Journal of Computational and Applied Mathematics 236 (2011) 1449–1463

Contents lists available at SciVerse ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Solving nonlinear integral equations of Fredholm type with high order iterative methods

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ARTICLE INFO

Article history:
Received 26 February 2010
Received in revised form 2 September 2011

MSC:
45G10
47H99
65J15

Keywords:
Iterative methods
Hybrid methods
Order of convergence
Efficiency
Semilocal convergence
Fredholm integral equation

ABSTRACT

The application of high order iterative methods for solving nonlinear integral equations is not usual in mathematics. But, in this paper, we show that high order iterative methods can be used to solve a special case of nonlinear integral equations of Fredholm type and second kind. In particular, those that have the property of the second derivative of the corresponding operator have associated with them a vector of diagonal matrices once a process of discretization has been done.

1. Introduction

In this paper, we are interested in solving the following nonlinear integral equation of Fredholm type and second kind:

$$x(s) = l(s) + \lambda \int_a^b K(s,t)H(x(t))dt, \quad s \in [a,b],\ \lambda \in \mathbb{R}, \tag{1}$$

where $-\infty < a < b < \infty$, f, H and K are known functions and x is a solution to be determined. The analysis and computation of this kind of Fredholm equation is justified by the dynamic model of a chemical reactor [1], which is governed by control equations [2].

To calculate an approximation of a solution x of Eq. (1) we can use a discrete scheme where a numerical quadrature is applied, so that solving (1) is then reduced to solve a system of nonlinear equations. If we use the Gauss–Legendre quadrature [3] to approximate an integral,

$$\int_a^b g(t)\,dt \simeq \frac{1}{2}\sum_{j=1}^{m} \beta_j\, g(t_j),$$

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0377-0427/$ – see front matter
doi:10.1016/j.cam.2011.09.009

where the nodes t_j and the weights β_j are well-known, (1) is transformed into the following nonlinear system of equations:

$$x(t_i) = l(t_i) + \left(\frac{\lambda}{2}\sum_{j=1}^{m}\beta_j K(t_i, t_j)H(x(t_j))\right), \quad i = 1, 2, \dots, m, \tag{2}$$

which is written in the form

$$F(\bar{x}) \equiv (F_1(\bar{x}), F_2(\bar{x}), \dots, F_m(\bar{x})) = \bar{0}, \tag{3}$$

where $F : \mathbb{R}^m \to \mathbb{R}^m, \bar{x} = (x_1, x_2, \dots, x_m)^t, \bar{l} = (l(t_1), l(t_2), \dots, l(t_m))^t, x_i = x(t_i)$ and

$$F_i(\bar{x}) = x_i - l(t_i) - \left(\frac{\lambda}{2}\sum_{j=1}^{m}\beta_j K(t_i, t_j)H(x_j)\right), \quad i = 1, 2, \dots, m.$$

The Newton method is usually applied to solve nonlinear systems of form (3) and the successive approximations are given by the following algorithm:

$$\begin{cases} \bar{x}_0 \text{ given}, \\ F'(\bar{x}_k)\,\bar{c}_k = -F(\bar{x}_k), \quad k \geq 0, \\ \bar{x}_{k+1} = \bar{x}_k + \bar{c}_k. \end{cases}$$

When we are interested in solving nonlinear systems of equations by means of one-point iterative methods, the efficiency of the methods depends fundamentally on the order of convergence (ρ) and the operational cost (σ) of doing an step of the algorithm. The order of convergence measures the speed of convergence of the method and the operational cost of doing a step of the algorithm is the number of operations (products and divisions) which are needed when it is applied. So, the efficiency of an iterative method can be measured by the computational efficiency index, $CE = \rho^{1/\sigma}$ (see [4]). For example, for system (3), the computational efficiency of the Newton method is $CE = 2^{3/(m^3+12m^2+2m)}$, since the order of convergence is $\rho = 2$ and the operational cost per step is $\sigma = (m^3 + 12m^2 + 2m)/3$ (see [5]).

In view of the expression of the well known one-point iterative processes of third order of convergence, in [6,7] we consider the family of iterative processes that includes most of them and is given by

$$\begin{cases} x_{n+1} = x_n - H(L_F(x_n))\Gamma_n F(x_n), \quad n \in \mathbb{N}, \\ H(L_F(x_n)) = I + \dfrac{1}{2}L_F(x_n) + \displaystyle\sum_{k\geq 2} A_k L_F(x_n)^k, \quad \{A_k\}_{k\geq 2} \subset \mathbb{R}^+, \end{cases} \tag{4}$$

where $L_F(x_n) = \Gamma_n F''(x_n)\Gamma_n F(x_n)$, $\Gamma_n = [F'(x_n)]^{-1}$ and $\{A_k\}_{k\geq 2}$ is a positive real sequence with $\sum_{k\geq 2} A_k t^k < +\infty$ for $|t| < r$. In a similar way, in [8] the family

$$\begin{cases} x_{n+1} = x_n - H(L_B(x_n), L_F(x_n))\Gamma_n F(x_n), \quad n \in \mathbb{N}, \\ H(L_B(x_n), L_F(x_n)) = I + \dfrac{1}{2}L_F(x_n) + L_F(x_n)\displaystyle\sum_{k\geq 2} A_k L_B(x_n)^{k-1}, \quad \{A_k\}_{k\geq 2} \subset \mathbb{R}^+, \end{cases}$$

is considered, where $B(x_n)$ is a bilinear operator from $X \times X$ to Y to determine and $L_B(x_n) = \Gamma_n B(x_n)\Gamma_n F(x_n)$, that obviously is reduced to family (4) in the case $B = F''$. As the application of iterative processes with an infinite series is not easily considerable, the series is truncated in practice, as we can see in [8].

Since the main aim of this paper is to present more efficient one-point iterative methods than the Newton method from (4), we also truncate the series of (4). Moreover, the parameters A_k that appear in the truncated series can be optimized to obtain more efficient iterative processes.

On the other hand, it is interesting to emphasize that in [8] the semilocal convergence study of the methods is carried out under conditions already studied for the operator F'' in [7], while in this paper we consider a milder condition to study the semilocal convergence of our methods.

To carry out the above, a first idea is to consider iterative methods with higher order of convergence than the Newton method, but in this situation the operational cost can be increased highly, since one-point iterative methods of order of convergence ρ depend explicitly on the first $\rho - 1$ derivatives of F [4].

In this paper, we consider third-order one-point iterative methods [9] that do not have the problem of increasing highly the operational cost when they are applied to solve a nonlinear system of equations. We then consider the family of

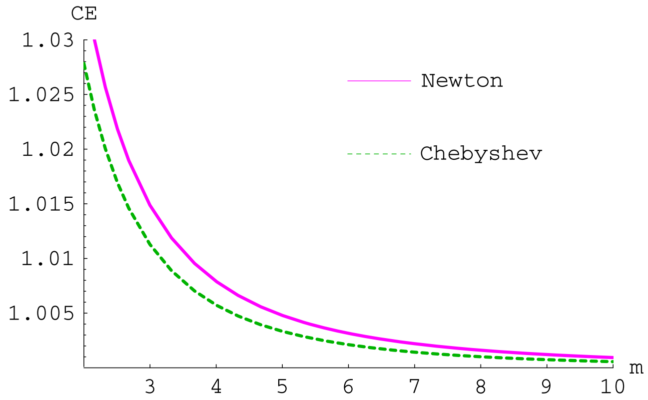

Fig. 1. The computational efficiencies of the Newton and the Chebyshev methods: $2^{3/(m^3+12m^2+2m)}$ and $3^{3/(4m^3+18m^2+8m)}$, respectively.

methods [7]

$$\begin{cases} \bar{x}_0 \text{ given}, \\ F'(\bar{x}_k)\,\bar{c}_k = -F(\bar{x}_k), \quad k \geq 0, \\ F'(\bar{x}_k)\,\bar{d}_k^{(1)} = -(\bar{c}_k)^t\,F''(\bar{x}_k)\,\bar{c}_k, \\ F'(\bar{x}_k)\,\bar{d}_k^{(2)} = -(\bar{c}_k)^t\,F''(\bar{x}_k)\,\bar{d}_k^{(1)}, \\ \vdots \\ F'(\bar{x}_k)\,\bar{d}_k^{(p)} = -(\bar{c}_k)^t\,F''(\bar{x}_k)\,\bar{d}_k^{(p-1)}, \\ \bar{x}_{k+1} = \bar{x}_k + \bar{c}_k + \displaystyle\sum_{i=1}^{p} \alpha_i\,\bar{d}_k^{(i)}, \quad \alpha_1 = \frac{1}{2},\ \alpha_i \in \mathbb{R}^+, \end{cases} \tag{5}$$

which have the significant feature of every lineal system has the same matrix, $F'(\bar{x}_k)$, so that we only have to do a factorization LU, as the Newton method, whose operational cost is $(m^3 - m)/3$ operations. Despite this feature, at first sight the number of operations related to $(\bar{c}_k)^t\,F''(\bar{x}_k)\,\bar{d}_k^{(p-1)}$, for each p, is $m^3 + m^2 + m$, which increases considerably the operational cost of the algorithm. In general, the iterative methods of the family are no more efficient than the Newton method, as we can observe in Fig. 1, where we compare the computational efficiencies of the Newton method, $2^{3/(m^3+12m^2+2m)}$, and the Chebyshev method [10] (method (5) with $p = 1$), $3^{3/(4m^3+18m^2+8m)}$.

In Section 2, we see that discretization (2) has the feature of reducing the operational cost of doing $(\bar{c}_k)^t\,F''(\bar{x}_k)\,\bar{d}_k^{(p-1)}$, so that we can consider (in terms of m) more efficient methods of family (5) than the Newton method for approximating a solution of the corresponding equation $F(\bar{x}) = 0$ given in (3). It is due to the fact that the second Fréchet derivative $F''(\bar{x})$ has associated with it a vector of diagonal matrices.

In Section 3, we analyze the local and semilocal convergence of family (5) for solving (3). The semilocal convergence is given under a new type of condition that adapts well to the above problem and we prove that the order of convergence of (5) is locally at least three. Domains of existence and uniqueness of solutions of (3) are also given.

In Section 4, we study the optimization of family (5) as a function of the parameter p when it is applied to solve (3) and it is quadratic. Moreover, we present a numerical experiment where a more efficient iterative method of (5) than the Newton method is applied to solve a quadratic integral equation of type (1).

To finish, in Section 5, we solve a negative common effect of third-order one-point iterative methods: the reduction in the region of accessibility of starting points (every starting point from which iterative methods are convergent) with respect to the Newton method (see [5]). In this case, we use hybrid iterative methods, which are predictor–corrector in the following way: from a starting point that does not satisfy the semilocal convergence conditions (prediction), a new starting point is obtained that guarantees the semilocal convergence of the iterative methods to the solution of the problem (correction).

2. On the scheme of discretization

On the one hand, when we consider how to solve nonlinear system (3), the operational cost of doing a step of the Newton method is $(m^3 + 12m^2 + 2m)/3$ operations and for an iterative method of (5) is $(m^3 + 3(p+5)m^2 + (6p+5)m)/3$. Notice that the operational cost of (5) is increased when p is increased. On the other hand, we see that discretization (2) has the feature of reducing the operational cost of computing $(\bar{c}_k)^t\,F''(\bar{x}_k)\,\bar{d}_k^{(p-1)}$, since the second derivative has associated with it a vector of diagonal matrices, so that we can consider in terms of m, more efficient third order methods of family (5) than the Newton method for approximating a solution of Eq. (3).

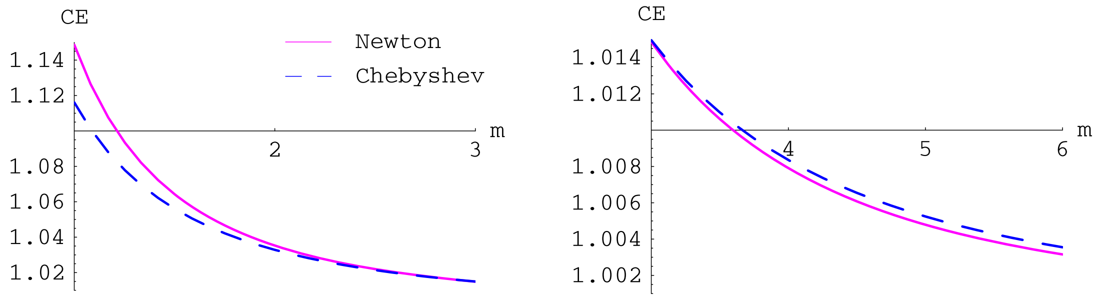

Fig. 2. The computational efficiencies of the Newton and the Chebyshev methods are respectively $2^{3/(m^3+12m^2+2m)}$ and $3^{3/(m^3+18m^2+11m)}$ when they are applied to solve (6).

Now, if we denote

$$a_{ij}=\frac{1}{2}\beta_j K(t_i,t_j),\quad i,j=1,2,\ldots,m,$$

and $A=(a_{ij})$, then we can write nonlinear system (3) in the following matrix form:

$$F(\bar{x})\equiv\bar{x}-\bar{l}-\lambda A(H(x_1),H(x_2),\ldots,H(x_m))^t=\bar{0}. \tag{6}$$

Therefore, the first and the second Fréchet-derivatives of the operator F are given as follows:

$$F'(\bar{x})\bar{y}=[I-\lambda A\mathrm{Diag}(H'(x_1),H'(x_2),\ldots,H'(x_m))]\bar{y},\quad \forall\bar{y}\in\mathbb{R}^m,$$

where $\mathrm{Diag}(\bar{x})$ denotes the diagonal matrix with the components of the vector $\bar{x}=(x_1,x_2,\ldots,x_m)$ in the diagonal, and F'' is the bilinear operator defined by

$$\bar{y}^tF''(\bar{x})\bar{z}=-\lambda A(H''(x_1)y_1z_1,H''(x_2)y_2z_2,\ldots,H''(x_m)y_mz_m)^t,$$

for $\bar{y}=(y_1,\ldots,y_m),\bar{z}=(z_1,\ldots,z_m)$.

Observe that in this case the number of operations related to $F(\bar{x}_k)$ and $F'(\bar{x}_k)$ is m^2+m and $2m^2$, respectively. In general, the number of operations related to $(\bar{c}_k)^t\,F''(\bar{x}_k)\,\bar{d}_k^{(p-1)}$ is m^3+m^2+m, but in the case of considering the previous process of discretization, the second Fréchet-derivative has associated with it a vector of diagonal matrices and the operational cost of computing $(\bar{c}_k)^t\,F''(\bar{x}_k)\,\bar{d}_k^{(p-1)}$ is then reduced to m^2+2m operations.

So, for solving nonlinear system (6), as method (5) requires $(m^3+3(p+5)m^2+(6p+5)m)/3$ operations to do a step, the particular cases of the Chebyshev and the Newton methods require $(m^3+18m^2+11m)/3$ and $(m^3+12m^2+2m)/3$ operations per step, respectively. As we can observe in Fig. 2, if $m\geq 3$, the computational efficiency of the Chebyshev method is better than that of the Newton method.

3. Analysis of the convergence and order of convergence

We now present the analysis of the convergence of the family of methods given in (5), which is now written as:

$$\begin{cases}\bar{x}_0\ \text{given},\\ \bar{x}_{n+1}=\bar{x}_n-\Psi(L_F(\bar{x}_n))[F'(\bar{x}_n)]^{-1}F(\bar{x}_n),\quad n\geq 0,\\ \Psi(L_F(\bar{x}))=\displaystyle\sum_{k=0}^{p}\alpha_kL_F(\bar{x})^k,\quad \alpha_0=1,\ \alpha_1=1/2,\ \alpha_k\in\mathbb{R}^+,\ k\geq 2,\end{cases} \tag{7}$$

where $L_F(\bar{x})$ is the degree of logarithmic convexity defined by

$$L_F(\bar{x})=F'(\bar{x})^{-1}F''(\bar{x})[F'(\bar{x})^{-1}F(\bar{x})]\in\mathcal{L}(\mathbb{R}^m),$$

and $\mathcal{L}(\mathbb{R}^m)$ is the set of bounded linear operators from $\mathbb{R}^m$ into $\mathbb{R}^m$, provided that the operator $[F'(\bar{x}_n)]^{-1}=\Gamma_n$ exists.

Notice that the method given in (7) with $p=0$ is the Newton method.

3.1. Local convergence

Firstly, we prove that the order of convergence of (7) is locally at least three.

Theorem 3.1. *Suppose that $F:\Omega\subseteq\mathbb{R}^m\to\mathbb{R}^m$ is the operator given in (3), where Ω is a non-empty open convex domain. If F has a simple root $\bar{x}^*\in\Omega\subset\mathbb{R}^m$, $[F'(\bar{x})]^{-1}$ exists in a neighborhood of $\bar{x}^*$ and $\bar{x}_0$ is sufficiently close to $\bar{x}^*$, then iterations (7) have order of convergence at least three.*

Proof. From Taylor's formula, we have

$$\bar{0} = F(\bar{x}^*) = F(\bar{x}_n) - F'(\bar{x}_n)\bar{e}_n + \frac{1}{2!}F''(\bar{x}_n)\bar{e}_n^2 - \frac{1}{3!}F'''(\bar{x}_n)\bar{e}_n^3 + O(\|\bar{e}_n\|^4),$$

where $\bar{e}_n = \bar{x}_n - \bar{x}^*$. Hence,

$$\Gamma_n F(\bar{x}_n) = \bar{e}_n - \frac{1}{2!}\Gamma_n F''(\bar{x}_n)\bar{e}_n^2 + \frac{1}{3!}\Gamma_n F'''(\bar{x}_n)\bar{e}_n^3 + O(\|\bar{e}_n\|^4)$$

and

$$L_F(\bar{x}_n) = \Gamma_n F''(\bar{x}_n)\bar{e}_n - \frac{1}{2!}\Gamma_n F''(\bar{x}_n)\Gamma_n F''(\bar{x}_n)\bar{e}_n^2 + \frac{1}{3!}\Gamma_n F''(\bar{x}_n)\Gamma_n F'''(\bar{x}_n)\bar{e}_n^3 + O(\|\bar{e}_n\|^4).$$

Moreover, from (7), it follows

$$\begin{aligned}
\bar{e}_{n+1} &= \bar{x}_{n+1} - \bar{x}^* = \bar{e}_n - \left(I + \frac{1}{2}L_F(\bar{x}_n) + \sum_{k=2}^{p}\alpha_k L_F(\bar{x}_n)^k\right)\Gamma_n F(\bar{x}_n)\\
&= \frac{1}{2}\Gamma_n F''(\bar{x}_n)\bar{e}_n^2 - \frac{1}{6}\Gamma_n F'''(\bar{x}_n)\bar{e}_n^3 + O(\|\bar{e}_n\|^4)\\
&\quad - \frac{1}{2}\Gamma_n F''(\bar{x}_n)\bar{e}_n^2 + \frac{1}{2}(\Gamma_n F''(\bar{x}_n))^2\bar{e}_n^3 + O(\|\bar{e}_n\|^4) - \alpha_2(\Gamma_n F''(\bar{x}_n))^2\bar{e}_n^3 + O(\|\bar{e}_n\|^4)\\
&= \left(-\frac{1}{6}\Gamma_n F'''(\bar{x}_n) + \left(\frac{1}{2} - \alpha_2\right)(\Gamma_n F''(\bar{x}_n))^2\right)\bar{e}_n^3 + O(\|\bar{e}_n\|^4).
\end{aligned}$$

Therefore, (7) has order of convergence at least three. □

3.2. Semilocal convergence

Secondly, an analysis of the semilocal convergence is given. Basic results concerning the convergence of (7) have been published under assumptions of Newton–Kantorovich type [14]; for instance, in [7], we assume the following conditions:

(K1) There exists a point $\bar{x}_0 \in \Omega$ where the operator $\Gamma_0 = [F'(\bar{x}_0)]^{-1} \in \mathcal{L}(\mathbb{R}^m)$ is defined and $\|\Gamma_0\| \leq \beta$,
(K2) $\|\Gamma_0 F(\bar{x}_0)\| \leq \eta$,
(K3) $\|F''(\bar{x})\| \leq M$, $\forall \bar{x} \in \Omega$,
(K4) $\|F''(\bar{x}) - F''(\bar{y})\| \leq K\|\bar{x} - \bar{y}\|,\ K \geq 0,\ \bar{x}, \bar{y} \in \Omega$.

According to these conditions, the number of problems that can be solved is limited. In general, we cannot analyze the convergence of (7) to a solution of (6), since in general F'' is not bounded. Moreover, it is not easy to locate a domain where the solution lies.

In order to consider more general situations as those of the nonlinear integral equations of type (1), we relax the previous convergence conditions such that they are adjusted better to this type of integral operators. An elegant alternative consists of relaxing the strong condition (K3) by

$$\|F''(\bar{x})\| \leq \omega(\|\bar{x}\|), \quad \bar{x} \in \Omega, \tag{8}$$

where $\omega : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+ \cup \{0\}$ is a continuous real function such that $\omega(0) \geq 0$ and ω is a monotone function. We then analyze the semilocal convergence of (7) in $\mathbb{R}^m$ by assuming only (K1), (K2) and (8), so that fewer convergence conditions are required.

According to expression (7), we obtain the following decomposition for any operator F.

Lemma 3.2. *Let $F : \Omega \subseteq \mathbb{R}^m \to \mathbb{R}^m$ be a sufficiently differentiable operator on a non-empty open convex domain Ω. Then,*

$$\begin{aligned}
F(\bar{x}_{n+1}) &= -F''(\bar{x}_n)\left(\Gamma_n F(\bar{x}_n)\left(\sum_{k=1}^{p}\alpha_k L_F(\bar{x}_n)^{k-1}\right)\Gamma_n F(\bar{x}_n)\right)\\
&\quad + \int_0^1 F''(\bar{x}_n + t(\bar{x}_{n+1} - \bar{x}_n))(\bar{x}_{n+1} - \bar{x}_n)^2(1-t)\,dt.
\end{aligned}\tag{9}$$

Proof. From Taylor's formula, we have

$$F(\bar{x}_{n+1}) = F(\bar{x}_n) + F'(\bar{x}_n)(\bar{x}_{n+1} - \bar{x}_n) + \int_{\bar{x}_n}^{\bar{x}_{n+1}} F''(\bar{x})(\bar{x}_{n+1} - \bar{x})\,d\bar{x},$$

for $\bar{x}_{n+1} \in \Omega$. Now, taking $\bar{x} = \bar{x}_n + t(\bar{x}_{n+1} - \bar{x}_n)$ and (7), we obtain (9), since

$$F'(\bar{x}_n)(\bar{x}_{n+1} - \bar{x}_n) = -F'(\bar{x}_n)\left(\Gamma_n F(\bar{x}_n) + \left(\sum_{k=1}^{p} \alpha_k L_F(\bar{x}_n)^k\right)\Gamma_n F(\bar{x}_n)\right)$$
$$= -F(\bar{x}_n) - F''(\bar{x}_n)\left(\Gamma_n F(\bar{x}_n)\sum_{k=1}^{p} \alpha_k L_F(\bar{x}_n)^{k-1}\right)\Gamma_n F(\bar{x}_n). \quad \square$$

From now on, the results are adapted to the particular case of F being given by (3). We convert conditions (K1), (K2) and (8) into this situation and construct a system of recurrence relations which allows us to establish the convergence of (7).

To do this, we use the following notation:

$$\|\bar{x} - \bar{l} - \lambda AH(\bar{x})\| = \phi(\bar{x}), \tag{10a}$$
$$1 - |\lambda|\,\|A\|\,\|H'(\bar{x})\| = \vartheta(\bar{x}), \tag{10b}$$
$$|\lambda|\,\|A\|\,\|H''(\bar{x})\| = \chi(\bar{x}). \tag{10c}$$

Lemma 3.3. *Let $\bar{x}_0 \in \Omega$ such that*

(C1) $|\lambda|\,\|A\|\,\|H'(\bar{x}_0)\| < 1$.

Then, the operator $[I - \lambda AH'(\bar{x}_0)]^{-1} \in \mathcal{L}(\mathbb{R}^m)$ is well-defined and

$$\|[I - \lambda AH'(\bar{x}_0)]^{-1}\| \le \frac{1}{\vartheta(\bar{x}_0)} = \beta.$$

Proof. Taking into account that

$$[I - [I - \lambda AH'(\bar{x}_0)]]\bar{y} = [\lambda AH'(\bar{x}_0)]\bar{y}, \quad \forall \bar{y} \in \mathbb{R}^m,$$

and

$$\|\lambda AH'(\bar{x}_0)\| \le |\lambda|\,\|A\|\,\|H'(\bar{x}_0)\| < 1,$$

the proof follows from Banach's lemma. $\square$

After that, we assume that the operator F given in (6) satisfies condition (C1) and the following:

(C2) $\frac{\phi(\bar{x}_0)}{\vartheta(\bar{x}_0)} = \eta$,

(C3) $\chi(\bar{x}) \le \omega(\|\bar{x}\|)$, $\forall \bar{x} \in \Omega$, where $\omega : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+ \cup \{0\}$ is a continuous real function such that $\omega(0) \ge 0$ and ω is a monotone function.

(C4) The equation

$$4t - 2\psi(\beta\eta\varphi(t))((1 + \beta\eta\varphi(t))t + \eta) - \beta\eta\varphi(t)\psi(\beta\eta\varphi(t))^2(t - 2\eta) = 0 \tag{11}$$

has at least one positive root, where

$$\varphi(t) = \begin{cases} \omega(\|\bar{x}_0\| + t) & \text{if } \omega \text{ is non-decreasing,} \\ \omega(\|\bar{x}_0\| - t) & \text{if } \omega \text{ is non-increasing,} \end{cases}$$

and

$$\psi(t) = \sum_{k=0}^{p} \alpha_k t^k, \qquad \alpha_0 = 1, \qquad \alpha_1 = \frac{1}{2}. \tag{12}$$

We denote the smallest root of the previous equation by R. Notice that R must be less than $\|\bar{x}_0\|$ if ω is non-increasing. Let $a_0 = \beta\eta\varphi(R)$ and define the scalar sequence:

$$a_{n+1} = a_n f(a_n)^2 g(a_n), \quad n \ge 1, \tag{13}$$

where

$$f(t) = \frac{1}{1 - t\psi(t)}, \qquad g(t) = \psi(t)\left(1 + \frac{t}{2}\psi(t)\right) - 1. \tag{14}$$

Using a technique based on recurrence relations, under conditions (C1)–(C4), we prove methods (7) are convergent to a solution of (3). Also, we find the domains where the solution is located and unique. So, we construct a system of recurrence relations which allows us to establish the convergence of iterations (7).

From initial conditions (C1)–(C4) and (10), we have

$$\|L_F(\bar{x}_0)\| \leq \frac{\phi(\bar{x}_0)}{\vartheta(\bar{x}_0)^2}\chi(\bar{x}_0) \leq \omega(\|\bar{x}_0\|)\beta\eta = a_0.$$

Then x_1 is well defined and

$$\|\Psi(L_F(\bar{x}_0))\| \leq \sum_{k=0}^{p}\alpha_k a_0^k = \psi(a_0).$$

Moreover,

$$\|\bar{x}_1 - \bar{x}_0\| \leq \psi(a_0)\eta.$$

In the following result we give the recurrence relations which are needed to prove the semilocal convergence of (7).

Lemma 3.4. *Suppose that $\bar{x}_0, \bar{x}_n \in \Omega$, for $n \in \mathbb{N}$. If $a_0\psi(a_0) < 1$ and $f(a_0)^2 g(a_0) < 1$. Then, the following relations are satisfied:*

(I) *There exists $[I - \lambda AH'(\bar{x}_n)]^{-1}$ and $\frac{1}{\vartheta(\bar{x}_n)} \leq \frac{f(a_{n-1})}{\vartheta(\bar{x}_{n-1})}$,*
(II) $\frac{\phi(\bar{x}_n)}{\vartheta(\bar{x}_n)} \leq f(a_{n-1})g(a_{n-1})\frac{\phi(\bar{x}_{n-1})}{\vartheta(\bar{x}_{n-1})}$,
(III) $\varphi(R)\frac{\phi(\bar{x}_n)}{\vartheta(\bar{x}_n)^2} \leq a_n$ *and there exists* $\Psi(L_F(\bar{x}_n))$,
(IV) $\|\bar{x}_{n+1} - \bar{x}_n\| \leq \psi(a_n)\frac{\phi(\bar{x}_n)}{\vartheta(\bar{x}_n)}$,
(V) $\|\bar{x}_{n+1} - \bar{x}_0\| \leq \left(\sum_{i=0}^{n}\psi(a_i)\left(\prod_{k=0}^{i-1} f(a_k)g(a_k)\right)\right)\eta$.

Proof. We prove the result by invoking induction of an inductive process for n. Firstly, we prove that (I)–(V) are satisfied for $n = 1$. From

$$A[H'(\bar{x}_1) - H'(\bar{x}_0)] = \int_0^1 AH''(\bar{x}_0 + t(\bar{x}_1 - \bar{x}_0))(\bar{x}_1 - \bar{x}_0)dt$$

and taking norms, it follows

$$|\lambda|\ \|A\|\ \|H'(\bar{x}_1) - H'(\bar{x}_0)\| \leq \varphi(R)\|\bar{x}_1 - \bar{x}_0\| \leq \varphi(R)\psi(a_0)\eta.$$

Notice that $\|I - [F'(\bar{x}_0)]^{-1}F'(\bar{x}_1)\| \leq \beta|\lambda|\ \|A\|\ \|H'(\bar{x}_1) - H'(\bar{x}_0)\|$. Therefore,

$$\beta|\lambda|\ \|A\|\ \|H'(\bar{x}_1) - H'(\bar{x}_0)\| \leq \beta\varphi(R)\psi(a_0)\eta = a_0\psi(a_0) < 1.$$

Thus, from Banach's Lemma, the operator $[F'(\bar{x}_1)]^{-1} = [I - \lambda AH'(\bar{x}_1)]^{-1}$ exists and

$$\frac{1}{\vartheta(\bar{x}_1)} \leq \frac{1}{1 - \|I - \Gamma_0 F'(x_1)\|}\frac{1}{\vartheta(\bar{x}_0)} \leq \frac{1}{1 - a_0\psi(a_0)}\frac{1}{\vartheta(\bar{x}_0)} = \frac{f(a_0)}{\vartheta(\bar{x}_0)}.$$

Now, from (9) and taking into account g, we have

$$\begin{aligned}\frac{\phi(\bar{x}_1)}{\vartheta(\bar{x}_1)} &\leq f(a_0)\left(\sum_{k=1}^{p}\alpha_k a_0^k + \int_0^1 a_0\psi(a_0)^2(1-t)\,dt\right)\frac{\phi(\bar{x}_0)}{\vartheta(\bar{x}_0)}\\ &= f(a_0)\left(\psi(a_0) - 1 + \frac{1}{2}a_0\psi(a_0)^2\right)\frac{\phi(\bar{x}_0)}{\vartheta(\bar{x}_0)} \leq f(a_0)g(a_0)\eta,\end{aligned}$$

$$\varphi(R)\frac{\phi(\bar{x}_1)}{\vartheta(\bar{x}_1)^2} \leq f(a_0)^2 g(a_0)a_0 = a_1,$$

and therefore (II) and (III) hold. From the condition $f(a_0)^2 g(a_0) < 1$, it is easy to prove that $\{a_n\}$ is a decreasing sequence. Thus,

$$\|\Psi(L_F(\bar{x}_1))\| \leq \sum_{k=0}^{p}\alpha_k a_1^k = \psi(a_1)$$

and (IV) and (V) hold, since

$$\|\bar{x}_2 - \bar{x}_1\| \leq \psi(a_1)\frac{\phi(\bar{x}_1)}{\vartheta(\bar{x}_1)},$$

$$\|\bar{x}_2 - \bar{x}_0\| \leq \|\bar{x}_2 - \bar{x}_1\| + \|\bar{x}_1 - \bar{x}_0\| \leq (\psi(a_0) + \psi(a_1)f(a_0)g(a_0))\eta.$$

We suppose that (I)–(V) are satisfied for n and prove them for $n+1$. By hypothesis, we have $\bar{x}_{j+1} \in \Omega, j = 1, 2, \ldots, n$. If

$$A[H'(\bar{x}_{n+1}) - H'(\bar{x}_n)] = \int_0^1 AH''(\bar{x}_n + t(\bar{x}_{n+1} - \bar{x}_n))\,(\bar{x}_{n+1} - \bar{x}_n)dt,$$

then

$$|\lambda|\,\|A\|\,\|H'(\bar{x}_{n+1}) - H'(\bar{x}_n)\| \le \varphi(R)\|\bar{x}_{n+1} - \bar{x}_n\| \le \varphi(R)\psi(a_n)\frac{\phi(\bar{x}_n)}{\vartheta(\bar{x}_n)}.$$

Notice that

$$\|I - [F'(\bar{x}_n)]^{-1}F'(\bar{x}_{n+1})\| \le \frac{1}{\vartheta(\bar{x}_n)}|\lambda|\,\|A\|\,\|H'(\bar{x}_{n+1}) - H'(\bar{x}_n)\|$$

and

$$\frac{1}{\vartheta(\bar{x}_n)}|\lambda|\,\|A\|\,\|H'(\bar{x}_{n+1}) - H'(\bar{x}_n)\| \le \varphi(R)\frac{\phi(\bar{x}_n)}{\vartheta(\bar{x}_n)^2}\psi(a_n) = a_n\psi(a_n) < 1.$$

Thus, from Banach's Lemma, the operator $[F'(\bar{x}_{n+1})]^{-1} = [I - \lambda AH'(\bar{x}_{n+1})]^{-1}$ exists and

$$\frac{1}{\vartheta(\bar{x}_{n+1})} \le \frac{1}{1 - \|I - [F'(\bar{x}_n)]^{-1}F'(\bar{x}_{n+1})\|}\frac{1}{\vartheta(\bar{x}_n)} \le \frac{1}{1 - a_n\psi(a_n)}\frac{1}{\vartheta(\bar{x}_n)} = \frac{f(a_n)}{\vartheta(\bar{x}_n)}.$$

Now, from (9), it follows that

$$\frac{\phi(\bar{x}_{n+1})}{\vartheta(\bar{x}_{n+1})} \le f(a_n)g(a_n)\frac{\phi(\bar{x}_n)}{\vartheta(\bar{x}_n)}.$$

Therefore,

$$\varphi(R)\frac{\phi(\bar{x}_{n+1})}{\vartheta(\bar{x}_{n+1})} \le f(a_n)^2 g(a_n)a_n = a_{n+1}$$

and

$$\|\Psi(L_F(\bar{x}_{n+1}))\| \le \sum_{k=0}^{p}\alpha_k a_{n+1}^k = \psi(a_{n+1}).$$

We also observe that

$$\|\bar{x}_{n+2} - \bar{x}_{n+1}\| \le \psi(a_{n+1})\frac{\phi(\bar{x}_{n+1})}{\vartheta(\bar{x}_{n+1})},$$

$$\begin{aligned}\|\bar{x}_{n+2} - \bar{x}_0\| &\le \|\bar{x}_{n+2} - \bar{x}_{n+1}\| + \|\bar{x}_{n+1} - \bar{x}_0\| \\ &\le \psi(a_{n+1})\left(\prod_{k=0}^{n} f(a_k)g(a_k)\right)\eta + \left(\sum_{i=0}^{n}\psi(a_i)\left(\prod_{k=0}^{i-1} f(a_k)g(a_k)\right)\right)\eta,\end{aligned}$$

and, by hypothesis, we have

$$\|\bar{x}_{n+2} - \bar{x}_0\| \le \left(\sum_{i=0}^{n+1}\psi(a_i)\left(\prod_{k=0}^{i-1} f(a_k)g(a_k)\right)\right)\eta,$$

and the proof of the lemma is complete. □

3.2.1. Main result

Now, we give results where some properties of the real functions defined in (14) and the real sequence $\{a_n\}$ are provided. We then establish the semilocal convergence of the third-order methods given in (7).

Lemma 3.5. *Let ψ, f and g be the three real functions given in (12) and (14). If $a_0\psi(a_0) < 1$, then $f(t)$ is an increasing function and $f(t) > 1$ for $t \in (0, a_0)$. Moreover, $g(t)$ is an increasing function for $t > 0$.*

Lemma 3.6. *Let ψ, f and g be the three real functions given in (12) and (14). If $a_0\psi(a_0) < 1$ and $f(a_0)^2 g(a_0) < 1$, then the sequence $\{a_n\}$ given in (13) is decreasing. Moreover, $f(a_0)g(a_0) < 1$.*

Proof. To prove the thesis we use mathematical induction. By hypothesis,

$$a_1 = f(a_0)^2 g(a_0) a_0 < a_0.$$

Besides, taking into account $f(a_0) > 1$, it follows $f(a_0)g(a_0) < 1$. We now suppose that $a_k < a_{k-1}$ for all $k \leq n$. As a consequence,

$$a_{n+1} < a_n \Leftrightarrow f(a_n)^2 g(a_n) < 1,$$

since $f(a_n)^2 g(a_n) < f(a_0)^2 g(a_0) < 1$. The proof is then complete. □

After that, we suppose that the operator $[F'(\bar{x}_0)]^{-1}$ exists at some $\bar{x}_0 \in \Omega$.

Theorem 3.7. *Let $F : \Omega \subseteq \mathbb{R}^m \to \mathbb{R}^m$ be the operator given in (3), where Ω is a non-empty open convex domain. We suppose that $[F'(\bar{x}_0)]^{-1}$ exists for some $\bar{x}_0 \in \Omega$ and conditions (C1)–(C4) hold. If*

$$a_0 \psi(a_0) < 1, \qquad f(a_0)^2 g(a_0) < 1 \tag{15}$$

and $B(\bar{x}_0, R) \subseteq \Omega$, then methods (7), starting from $\bar{x}_0$, generate a sequence $\{\bar{x}_n\}$ which converges to a solution $\bar{x}^* \in B(\bar{x}_0, R)$ of Eq. (3).

Proof. Taking into account (C1)–(C4), we obtain $\bar{x}_1 \in B(\bar{x}_0, R)$. Besides, from Lemma 3.4 and following mathematical induction, we obtain $\bar{x}_n \in B(\bar{x}_0, R) \subseteq \Omega$, for all $n \in \mathbb{N}$.

To establish the convergence of (7) we prove that $\{\bar{x}_n\}$ is a Cauchy sequence. To do this, we consider $n, m \in \mathbb{N}$ and

$$\begin{aligned} \|\bar{x}_{n+m} - \bar{x}_n\| &\leq \sum_{k=n}^{n+m-1} \|\bar{x}_{k+1} - \bar{x}_k\| \leq \sum_{k=n}^{n+m-1} \psi(a_0)(f(a_0)g(a_0))^k \eta \\ &\leq \psi(a_0)(f(a_0)g(a_0))^n \left(\sum_{k=0}^{m-1} (f(a_0)g(a_0))^k \right) \eta \\ &= \psi(a_0)(f(a_0)g(a_0))^n \frac{1 - (f(a_0)g(a_0))^m}{1 - f(a_0)g(a_0)} \eta, \end{aligned} \tag{16}$$

since $\{a_n\}$ is a non-increasing sequence. Therefore, $\{\bar{x}_n\}$ is a Cauchy sequence and we have that $\lim_{n\to\infty} \bar{x}_n = \bar{x}^*$. On the other hand, we note that $\{\|I - \lambda A H'(\bar{x}_n)\|\}$ is a bounded sequence, since

$$\begin{aligned} \|I - \lambda A H'(\bar{x}_n)\| &\leq |\lambda| \, \|A\| \, \|H'(\bar{x}_n) - H'(\bar{x}_0)\| + \|I - \lambda A H'(\bar{x}_0)\| \\ &\leq R \int_0^1 \varphi(R) \, dt + \|I - \lambda A H'(\bar{x}_0)\|. \end{aligned}$$

Thus, from

$$\frac{1}{\vartheta(\bar{x}_n)} \leq \|I - \lambda A H'(\bar{x}_n)\| \frac{\phi(\bar{x}_n)}{\vartheta(\bar{x}_n)}$$

and $\lim_{n\to\infty} \frac{\phi(\bar{x}_n)}{\vartheta(\bar{x}_n)} = 0$, we have that $\phi(\bar{x}^*) = 0$. By the continuity of H, we obtain $\bar{x}^* - \bar{f} - \lambda A H(\bar{x}^*) = \bar{0}$ and $\bar{x}^*$ is therefore a solution of the equation $F(\bar{x}^*) = \bar{0}$ with F given in (6). □

3.2.2. Uniqueness of the solution

Now, we provide a result about the uniqueness of the solution x^* of $F(\bar{x}^*) = \bar{0}$ with F given in (3).

Theorem 3.8. *Let F be the operator given in (3). We suppose that conditions (C1)–(C4) hold. Then, the solution $\bar{x}^*$ of the equation $F(\bar{x}^*) = \bar{0}$ is unique in $B(\bar{x}_0, \tilde{R})$, where $\tilde{R}$ is the biggest positive root of the equation*

$$\beta \int_0^1 \int_0^1 \varphi(s(R + t(\xi - R))) \, ds (R + t(\xi - R)) \, dt = 1. \tag{17}$$

Proof. To show the uniqueness of $\bar{x}^*$, we suppose that $\bar{z}^*$ is another solution of $F(\bar{x}) = \bar{0}$ in $B(\bar{x}_0, \tilde{R})$. From the approximation

$$0 = [F'(\bar{x}_0)]^{-1}[F(\bar{z}^*) - F(\bar{x}^*)] = \int_0^1 [F'(\bar{x}_0)]^{-1}[I - \lambda A H'(\bar{x}^* + t(\bar{z}^* - \bar{x}^*)) \, dt] \, (\bar{z}^* - \bar{x}^*),$$

Table 1
Order of convergence of methods (7) and the number m of equations of system (3) when it is quadratic.

p	Order of methods (7)	m
1	3	3
2	4	3
3	4	5
4	4	7
5	4	10
6	4	13

and the fact that the operator $P = \int_0^1 [F'(\bar{x}_0)]^{-1}[I - \lambda A H'(\bar{x}^* + t(\bar{z}^* - \bar{x}^*))\, dt]$ is invertible, it follows $\bar{z}^* = \bar{x}^*$. Observe that P is invertible by Banach's lemma, since

$$\begin{aligned}\|I - P\| &\le \beta \left(|\lambda|\, \|A\| \int_0^1 \|H'(\bar{x}^* + t(\bar{z}^* - \bar{x}^*)) - H'(\bar{x}_0)\|\, dt \right) \\ &\le \beta |\lambda|\, \|A\| \int_0^1 \int_0^1 \|H''\left(\bar{x}_0 + s\left(\bar{x}^* + t(\bar{z}^* - \bar{x}^*)\right)\right)\| \left((1-t)\|\bar{x}^* - \bar{x}_0\| + t\|\bar{z}^* - \bar{x}_0\|\right)\, ds\, dt \\ &< \beta \int_0^1 \int_0^1 \varphi(s(R + t(\xi - R)))(R + t(\xi - R))\, ds\, dt = 1.\end{aligned}$$

From (17), it is easy to see that the uniqueness of the solution is then guaranteed in $B(\bar{x}_0, R)$ if $\int_0^1 \varphi(sR) < 1/(R\beta)$. □

4. An optimization of methods (7)

Notice that methods (7) can be optimized as a function of the parameter p when they are applied to solve system (3). Due to the computational efficiency, the choice of the parameter p depends on the number m of equations of (3). In the particular case of (3) is a quadratic system, it is possible to optimize the computational efficiency of (7). Thus, if we choose $\alpha_2 = 1/2$, methods (7) have R-order of convergence at least 4 (see [11]) and the computational efficiency is $CE = 4^{3/(m^3+3(p+5)m^2+(6p+5)m)}$.

We indicate in Table 1 the number m of equations of system (3) based on methods (7), order of convergence four, has better computational efficiency than the Newton method. For instance, the Chebyshev method, (7) with $p = 1$, has better CE than the Newton method when $m \ge 3$, method (7) with $p = 2$ has order of convergence four and better CE than the Newton method when $m \ge 3$, and so on. Thus, to improve the CE of the Newton method from methods (7), we have to choose an optimum value of p according to the number m of equations of (3) when it is quadratic.

Moreover, from a certain number m of equations of system (3), we can also realize an optimization between the methods of (7). For instance for $m \ge 2$, the Chebyshev-like method given by (7) with $p = 2$ [12] has better CE than the Chebyshev method.

We now illustrate the above-mentioned with an example. The aim is to find the optimum value of p, fixed the number m of equations of system (3), when (7) is applied to solve a quadratic system. We then consider the following quadratic integral operator:

$$F(x)(s) = x(s) - 1 - \frac{1}{2}\int_0^1 G(s,t)x(t)^2\, dt, \quad s \in [0,1],$$

where $x \in C[0,1]$, $s, t \in [0,1]$, and the kernel G is the Green function

$$G(s,t) = \begin{cases} (1-s)t, & t \le s, \\ s(1-t), & s \le t. \end{cases} \tag{18}$$

$F : C^+[0,1] \subseteq C[0,1] \to C[0,1]$ and $C^+[0,1] = \{x \in C[0,1] \mid x(t) \ge 0,\ t \in [0,1]\}$.

In this case, if we fix the number m of equations, for instance $m = 8$, the optimum value of p is two, see Fig. 3. Then, we choose (7) with $p = 2$,

$$\begin{cases} \bar{x}_0 \text{ given}, \\ \bar{x}_{n+1} = \bar{x}_n - \Psi(L_F(\bar{x}_n))[F'(\bar{x}_n)]^{-1}F(\bar{x}_n), \quad n \ge 0, \\ \Psi(L_F(\bar{x})) = \bar{1} + \dfrac{1}{2}L_F(\bar{x}) + \dfrac{1}{2}L_F(\bar{x})^2, \end{cases} \tag{19}$$

which has order of convergence four, for solving the nonlinear system

$$F(\bar{x}) = \bar{x} - \bar{1} - A\frac{\bar{x}^2}{2} = \bar{0}, \tag{20}$$

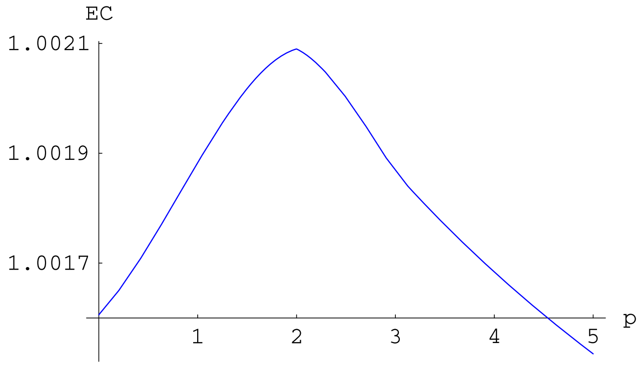

Fig. 3. The computational efficiency of methods (7) when $m = 8$.

Table 2
Numerical solution $\bar{x}^*$ of system (20).

x_1^*	1.00545...
x_2^*	1.02581...
x_3^*	1.05162...
x_4^*	1.06936...
x_5^*	1.06936...
x_6^*	1.05162...
x_7^*	1.02581...
x_8^*	1.00545...

where $\bar{x} = (x_1, \dots, x_{18})^t$, $\bar{1} = (1, \dots, 1)^t$, $A = (a_{ij})$ with

$$a_{ij} = \begin{cases} \frac{1}{2}(1-t_i)t_j\beta_j, & j \le i, \\ \frac{1}{2}t_i(1-t_j)\beta_j, & i \le j, \end{cases} \tag{21}$$

and the weights β_j and the nodes t_j. Therefore,

$$F'(\bar{x})\bar{y} = [I - AD_1(\bar{x})]\bar{y}, \quad \forall \bar{y} \in \mathbb{R}^8,$$

where $D_1(\bar{x})$ denotes the diagonal matrix with the components of the vector $(x_1, x_2, \dots, x_n)$ in the diagonal, and F'' is the bilinear operator defined by

$$\bar{y}^t F''(\bar{x})\bar{z} = -A(y_1z_1, \dots, y_8z_8)^t, \quad \forall \bar{y}, \bar{z} \in \mathbb{R}^8.$$

We denote the n-th iteration of (19) by $\bar{x}_n = (x_1^{(n)}, x_2^{(n)}, \dots, x_8^{(n)})^t$. If we choose $\bar{x}_0 = (2, 2, \dots, 2)^t$, we obtain $\beta = 1.32822\dots$ and $\eta = 1.30185\dots$. Notice that in this case the real function ω is constant, $\omega(\bar{x}) = \|A\|$, and Eqs. (11) and (17) are reduced to the linear equations $(1.47064\dots) - (0.649409\dots)\,t = 0$ and $(0.185826\dots) + (0.0820573\dots)\,\xi = 1$, respectively. Therefore, the radii of the domains of existence and uniqueness are $R = 2.26459\dots$ and $\tilde{R} = 9.92201\dots$, respectively. Moreover, $a_0 = \beta\eta\varphi(R) = 0.213653\dots$, so that $a_0\psi(a_0) = 0.241354\dots < 1, f(a_0)^2 g(a_0) = 0.462127\dots < 1$, and consequently conditions (15) of Theorem 3.7 are satisfied.

After applying two iterations of method (19) and using the stopping criterion $\|\bar{x}_n - \bar{x}_{n-1}\| < 10^{-180}$, we obtain the numerical solution $\bar{x}^* = (x_1^*, x_2^*, \dots, x_8^*)$ of system (20), which is given in Table 2 and shown in Fig. 4 once it is interpolated.

Considering the same stopping criterion in Table 3 as above, we obtain the errors $\|\bar{x}_n - \bar{x}^*\|$. In the case, $m = 8$, when method (19) is used, the operational cost (products and divisions) is 1992 after three iterations and, when the Newton method is applied, the operational cost is 2592 after six iterations. Thus, we approximate the solution $\bar{x}^*$ of (20) by means of a method of family (7) which is more efficient than the Newton method, the most common method for solving nonlinear systems.

On the other hand, the study of the optimization of p is open for another type of nonlinear systems.

5. Predictor–corrector methods

It is well-known that the higher the order of convergence of an iterative method, the smaller its region of accessibility (the region where we can guarantee the convergence of the method). In this section, we analyze how to improve the region of accessibility of methods (7), so that it is increased to the region of accessibility of the Newton method.

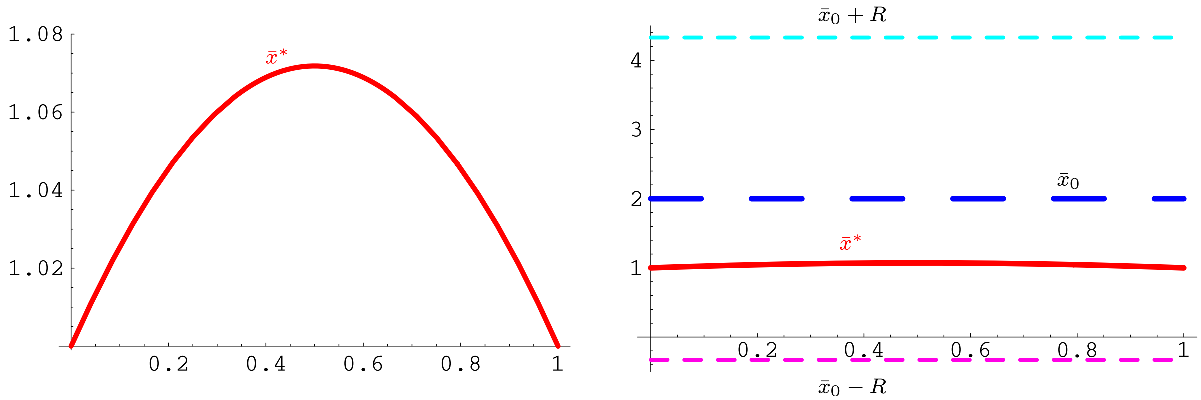

Fig. 4. Approximated solution of (20) and the domain of existence of the solution ($m = 8$).

Table 3
Errors $\|\bar{x}_n - \bar{x}^*\|$ for the Newton method and method (19).

n	The Newton method	Method (19)
0	$0.994549\ldots$	$0.994549\ldots$
1	$6.93651\ldots \times 10^{-2}$	$1.47340\ldots \times 10^{-3}$
2	$2.58022\ldots \times 10^{-4}$	$3.32053\ldots \times 10^{-15}$
3	$3.44598\ldots \times 10^{-9}$	$8.23120\ldots \times 10^{-62}$
4	$6.10030\ldots \times 10^{-19}$	
5	$1.90917\ldots \times 10^{-38}$	
6	$1.86951\ldots \times 10^{-77}$	

We consider the following nonlinear integral operator of type (1):

$$F(x)(s) = x(s) - 1 - \frac{1}{20}\int_0^1 G(s,t)\left(x(t)^{5/2} + x(t)^3/5\right)dt, \quad s \in [0,1],$$

where $x \in C[0,1]$, $s, t \in [0,1]$ and the kernel G is function (18).

In this example, to consider a different situation as above, we choose $m = 48$. In this case, methods (7) with $1 \le p \le 11$, are more efficient than the Newton method, and (7) with $p = 1$, the Chebyshev method, is the most efficient of the methods of (7).

Nonlinear system (6) is in this case reduced to

$$F(\bar{x}) = \bar{x} - \bar{1} - A(\bar{x}^{5/2}/20 + \bar{x}^3/100) = \bar{0}, \tag{22}$$

where $\bar{x} = (x_1, x_2, \ldots, x_{48})^t$, $\bar{1} = (1, 1, \ldots, 1)^t$ and $A = (a_{ij})$ with a_{ij} as in (21). Therefore,

$$F'(\bar{x})\bar{y} = \left[I - A\left(\frac{1}{8}D_{3/2}(\bar{x}) + \frac{3}{100}D_2(\bar{x})\right)\right]\bar{y}, \quad \forall \bar{y} \in \mathbb{R}^{48},$$

where $D_k(\bar{x})$, $k = 3/2$ and $k = 2$, denotes the diagonal matrix with the components of the vector $(x_1^k, x_2^k, \ldots, x_n^k)$ in the diagonal, and F'' is the bilinear operator defined by

$$\bar{y}^t F''(\bar{x})\bar{z} = -A\left(\left(\frac{3}{16}x_1^{1/2} + \frac{3}{50}x_1\right)z_1 y_1, \ldots, \left(\frac{3}{16}x_{48}^{1/2} + \frac{3}{50}x_{48}\right)z_{48}y_{48}\right)^t, \quad \forall \bar{y}, \bar{z} \in \mathbb{R}^{48}.$$

So, we have $\omega(\bar{x}) = \|A\|\left(\frac{3}{16}\bar{x}^{1/2} + \frac{3}{50}\bar{x}\right)$. On the other hand, we denote the n-th iteration by $\bar{x}_n = (x_1^{(n)}, x_2^{(n)}, \ldots, x_{48}^{(n)})^t$. If $\bar{x}_0 = (2.75, 2.75, \ldots, 2.75)^t$, we obtain $\beta = 1.110593\ldots$, $\eta = 1.94325\ldots$. If we choose the Chebyshev method to approximate a solution of (22), in a similar way to the previous example, we obtain that the smallest roots of Eqs. (11) and (17) are $R = 3.18615\ldots$ and $\tilde{R} = 12.5072\ldots$, respectively. Moreover, $a_0 = \beta\eta\varphi(R) = 0.219246\ldots$, $a_0\psi(a_0) = 0.243281\ldots < 1$, $f(a_0)^2 g(a_0) = 0.427153\ldots < 1$, so that conditions (15) of Theorem 3.7 are satisfied. Furthermore, the domain of existence of solutions is $\{x \in \mathbb{R}^m;\ \|\bar{x} - \bar{x}_0\| \le 3.18615\ldots\}$ and the domain of uniqueness is $\{x \in \mathbb{R}^m;\ \|\bar{x} - \bar{x}_0\| < 12.5072\ldots\}$.

After applying three iterations of the Chebyshev method and using the stopping criterion $\|\bar{x}_n - \bar{x}_{n-1}\| < 10^{-180}$, we obtain the numerical solution $\bar{x}^* = (x_1^*, x_2^*, \ldots, x_{48}^*)^t$ of (22) given in Table 4. Considering the same stopping criterion in Table 5, we obtain the errors $\|\bar{x}_n - \bar{x}^*\|$.

Notice that after five iterations of the Chebyshev method, the operational cost (products and divisions) is 254,320, and after nine iterations of the Newton method the operational cost is 415,008. Consequently, the Chebyshev method is more efficient than the Newton method.

In the previous example, if we choose $\bar{x}_0 = (3.5, 3.5, \ldots, 3.5)^t$ as the starting vector, the convergence conditions of Theorem 3.7 are not satisfied for any iterative method of (7). Thus, starting from the vector $\bar{x}_0$, the convergence of (7) to

Table 4
Numerical solution $\bar{x}^*$ of system (22).

x_1^*	1.00001...	x_{17}^*	1.00596...	x_{33}^*	1.00553...
x_2^*	1.00009...	x_{18}^*	1.00635...	x_{34}^*	1.00508...
x_3^*	1.00023...	x_{19}^*	1.00669...	x_{35}^*	1.00460...
x_4^*	1.00044...	x_{20}^*	1.00699...	x_{36}^*	1.00411...
x_5^*	1.00069...	x_{21}^*	1.00724...	x_{37}^*	1.00362...
x_6^*	1.00100...	x_{22}^*	1.00743...	x_{38}^*	1.00313...
x_7^*	1.00136...	x_{23}^*	1.00755...	x_{39}^*	1.00265...
x_8^*	1.00176...	x_{24}^*	1.00762...	x_{40}^*	1.00219...
x_9^*	1.00219...	x_{25}^*	1.00762...	x_{41}^*	1.00176...
x_{10}^*	1.00265...	x_{26}^*	1.00755...	x_{42}^*	1.00136...
x_{11}^*	1.00313...	x_{27}^*	1.00743...	x_{43}^*	1.00100...
x_{12}^*	1.00362...	x_{28}^*	1.00724...	x_{44}^*	1.00069...
x_{13}^*	1.00411...	x_{29}^*	1.00699...	x_{45}^*	1.00044...
x_{14}^*	1.00460...	x_{30}^*	1.00669...	x_{46}^*	1.00023...
x_{15}^*	1.00508...	x_{31}^*	1.00635...	x_{47}^*	1.00009...
x_{16}^*	1.00553...	x_{32}^*	1.00596...	x_{48}^*	1.00001...

Table 5
Errors $\|\bar{x}_n - \bar{x}^*\|$ for the Newton and the Chebyshev methods.

n	Newton	Chebyshev
0	1.74998 ...	1.749998 ...
1	$8.38662\ldots \times 10^{-2}$	$2.27561\ldots \times 10^{-2}$
2	$7.85914\ldots \times 10^{-5}$	$2.79589\ldots \times 10^{-8}$
3	$6.72419\ldots \times 10^{-11}$	$4.76096\ldots \times 10^{-26}$
4	$6.72419\ldots \times 10^{-11}$	$2.30678\ldots \times 10^{-79}$
5	$4.86404\ldots \times 10^{-23}$	
6	$2.53980\ldots \times 10^{-47}$	
7	$6.92228\ldots \times 10^{-96}$	

a solution $\bar{x}^*$ of Eq. (22) is not guaranteed. This is a problem that iterative methods of high order usually have, since the convergence conditions are more demanding. In this section we avoid this problem by using predictor–corrector methods.

The idea is to approximate a solution of (3) by the Newton method for a finite number of steps, N_0, and starting from $\bar{x}_0 \in \Omega$. After that, we take $\bar{x}_{N_0}$ as a starting point for a method of (7). This is to choose a starting point, $\bar{x}_0$, which is in the region of accessibility of the Newton method, and iterate a number of steps N_0 until $\bar{x}_{N_0}$ is in the region of accessibility of a method of (7).

Note that the Newton method is convergent from $\bar{x}_0$ to a solution $\bar{x}^*$ of Eq. (3) under the following conditions (see [13]):

$\widetilde{(C1)}$ there exists a point $\bar{x}_0 \in \mathbb{R}^m$ such that the operator $[I - \lambda AH'(\bar{x}_0)]^{-1} \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$ is well-defined and

$$\|[I - \lambda AH'(\bar{x}_0)]^{-1}\| \leq \frac{1}{\vartheta(\bar{x}_0)} = \tilde{\beta},$$

$\widetilde{(C2)}$ $\frac{\phi(\bar{x}_0)}{\vartheta(\bar{x}_0)} = \tilde{\eta}$,

$\widetilde{(C3)}$ $\chi(\bar{x}) \leq \omega(\|\bar{x}\|), \forall x \in \Omega$, where $\omega : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+ \cup \{0\}$ is a continuous real function such that $\omega(0) \geq 0$ and ω is a monotone function

$\widetilde{(C4)}$ the equation

$$3\tilde{\beta}\tilde{\eta}\varphi(t)t - 2\tilde{\beta}\tilde{\eta}^2\varphi(t) - 2t + 2\tilde{\eta} = 0$$

has at least one positive root, where

$$\varphi(t) = \begin{cases} \omega(\|\bar{x}_0\| + t) & \text{if } \omega \text{ is non-decreasing,} \\ \omega(\|\bar{x}_0\| - t) & \text{if } \omega \text{ is non-increasing.} \end{cases}$$

We denote the smallest root of the previous equation by R_1. Notice that R_1 must be less than $\|\bar{x}_0\|$ if ω is non-increasing.

If

$$\widetilde{a_0} = \tilde{\beta}\tilde{\eta}\varphi(R_1) < 1/2, \tag{23}$$

then the Newton method, starting from $\bar{x}_0$, generates a sequence $\{\bar{x}_n\}$ that converges to a solution $\bar{x}^* \in B(\bar{x}_0, R)$ of Eq. (3).

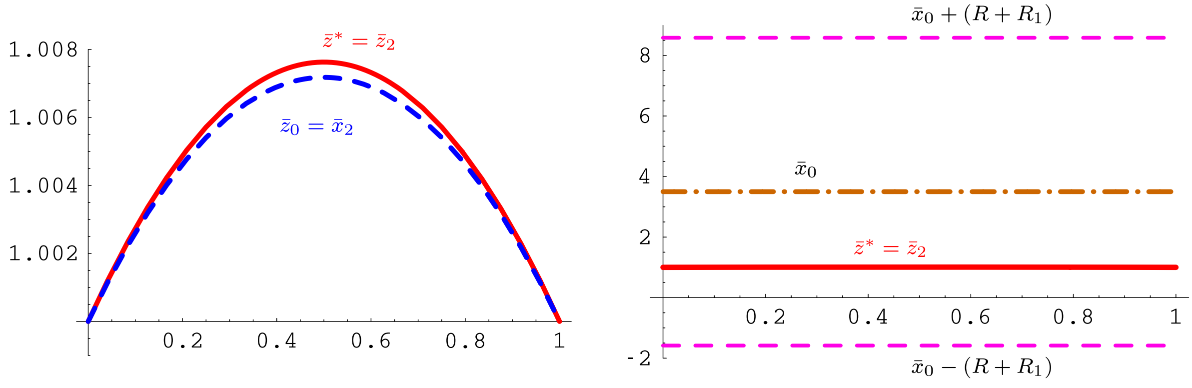

Fig. 5. Approximated solution of (22) and the domain of existence of the solution.

Table 6
Errors $\|\bar{x}_n - \bar{x}^*\|$ for the Newton method and method (24) with $p = 1$.

n	The Newton method	Method (24) with $p = 1$
0	$2.49998\ldots$	$2.49998\ldots$
1	$2.05774\ldots \times 10^{-1}$	$2.05774\ldots \times 10^{-1}$
2	$4.48357\ldots \times 10^{-4}$	$4.48357\ldots \times 10^{-4}$
3	$2.18982\ldots \times 10^{-9}$	$2.01938\ldots \times 10^{-13}$
4	$5.15933\ldots \times 10^{-20}$	$1.77070\ldots \times 10^{-41}$
5	$2.85760\ldots \times 10^{-41}$	$1.18362\ldots \times 10^{-125}$
6	$8.76309\ldots \times 10^{-84}$	
7	$8.24023\ldots \times 10^{-169}$	

Then, we consider the following algorithm:

$$\left\{\begin{array}{l}\left\{\begin{array}{l}\bar{x}_0 \in \Omega,\\ F'(\bar{x}_n)\,\bar{c}_n = -F(\bar{x}_n), \quad n \geq 0,\\ \bar{x}_{n+1} = \bar{x}_n + \bar{c}_n, \quad n = 1, 2, \ldots, N_0 - 1,\end{array}\right.\\ \left\{\begin{array}{l}\bar{z}_0 = \bar{x}_{N_0},\\ F'(\bar{z}_n)\,\bar{c}_n = -F(\bar{z}_n),\\ F'(\bar{z}_n)\,\bar{d}_n^{(1)} = -(\bar{c}_n)^t\,F''(\bar{z}_n)\,\bar{c}_n,\\ F'(\bar{z}_n)\,\bar{d}_n^{(2)} = -(\bar{c}_n)^t\,F''(\bar{z}_n)\,\bar{d}_n^{(1)},\\ \vdots\\ F'(\bar{z}_n)\,\bar{d}_n^{(p)} = -(\bar{c}_n)^t\,F''(\bar{z}_n)\,\bar{d}_n^{(p-1)},\\ \bar{z}_{n+1} = \bar{z}_n + \bar{c}_n + \displaystyle\sum_{i=1}^{p} \alpha_i\,\bar{d}_n^{(i)}, \quad \alpha_1 = \frac{1}{2},\ \alpha_i \in \mathbb{R}^+,\end{array}\right.\end{array}\right. \tag{24}$$

where $\bar{x}_0$ must satisfy condition (23) and $\bar{z}_0 = \bar{x}_{N_0}$ must satisfy conditions (15). Then, if $\widetilde{a_0} < 1/2$, we can use the Newton method for N_0 number of steps, so that $\bar{x}_{N_0} = \bar{z}_0$ satisfies (15), and then we apply (7). The key of the problem is to guarantee the existence of N_0 that we estimate in the following result and whose proof is similar to that which is written in [5].

Theorem 5.1. *Let $F : \Omega \subseteq \mathbb{R}^m \to \mathbb{R}^m$ be the operator given in (3), where Ω is a non-empty open convex domain. We suppose that $[F'(\bar{x}_0)]^{-1}$ exists for some $\bar{x}_0 \in \Omega$ and conditions $\widetilde{(C1)}$–$\widetilde{(C4)}$ hold. If a_0 does not satisfy any condition of (15) and $\widetilde{a_0} < 1/2$, then we take in (24) $\bar{z}_0 = \bar{x}_{N_0}$ with $N_0 = \max\{N_1, N_2\}$, where $N_1 = 1 + [\frac{\log \widetilde{a_0} + \log \psi(\widetilde{a_0})}{\log(2(1-\widetilde{a_0})^2) - \log \widetilde{a_0}}]$, $N_2 = 1 + [\frac{\log s - \log \widetilde{a_0}}{\log \widetilde{a_0} - \log(2(1-\widetilde{a_0})^2)}]$, when these values are positive and null in another case, s is the smallest positive root of equation $f(t)^2 g(t) = 1$ with f, g given in (14) and $[t]$ is the integer part of the real number t, so that $\bar{z}_0$ satisfies the corresponding conditions given in (15). If $B(x_0, R_1 + R) \subseteq \Omega$, then the sequence $\{\bar{z}_n\}$ defined in (24), which starts at $\bar{x}_0$, converges to a solution $\bar{x}^*$ of Eq. (3) and $\bar{x}_n, \bar{z}_n, \bar{x}^* \in \overline{B(x_0, R_1 + R)}$.*

Taking into account the last result for the initial vector considered in the previous example $\bar{x}_0 = \overline{3.5}$, we obtain $N_0 = 2$. We then apply method (24) with $p = 1$ and $\bar{z}_0 = \bar{x}_2$. Then, from $\bar{z}_0$, we can guarantee the convergence of the Chebyshev method to the solution of system (3) (see Table 4), which is obtained after two iterations. In Table 6 we show the errors $\|\bar{x}_n - \bar{x}^*\|$ when the stopping criterion $\|\bar{x}_n - \bar{x}^*\| < 10^{-180}$ is used.

Finally, we show in Fig. 5 the interpolated approximation which is obtained when method (24) with $p = 1$ is applied to solve Eq. (22) along with the starting vector $\bar{x}_0 = \overline{3.5}$.

Remark 5.2. Notice that we can extend the previous results to the following more general integral equations of type (1):

$$x(s) = l(s) + \sum_{j=1}^{n} \lambda_j \int_a^b K_j(s,t)\, H_j(x(t))\, dt, \quad s \in [a,b],\ \lambda_j \in \mathbb{R},$$

since the generalization of the results obtained for (1) is immediate.

Acknowledgments

Preparation of this paper was partly financed by the Ministry of Science and Innovation (MTM 2008-01952) and the Riojan Autonomous Community (Colabora 2009/04).

References

[1] D.D. Bruns, J.E. Bailey, Nonlinear feedback control for operating a nonisothermal CSTR near an unstable steady state, Chem. Eng. Sci. 32 (1977) 257–264.
[2] M. Ganesh, M.C. Joshi, Numerical solvability of Hammerstein integral equations of mixed type, IMA J. Numer. Anal. 11 (1991) 21–31.
[3] J.D. Faires, R. Burden, Numerical Methods, second ed., Brooks/Cole Publishing Co., Pacific Grove, CA, 1998.
[4] J.F. Traub, Iterative Methods for the Solution of Equations, Prentice Hall, Englewood Cliffs, NJ, 1964.
[5] J.A. Ezquerro, M.A. Hernández, An optimization of Chebyshev's method, J. Complexity 25 (2009) 343–361.
[6] M.A. Hernández, N. Romero, Application of iterative processes of *R*-order at least three to operators with unbounded second derivative, Appl. Math. Comput. 185 (1) (2007) 737–747.
[7] M.A. Hernández, N. Romero, On a characterization of some Newton-like methods of *R*-order at least three, J. Comput. Appl. Math. 183 (2005) 53–66.
[8] S. Amat, S. Busquier, J.M. Gutiérrez, Third-order iterative methods with applications to Hammerstein equations: a unified approach, J. Comput. Appl. Math. 235 (2011) 2936–2943.
[9] S. Amat, S. Busquier, Geometry and convergence of some third-order methods, Southwest J. Pure Appl. Math. 2 (2001) 61–72.
[10] I.K. Argyros, D. Chen, Results on the Chebyshev method in Banach spaces, Proyecciones 12 (2) (1993) 119–128.
[11] J.A. Ezquerro, M.A. Hernández, N. Romero, A modification of Cauchy's method for quadratic equations, J. Math. Anal. Appl. 339 (2) (2008) 954–969.
[12] S. Amat, S. Busquier, J.M. Gutiérrez, Geometric constructions of iterative functions to solve nonlinear equations, J. Comput. Appl. Math. 157 (1) (2003) 197–205.
[13] J.E. Ezquerro, M.A. Hernández, On a application of Newton's method to nonlinear operators with ω-conditioned second derivative, BIT 42 (3) (2002) 519–530.
[14] L.V. Kantorovich, G.P. Akilov, Functional Analysis, Pergamon Press, Oxford, 1982.