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Majorizing sequences for Newton's method from initial value problems

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ABSTRACT

The most restrictive condition used by Kantorovich for proving the semilocal convergence of Newton's method in Banach spaces is relaxed in this paper, providing we can guarantee the semilocal convergence in situations that Kantorovich cannot. To achieve this, we use Kantorovich's technique based on majorizing sequences, but our majorizing sequences are obtained differently, by solving initial value problems.

1. Introduction

We are interested in approximating a solution x^* of the equation $F(x) = 0$, where F is a nonlinear operator defined on a non-empty open convex domain Ω of a Banach space X with values in a Banach space Y, by means of the best-known iterative method, Newton's method, whose algorithm is

$$\begin{cases} x_0 \quad \text{given in } \Omega, \\ x_n = x_{n-1} - [F'(x_{n-1})]^{-1} F(x_{n-1}), \quad n \in \mathbb{N}. \end{cases} \tag{1}$$

The generalization of Newton's method to Banach spaces is due to the Russian mathematician Kantorovich, who published some papers in the mid-twenty century. Initially (see [1]), Kantorovich proves the semilocal convergence of Newton's method under the following conditions:

(C_1) $\|\Gamma_0\| \leq \beta$,
(C_2) $\|\Gamma_0 F(x_0)\| \leq \eta$,
(C_3) $\|F''(x)\| \leq k$, $x \in \Omega$,
(C_4) $k\beta\eta \leq \frac{1}{2}$,
(C_5) $\overline{B(x_0, s^*)} = \{x \in \Omega;\ \|x - x_0\| \leq s^*\} \subseteq \Omega$ with $s^* = \frac{1-\sqrt{1-2k\beta\eta}}{k\beta}$,

where the operator $\Gamma_0 = [F'(x_0)]^{-1} \in \mathcal{L}(Y, X)$ exists for some $x_0 \in \Omega$ and $\mathcal{L}(Y, X)$ is the set of bounded linear operators from Y to X.

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There are several techniques to prove the semilocal convergence of Newton's method. In this paper, we use the majorant principle to prove it, which is based on the concept of majorizing sequence. This technique was first developed in [1] and used later by many authors to analyse the semilocal convergence of various iterative methods (see [2–4]). The majority of results presented in the mathematical literature demand that the operator F'' is bounded in the domain Ω, where the solution x^* must exist. According to this, the number of equations that can be solved by Newton's method is limited, since it is not easy to see that F'' is bounded in a general domain Ω. It is not easy either to locate a domain where F'' is bounded and the solution x^* is contained.

The main aim of this paper is to generalize the semilocal convergence conditions given by Kantorovich for Newton's method, so that condition (C_3) is relaxed in order to Newton's method, can be applied to solve more equations. To do this, we follow a variation of Kantorovich's technique. In particular, we construct majorizing sequences *ad hoc*, so that these are adapted for particular problems, since the proposed modification of condition (C_3) gives more information about the operator F, not just that F'' is bounded, as (C_3) does.

We begin remembering in Section 2 what a majorizing sequence is and the classical result of Kantorovich. In this section, we introduce our modification of condition (C_3) and indicate how we can construct the new majorizing sequence. In Section 3, we give the new semilocal convergence of Newton's method which is based on the majorizing sequence constructed previously. We present in Section 4 how the domain of starting points, the domains of existence and uniqueness of solution and the a priori error bounds are improved from the variation of Kantorovich's technique with a simple example. Section 5 is dedicated to prove that the R-order of convergence [5] of Newton's method is at least two under our modification of condition (C_3). Next, some a priori error bounds are given in Section 6 that also prove the R-order of convergence at least two of Newton's method. Finally, an application is analysed from Newton's method in Section 7, where a well-known nonlinear integral equation is shown.

Throughout the paper, we denote $\overline{B(x,\sigma)} = \{y \in X;\ \|y-x\| \leq \sigma\}$ and $B(x,\sigma) = \{y \in X;\ \|y-x\| < \sigma\}$.

2. Preliminary analysis

In 1948, Kantorovich established, using recurrence relations [6], a result for Newton's method that is known as Kantorovich's theorem, that summarizes the basic result in Banach spaces about the semilocal convergence, error estimates and existence and uniqueness of solution. A year later Kantorovich gave another proof of the same result by using the majorant principle. This principle, based on majorizing sequences, is not the unique technique for studying the semilocal convergence of Newton's method in Banach spaces. Later, other authors have studied Newton's method in different contexts, as we can see in [7–11].

In this paper, we deal with the study of Kantorovich's result from majorant principle. Therefore, we begin introducing the concept of majorizing sequence and remembering how it is used to prove the convergence of sequences in Banach spaces.

Definition 1. If $\{x_n\}$ is a sequence in a Banach space X and $\{u_n\}$ is a scalar sequence, then $\{u_n\}$ is a *majorizing sequence* of $\{x_n\}$ if $\|x_n - x_{n-1}\| \leq u_n - u_{n-1}$, for all $n \in \mathbb{N}$.

Observe that it follows from the last inequality that the sequence $\{u_n\}$ is non-decreasing. The interest of the majorizing sequence is that the convergence of the sequence $\{x_n\}$ in X is deduced from the convergence of the scalar sequence $\{u_n\}$, as we can see in the following result [1].

Theorem 2. *Let $\{x_n\}$ be a sequence in a Banach space X and $\{u_n\}$ a majorizing sequence of $\{x_n\}$. Then, if $\{u_n\}$ converges to $u^* < \infty$, there exists $x^* \in X$ such that $x^* = \lim_n x_n$ and $\|x^* - x_n\| \leq u^* - u_n$, for $n = 0, 1, 2, \ldots$.*

From the definition of majorizing sequence and the last result, Kantorovich proved the following result for Newton's method.

Theorem 3 (*Kantorovich's Theorem*)**.** *Let $F : \Omega \subseteq X \to X$ be a twice continuously differentiable operator defined on a non-empty open convex domain Ω of a Banach space X with values in a Banach space Y. Suppose that conditions (C_1)–(C_5) are satisfied. Then, Newton's sequence defined in (1) and starting at x_0 converges to a solution x^* of the equation $F(x) = 0$. Moreover, $x_n, x^* \in B(x_0, s^*)$, for all $n = 0, 1, 2, \ldots$. Furthermore, if $k\beta\eta < \frac{1}{2}$, the solution x^* is unique in $B(x_0, s^{**}) \cap \Omega$, where $s^{**} = \frac{1+\sqrt{1-2k\beta\eta}}{k\beta}$, and if $k\beta\eta = \frac{1}{2}$, x^* is unique in $\overline{B(x_0, s^*)}$.*

As we have written in the Introduction, condition (C_3) limits the application of the previous theorem. In Section 7, we can see this restriction from an application. To improve what Kantorovich does, we replace condition (C_3) with the following milder condition

$$\|F''(x)\| \leq \omega(\|x\|), \quad x \in \Omega, \tag{2}$$

where $\omega : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}$ is a non-decreasing continuous function. Obviously, condition (2) generalizes (C_3).

On the other hand, from conditions (C_1)–(C_3), Kantorovich constructs a majorizing sequence by applying Newton's method,

$$s_0 \quad \text{given}, \qquad s_{n+1} = s_n - \frac{f(s_n)}{f'(s_n)}, \quad n = 0, 1, 2, \ldots,$$

to a scalar function f to be determined and such that

$$-\frac{1}{f'(s_0)} = \beta, \quad -\frac{f(s_0)}{f'(s_0)} = \eta \quad \text{and} \quad f''(s) = k, \quad s \in [s_0, s'],$$

where $s_0 \geq 0$, $s' \in \mathbb{R}$, $s' - s_0 \leq s^* - s_0$ and s^* is the smallest positive root of the equation $f(s) = 0$.

To do this, Kantorovich considers that f is a second-degree polynomial and fixes the coefficient of the polynomial by means of the three previous equalities to obtain that $f(s)$ is

$$p(s) = \frac{k}{2}(s - s_0)^2 - \frac{s - s_0}{\beta} + \frac{\eta}{\beta}. \tag{3}$$

Observe that this problem is of interpolation fitting.

In our case, if we consider (C_1), (C_2) and (2), we cannot obtain a scalar function f by interpolation fitting, as Kantorovich does, since (2) does not allow determining the class of functions where (C_1) and (C_2) can be applied. To solve this problem, we proceed differently. Observe that polynomial (3) can be obtained otherwise, without interpolation fitting, by solving the following initial value problem:

$$\begin{cases} p''(s) - k = 0, \\ p(s_0) = \dfrac{\eta}{\beta}, \qquad p'(s_0) = -\dfrac{1}{\beta}. \end{cases}$$

This new way of getting polynomial (3) has the advantage of being able to be generalized to conditions (C_1), (C_2) and (2). To do this, we first note that we have

$$\|F''(x)\| \leq \omega(\|x\|) \leq \omega(t - t_0 + \|x_0\|) \equiv \omega(t; t_0), \tag{4}$$

provided that $\|x\| - \|x_0\| \leq \|x - x_0\| \leq t - t_0$, since ω is non-decreasing. In consequence, instead of (2), we consider

$$\|F''(x)\| \leq \omega(t; t_0), \quad \text{when } \|x - x_0\| \leq t - t_0,$$

where $\omega : [t_0, +\infty) \to \mathbb{R}$ is a continuous non-decreasing function such that $\omega(t_0; t_0) \geq 0$. The corresponding initial value problem to solve is then

$$\begin{cases} y''(t) - \omega(t; t_0) = 0, \\ y(t_0) = \dfrac{\eta}{\beta}, \qquad y'(t_0) = -\dfrac{1}{\beta}. \end{cases} \tag{5}$$

In the next section, we calculate the solution function of problem (5), from which we construct the majorizing sequence involved in our study.

3. Semilocal convergence result

From the above-mentioned, we can establish the following result.

Theorem 4. *We suppose that $\omega(t; t_0)$ is continuous for all $t \in [t_0, t']$. Then, for any real numbers $\beta \neq 0$ and η, there exists only one solution $y(t)$ of initial value problem (5) in $[t_0, t']$, that is,*

$$f(t) = \int_{t_0}^{t} \int_{t_0}^{\theta} \omega(\xi; t_0)\, d\xi\, d\theta - \frac{t - t_0}{\beta} + \frac{\eta}{\beta}, \tag{6}$$

where ω is the function defined in (4).

Observe that (6) with $t_0 = s_0$ is reduced to polynomial (3) if ω is constant.

We can see in [1] that Kantorovich constructs a majorizing sequence $\{s_n\}$ from the application of Newton's method to polynomial (3) with $s_0 = 0$, so that $\{s_n\}$ converges to the smallest positive root s^* of the equation $p(s) = 0$. The convergence of the sequence is obvious, since the polynomial $p(s)$ is a decreasing convex function in $[s_0, s']$. Therefore, by analogy with Kantorovich, if we want to apply the technique of majorizing sequence to our particular problem, the equation $f(t) = 0$, where f is defined in (6), must have at least one root $>t_0$, so that we have to guarantee the convergence of the scalar sequence

$$t_n = t_{n-1} - \frac{f(t_{n-1})}{f'(t_{n-1})}, \quad n \in \mathbb{N}, \tag{7}$$

from t_0, to this root, for obtaining a majorizing sequence under conditions (C_1), (C_2) *y* (2). Clearly, the first we need is to analyse the function f defined in (6). Then, we give some properties of the function f.

Theorem 5. *Let f and ω be the functions defined respectively in (6) and (4).*

(*a*) *If there exists a solution $\alpha > t_0$ of the equation*

$$f'(t) = \int_{t_0}^{t} \omega(\xi; t_0)\, d\xi - \frac{1}{\beta} = 0, \tag{8}$$

then α is the unique minimum of f in $[t_0, +\infty)$ and f is non-increasing in $[t_0, \alpha)$.

(*b*) *If $f(\alpha) \le 0$, then the equation $f(t) = 0$ has at least one root in $[t_0, +\infty)$. Moreover, if t^* is the smallest root of $f(t) = 0$ in $[t_0, \infty)$, we have $t_0 < t^* \le \alpha$.*

As we are interested in the fact that (7) is a majorizing sequence of the sequence $\{x_n\}$ defined in (1), we establish the convergence of $\{t_n\}$ in the next result.

Theorem 6. *Let $\{t_n\}$ be the scalar sequence defined in (7), where the function f is given in (6). Suppose that there exist a solution $\alpha > t_0$ of Eq. (8) such that $f(\alpha) \le 0$. Then, the sequence $\{t_n\}$ is non-decreasing and converges to the root t^* of the equation $f(t) = 0$.*

The following is to prove that (7) is a majorizing sequence of sequence (1) and (1) is well-defined, provided that $B(x_0, t^* - t_0) \subseteq \Omega$. Previously, from (C_2), we observe that

$$\|x_1 - x_0\| = \|\Gamma_0 F(x_0)\| \le \eta = t_1 - t_0 < t^* - t_0.$$

Theorem 7. *Let f be the function defined in (6). Suppose that conditions (C_1), (C_2) and (2) are satisfied. Suppose also that $f(\alpha) \le 0$, where α is a solution of Eq. (8) such that $\alpha > t_0$, and $\overline{B(x_0, t^* - t_0)} \subseteq \Omega$. Then, $x_n \in B(x_0, t^* - t_0)$, for all $n \in \mathbb{N}$. Moreover, (7) is a majorizing sequence of sequence (1), namely*

$$\|x_n - x_{n-1}\| \le t_n - t_{n-1}, \quad \textit{for all } n \in \mathbb{N}.$$

Proof. We prove the theorem from the next four recurrence relations (for $n \in \mathbb{N}$).

(I_n) There exists $\Gamma_n = [F'(x_n)]^{-1}$ and $\|\Gamma_n\| \le -\frac{1}{f'(t_n)}$,
(II_n) $\|F(x_n)\| \le f(t_n)$,
(III_n) $\|x_{n+1} - x_n\| \le t_{n+1} - t_n$,
(IV_n) $\|x_{n+1} - x_0\| \le t^* - t_0$.

We begin proving (I_1)–(IV_1).

(I_1): from $x = x_0 + \tau(x_1 - x_0)$ and $t = t_0 + \tau(t_1 - t_0)$, where $0 \le \tau \le 1$, it follows $\|x - x_0\| = \tau\|x_1 - x_0\| \le \tau(t_1 - t_0) = t - t_0$, so that

$$\begin{aligned}
\|I - \Gamma_0 F'(x_1)\| &= \left\| \int_0^1 \Gamma_0 F''\Big(x_0 + t(x_1 - x_0)\Big)(x_1 - x_0)\, dt \right\| \\
&\le \|\Gamma_0\| \int_0^1 \omega\Big(t_0 + \tau(t_1 - t_0); t_0\Big)\, d\tau\, (t_1 - t_0) \\
&= \beta \int_{t_0}^{t_1} \omega(t; t_0)\, dt = 1 - \frac{f'(t_1)}{f'(t_0)} < 1,
\end{aligned}$$

since $\beta = -\frac{1}{f'(t_0)}$ and $\|x_1 - x_0\| \le t_1 - t_0$. Then, from Banach's lemma, we obtain that there exists Γ_1 and $\|\Gamma_1\| \le -\frac{1}{f'(t_1)}$.

(II_1): from Taylor's series, $\|x_1 - x_0\| \le t_1 - t_0$ and (1), we have

$$F(x_1) = \int_{x_0}^{x_1} F''(x)(x - x_0)\, dx = \int_0^1 F''\Big(x_0 + \tau(x_1 - x_0)\Big)(1 + \tau)\, d\tau\, (x_1 - x_0)^2$$

and

$$\begin{aligned}
\|F(x_1)\| &\le \int_0^1 \omega\Big(t_0 + \tau(t_1 - t_0); t_0\Big)(1 + \tau)\, d\tau\, (t_1 - t_0)^2 \\
&= \int_0^1 f''\Big(t_0 + \tau(t_1 - t_0)\Big)(1 + \tau)\, d\tau\, (t_1 - t_0)^2 = f(t_1).
\end{aligned}$$

(III_1): $\|x_2 - x_1\| \le \|\Gamma_1\|\|F(x_1)\| \le -\frac{f(t_1)}{f'(t_1)} = t_2 - t_1$.

(IV_1): $\|x_2 - x_0\| \le \|x_2 - x_1\| + \|x_1 - x_0\| \le t_2 - t_0 \le t^* - t_0$.

If we now suppose that (I_k)–(IV_k) are true for $k = 1, 2, \ldots, m$, we can prove that (I_{m+1})–(IV_{m+1}) are also true, so that (I_n)–(IV_n) are true for all $n \in \mathbb{N}$ by mathematical induction. □

We are then ready to prove the following semilocal convergence result for Newton's method under conditions (C_1), (C_2) and (2).

Theorem 8. *Let X and Y be two Banach spaces and $F : \Omega \subseteq X \to Y$ a nonlinear twice continuously differentiable operator on a non-empty open convex domain Ω. Suppose that (C_1), (C_2) and (2) are satisfied. Suppose also that $f(\alpha) \le 0$, where α is a solution of Eq. (8) such that $\alpha > t_0$, and $\overline{B(x_0, t^* - t_0)} \subseteq \Omega$. Then, Newton's sequence $\{x_n\}$, given by (1), converges to a solution x^* of $F(x) = 0$ starting at x_0. Moreover, $x_n, x^* \in \overline{B(x_0, t^* - t_0)}$ and*

$$\|x^* - x_n\| \le t^* - t_n, \quad n = 0, 1, 2, \ldots$$

Proof. Observe that $\{x_n\}$ is convergent, since $\{t_n\}$ is a majorizing sequence of $\{x_n\}$ and convergent. Moreover, as $\lim_{n\to+\infty} t_n = t^*$, if $x^* = \lim_{n\to+\infty} x_n$, then $\|x^* - x_n\| \le t^* - t_n$, for all $n = 0, 1, 2, \ldots$ Furthermore,

$$\begin{aligned}
\|F'(x_n) - F'(x_0)\| &= \left\| \int_0^1 F''\Big(x_0 + \tau(x_n - x_0)\Big)\, d\tau (x_n - x_0) \right\| \\
&\le \int_0^1 \omega\Big(t_0 + \tau(t_n - t_0); t_0\Big)\, d\tau (t^* - t_0) \\
&\le \omega(t^*; t_0)(t^* - t_0),
\end{aligned}$$

since $\|x - x_0\| \le t - t_0$ and $\|x_n - x_0\| \le t^* - t_0$, so that

$$\|F'(x_n)\| \le \|F'(x_0)\| + \omega(t^*; t_0)(t^* - t_0),$$

and consequently, the sequence $\{\|F'(x_n)\|\}$ is bounded. Therefore, from

$$\|F(x_n)\| \le \|F'(x_n)\| \|x_{n+1} - x_n\|,$$

it follows that $\lim_{n\to+\infty} \|F(x_n)\| = 0$, and, by the continuity of F, we obtain $F(x^*) = 0$. □

Once we have proved the semilocal convergence of Newton's method and located the solution x^*, we prove the uniqueness of x^*. First, we note that if $\omega(0) > 0$, then f' is increasing and $f'(t) > 0$ in $(\alpha, +\infty)$, so that f is strictly increasing in $(\alpha, +\infty)$. The last ensures that f has two real zeros t^* and t^{**} such that $t_0 < t^* \le t^{**}$. If $\omega(0) = 0$, ω must be strictly increasing for the function f to have two zeros. Note that the latter is not restrictive because only the lineal case is eliminated.

Theorem 9. *Under the conditions of Theorem 8, the solution x^* is unique in $B(x_0, t^{**} - t_0) \cap \Omega$ if $t^* < t^{**}$ or in $\overline{B(x_0, t^* - t_0)}$ if $t^* = t^{**}$.*

Proof. Suppose that $t^* < t^{**}$ and y^* is another solution of $F(x) = 0$ in $\overline{B(x_0, t^{**} - t_0)} \cap \Omega$. Then,

$$\|y^* - x_0\| \le \rho(t^{**} - t_0) \quad \text{with } \rho \in (0, 1).$$

We now suppose that $\|y^* - x_k\| \le \rho^{2^k}(t^{**} - t_k)$ for $k = 0, 1, \ldots, n$. Therefore,

$$\begin{aligned}
\|y^* - x_{n+1}\| &\le \|\Gamma_n\| \left\| \Big(F(y^*) - F(x_n) - F'(x_n)(y^* - x_n)\Big) \right\| \\
&\le \|\Gamma_n\| \int_0^1 \left\| F''\Big(x_n + t(y^* - x_n)\Big) \right\| (1 - t)\|y^* - x_n\|^2\, dt.
\end{aligned}$$

Since

$$\|x_n + t(y^* - x_n) - x_0\| \le \|x_n - x_0\| + t\|y^* - x_n\| \le t_n + t(t^{**} - t_n) - t_0$$

and $\|y^* - x_n\| \le \rho^{2^n}(t^{**} - t_n) < t^{**} - t_n$, it follows that

$$\begin{aligned}
\|y^* - x_{n+1}\| &\le -\frac{1}{f'(t_n)} \int_0^1 \omega\Big(t_n + t(t^{**} - t_0) - t_0 + \|x_0\|\Big)(1 - t)\, dt\, \|y^* - x_n\|^2 \\
&= -\frac{M}{f'(t_n)} \|y^* - x_n\|^2,
\end{aligned}$$

where $M = \int_0^1 \omega\,(t_n + t(t^{**} - t_n) - t_0 + \|x_0\|)\,(1 - t)\, dt = \int_0^1 \omega\,(t_n + t(t^{**} - t_n); t_0)\,(1 - t)\, dt$. On the other hand, since

$$\begin{aligned}
t^{**} - t_{n+1} &= -\frac{1}{f'(t_n)} \Big(f(t^{**}) - f(t_n) - f'(t_n)(t^{**} - t_n)\Big) \\
&= -\frac{1}{f'(t_n)} \int_0^1 f''\Big(t_n + t(t^{**} - t_n)\Big)(1 - t)(t^{**} - t_n)^2\, dt
\end{aligned}$$

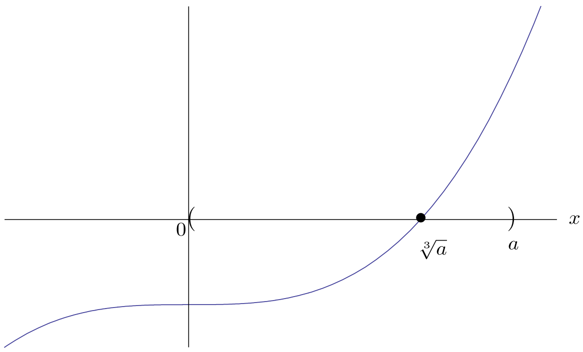

Fig. 1. $F(x) = x^3 - a$.

$$= -\frac{1}{f'(t_n)} \int_0^1 \omega\Big(t_n + t(t^{**} - t_n); t_0\Big)(1-t)\,dt\,(t^{**} - t_n)^2$$
$$= -\frac{M}{f'(t_n)}(t^{**} - t_n)^2,$$

we obtain

$$\|y^* - x_{n+1}\| \le \frac{t^{**} - t_{n+1}}{(t^{**} - t_n)^2}\|y^* - x_n\|^2 \le \rho^{2^{n+1}}(t^{**} - t_{n+1}),$$

so that $y^* = x^*$.

The case $t^* = t^{**}$ follows similarly to the previous one. □

Remark 10. Note that the function given in (6) is such that $f(t + t_0) = \varphi(t)$, where

$$\varphi(t) = \int_0^t \int_0^\theta \omega(\xi + \|\bar{x}_0\|)\,d\xi\,d\theta - \frac{t}{\beta} + \frac{\eta}{\beta}.$$

Therefore, the scalar sequences given by Newton's method with f and φ can be obtained, one from the other, by translation. In consequence, the last results are independent of the value t_0. For this reason, we always choose $t_0 = 0$, which simplifies considerably the expressions used. Observe that Kantorovich's polynomial (3) also has this property, so that Kantorovich always considers $s_0 = 0$; see [1].

4. Improvement of the domain of starting points, the domains of existence and uniqueness of solution and the a priori error bounds

In this section, by means of a simple example, we show that we can improve the domain of starting points, the domains of existence and uniqueness of solution and the a priori error bounds for Newton's method if we use condition (2) instead of condition (C_3).

Consider the equation $F(x) = 0$, where $F : \Omega = (0, a) \to \mathbb{R}$ and $F(x) = x^3 - a$ with $a > 1$. Then,

$$\|\Gamma_0\| = \frac{1}{3x_0^2} = \beta, \quad \|\Gamma_0 F(x_0)\| = \frac{|x_0^3 - a|}{3x_0^2} = \eta, \quad \|F''(x)\| = 6|x|.$$

In consequence, we have $k = 6a$ for Theorem 3 and $\omega(t; t_0) = \omega(t; 0) = \omega(t + \|x_0\|) = 6(t + |x_0|)$ for Theorem 8.

When analysing the domain of starting points for Newton's method from Theorems 3 and 8, we will only pay attention to the interval $(0, \sqrt[3]{a})$, since Newton's method always converges if we choose x_0 in the interval $(\sqrt[3]{a}, a)$, since F is increasing and convex in $(\sqrt[3]{a}, a)$; see Fig. 1.

For Theorem 3, we need that $k\beta\eta \le \frac{1}{2}$, which is equivalent to $3x_0^4 + 4ax_0^3 - 4a^2 \equiv g(x_0) \ge 0$, since $x_0 \in (0, \sqrt[3]{a})$. In addition, $x_0 \in (r^*, \sqrt[3]{a})$, where r^* is such that $g(r^*) = 0$. For Theorem 8, we need that

$$0 \ge f(\alpha) = (5 - 4\sqrt{2})|x_0|^3 + |x_0^3 - a|,$$

which is equivalent to $4(1 - \sqrt{2})x_0^3 + a \le 0$, since $x_0 \in (0, \sqrt[3]{a})$. Consequently, $x_0 \ge \sqrt[3]{\frac{a}{4(\sqrt{2}-1)}}$.

If we consider the particular case $a = 2011$, we obtain $x_0 \in (12.6026\ldots, \sqrt[3]{2011})$ for Theorem 3 and $x_0 \in (10.6670\ldots, \sqrt[3]{2011})$ for Theorem 8. Therefore, we improve the domain of starting points by Theorem 8 with respect to Kantorovich's Theorem 3.

Table 1
Absolute error and a priori error bounds for $x^* = \sqrt[3]{2011}$.

n	$\lvert x^* - x_n \rvert$	$\lvert t^* - t_n \rvert$	$\lvert s^* - s_n \rvert$
0	0.0122668...	0.0122908...	0.0152011...
1	0.0000119369...	0.0000119835...	0.00292236...
2	1.12905×10^{-11}	1.14214×10^{-11}	0.000156661 ...

Taking then, for example, $x_0 = 12.61$, we obtain $s^* = 0.01520\ldots$ and $s^{**} = 0.06387\ldots$ for Theorem 3, so that the domains of existence and uniqueness of solution are respectively

$$\{z \in (0, 2011);\ |z - x_0| \le 0.01520\ldots\} \quad \text{and} \quad \{z \in (0, 2011);\ |z - x_0| < 0.06387\ldots\}.$$

For Theorems 8 and 9, we have that $t^* = 0.01229\ldots$ and $t^{**} = 9.96796\ldots$ are the roots of the equation $f(t) = 0$, so that the domains of existence and uniqueness of solution are respectively

$$\{z \in (0, 2011);\ |z - x_0| \le 0.01229\ldots\} \quad \text{and} \quad \{z \in (0, 2011);\ |z - x_0| < 9.96796\ldots\}.$$

Therefore the domains of existence and uniqueness of solution that we have obtained from Theorems 8 and 9 are better than those obtained from Kantorovich's Theorem 3.

Finally, we also obtain better a priori error bounds (see Table 1), where $\{s_n\}$ denotes the majorizing sequence obtained from Kantorovich's polynomial (3) with $s_0 = 0$ and $\{t_n\}$ denotes the majorizing sequence obtained from the function f defined in (6) with $t_0 = 0$.

5. Local convergence and order of convergence

From the ideas of Dennis and Schnabel in [12], we obtain a local convergence result that leads to quadratic convergence of Newton's method under condition (2).

Theorem 11. *Let $F : \Omega \subseteq X \to Y$ be a nonlinear twice continuously differentiable operator on a non-empty open convex domain Ω of a Banach space X with values in a Banach space Y. Let x^* be a solution of $F(x) = 0$ such that the operator $[F'(x^*)]^{-1}$ exists, $B(x^*, r) \subseteq \Omega$ and $\|[F'(x^*)]^{-1}\| \le \gamma$, with $r, \gamma > 0$. Suppose that condition (2) is satisfied and there exists the smallest positive root R of the equation*

$$2\gamma\omega(\|x^*\| + t)t - 1 = 0. \tag{9}$$

Then, there exists $\varepsilon > 0$ such that Newton's sequence $\{x_n\}$ is well-defined and converges to x^* for every $x_0 \in B(x^*, \varepsilon)$. Moreover,

$$\|x^* - x_n\| < \frac{1}{2\varepsilon}\|x^* - x_{n-1}\|^2, \quad n \in \mathbb{N}. \tag{10}$$

Proof. Let $\varepsilon = \min\{r, R\}$. As $\|\widetilde{x} + t(x^* - \widetilde{x})\| = \|x^* + (1-t)(\widetilde{x} - x^*)\| \le \|x^*\| + \varepsilon$, we have

$$\begin{aligned}\|I - [F'(x^*)]^{-1}F'(\widetilde{x})\| &\le \|[F'(x^*)]^{-1}\| \int_0^1 \left\|F''\left(\widetilde{x} + t(x^* - \widetilde{x})\right)\right\| dt\,\|x^* - \widetilde{x}\| \\ &\le \|[F'(x^*)]^{-1}\|\omega(\|x^*\| + \varepsilon)\varepsilon,\end{aligned}$$

so that

$$\|I - [F'(x^*)]^{-1}F'(\widetilde{x})\| < \gamma\omega(\|x^*\| + R)R = \frac{1}{2} < 1,$$

as a consequence of $\varepsilon \le R$ and R satisfies (9). In consequence, by Banach's lemma, the operator $[F'(\widetilde{x})]^{-1}$ exists and $\|[F'(\widetilde{x})]^{-1}\| < 2\|[F'(x^*)]^{-1}\| \le 2\gamma$.

Therefore, if $x_0 \in B(x^*, \varepsilon)$, there exists $\Gamma_0 = [F'(x_0)]^{-1}$, $\|\Gamma_0\| \le 2\gamma$ and x_1 is well-defined. Moreover, as

$$x_1 - x^* = \Gamma_0 \int_0^1 F''(x_0 + t(x^* - x_0))(1-t)(x^* - x_0)^2\, dt,$$

then

$$\begin{aligned}\|x^* - x_1\| &\le \|\Gamma_0\| \int_0^1 \|F''(x_0 + t(x^* - x_0))\|(1-t)\, dt\,\|x^* - x_0\|^2 \\ &\le \|\Gamma_0\|\omega(\|x^*\| + \varepsilon)\frac{1}{2}\|x^* - x_0\|^2 \\ &< \gamma\omega(\|x^*\| + R)R\|x^* - x_0\| \\ &\le \frac{1}{2}\|x^* - x_0\|.\end{aligned}$$

Following now an inductive argument, we have

$$\|x^* - x_n\| < \gamma\omega(\|x^*\| + R)\|x^* - x_{n-1}\|^2 < \frac{1}{2}\|x^* - x_{n-1}\|, \quad n \in \mathbb{N},$$

and then, $\|x^* - x_n\| < \frac{1}{2^n}\|x^* - x_0\|$, $n \in \mathbb{N}$, so that $\lim_{n\to+\infty} x_n = x^*$.

On the other hand, (10) follows from $\gamma\omega(\|x^*\| + R) = \frac{1}{2R}$. □

Remark 12. From (10), it follows that Newton's method has Q-order of convergence [5] at least two. Moreover, if $\varepsilon < 2$, then

$$\|x_n - x^*\| < \frac{1}{2\varepsilon}\|x_{n-1} - x^*\|^2 \le \left(\frac{1}{2\varepsilon}\right)^{1+2+\cdots+2^{n-1}} \|x_0 - x^*\|^{2^n} = \left(\sqrt{\frac{\varepsilon}{2}}\right)^{2^n} \sqrt{2\varepsilon},$$

and consequently, Newton's method has R-order of convergence [5] at least two.

Remark 13. Note that if ω is constant (Kantorovich's case), R exists and is $R = \frac{1}{2\gamma k}$, which generalizes the result given by Dennis and Schnabel in [12].

After that, we illustrate the previous result with the following example given in [12].

Example 14. Let $F(x, y, z) = 0$ be a nonlinear system, where $F : \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ and $F(x, y, z) = (x, y^2 + y, e^z - 1)$. It is obvious that $(0, 0, 0) = \overline{x}^*$ is a solution of the system.

From F, we deduce

$$F'(\overline{x}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2y+1 & 0 \\ 0 & 0 & e^z \end{pmatrix} \quad \text{and} \quad F'(\overline{x}^*) = \operatorname{diag}\{1, 1, 1\},$$

where $\overline{x} = (x, y, z)$. Hence, $[F'(\overline{x}^*)]^{-1} = \operatorname{diag}\{1, 1, 1\}$ and $\gamma = 1$. Moreover,

$$F''(\overline{x}) = \left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^z \end{array}\right),$$

and consequently, $\|F''(\overline{x})\| \le \max\{2, e^{\|\overline{x}\|}\}$.

Now, we can consider two situations. First, if $\Omega = B(\overline{x}^*, r)$ with $r < \ln 2$, then $\|F''(\overline{x})\| \le 2$ and $\omega(t) = 2$, so that $R = \frac{1}{4}$. Therefore, Newton's method is convergent from every starting point $\overline{x} \in B\left(\overline{x}^*, \frac{1}{4}\right)$. And second, if $\Omega = B(\overline{x}^*, r)$ with $r \ge \ln 2$, then $\|F''(\overline{x})\| \le e^{\|\overline{x}\|}$, $\omega(t) = e^t$ and Eq. (9) is reduced to $2e^t t - 1 = 0$, whose unique solution is $R = 0.351734\ldots$ Therefore, Newton's method is convergent from every starting point $\overline{x} \in B(\overline{x}^*, 0.351734\ldots)$.

If we compare the last results with those of Dennis and Schnabel in [12], we can emphasize two things. First, if $r < \ln 2$, we obtain the same domain of starting points as Dennis and Schnabel. And second, if $r \ge \ln 2$, our results have two advantages with respect to Dennis and Schnabel. The first and most important is that our result is independent of the value r, while that of Dennis and Schnabel is not. The second is that we extend the domain of starting points obtained by Dennis and Schnabel, since $k = e^r$ and $\frac{1}{2\gamma k} = \frac{1}{2e^r} < 0.351734\ldots$ for all $r \ge \ln 2$.

6. A priori error estimates

If f has two real positive zeros t^* and t^{**} such that $t^* \le t^{**}$, we can then write

$$f(t) = (t^* - t)(t^{**} - t)g(t)$$

with $g(t^*) \ne 0$ and $g(t^{**}) \ne 0$. Next, we give a result which provides some a priori error estimates for Newton's method. Remember that we have written above how the function ω should be for f to have two real positive roots.

Theorem 15. *Suppose that the function f defined in (6) has two real positive roots t^* and t^{**}.*

(i) *If $t^* < t^{**}$, then*

$$\frac{(t^{**} - t^*)\theta^{2^n}}{\sqrt{m_1} - \theta^{2^n}} < t^* - t_n < \frac{(t^{**} - t^*)\Delta^{2^n}}{\sqrt{M_1} - \Delta^{2^n}}, \quad n \ge 0,$$

where $\theta = \frac{t^*}{t^{**}}\sqrt{m_1}$, $\Delta = \frac{t^*}{t^{**}}\sqrt{M_1}$, $m_1 = \min\{H_1(t);\, t \in [0, t^*]\}$, $M_1 = \max\{H_1(t);\, t \in [0, t^*]\}$, $H_1(t) = \frac{(t^{**}-t)g'(t)-g(t)}{(t^*-t)g'(t)-g(t)}$ and provided that $\theta < 1$ and $\Delta < 1$.

(ii) *If* $t^* = t^{**}$, *then*

$$m_2^n t^* \le t^* - t_n \le M_2^n t^*,$$

where $m_2 = \min\{H_2(t);\, t \in [0, t^*]\}$, $M_2 = \max\{H_2(t);\, t \in [0, t^*]\}$ *and* $H_2(t) = \frac{(t^*-t)g'(t)-g(t)}{(t^*-t)g'(t)-2g(t)}$ *and provided that* $m_2 < 1$ *and* $M_2 < 1$.

Proof. Let $t^* < t^{**}$ and denote $a_n = t^* - t_n$ and $b_n = t^{**} - t_n$ for all $n = 0, 1, 2, \ldots$. Then

$$f(t_n) = a_n b_n g(t_n), \qquad f'(t_n) = a_n b_n g'(t_n) - (a_n + b_n) g(t_n)$$

and

$$a_{n+1} = t^* - t_{n+1} = t^* - t_n + \frac{f(t_n)}{f'(t_n)} = \frac{{a_n}^2\Big(b_n g'(t_n) - g(t_n)\Big)}{a_n b_n g'(t_n) - (a_n + b_n) g(t_n)}.$$

From $\frac{a_{n+1}}{b_{n+1}} = \frac{a_n^2\left(b_n g'(t_n)-g(t_n)\right)}{{b_n}^2\left(a_n g'(t_n)-g(t_n)\right)}$, it follows

$$m_1 \left(\frac{a_n}{b_n}\right)^2 \le \frac{a_{n+1}}{b_{n+1}} \le M_1 \left(\frac{a_n}{b_n}\right)^2.$$

In addition,

$$\frac{a_{n+1}}{b_{n+1}} \le M_1^{\frac{2^{n+1}-1}{2}} \left(\frac{a_0}{b_0}\right)^{2^{n+1}} = \frac{\Delta^{2^{n+1}}}{\sqrt{M_1}} \qquad \text{and} \qquad \frac{a_{n+1}}{b_{n+1}} \ge m_1^{\frac{2^{n+1}-1}{2}} \left(\frac{a_0}{b_0}\right)^{2^{n+1}} = \frac{\theta^{2^{n+1}}}{\sqrt{m_1}}.$$

Taking then into account that $b_{n+1} = (t^{**} - t^*) + a_{n+1}$, it follows:

$$\frac{(t^{**} - t^*)\theta^{2^{n+1}}}{\sqrt{m_1} - \theta^{2^{n+1}}} < t^* - t_{n+1} < \frac{(t^{**} - t^*)\Delta^{2^{n+1}}}{\sqrt{M_1} - \Delta^{2^{n+1}}}.$$

If $t^* = t^{**}$, then $a_n = b_n$ and

$$a_{n+1} = \frac{a_n\Big(a_n g'(t) - g(t)\Big)}{a_n g'(t) - 2g(t_n)}.$$

Consequently, $m_2 a_n \le a_{n+1} \le M_2 a_n$ and

$$m_2^{n+1} t^* \le t^* - t_{n+1} \le M_2^{n+1} t^*. \quad \square$$

Remark 16. From $t^* < t^{**}$ in the last theorem, it follows that the order of convergence of Newton's method is two, while it is one if $t^* = t^{**}$.

7. Application

We illustrate the theory developed in this paper with the following Bratu's equation

$$x(s) = \int_0^1 G(s, t) e^{x(t)}\, dt, \quad s \in [0, 1], \tag{11}$$

where $G(s, t)$ is the Green function

$$G(s, t) = \begin{cases} (1 - s)t, & t \le s, \\ s(1 - t), & s \le t. \end{cases}$$

Problems of this type are usual in science and engineering for modelling complicated problems, as we can see in [13] and in its references. For example, Bratu's equation is usually used to model a combustion problem in a numerical slab. It is well-known that Eq. (11) has two real and distinct solutions (see [14]), which are approximated later.

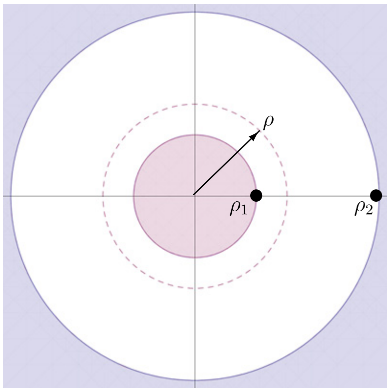

Fig. 2. $\|\bar{x}^*\|_\infty \in [0, \rho_1] \cup [\rho_2, +\infty)$.

To illustrate the theory developed in the previous sections we first discretize Eq. (11) to transform it into a finite-dimension problem. To do this, we approximate the integral of (11) by the following Gauss–Legendre quadrature formula with m nodes:

$$\int_a^b f(t)\,dt \simeq \sum_{i=1}^{m} c_i f(t_i),$$

where the nodes t_i and the weights c_i are known. Next, we denote the approximation of $x(t_i)$ by x_i, $i = 1, 2, \ldots, m$, and Eq. (11) is now equivalent to the following nonlinear system of equations:

$$x_i = \sum_{j=1}^{m} a_{ij} e^{x_j}, \quad i = 1, 2, \ldots, m, \text{ where } a_{ij} = \begin{cases} c_j t_j (1 - t_i) & \text{if } j \le i, \\ c_j t_i (1 - t_j) & \text{if } j > i, \end{cases}$$

that can be written as

$$F(\bar{x}) = \bar{x} - A\bar{v}_x = 0,$$

where $\bar{x} = (x_1, \ldots, x_m)^T$, $A = (a_{ij})$ and $\bar{v}_x = (e^{x_1}, \ldots, e^{x_m})^T$.

For this operator F we obtain that

$$F'(\bar{x}) = I - AD(\bar{x}), \qquad F''(\bar{x})\bar{y}\bar{z} = -A(e^{x_1} y_1 z_1, \ldots, e^{x_m} y_m z_m)^T,$$

where $D(\bar{x}) = \operatorname{diag}\{e^{x_1}, e^{x_2}, \ldots, e^{x_n}\}$, $\bar{y} = (y_1, y_2, \ldots, y_m)^T$ and $\bar{z} = (z_1, z_2, \ldots, z_m)^T$. Observe that $\|F''(\bar{x})\|_\infty$ is not bounded, since $\|F''(\bar{x})\|_\infty \le \|A\|_\infty e^{\|\bar{x}\|_\infty}$ and the function $e^{\|\bar{x}\|_\infty}$ is increasing. In consequence, Kantorovich's Theorem 3 cannot be applied. Even if F' were of Lipschitz-type, Theorem 3 could not be either applied.

If we were interested in applying Theorem 3, we could locate a root in some domain and look for a bound for $\|F''(\bar{x})\|_\infty$ there (see [15]). In this case, if $\bar{x}^*$ is a root of $F(\bar{x}) = 0$, we have $\|\bar{x}^*\|_\infty \in [0, \rho_1] \cup [\rho_2, +\infty]$, where ρ_1 and ρ_2 $(0 < \rho_1 < \rho_2)$ are the two positive real roots of the scalar equation $t - \|A\|_\infty e^t = 0$. See Fig. 2.

Observe that we can only approximate the solution $\bar{x}^*$ such that $\|\bar{x}^*\|_\infty \in [0, \rho_1]$ from Theorem 3. To do this, we choose a starting point $\bar{x}_0$ such that $\bar{x}_0 \in B(0, \rho)$ with $\rho \in (\rho_1, \rho_2)$.

For Eq. (11), we have $\rho_1 = 0.14248\ldots$ and $\rho_2 = 3.27839\ldots$. If we choose $\rho = 3$, $m = 8$ and $\bar{x}_0 = \bar{0} = (0, \ldots, 0)^T$, we see that the conditions of Theorem 3 hold, since

$$\|F''(\bar{x})\|_\infty \le (0.123559\ldots)e^{t+\|\bar{x}_0\|_\infty} = 2.48175\ldots = k,$$

$$\beta = 1.13821\ldots, \qquad \eta = 0.138214\ldots \quad \text{y } k\beta\eta = 0.390423\ldots < \frac{1}{2}.$$

Moreover, $s^* = 0.1882\ldots$ and $B(\bar{x}_0, s^*) \subseteq \Omega = B(\bar{0}, \rho)$. Therefore, Newton's sequence converges to the solution $\bar{x}^* = (x_1^*, x_2^*, \ldots, x_8^*)^T$, which is given in Table 2 and obtained after three iterations of Newton's method.

If we now interpolate the values of Table 2 and take into account that Eq. (11) satisfies $x(0) = x(1) = 0$, we obtain the solution drawn in Fig. 3.

On the other hand, if we consider Theorem 8 and choose $t_0 = 0$, we obtain

$$\|F''(\bar{x})\|_\infty \le (0.123559\ldots)\, e^{t+\|\bar{x}_0\|_\infty},$$

and consequently,

$$f(t) = (0.123559\ldots)\, e^{\|\bar{x}_0\|_\infty}(e^t - 1 - t) - \frac{t}{(1.13821\ldots)} + (0.121431\ldots)$$

Moreover, as the condition $f(\alpha) \le 0$, where $\alpha = 2.09317\ldots$, is satisfied, the convergence of Newton's sequence is also guaranteed from Theorem 8.

Table 2
Numerical solution $\overline{x}^*$ of (11).

n	x_n^*
1	0.010934250179...
2	0.051801839563...
3	0.103646768847...
4	0.139301422764...
5	0.139301422764...
6	0.103646768847...
7	0.051801839563...
8	0.010934250179...

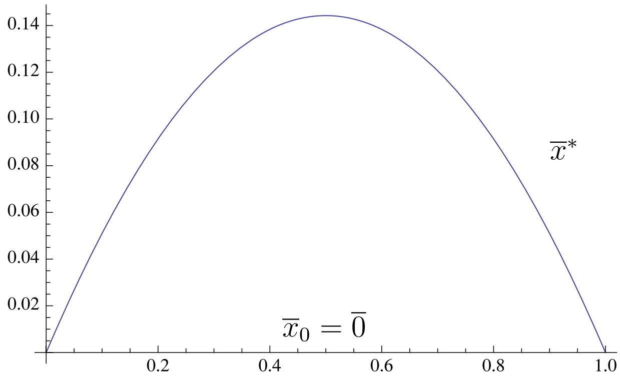

Fig. 3. Solution $\overline{x}^*$ of (11).

After that, we have $s^* = 0.188285\ldots$ and $s^{**} = 0.519739\ldots$ for Theorem 3, so that the domains of existence and uniqueness of solution are respectively

$$\{\overline{\zeta} \in \mathbb{R}^8;\ \|\overline{\zeta}\|_\infty \le 0.188285\ldots\} \quad \text{and} \quad \{\overline{\zeta} \in \mathbb{R}^8;\ \|\overline{\zeta}\|_\infty < 0.519739\ldots\}.$$

For Theorems 8 and 9, we obtain $t^* = 0.139652\ldots$ and $t^{**} = 3.28237\ldots$. In addition, the domains of existence and uniqueness of solution are respectively

$$\{\overline{\zeta} \in \mathbb{R}^8;\ \|\overline{\zeta}\|_\infty \le 0.139652\ldots\} \quad \text{and} \quad \{\overline{\zeta} \in \mathbb{R}^8;\ \|\overline{\zeta}\|_\infty < 3.28237\ldots\}.$$

We then see that the domains obtained from Kantorovich's Theorem 3 are improved by Theorems 8 and 9, since the domain of existence of solution is smaller and the domain of uniqueness of solution is bigger. We also see that the domain of existence of solution obtained from Theorem 8 is very precise because three decimal figures are fixed.

About the second solution, denoted by $\overline{x}^{**}$ and such that $\|\overline{x}^{**}\|_\infty \ge \rho_2 = 3.27839\ldots$, we cannot locate a domain where $\overline{x}^{**}$ lies by Kantorovich's Theorem 3, as we can see in the following. If for example we choose the point $\overline{x}_0 = \overline{4} = (4, 4, \ldots, 4)^T$, which is outside of the corona shown in Fig. 2, we cannot apply Kantorovich's Theorem 3 to guarantee the convergence of Newton's sequence to $\overline{x}^{**}$, since we cannot choose a domain where $\overline{x}^{**}$ lies, so that we cannot bound $\|F''(\overline{x})\|$. If we try to apply Theorem 8, we see that condition $f(\alpha) \le 0$ is not satisfied, since $\alpha = 0.03142\ldots$ and $f(\alpha) = 1.17917\ldots > 0$, so that we cannot apply Theorem 8 either. However, it seems clear that the conditions of Theorem 8 can be satisfied if the starting point is improved. So, after six iterations of Newton's method, we obtain the point

$$\overline{x}_6 = \begin{pmatrix} 0.206543500522\ldots \\ 1.052503463129\ldots \\ 2.411149655559\ldots \\ 3.827142962625\ldots \\ 3.827142962625\ldots \\ 2.411149655559\ldots \\ 1.052503463129\ldots \\ 0.206543500522\ldots \end{pmatrix},$$

which is used as new starting point $\overline{z}_0$ for Newton's method, so that the hypotheses of Theorem 8 are now satisfied for $\overline{z}_0$:

$$\beta = 3.26003\ldots, \qquad \eta = 0.00820\ldots, \qquad \alpha = 0.05264\ldots \quad \text{and} \quad f(\alpha) = -0.00567\ldots \le 0.$$

Choosing $\widetilde{t}_0 = 0$, the new scalar function given in (6) for Theorem 8 is

$$\widetilde{f}(t) = (0.123559\ldots)\, \mathrm{e}^{(3.827142\ldots)}(\mathrm{e}^t - 1 - t) - \frac{t}{(3.26003\ldots)} + (0.002515\ldots),$$

Table 3
Numerical solution $\overline{x}^{**}$ of (11).

n	x_n^{**}
1	0.206096559495...
2	1.050217105725...
3	2.405892177578...
4	3.818891992567...
5	3.818891992567...
6	2.405892177578...
7	1.050217105725...
8	0.206096559495...

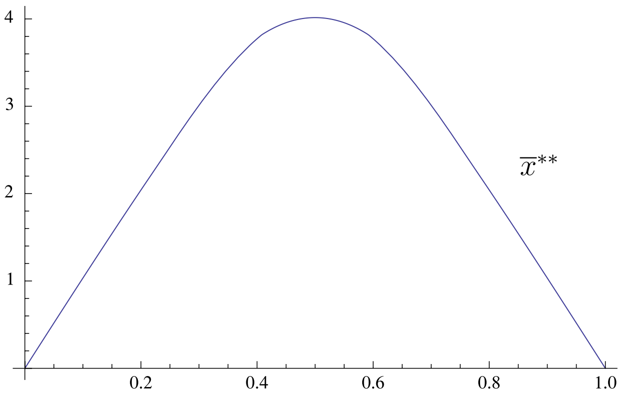

Fig. 4. Solution $\overline{x}^{**}$ of (11).

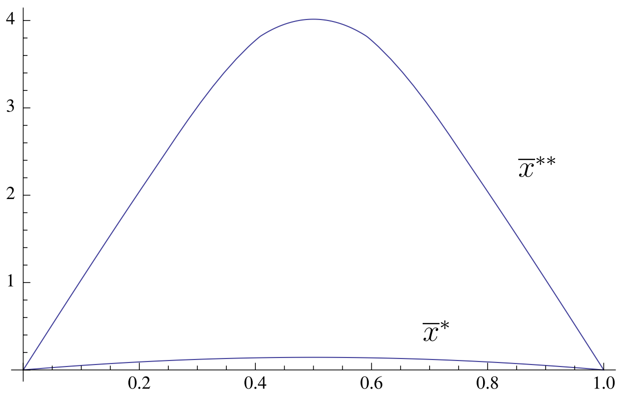

Fig. 5. Solutions $\overline{x}^{*}$ and $\overline{x}^{**}$ of (11).

which has two real positive zeros $\widetilde{t}^* = 0.00894867\ldots$ and $\widetilde{t}^{**} = 0.09571\ldots$. Then, the domains of existence and uniqueness of solution obtained from Theorems 8 and 9 are respectively

$$\{\overline{\zeta} \in \mathbb{R}^8;\ \|\overline{\zeta} - \overline{z}_0\|_\infty \leq 0.00894\ldots\} \quad \text{and} \quad \{\overline{\zeta} \in \mathbb{R}^8;\ \|\overline{\zeta} - \overline{z}_0\|_\infty < 0.09571\ldots\}.$$

Observe again the precision of the domain of existence which fixes two decimal figures. Moreover, the domain of uniqueness is very small due to the proximity of $\overline{z}_0$ to the solution $\overline{x}^{**}$ in the vicinity of the points $\overline{x} = 0$ and $\overline{x} = 1$.

Note that $\|\overline{z}_0\|_\infty = 3.82714\ldots > \rho_2 = 3.27839\ldots$. Taking then $\overline{z}_0$ as the new starting point for Newton's method, the second solution $\overline{x}^{**} = (x_1^{**}, x_2^{**}, \ldots, x_8^{**})^T$ of (11) is approximated after three more iterations and given in Table 3.

By interpolating the values of Table 3 and taking into account that Eq. (11) is zero in $s = 0$ y $s = 1$, the solution drawn in Fig. 4 is obtained.

Finally, both solutions $\overline{x}^{*}$ and $\overline{x}^{**}$ of Eq. (11) are drawn together in Fig. 5.

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References

[1] L.V. Kantorovich, G.P. Akilov, Functional Analysis, Pergamon Press, Oxford, 1982.
[2] I.K. Argyros, An improved convergence analysis and applications for Newton-like methods in Banach space, Numerical Functional Analysis and Optimization 24 (7–8) (2003) 653–672.
[3] I.K. Argyros, On the Newton–Kantorovich hypothesis for solving equations, Journal of Computational and Applied Mathematics 169 (2) (2004) 315–332.
[4] T. Yamamoto, Convergence theorem for Newton-like methods in Banach spaces, Numerische Mathematik 51 (1987) 545–557.
[5] F.A. Potra, V. Pták, Nondiscrete Induction and Iterative Methods, Pitman Publishing Limited, London, 1994.
[6] L.V. Kantorovich, On Newton's method for functional equations, Doklady Akademii Nauk SSSR 59 (1948) 1237–1240 (in Russian).
[7] G. Alefeld, J. Herzberger, Introduction to Interval Computations, Academic Press, London, 1983.
[8] F. Cianciaruso, E. De Pascale, P.P. Zabreiko, Some remarks on the Newton–Kantorović approximations, Atti del Seminario Matematico e Fisico dell'Università di Modena 48 (2000) 207–215.
[9] R.E. Moore, Methods and Applications of Interval Analysis, in: SIAM Studies in Applied Mathematics, 1979.
[10] F.A. Potra, V. Pták, Sharp error bounds for Newton process, Numerische Mathematik 34 (1980) 63–72.
[11] W.C. Rheinboldt, A unified convergence theory for a class of iterative processes, SIAM Journal on Numerical Analysis 5 (1968) 42–63.
[12] J.E. Dennis, R.B. Schnabel, Numerical Methods for Uunconstrained Optimization and Nonlinear Equations, SIAM, Philadelphia, 1996.
[13] H. Caglar, N. Caglar, M. Özer, A. Valaristos, A.N. Miliou, A.N. Anagnostopoulos, Dynamics of the solution of Bratu's equation, Nonlinear Analysis 71 (2007) 672–678.
[14] H.T. Davis, Introduction to Nonlinear Differential and Integral Equations, Dover Publications, New York, 1962.
[15] J.A. Ezquerro, M.A. Hernández, Halley's method for operators with unbounded second derivative, Applied Numerical Mathematics 57 (3) (2007) 354–360.