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Poincaré and Opial inequalities for vector-valued convolution products

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ABSTRACT

Various L^p form Poincaré and Opial inequalities are given for vector-valued convolution products. We apply our results to infinitesimal generators of C_0-semigroups and cosine functions. Typical examples of these operators are differential operators in Lebesgue spaces.

1. Introduction

Recently in [1] the authors have proven the following: let $1 < q < \infty$ and $-\infty < a < b < \infty$. The best constant C (independent of a and b) for which the Poincaré inequality

$$\left\| f - \frac{1}{b-a}\int_a^b f(t)dt \right\|_{L^1([a,b])} \leq C(b-a)^{2-\frac{1}{q}} \|f\|_{L^q([a,b])}$$

holds for all Lipschitz continuous functions f is $C = \frac{1}{2}(1+p)^{\frac{-1}{p}}$ where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. In [2] the author has proved analogous Poincaré like inequalities for natural powers of infinitesimal generators of semigroups and cosine functions.

In 1960, Opial [3] shown the following inequality: let $f \in C^{(1)}([0,h])$ be such that $f(0) = f(h) = 0$ and $f(t) > 0$ in $(0,h)$. Then the following inequality holds

$$\int_0^h |f(t)|\,|f'(t)|dt \leq \frac{h}{4}\int_0^h |f'(t)|^2 dt.$$

It is known that $\frac{h}{4}$ is the best constant in the previous inequality. In [4,5], Opial like inequalities for natural powers of infinitesimal generators of semigroups and cosine functions are given respectively.

In this paper we consider a new point of view in these problems. We work with scalar and vector-valued functions and we prove Poincaré (in the second section) and Opial inequalities (in the third one) for convolution products. In the third and fourth sections, we apply our results to fractional powers of infinitesimal generators of C_0-semigroups and cosine functions. We extend some theorems which appear in [4,5,2].

We are reminded that the Lebesgue space $L^p([a,b])$ with the norm

$$\|f\|_{L^p([a,b])} := \left(\int_a^b |f(t)|^p\right)^{\frac{1}{p}} dt, \quad f \in L^p([a,b]),$$

for $p \geq 1$, is a Banach space and the usual convolution product, $*$, is given by

$$f * g(t) := \int_0^t f(t-s)g(s)ds, \quad t \geq 0,\ f, g \in L^1(\mathbb{R}^+).$$

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2. Poincaré like inequalities for convolution products

In the next definition, we fix a notation which we follow in the paper. In fact the function δ_α is the α-fractional integral (in the sense of Riemann–Liouville) of the function δ.

Definition 1. Let $\delta : [0, +\infty) \to X$ (or $(\delta(u))_{u\geq 0} \subset X$) a continuous function. Then we define

$$\delta_\alpha(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1}\delta(u)du, \quad t > 0,$$

for $\alpha > 0$.

The operator δ_α may be written by the usual convolution $\delta_\alpha = j_\alpha * \delta$, where $j_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$. The operator δ_α has been widely studied for $\alpha \in \mathbb{N}$ and $\alpha > 0$, see for example [6].

Now consider $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 2. *Let $(\delta(u))_{u\geq 0} \subset X$ be a continuous map and $\alpha > 0$. Then*

$$\|\delta_\alpha(t)\|_X \leq \frac{t^{\alpha-1+\frac{1}{p}}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \left(\int_0^t \|\delta(s)\|_X^q ds\right)^{\frac{1}{q}}, \quad t \geq 0,$$

for $0 < \alpha < 1$ and $1 < p < \frac{1}{1-\alpha}$; and for $\alpha \geq 1$ and $1 < p < \infty$.

Proof. Take $0 < t$, and $1 < p$ such that $p(\alpha - 1) > -1$. By the Hölder inequality, we get that

$$\begin{aligned}\|\delta_\alpha(t)\|_X &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\|\delta(s)\|_X ds \\ &\leq \frac{1}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{p(\alpha-1)}ds\right)^{\frac{1}{p}} \left(\int_0^t \|\delta(s)\|_X^q ds\right)^{\frac{1}{q}} \\ &\leq \frac{t^{\alpha-1+\frac{1}{p}}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \left(\int_0^t \|\delta(s)\|_X^q ds\right)^{\frac{1}{q}},\end{aligned}$$

and we conclude the proof. □

The next result is the main one of this section. They are Poincaré type inequalities for certain convolution products. We are reminded that

$$\| \|\delta_\alpha(t)\|_X \|_{L^\nu([0,a])} := \left(\int_0^a \|\delta_\alpha(t)\|_X^\nu dt\right)^{\frac{1}{\nu}}.$$

Theorem 3. *Let $(\delta(u))_{u\geq 0} \subset X$ be a continuous map and $\alpha > 0$. Then*

$$\| \|\delta_\alpha(t)\|_X \|_{L^\nu([0,a])} \leq \frac{a^{\alpha-1+\frac{1}{p}+\frac{1}{\nu}}}{\Gamma(\alpha)(1-(1-\alpha)p)^{\frac{1}{p}} \left(\nu\left(\frac{1}{p}+\alpha-1\right)+1\right)^{\frac{1}{\nu}}} \| \|\delta(t)\|_X \|_{L^q([0,a])}$$

for $a > 0$, in the case $0 < \alpha \leq 1$ with $1 < p < \frac{1}{1-\alpha}$; and in the case $\alpha \geq 1$ with $1 < p < \infty$; $\frac{1}{p} + \frac{1}{q} = 1$, and $\nu \geq 1$.

Proof. Take $0 < t \leq a$, and $1 < p$ such that $p(\alpha - 1) > -1$ with $\frac{1}{p} + \frac{1}{q} = 1$. By Lemma 2, we get that

$$\|\delta_\alpha(t)\|_X \leq \frac{t^{\alpha-1+\frac{1}{p}}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \left(\int_0^a \|\delta(s)\|_X^q ds\right)^{\frac{1}{q}}.$$

For $\nu \geq 1$, we have that

$$\begin{aligned}\| \|\delta_\alpha(t)\|_X \|_{L^\nu([0,a])}^\nu &= \int_0^a \|\delta_\alpha(t)\|_X^\nu dt \\ &\leq \frac{a^{\nu(\alpha-1+\frac{1}{p})+1}}{\Gamma(\alpha)^\nu (p(\alpha-1)+1)^{\frac{\nu}{p}} (\nu(\alpha-1+\frac{1}{p})+1)} \| \|\delta(t)\|_X \|_{L^q([0,a])}^\nu,\end{aligned}$$

and we conclude the proof. □

In the case $\nu = q$ in the previous theorem, we obtain the following result.

Corollary 4. *Let $(\delta(u))_{u\geq 0} \subset X$ be a continuous map, $q > 1$ and $\alpha > \frac{1}{q}$. Then*

$$\| \|\delta_\alpha(t)\|_X \|_{L^q([0,a])} \leq \frac{a^\alpha}{\Gamma(\alpha)\left(\frac{q\alpha-1}{q-1}\right)^{\frac{q-1}{q}}(q\alpha)^{\frac{1}{q}}} \| \|\delta(t)\|_X \|_{L^q([0,a])},$$

for $a > 0$.

The following result is similar to [2, Corollary 7].

Corollary 5. *In the case $\alpha > \frac{1}{2}$, and $\nu \geq 1$ we get that*

$$\| \|\delta_\alpha(t)\|_X \|_{L^\nu([0,a])} \leq \frac{a^{\alpha-\frac{1}{2}+\frac{1}{\nu}}}{\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}(\nu(\alpha-\frac{1}{2})+1)^{\frac{1}{\nu}}} \| \|\delta(t)\|_X \|_{L^2([0,a])};$$

and if $\nu = 2$, then

$$\| \|\delta_\alpha(t)\|_X \|_{L^2([0,a])} \leq \frac{a^\alpha}{\Gamma(\alpha)(2\alpha(2\alpha-1))^{\frac{1}{2}}} \| \|\delta(t)\|_X \|_{L^2([0,a])}.$$

Now, we consider the L^1-case. Note that the techniques that we use are different from [2, Theorem 8].

Theorem 6. *If $1 \leq \nu < \frac{1}{1-\alpha}$ for $0 < \alpha < 1$; and $1 \leq \nu < \infty$ if $\alpha \geq 1$, we have that*

$$\| \|\delta_\alpha(t)\|_X \|_{L^\nu([0,a])} \leq \frac{a^{\alpha-1+\frac{1}{\nu}}}{\Gamma(\alpha)(\nu(\alpha-1)+1)^{\frac{1}{\nu}}} \| \|\delta(t)\|_X \|_{L^1([0,a])}.$$

Proof. By Minkowski's inequality, we have that

$$\begin{aligned}
\| \|\delta_\alpha(t)\|_X \|_{L^\nu([0,a])} &= \left(\int_0^a \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\delta(s)ds \right\|_X^\nu dt\right)^{\frac{1}{\nu}} \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^a \|\delta(s)ds\|_X \left(\int_s^a (t-s)^{\nu(\alpha-1)}dt\right)^{\frac{1}{\nu}} ds \\
&\leq \frac{a^{\alpha-1+\frac{1}{\nu}}}{\Gamma(\alpha)(\nu(\alpha-1)+1)^{\frac{1}{\nu}}} \| \|\delta(t)\|_X \|_{L^1([0,a])},
\end{aligned}$$

and we conclude the proof. □

In the case of $\nu = 1$ in the previous theorem, we obtain the following result.

Corollary 7. *For $\alpha, a > 0$, we have that*

$$\| \|\delta_\alpha(t)\|_X \|_{L^1([0,a])} \leq \frac{a^\alpha}{\Gamma(\alpha+1)} \| \|\delta(t)\|_X \|_{L^1([0,a])}.$$

3. Opial type inequalities for convolution products

The next result is an Opial type inequality for vector-valued convolution products and is inspired by [4, Theorem 8].

Theorem 8. *Let $(\delta(u))_{u\geq 0} \subset X$ be a continuous function and $\alpha > 0$. Then*

$$\int_0^t \|\delta_\alpha(s)\|_X \|\delta(s)\|_X ds \leq \frac{t^{\alpha-1+\frac{2}{p}}}{2^{\frac{1}{q}}\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}(p(\alpha-1)+2)^{\frac{1}{p}}} \left(\int_0^t \|\delta(s)\|_X^q ds\right)^{\frac{2}{q}}$$

for $t > 0$, $1 < p$, $p(\alpha-1) > -1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By Lemma 2, we get

$$\|\delta_\alpha(t)\|_X \leq \frac{t^{\alpha-1+\frac{1}{p}}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}}\,(z(t))^{\frac{1}{q}}$$

where $z(t) := \int_0^t \|\delta(s)\|_X^q ds$, for $t > 0$. Note that $(z'(t))^{\frac{1}{q}} = \|\delta(t)\|_X$, for $t > 0$.

Consequently, we have that

$$\int_0^t \|\delta_\alpha(s)\|_X \|\delta(s)\|_X ds \leq \frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \int_0^t s^{\alpha-1+\frac{1}{p}} \left(z(s)z'(s)\right)^{\frac{1}{q}} ds,$$

for $t > 0$. Now we apply Minkowski's inequality to get

$$\begin{aligned}\int_0^t s^{\alpha-1+\frac{1}{p}} \left(z(s)z'(s)\right)^{\frac{1}{q}} ds &\leq \left(\int_0^t s^{p(\alpha-1)+1}\right)^{\frac{1}{p}} \left(\int_0^t z(s)z'(s)ds\right)^{\frac{1}{q}} \\ &\leq \frac{t^{\alpha-1+\frac{2}{p}}}{(p(\alpha-1)+2)^{\frac{1}{p}}2^{\frac{1}{q}}} \left(z^2(t)-z^2(0)\right)^{\frac{1}{q}} = \frac{t^{\alpha-1+\frac{2}{p}}}{(p(\alpha-1)+2)^{\frac{1}{p}}2^{\frac{1}{q}}}\,(z(t))^{\frac{2}{q}},\end{aligned}$$

and we conclude the result. □

Corollary 9. *Take $\alpha > \frac{1}{2}$. Then*

$$\int_0^t \|\delta_\alpha(s)\|_X \|\delta(s)\|_X ds \leq \frac{t^\alpha}{2\Gamma(\alpha)(\alpha(2\alpha-1))^{\frac{1}{2}}} \left(\int_0^t \|\delta(s)\|_X^2 ds\right)$$

for $t > 0$.

Now we consider the L^∞-case and the norm given by

$$\|\,\|\delta(t)\|_X\,\|_\infty := \sup_{t>0} \|\delta(t)\|_X.$$

The next result is similar to [4, Theorem 10].

Theorem 10. *Take $\alpha > 0$. Then*

$$\int_0^t \|\delta_\alpha(s)\|_X \|\delta(s)\|_X ds \leq \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\|\,\|\delta(t)\|_X\,\|_\infty,$$

for $t > 0$.

Proof. It is straightforward to check that

$$\|\delta_\alpha(t)\|_X \leq \frac{t^\alpha}{\Gamma(\alpha+1)}\|\,\|\delta(t)\|_X\,\|_\infty,$$

and then

$$\int_0^t \|\delta_\alpha(s)\|_X \|\delta(s)\|_X ds \leq \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\|\,\|\delta(t)\|_X\,\|_\infty,$$

for $t > 0$. □

4. Applications to fractional powers of infinitesimal generators of C_0-semigroups

Take $0 < \alpha < 1$. Functions $(f_{t,\alpha})_{t>0}$ are defined by the identity

$$f_{t,\alpha}(s) := \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{zs}e^{-tz^\alpha}dz, \quad s > 0,\ \sigma > 0,$$

i.e. $\mathcal{L}(f_{t,\alpha})(z) = e^{-tz^\alpha}$ for $z \in \mathbb{C}^+$, see[7, Section IX,11] (where $\mathcal{L}$ is the usual Laplace transform). These functions are sometimes called *Levy stable density functions*. The case $\alpha = \frac{1}{2}$ is the well-known *backwards heat semigroup*, $f_{t,\frac{1}{2}} = c^t$,

$$c^t(s) = \frac{t}{2\sqrt{\pi}s^{3/2}}e^{\frac{-t^2}{4s}}, \quad t, s > 0$$

see more details in [7]. Functions $(f_{t,\alpha})_{t>0}$ satisfy $f_{t,\alpha}(\lambda) \geq 0$, $(f_{t,\alpha})_{t>0} \subset L^1(\mathbb{R}^+)$, $\|f_{t,\alpha}\|_1 = 1$ and

$$f_{t,\alpha} * f_{s,\alpha} = f_{t+s,\alpha}, \quad t, s \geq 0,$$

with $0 < \alpha < 1$ [7].

A C_0-semigroup is a family $(T(t))_{t\geq 0}$ of bounded linear operators from a Banach space X into itself satisfying $T(0) = I$,

$$T(t)T(s) = T(t+s), \quad t, s > 0,$$

and the map $t \mapsto T(t)x$ is continuous for $x \in X$. The C_0-semigroup is exponentially bounded and its infinitesimal generator $(A, D(A))$ is the closed operator from X into itself defined as

$$Ax := \lim_{t\to 0} \frac{1}{t}(T(t)x - x), \quad x \in D(A).$$

The set $D(A)$ is dense. Usually we write $A = T'(0)$, see details in [8].

Given a strongly continuous semigroup $(T(t))_{t\geq 0}$ of uniformly bounded operators on a Banach space $(X, \| \|_X)$, one may define a family of operators $(T^\alpha(t))_{t\geq 0}$ by $T^\alpha(0) = I_X$ and

$$T^\alpha(t)x := \int_0^\infty f_{t,\alpha}(s)T(s)x\,ds, \quad x \in X.$$

The family $(T^\alpha(t))_{t\geq 0}$ forms a strongly continuous semigroup of bounded operators, the so-called semigroup subordinated to $(T(t))_{t\geq 0}$ with respect to $(f_{t,\alpha})_{t>0}$. Let $(A, D(A))$ be the infinitesimal generator of $(T(t))_{t\geq 0}$, then the generator of $(T^\alpha(t))_{t\geq 0}$ is the fractional power $-(-A)^\alpha$ and $D(A) \subset D((-A)^\alpha)$, see more details in [9,7]. Note that this operator can be expressed in terms of $(T(t))_{t\geq 0}$ by

$$-(-A)^\alpha x = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{T(t)x - x}{t^{1+\alpha}}\,dt, \quad x \in D(A).$$

For $n+1 \geq \beta > n$, and $n \in \mathbb{N} \cup \{0\}$ we define the operator $((-A)^\beta, D((-A)^\beta))$ in the usual way, $D((-A)^\beta) := \{x \in D(A^n) \mid A^n x \in D((-A)^{\beta-n})\}$, and

$$(-A)^\beta x := (-A)^{\beta-n}((-A)^n)x, \quad x \in D((-A)^\beta).$$

Definition 11. Let $(T(t))_{t\geq 0}$ be a C_0-semigroup generated by $(A, D(A))$. Then we define

$$\Delta_\alpha(t)x := \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-u)^{\alpha-1} T(u)(-A)^\alpha x\,du, \quad x \in D((-A)^\alpha),$$

for $\alpha > 0$.

The operator Δ_α has been studied in [4] for $\alpha \in \mathbb{N}$.

Theorem 12. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup generated by $(A, D(A))$. Then*

$$\| \|\Delta_\alpha(t)x\|_X \|_{L^\nu([0,a])} \leq \frac{a^{\alpha-1+\frac{1}{p}+\frac{1}{\nu}}}{\Gamma(\alpha)(1-(1-\alpha)p)^{\frac{1}{p}} \left(\nu\left(\frac{1}{p}+\alpha-1\right)+1\right)^{\frac{1}{\nu}}} \| \|T(t)(-A)^\alpha x\|_X \|_{L^q([0,a])},$$

$$\int_0^t \|\Delta_\alpha(s)x\|_X \|T(s)(-A)^\alpha x\|_X\,ds \leq \frac{t^{\alpha-1+\frac{2}{p}}}{2^{\frac{1}{q}}\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}(p(\alpha-1)+2)^{\frac{1}{p}}} \left(\int_0^t \|T(s)(-A)^\alpha x\|_X^q\,ds\right)^{\frac{2}{q}}$$

for $x \in D((-A)^\alpha)$, $1 < p$, $p(\alpha-1) > -1$, $\frac{1}{p}+\frac{1}{q} = 1$, and $\nu \geq 1$.

Proof. We define $\delta(t) := T(t)(-A)^\alpha x$ for $x \in D((-A)^\alpha)$ and $t > 0$. We apply Theorems 3 and 8 to obtain the inequalities. □

Examples. (1) We consider $X = L^r(\mathbb{R}^+)$ with $1 \leq r < \infty$,

$$\|f\|_r := \left(\int_0^\infty |f(t)|^r\,dt\right)^{\frac{1}{r}}, \quad f \in L^r(\mathbb{R}^+),$$

and the shift semigroup $(T(t))_{t\geq 0} \subset \mathcal{B}(L^r(\mathbb{R}^+))$, given by

$$T(t)f(s) = f(s-t)\chi_{(t,\infty)}(s), \quad s > 0.$$

The infinitesimal generator of the shift semigroup is the usual derivative operator $\frac{d}{ds}$. Taking $\alpha = 1$ in the Theorem 12, we deduce the usual Poincaré inequality

$$\left\| f - \frac{1}{a}\int_0^a f(s)ds \right\|_{L^\nu([0,a])} \le \frac{a^{\alpha-1+\frac{1}{p}+\frac{1}{\nu}}}{\Gamma(\alpha)(1-(1-\alpha)p)^{\frac{1}{p}}\left(\nu\left(\frac{1}{p}+\alpha-1\right)+1\right)^{\frac{1}{\nu}}}\|f'\|_{L^q([0,a])},$$

for $\nu \ge 1$ and $\frac{1}{p}+\frac{1}{q} = 1$, see Section 1.

(2) We consider $X = L^r(\mathbb{R}^n)$ with $1 \le r < \infty$ with the usual norm given by

$$\|f\|_r := \left(\int_{\mathbb{R}^n} |f(y)|^r dy\right)^{\frac{1}{r}}, \quad f \in L^r(\mathbb{R}^n).$$

We consider the Gaussian semigroup $(G^t)_{t>0} \subset \mathcal{B}(L^r(\mathbb{R}^n))$ given

$$G^t(f)(y) = \frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-|x|^2/4t} f(y-x)dx, \quad f \in L^r(\mathbb{R}^n),\ y \in \mathbb{R}^n.$$

The infinitesimal generator of $(G^t)_{t>0}$ is the Laplacian operator $L_r = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$. We consider the fractional powers $(-L_r)^\alpha$ for $\alpha > 0$ studied in [6, Section 12.3] and to obtain the inequalities

$$\| \|\Delta_\alpha(t)f\|_r \|_{L^\nu([0,a])} \le \frac{a^{\alpha-1+\frac{1}{p}+\frac{1}{\nu}}}{\Gamma(\alpha)(1-(1-\alpha)p)^{\frac{1}{p}}\left(\nu\left(\frac{1}{p}+\alpha-1\right)+1\right)^{\frac{1}{\nu}}}\| \|G^t((-L_r)^\alpha f)\|_r \|_{L^q([0,a])},$$

$$\int_0^t \|\Delta_\alpha(s)f\|_r \|G^s((-L_r)^\alpha f)\|_r ds$$
$$\le \frac{t^{\alpha-1+\frac{2}{p}}}{2^{\frac{1}{q}}\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}(p(\alpha-1)+2)^{\frac{1}{p}}}\left(\int_0^t \|G^s((-L_r)^\alpha f)\|_r^q ds\right)^{\frac{2}{q}}$$

for $f \in D((-L_r)^\alpha)$, $1 < p$, $p(\alpha-1) > -1$, $\frac{1}{p}+\frac{1}{q} = 1$, and $\nu \ge 1$.

5. Applications to fractional powers of infinitesimal generators of cosine functions

A cosine operator function is a family $(C(t))_{t\ge 0}$ of bounded linear operators from a Banach space X into itself satisfying $C(0) = I$,

$$C(t+s) + C(t-s) = 2C(t)C(s), \quad t > s > 0,$$

and the map $t \mapsto C(t)x$ is continuous for $x \in X$. The associated sine operator function $S : \mathbb{R}^+ \to \mathcal{B}(X)$ is defined by

$$S(t)x := \int_0^t C(s)xds, \quad x \in X, t > 0.$$

The cosine function operator (and the sine operator function) is exponentially bounded. The infinitesimal generator A of $(C(t))_{t>0}$ is the closed operator from X into itself defined as

$$Ax := \lim_{t\to 0}\frac{2}{t^2}\left(C(t)x - x\right), \quad x \in D(A).$$

The set $D(A)$ is dense. Usually we write $A = C''(0)$.

Let $(A, D(A))$ be the infinitesimal generator of a uniformly bounded cosine function $(C(t))_{t>0} \subset \mathcal{B}(X)$. Then $(A, D(A))$ is the infinitesimal generator of uniformly bounded C_0-semigroup $(T(t))_{t>0} \subset \mathcal{B}(X)$ given by

$$T(t)x = \frac{1}{(4\pi t)^{\frac{1}{2}}}\int_0^\infty e^{\frac{-s^2}{4t}} C(s)xds, \quad x \in X,\ t > 0,$$

[8]. In fact, $((-A)^\alpha, D((-A)^\alpha))$is the infinitesimal generator of a uniformly bounded C_0-semigroup $(T^\alpha(t))_{t>0} \subset \mathcal{B}(X)$ for $0 < \alpha \le 1$, and we may define the closed operator $((-A)^\alpha, D((-A)^\alpha))$for $\alpha > 0$.

The following operators have been studied in [4,5] for $\alpha \in \mathbb{N}$.

Definition 13. Let $(C(t))_{t\geq 0}$ be a uniformly bounded cosine function generated by $(A, D(A))$ and $\alpha > 0$. Then we define

$$\mathcal{T}_\alpha(t)x := \frac{1}{\Gamma(2\alpha)}\int_0^t (t-u)^{2\alpha-1}C(u)(-A)^\alpha x du, \quad x \in D((-A)^\alpha),$$
$$\mathcal{M}_\alpha(t)x := \frac{1}{\Gamma(2\alpha+1)}\int_0^t (t-u)^{2\alpha}C(u)(-A)^\alpha x du, \quad x \in D((-A)^\alpha),$$

for $t > 0$.

The next result was proved in [2, Theorem 10,12] for $\alpha \in \mathbb{N}$.

Theorem 14. *Let $(C(t))_{t\geq 0}$ be a cosine function generated by $(A, D(A))$ and let $\alpha > 0$. Then*

$$\| \, \|\mathcal{T}_\alpha(t)x\|_X \, \|_{L^\nu([0,a])} \leq \frac{a^{2\alpha-1+\frac{1}{p}+\frac{1}{\nu}}}{\Gamma(2\alpha)(1-(1-2\alpha)p)^{\frac{1}{p}}\left(\nu\left(\frac{1}{p}+2\alpha-1\right)+1\right)^{\frac{1}{\nu}}} \| \, \|C(t)(-A)^\alpha x\|_X \, \|_{L^q([0,a])},$$

for $x \in D((-A)^\alpha)$, $1 < p$, $p(2\alpha-1) > -1$, $\frac{1}{p}+\frac{1}{q} = 1$, and $\nu \geq 1$; and

$$\| \, \|\mathcal{M}_\alpha(t)x\|_X \, \|_{L^\nu([0,a])} \leq \frac{a^{2\alpha+\frac{1}{p}+\frac{1}{\nu}}}{\Gamma(2\alpha+1)(1+2\alpha p)^{\frac{1}{p}}\left(\nu\left(\frac{1}{p}+2\alpha\right)+1\right)^{\frac{1}{\nu}}} \| \, \|C(t)(-A)^\alpha x\|_X \, \|_{L^q([0,a])},$$

for $x \in D((-A)^\alpha)$, $1 < p$, $\frac{1}{p}+\frac{1}{q} = 1$, and $\nu \geq 1$.

Proof. We take $\alpha' = 2\alpha$, $\delta(t) := T(t)(-A)^\alpha x$ for $x \in D((-A)^\alpha)$ and $t > 0$ and consider $\delta_{\alpha'}$ given in Definition 1. We apply Theorem 3 to obtain the first inequality. Similarly we show the second inequality. □

Now we may prove similar Opial inequalities for operators $\mathcal{T}_\alpha$ and $\mathcal{M}_\alpha$ which were proved in [5, Theorem 2] for $\alpha \in \mathbb{N}$. To show we apply Theorem 8. The proof is left to the reader.

Theorem 15. *Let $(C(t))_{t\geq 0}$ be a cosine function generated by $(A, D(A))$. Then*

$$\int_0^t \|\mathcal{T}_\alpha(s)x\|_X\|C(s)(-A)^\alpha x\|_X ds \leq \frac{t^{2\alpha-1+\frac{2}{p}}}{2^{\frac{1}{q}}\Gamma(2\alpha)(p(2\alpha-1)+1)^{\frac{1}{p}}(p(2\alpha-1)+2)^{\frac{1}{p}}}\left(\int_0^t \|C(s)(-A)^\alpha x\|_X^q ds\right)^{\frac{2}{q}}$$

for $x \in D((-A)^\alpha)$, $1 < p$, $p(2\alpha-1) > -1$, $\frac{1}{p}+\frac{1}{q} = 1$, and $\nu \geq 1$; and

$$\int_0^t \|\mathcal{M}_\alpha(s)x\|_X\|C(s)(-A)^\alpha x\|_X ds \leq \frac{t^{2\left(\alpha+\frac{1}{p}\right)}}{2\Gamma(2\alpha+1)(p(2\alpha)+1)^{\frac{1}{p}}(p\alpha+1)^{\frac{1}{p}}}\left(\int_0^t \|C(s)(-A)^\alpha x\|_X^q ds\right)^{\frac{2}{q}}$$

for $x \in D((-A)^\alpha)$, $1 < p$, $\frac{1}{p}+\frac{1}{q} = 1$, and $\nu \geq 1$.

Example. Let X be the Banach space of odd, 2π-periodic real functions in the space of bounded uniformly continuous functions from $\mathbb{R}$ into itself: $(\mathrm{BUC}(\mathbb{R}), \| \, \|_\infty)$. Let $A = \frac{d^2}{dx^2}$ with $D(A)$ given by

$$D(A) = \{f \in X \mid f^{(2)} \in X\}.$$

The operator A generates a cosine function $(C(t))_{t\geq 0}$ given by

$$C(t)f(s) = \frac{1}{2}\left(f(s+t)+f(s-t)\right), \quad s \in \mathbb{R}, t \geq 0,$$

see more details in [8, p. 121]. Then we may apply Theorems 14 and 15 to obtain the following inequality

$$\int_0^t \|\mathcal{T}_n(s)x\|_\infty\|C(s)f^{(2n)}\|_\infty ds \leq \frac{t^{2n-1+\frac{2}{p}}}{2^{\frac{1}{q}}(2n-1)!(p(2n-1)+1)^{\frac{1}{p}}(p(2n-1)+2)^{\frac{1}{p}}}\left(\int_0^t \|C(s)f^{(2n)}\|_\infty^q ds\right)^{\frac{2}{q}}$$

for $f \in D(A^n)$, $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, and $\nu \geq 1$, see [5, Proposition 7].

6. Concluding remarks

In this paper we have proven several Poincaré and Opial inequalities for vector-valued convolution products. We have applied our results to orbits of C_0-semigroups and cosine functions. Differential operators in Lebesgue spaces are generators of these families of operators. These results recover, extend and generalize previous results in the scalar and vector-valued setting.

Possible extensions of our results may be done taking into account the function $\delta * \kappa$ (with $\kappa \in L^1_{loc}(\mathbb{R}^+)$) instead of the function δ_α (which we have studied in Sections 2 and 3). These future results may be applied to other families of operators, i.e., convoluted semigroups and convoluted cosine functions (which include α-times integrated semigroups and α-times integrated cosine functions).

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