] $\mathcal{L}\mathrm{ie}_0(\mathcal{J})$ stands for the derived algebra of the Lie multiplication algebra attached to the Jordan algebra $\mathcal{J}$ and $\mathcal{L}_0(\mathcal{T})$ is as defined in Theorem 2.2 for the Jordan triple $\mathcal{T}$.
[b] $\mathcal{L}$ stands for a simple Lie algebra different from $\mathfrak{sl}_n(k)$.
[c] $\mathcal{T}_{(\mathfrak{g},\mathfrak{s})}$ stands for a simple Lie triple system attached to one of the exceptional symmetric pairs $(\mathfrak{g}, \mathfrak{s}) = (F_4, B_4), (E_6, C_4), (E_7, A_7)$ or (E_8, D_8).
[d] $\mathcal{T}_{\mathcal{J}}$ stands for a simple symplectic Lie triple attached to a Jordan simple algebra $\mathcal{J} = k, \mathcal{H}_3(k), \mathcal{H}_3(k \times k), \mathcal{H}_3(\mathcal{Q})$ or $\mathcal{H}_3(\mathcal{O})$.

Acknowledgements

Supported by the Spanish Ministerio de Educación y Ciencia and FEDER (MTM 2007-67884-C04-02,03). Pilar Benito and Fabián Martín-Herce also acknowledge support from the Comunidad Autónoma de La Rioja (ANGI2005/05,06), and Alberto Elduque from the Diputación General de Aragón (Grupo de Investigación de Álgebra).

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