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On Steffensen's method on Banach spaces

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ABSTRACT

We present a modification of Steffensen's method as a predictor–corrector iterative method, so that we can use Steffensen's method to approximate a solution of a nonlinear equation in Banach spaces from the same starting points from which Newton's method converges. We study the semilocal convergence of the predictor–corrector method by using the majorant principle. We illustrate the method with an application to a discrete problem.

1. Introduction

We consider the problem of approximating locally a solution x^* of the equation

$$F(x) = 0, \tag{1}$$

where F is a nonlinear operator defined on a non-empty open convex subset Ω of a Banach space X with values in X.

It is well-known that Newton's method,

$$\begin{cases} x_0 \in \Omega, \\ x_{n+1} = x_n - [F'(x_n)]^{-1}F(x_n), \quad n \geq 0. \end{cases} \tag{2}$$

is the one of the most used iterative methods to approximate the solution x^* of (1). The quadratic convergence and the low operational cost of (2) ensure that Newton's method has a good computational efficiency. In particular, if (1) represents an m-dimensional system of equations such that $F(x_1, x_2, \ldots, x_m) = \overline{0}$, where $F : \Omega \subseteq \mathbb{R}^m \to \mathbb{R}^m$ and $F \equiv (F_1, F_2, \ldots, F_m)$, then the computational efficiency of Newton's method is $EC_N = 2^{\frac{3}{m(m^2+m-4)}}$, see [1].

On the other hand, it is known that a bounded linear operator on X, denoted by $[u, v; F]$, $u, v \in \Omega$, $u \neq v$, such that

$$[u, v; F] : \Omega \subset X \longrightarrow X \quad \text{and} \quad [u, v; F](u - v) = F(u) - F(v),$$

is called the divided difference of the first order of F in the points u and v, see [2].

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So, if we approximate the operator $F'(x) \in \mathcal{L}(X, X)$, where $\mathcal{L}(X, X)$ denotes the space of bounded linear operators from X to X, by a divided difference of first order, we save the evaluation of $F'(x)$ in each step x_n of Newton's method. In particular, if we approximate $F'(x_n)$ by $[x_n, x_{n-1}; F]$ in (2), we obtain the known secant method:

$$\begin{cases} x_{-1}, \quad x_0 \in \Omega, \\ x_{n+1} = x_n - [x_n, x_{n-1}; F]^{-1} F(x_n), \quad n \geq 0. \end{cases} \tag{3}$$

Obviously, method (3) has the same operational cost as method (2), but has lower order of convergence, since method (3) has only superlinear convergence $\frac{1+\sqrt{5}}{2}$ [3], so that the computational efficiency of (3) is $EC_S = \left(\frac{1+\sqrt{5}}{2}\right)^{\frac{3}{m(m^2+m-4)}}$, when it is used to solve m-dimensional systems of equations. In consequence, the computational efficiency of the secant method is lower than that of Newton's method.

Next, if we now approximate $F'(x)$ by $[x_n, x_n+F(x_n); F]$ in each step x_n of Newton's method, we obtain the known method of Steffensen:

$$\begin{cases} x_0 \in \Omega, \\ y_n = x_n + F(x_n), \\ x_{n+1} = x_n - [x_n, y_n; F]^{-1} F(x_n), \quad n \geq 0, \end{cases} \tag{4}$$

which has quadratic convergence [4] and the same computational efficiency as Newton's method, when it is used to solve m-dimensional systems of equations.

Although Steffensen's method is less used than Newton's method, its use is interesting, since Steffensen's method does not require the evaluation of $[F'(x_n)]^{-1}$ in each step x_n and has the same order of convergence as Newton's method. Our main aim in this paper is to improve the applicability of Steffensen's method, which is its weakest feature, as we will see later.

An important aspect to consider when studying the applicability of an iterative method is the set of starting points that we can take into account, so that the iterative method converges to a solution of an equation from any point of the set, what we call accessibility of the iterative method. We can observe this experimentally by means of the attraction basin of the iterative method. The attraction basin of an iterative method is the set of all starting points from which the iterative method converges to a solution of an equation, once we fix some tolerance or a maximum number of iterations.

In Figs. 1 and 2, we show the attraction basins associated with the three solutions, $z^* = 1, z^{**} = \exp\frac{2\pi i}{3}$ and $z^{***} = \exp\frac{-2\pi i}{3}$, of the complex equation $F(z) = z^3 - 1 = 0$, where $F : \mathbb{C} \longrightarrow \mathbb{C}$, when they are approximated respectively by Newton's method and Steffensen's method. To do this, we take a rectangle $D \subset \mathbb{C}$ to represent the regions such that iterations start at every $z_0 \in D$. In every case, a grid of 512×512 points in D is considered and these points are chosen as z_0. We use the rectangle $[-2.5, 2.5] \times [-2.5, 2.5]$ which contains the three solutions. The chosen iterative method, starting in $z_0 \in D$, can converge or diverge to any solution. In all the examples, a tolerance 10^{-3} and a maximum of 50 iterations are used. We do not continue if the required tolerance is not obtained with 50 iterations and we then decide that the iterative method does not converge to any solution starting from z_0. The pictures of the attraction basins are painted using the following strategy. A colour is assigned to each attraction basin according to the root at which an iterative method converges starting from z_0. The colour is made lighter or darker according to the number of iterations needed to reach the root with fixed precision. In particular, cyan, magenta and yellow are assigned respectively for the solutions z^*, z^{**} and z^{***}. Finally, black is assigned if the method does not converge to any solution with a fixed tolerance and a maximum number of iterations. The graphics have been generated with Mathematica 5.1. For other strategies, Ref. [5] can be consulted and the references therein given.

We can see in Figs. 1 and 2 the behaviour of Newton's method and Steffensen's method. Note that Steffensen's method is much more demanding with respect to the initial points than Newton's method, see the black colour. This clearly justifies that Steffensen's method is less used than Newton's method to approximate solutions of equations.

We can also study the accessibility of an iterative method from the convergence conditions required to the iterative method. In addition, we see that the accessibility of Newton's method is much better than that of Steffensen's method. For this, we consider the semilocal convergence results of both methods. It is well-known that the semilocal convergence conditions are of two kinds: conditions required to the starting point x_0 and conditions required to the operator F involved.

In Section 2, we present the semilocal convergence result given by Kantorovich for Newton's method and obtain a semilocal convergence result for Steffensen's method. After that, from both results, we compare the accessibility of the two methods. We anticipate that the differences observed experimentally with the attractions basins associated with the three solutions of the complex equation $F(z) = z^3 - 1 = 0$ are confirmed with the theoretical study.

On the other hand, in Section 3, we consider a modified Newton's method:

$$\begin{cases} x_0 \in \Omega, \\ x_{n+1} = x_n - [F'(x_0)]^{-1} F(x_n), \quad n \geq 0. \end{cases} \tag{5}$$

Observe that method (5) does not evaluate F' in a new point in each step, but F' is only evaluated in a point $x_0 \in \Omega$ such that $F'(x_0)$ exists. Like method (3), method (5) has no quadratic convergence. However, we prove that method (5), unlike method (3), has the same accessibility as Newton's method.

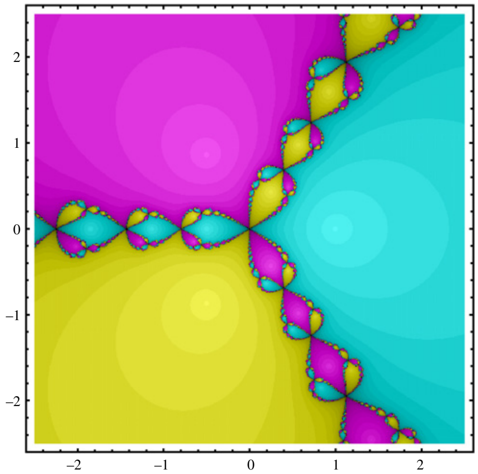

Fig. 1. Attraction basins of Newton's method when it is used to approximate the three solutions of the equation $F(z) = z^3 - 1 = 0$.

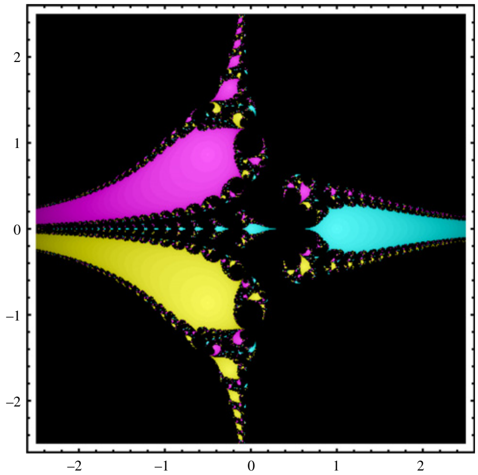

Fig. 2. Attraction basins of Steffensen's method when it is used to approximate the three solutions of the equation $F(z) = z^3 - 1 = 0$.

From the above-mentioned, in Section 4, we construct a predictor–corrector iterative method with the same efficiency as Newton's method, that evaluates no derivatives in each step and has the same accessibility as Newton's method. So, from the same starting points of the modified Newton's method, the predictor–corrector iterative method converges with the same rate of convergence as Steffensen's method. In consequence, the accessibility of Steffensen's method is improved by means of a method with a similar efficiency to that of Newton's method and without the requirement that the method evaluates derivatives in each step.

Finally, in Section 5, we consider an application where a discrete solution of a nonlinear integral equation is approximated by the predictor–corrector iterative method, but it is not by Steffensen's method.

Throughout the paper we denote $\overline{B(x, \varrho)} = \{y \in X;\ \|y - x\| \le \varrho\}$ and $B(x, \varrho) = \{y \in X;\ \|y - x\| < \varrho\}$.

2. Newton's method versus Steffensen's method

To do the study set out in the introduction, we standardize the initial convergence conditions required to the starting point and the operator involved. In this way, we guarantee the convergence of Newton's and Steffensen's methods and will be able to compare their accessibilities.

Then, throughout the paper, we consider that $F : \Omega \subseteq X \to X$ is a once continuously differentiable operator defined on a non-empty open convex subset Ω of a Banach space X. Suppose that the operator $\Gamma_0 = [F'(x_0)]^{-1} \in \mathcal{L}(X, X)$ exists for some $x_0 \in \Omega$ and the following conditions:

(C1) $\|F(x_0)\| \le \alpha_0$,
(C2) $\|[F'(x_0)]^{-1}\| \le \beta_0$,
(C3) $\|F'(x) - F'(y)\| \le k\|x - y\|$, $x, y \in \Omega$, $k \in \mathbb{R}^+$.

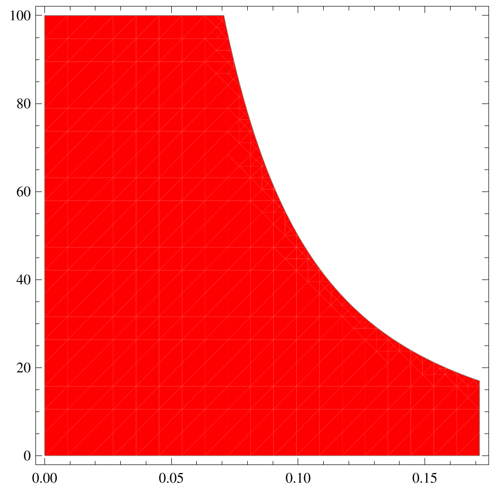

Fig. 3. Domain of parameters associated with Newton's method.

Note that if the operator F is differentiable and F' is continuous in the segment $[x, y] = \{tx + (1-t)y;\ t \in [0, 1]\}$, then the linear operator

$$[x, y; F] = \int_0^1 F'(x + t(y - x))\, dt$$

is a divided difference of first order of F in the points x and y. Moreover, if $x = y$, then $[x, x; F] = F'(x)$. See [2] for more detail.

2.1. Semilocal convergence of Newton's method

First, we remember the classic result of semilocal convergence given by Kantorovich for Newton's method under conditions $(C1)$–$(C3)$. The proof can be found in [6].

Theorem 1 (*The Newton–Kantorovich Theorem*)**.** *Let $F : \Omega \subseteq X \to X$ be a once continuously differentiable operator defined on a non-empty open convex domain Ω of a Banach space X. Suppose that conditions $(C1)$–$(C3)$, $k\alpha_0\beta_0^2 \le \frac{1}{2}$ and $\overline{B(x_0, t^*)} \subseteq \Omega$, where $t^* = \frac{1-\sqrt{1-2k\alpha_0\beta_0^2}}{k\beta_0}$, are satisfied. Then Newton's sequence, given by (2), converges to a solution x^* of the equation $F(x) = 0$, starting at x_0, and $x_n, x^* \in \overline{B(x_0, t^*)}$, for all $n = 0, 1, 2, \ldots$. Moreover, if $k\alpha_0\beta_0^2 < \frac{1}{2}$, x^* is the unique solution of $F(x) = 0$ in $B(x_0, t^{**}) \cap \Omega$, where $t^{**} = \frac{1+\sqrt{1-2k\alpha_0\beta_0^2}}{k\beta_0}$, and if $k\alpha_0\beta_0^2 = \frac{1}{2}$, x^* is unique in $\overline{B(x_0, t^*)}$.*

If we want to study the accessibility of Newton's method from the last theorem, we consider the pair of parameters (α_0, β_0) given in conditions $(C1)$ and $(C2)$, the condition $k\alpha_0\beta_0^2 \le \frac{1}{2}$ and take into account what we call the domain of parameters associated with Newton's method: $\left\{(\alpha_0, \beta_0) \in \mathbb{R}^2;\ k\alpha_0\beta_0^2 \le \frac{1}{2}\right\}$. Observe that the parameter k is always a fixed value, so that k will not affect the domain of parameters.

In Fig. 3, we show the domain of parameters associated with Newton's method. Note the values of β_0 are represented on the horizontal axis and the values of $k\alpha_0$ on the vertical axis.

Next, we establish a semilocal convergence result for Steffensen's method, take into account the domain of parameters associated with this method and do a comparative study of the domains of parameters associated with Newton's and Steffensen's methods.

2.2. Semilocal convergence of Steffensen's method

We establish the semilocal convergence of Steffensen's method from a point of view similar to that given by Kantorovich for Newton's method. For this, we use the majorant principle (see [6]).

Suppose the following initial conditions:

$(\widetilde{C1})$ $\|F(x_0)\| \le \tilde{\alpha}$,
$(\widetilde{C2})$ $\|\Gamma_0\| = \|[F'(x_0)]^{-1}\| \le \tilde{\beta}$,
$(\widetilde{C3})$ $\|F'(x) - F'(y)\| \le k\|x - y\|$, $x, y \in \Omega$, $k \in \mathbb{R}^+$.

The first thing that we are going to do is to prove that the operator $[x_0, y_0; F]^{-1} \in \mathcal{L}(X, X)$ exists for any $x_0, y_0 \in \Omega$. From,

$$\|I - \Gamma_0[x_0, y_0; F]\| \leq \|\Gamma_0\| \int_0^1 \|F'(x_0 + \tau(y_0 - x_0)) - F'(x_0)\| \, d\tau \leq \frac{1}{2} k\tilde{\alpha}\tilde{\beta},$$

if $k\tilde{\alpha}\tilde{\beta} < 2$, then the operator $[x_0, y_0; F]^{-1} \in \mathcal{L}(X, X)$ exists by the Banach lemma on invertible operators and

$$\|[x_0, y_0; F]^{-1}\| \leq \frac{2\tilde{\beta}}{2 - k\tilde{\alpha}\tilde{\beta}} = b.$$

Now, we define the polynomial

$$q(s) = \frac{M}{2}s^2 - \frac{s}{b} + \tilde{\alpha}, \quad M = k\left(1 + \frac{1}{b}\right), \; s \in [0, s'], \tag{6}$$

and denote the smallest positive zero of (6) by $s^* = \frac{1-\sqrt{1-2M\tilde{\alpha}b^2}}{Mb}$ and the largest positive zero by $s^{**} = \frac{1+\sqrt{1-2M\tilde{\alpha}b^2}}{Mb}$. Next, we define the scalar sequence $\{s_n\}$ by

$$\begin{cases} s_0 = 0, \\ s_{n+1} = s_n - \dfrac{q(s_n)}{q'(s_n)}, \quad n \geq 0. \end{cases} \tag{7}$$

Note that (7) is an increasing sequence that converges to s^*.

Theorem 2. *Let $F : \Omega \subseteq X \to X$ be a once continuously differentiable operator defined on a non-empty open convex domain Ω of a Banach space X. Suppose that conditions $(\widetilde{C1})$–$(\widetilde{C3})$*

$$k\tilde{\alpha}\tilde{\beta} \leq 2, \qquad M\tilde{\alpha}b^2 \leq \frac{1}{2} \tag{8}$$

and $B(x_0, s^* + \tilde{\alpha}) \subseteq \Omega$ are satisfied. Then Steffensen's sequence, defined by (4), converges to a solution x^* of the equation $F(x) = 0$, starting at x_0, and $x_n, y_n, x^* \in \overline{B(x_0, t^*)}$, for all $n = 0, 1, 2, \ldots$. Moreover, the solution x^* is unique in $B(x_0, x^{**} + \tilde{\alpha}) \cap \Omega$, where $s^{**} = \frac{1+\sqrt{1-2M\tilde{\alpha}b^2}}{Mb}$.

Proof. As

$$\|x_1 - x_0\| \leq b\tilde{\alpha} = s_1 - s_0 < s^* + \tilde{\alpha},$$

then $x_1 \in B(x_0, s^* + \tilde{\alpha}) \subseteq \Omega$. Now observe that

$$F(x_1) = \int_0^1 (F'(x_0 + \tau(x_1 - x_0)) - F'(x_0)) \, d\tau \, (x_1 - x_0) + \left(F'(x_0) - [x_0, y_0; F]\right)(x_1 - x_0),$$

$$\|F(x_1)\| \leq \frac{k}{2}\|x_1 - x_0\|^2 + \frac{k}{2}\|F(x_0)\| \, \|x_1 - x_0\| \leq \frac{k}{2}(s_1 - s_0)^2 + \frac{k}{2}q(s_0)(s_1 - s_0) = q(s_1)$$

and taking into account that sequence (7) is increasing and polynomial (6) is decreasing in $[0, s^*]$, it follows that

$$\|y_1 - x_0\| \leq \|x_1 - x_0\| + \|F(x_1)\| < s^* + \tilde{\alpha},$$

so that $y_1 \in B(x_0, s^* + \tilde{\alpha}) \subseteq \Omega$.

We now suppose that $x_j, y_{j-1} \in B(x_0, s^* + \tilde{\alpha}) \subseteq \Omega$, for $j = 2, 3, \ldots, n$.

Next, from

$$\begin{aligned} F(x_n) &= \int_0^1 (F'(x_{n-1} + \tau(x_n - x_{n-1})) - F'(x_{n-1})) \, \tau(x_n - x_{n-1}) \\ &\quad + \int_0^1 \left(F'(x_{n-1}) - F'(x_{n-1} + \tau(y_{n-1} - x_{n-1}))\right) d\tau \, (x_n - x_{n-1}) \end{aligned}$$

and

$$\begin{aligned} q(s_n) &= \int_0^1 (q'(s_{n-1} + \tau(s_n - s_{n-1})) - q'(s_{n-1})) \, d\tau \, (s_n - s_{n-1}) \\ &= M \int_0^1 \tau(s_n - s_{n-1})^2 \, d\tau = \frac{M}{2}(s_n - s_{n-1})^2, \end{aligned}$$

it follows that $\|F(x_n)\| \le q(s_n)$, for all $n \in \mathbb{N}$, since

$$\begin{aligned}\|F(x_n)\| &\le \int_0^1 k\tau\,\|x_n - x_{n-1}\|^2\,d\tau + \int_0^1 k\tau\,\|F(x_{n-1})\|\,\|x_n - x_{n-1}\|\,d\tau\\ &\le \frac{k}{2}(s_n - s_{n-1})^2 + \frac{k}{2}q(s_{n-1})(s_n - s_{n-1})\\ &\le \frac{M}{2}(s_n - s_{n-1})^2\\ &= q(s_n).\end{aligned}$$

Then, as polynomial (6) is decreasing in $[0, s^*]$, we have

$$\|y_n - x_0\| \le \|x_n - x_0\| + \|F(x_n)\| < s^* + \tilde{\alpha},$$

so that $y_n \in B(x_0, s^* + \tilde{\alpha}) \subseteq \Omega$.

After that, as

$$\begin{aligned}\|I - \Gamma_0[x_n, y_n; F]\| &\le \|\Gamma_0\| \left(k\|x_n - x_0\| + \int_0^1 \|F'(x_n + \tau(y_n - x_n)) - F'(x_n)\|\,d\tau \right)\\ &\le \tilde{\beta}\left(q'(s_n) + \frac{1}{b} \right) < 1,\end{aligned}$$

then the operator $[x_n, y_n; F]^{-1}$ exists and

$$\|[x_n, y_n; F]^{-1}\| \le \frac{\tilde{\beta}}{1 - \|I - \Gamma_0[x_n, y_n; F]\|} \le -\frac{1}{q'(s_n)}.$$

In consequence,

$$\begin{aligned}&\|x_{n+1} - x_n\| \le \|[x_n, y_n; F]^{-1}\|\,\|F(x_n)\| \le -\frac{q(s_n)}{q'(s_n)} = s_{n+1} - s_n,\\ &\|x_{n+1} - x_0\| \le \|x_{n+1} - x_n\| + \|x_n - x_0\| \le s_{n+1} - s_0 < s^* < s^* + \tilde{\alpha},\end{aligned}$$

and $x_{n+1} \in B(x_0, s^* + \tilde{\alpha}) \subseteq \Omega$.

Since $\{s_n\}$ is a Cauchy sequence, so is the sequence $\{x_n\}$ and then $\{x_n\}$ converges to a point $x^* \in \overline{B(x_0, s^* + \tilde{\alpha})}$. The fact that x^* is a solution of $F(x) = 0$ follows from the inequality $\|F(x_n)\| \le q(s_n)$ and the continuity of the operator F.

Finally, we prove the uniqueness of the solution x^* in $B(x_0, s^{**} + \tilde{\alpha}) \cap \Omega$. Suppose that z^* is another solution of $F(x) = 0$ in $B(x_0, s^{**} + \tilde{\alpha}) \cap \Omega$. Let

$$F(z^*) - F(x^*) = \int_{x^*}^{z^*} F'(x)\,dx = \int_0^1 F'(x^* + t(z^* - x^*))(z^* - x^*)\,dt = 0,$$

and the operator $N = \int_0^1 F'(x^* + t(z^* - x^*))\,dt$. From

$$\begin{aligned}\|\Gamma_0 N - I\| &\le \|\Gamma_0\| \int_0^1 \|F'(x^* + t(z^* - x^*)) - F'(x_0)\|\,dt\\ &\le k\beta \int_0^1 \big((1-t)\|x^* - x_0\| + t(\|z^* - x_0\|)\big)\,dt\\ &< \frac{k\beta}{2}\left(\frac{2}{Mb} + \tilde{\alpha} \right) < 1,\end{aligned}$$

since $k\tilde{\alpha}\tilde{\beta} \le 2$, and the Banach lemma on invertible operators, it follows that the operator N is invertible, and therefore $z^* = x^*$. □

In Fig. 4, we show the domain of parameters associated with Steffensen's method. Observe that this domain is obtained from the convergence conditions given in (8). Note the values of $\tilde{\beta}$ are represented on the horizontal axis and the values of $k\tilde{\alpha}$ on the vertical axis.

In Fig. 5, we show the domains of parameters associated with Newton's and Steffensen's methods. As we can see, the accessibility of Steffensen's method is much less than that of Newton's method.

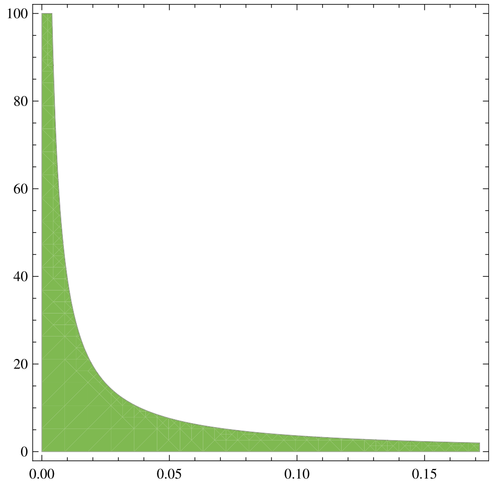

Fig. 4. Domain of parameters associated with Steffensen's method.

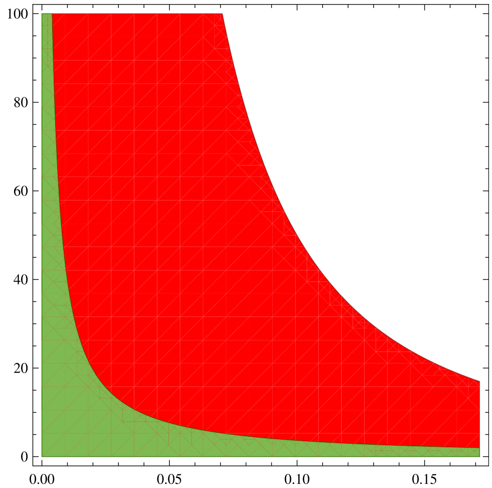

Fig. 5. Comparison between the domain of parameters associated with Newton's method (red) and Steffensen's method (green). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

2.3. Accessibility of solution by means of Steffensen's method

Looking for some parallelism with the attraction basins, we can also consider the following experimental form of studying the accessibility of an iterative method. We know that a given point $x \in \Omega$ has associated certain parameters of convergence. If the parameters of convergence satisfy the convergence conditions, we colour the point x; otherwise, we do not. So, the region that is finally coloured is what we call a region of accessibility of the iterative method.

In consequence, the region of accessibility of an iterative method provides the domain of starting points from which we have guaranteed the convergence of the iterative method. In other words, the region of accessibility represents the domain of starting points that satisfy the convergence conditions required by the iterative method that we want to apply.

In Figs. 6 and 7, we show the regions of accessibility of Newton's and Steffensen's methods when they are applied to solve the complex equation $F(z) = z^3 - 1 = 0$ once the value $k = 6|1.6 + 0.2i|$ is fixed.

Observe the big difference that exists between the two regions of accessibility, as was the case with the attraction basins associated with the solutions of the equation $F(z) = z^3 - 1 = 0$. This shows once again that the accessibility of Steffensen's method is reduced.

3. An alternative to Steffensen's method: a modified Newton's method

We have seen in the introduction that Steffensen's method has the same computational efficiency as Newton's method when it is used to solve m-dimensional systems of equations, but its accessibility is more reduced. In the following, we see

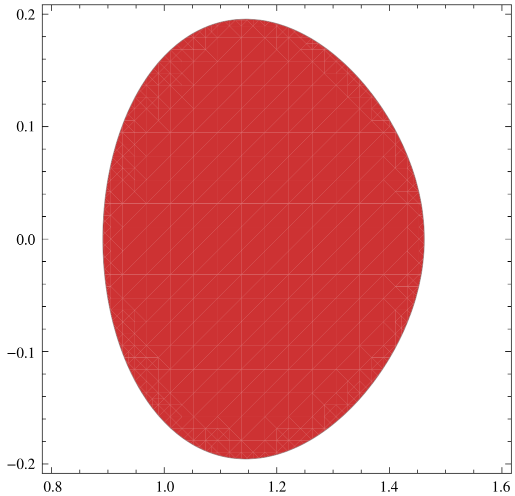

Fig. 6. Region of accessibility of Newton's method when it is applied to solve $F(z) = z^3 - 1 = 0$ (and $k = 6|1.6 + 0.2i|$).

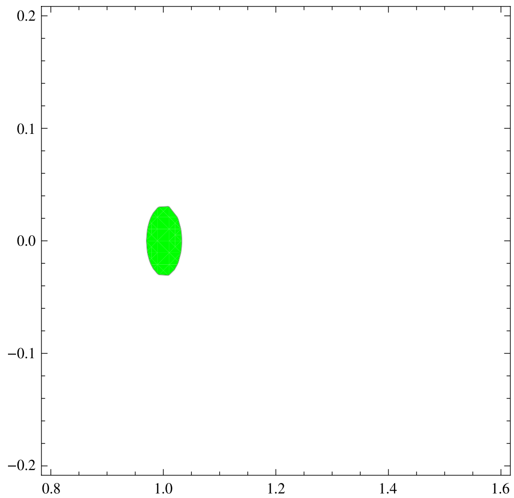

Fig. 7. Region of accessibility of Steffensen's method when it is applied to solve $F(z) = z^3 - 1 = 0$ (and $k = 6|1.6 + 0.2i|$).

that a modified Newton's method has the same region of accessibility as Newton's method, although it has the disadvantage of having less efficiency than Newton's method. This leads us to think of combining the modified Newton's method and Steffensen's method in order to obtain an iterative method with a good region of accessibility and a good efficiency. So, we construct an hybrid iterative method where the modified Newton's method works as a predictor method and Steffensen's method does as a corrector method.

3.1. Semilocal convergence of the modified Newton's method

We start seeing that the modified Newton's method has the same region of accessibility as Newton's method. Then, we establish the conditions for the operator F and the starting point x_0 under which the modified Newton's method converges to a solution of the equation $F(x) = 0$.

We define the polynomial

$$p(t) = \frac{k}{2}t^2 - \frac{t}{\beta_0} + \alpha_0, \quad t \in [t_0, t'], \tag{9}$$

and denote the smallest and the largest positive zeros of (9) by t^* and t^{**} respectively. We also define the scalar function

$$h(t) = t + \beta_0 p(t) \tag{10}$$

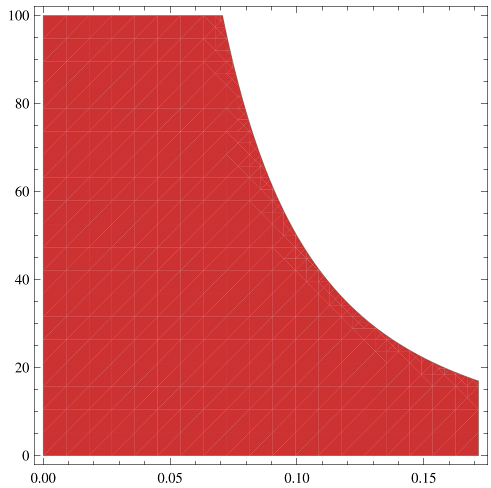

Fig. 8. Domain of parameters associated with the modified Newton's method.

and the scalar sequence

$$\begin{cases} t_0 = 0, \\ t_{n+1} = h(t_n) = t_n + \beta_0 p(t_n), \quad n \geq 0. \end{cases} \tag{11}$$

Note that (11) is an increasing sequence that converges to t^*.

Theorem 3. *Let $F : \Omega \subseteq X \to X$ be a once continuously differentiable operator defined on a non-empty open convex domain Ω of a Banach space X. Suppose that conditions $(C1)$–$(C3)$,*

$$k\alpha_0\beta_0^2 \leq 1/2 \tag{12}$$

and $\overline{B(x_0, t^*)} \subseteq \Omega$, where $t^* = \frac{1-\sqrt{1-2k\alpha_0\beta_0^2}}{k\beta_0}$, are satisfied. Then the modified Newton's sequence, given by (5), converges to a solution x^* of the equation $F(x) = 0$, starting at x_0, and $x_n, x^* \in \overline{B(x_0, t^*)}$, for all $n = 0, 1, 2, \ldots$. Moreover, if $k\alpha_0\beta_0^2 < \frac{1}{2}$, x^* is the unique solution of $F(x) = 0$ in $B(x_0, t^{**}) \cap \Omega$, where $t^{**} = \frac{1+\sqrt{1-2k\alpha_0\beta_0^2}}{k\beta_0}$, and if $k\alpha_0\beta_0^2 = \frac{1}{2}$, x^* is unique in $\overline{B(x_0, t^*)}$.

The proof of the last theorem is similar to that of Theorem 1, the Newton–Kantorovich theorem. We then omit it and suggest to the reader to see [6].

In Fig. 8, we show the domain of parameters associated with the modified Newton's method, which is obtained from the convergence condition given in (12). Note the values of β_0 are represented on the horizontal axis and the values of $k\alpha_0$ on the vertical axis. Observe that the domain of parameters associated with the modified Newton's method is the same as that of Newton's method.

In the next theorem, we give some a priori error estimates for the modified Newton's method, that are obtained by Ostrowski's technique, see [7]. This technique allows bounding the error made by (5) based on the zeros of polynomial (9). Moreover, this result allows us to study the semilocal convergence of the iterative predictor–corrector method.

Theorem 4. *Under the conditions of Theorem 3, we consider polynomial (9) and the two positive zeros t^* and t^{**} of (9) such that $t^* \leq t^{**}$. Then, we obtain the following error estimates for method (5):*

(i) *If $t^* < t^{**}$, then*

$$\frac{((t^{**}-t^*)t^*)^{n+1}}{(t^{**})^{n+1}-(t^*)^{n+1}} < t^* - t_n < \frac{t^*(t^{**}-t^*)(t^*k\beta_0)^n}{t^{**}-t^*(t^*k\beta_0)^n}.$$

(ii) *If $t^* = t^{**}$, then*

$$\left(\frac{1}{2}\right)^n t^* \leq t^* - t_n \leq t^*.$$

Proof. First, we prove item (i). Since $t^* < t^{**}$, then we can write

$$p(t) = \frac{k}{2}(t^* - t)(t^{**} - t).$$

If we denote $a_n = t^* - t_n$ and $b_n = t^{**} - t_n$ for all $n \geq 0$, then $p(t_n) = \frac{k}{2}a_n b_n$. As $p'(t_0) = -\frac{1}{\beta_0}$, then

$$a_{n+1} = t^* - t_{n+1} = a_n - \frac{k}{2}\beta_0 a_n b_n$$
$$b_{n+1} = t^{**} - t_{n+1} = b_n - \frac{k}{2}\beta_0 a_n b_n.$$

Moreover,

$$\frac{a_{n+1}}{b_{n+1}} = \frac{a_n}{b_n}\frac{2 - k\beta_0(t^{**} - t_n)}{2 - k\beta_0(t^* - t_n)},$$

and, since the function

$$Q(t) = \frac{2 - k\beta_0(t^{**} - t)}{2 - k\beta_0(t^* - t)}$$

is strictly increasing, it follows

$$\frac{t^*}{t^{**}} = Q(t_0) \leq Q(t) \leq Q(t^*) = k\beta_0 t^*.$$

In consequence,

$$\frac{a_{n+1}}{b_{n+1}} = \frac{a_n}{b_n}Q(t_n) \leq \frac{a_0}{b_0}Q(t^*)^{n+1} = \frac{t^*}{t^{**}}(k\beta_0 t^*)^{n+1},$$
$$\frac{a_{n+1}}{b_{n+1}} = \frac{a_n}{b_n}Q(t_n) \geq \frac{a_0}{b_0}Q(t_0)^{n+1} = \left(\frac{t^*}{t^{**}}\right)^{n+2}.$$

Finally, from $b_{n+1} = (t^{**} - t^*) + a_{n+1}$, we have

$$\frac{(t^{**} - t^*)(t^*)^{n+2}}{(t^{**})^{n+2} - (t^*)^{n+2}} < t^* - t_{n+1} < \frac{t^*(t^{**} - t^*)(t^* k\beta_0)^{n+1}}{t^{**} - t^*(t^* k\beta_0)^{n+1}}.$$

Second, we prove (ii). Since $t^* = t^{**}$, then $a_n = b_n$, $p(t_n) = \frac{k}{2}a_n^2$ and

$$\frac{a_{n+1}}{a_n} = 1 - \frac{k}{2}\beta_0 a_n.$$

Taking now into account that the function

$$R(t) = 1 - \frac{k}{2}\beta_0(t^* - t)$$

is strictly increasing, it follows that

$$\frac{1}{2} = R(t_0) \leq R(t) \leq R(t^*) = 1.$$

In consequence, from $a_{n+1} = R(t_n)a_n$, we have

$$a_{n+1} \leq a_0 \leq t^* - t_0 = t^*,$$
$$a_{n+1} \geq \left(\frac{1}{2}\right)^{n+1} a_0 = \left(\frac{1}{2}\right)^{n+1} t^*$$

and

$$\left(\frac{1}{2}\right)^{n+1} t^* \leq t^* - t_{n+1} \leq t^*.$$

This completes the proof. □

From the last theorem, we notice that the modified Newton's method has order of convergence at least one, since

$$\begin{cases} t^* - t_n \leq t^*(k\beta_0 t^*)^n & \text{if } t^* < t^{**}, \\ t^* - t_n \leq t^* & \text{if } t^* = t^{**}. \end{cases}$$

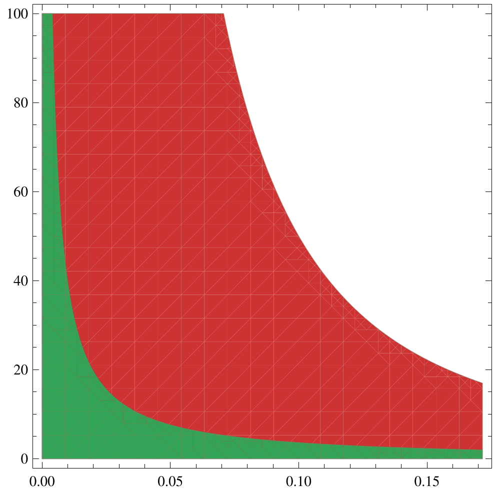

Fig. 9. Comparison between the domain of parameters associated with the modified Newton's method (red) and Steffensen's method (green). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

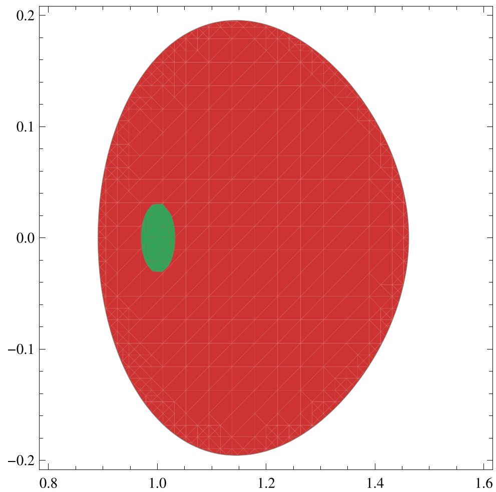

Fig. 10. Comparison between the regions of accessibility of the modified Newton's method (red) and Steffensen's method (green) when they are applied to solve $F(z) = z^3 - 1 = 0$ (and $k = 6|1.6 + 0.2i|$). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

3.2. Accessibility of solution by means of the modified Newton's method

We can see in Fig. 9 the same fact that happens with the domains of parameters associated with Newton's and Steffensen's methods, the big difference that exists between the domains of parameters associates with the modified Newton's method and Steffensen's method.

In Fig. 10, we fixed the value $k = 6|1.6+0.2i|$, and we show the regions of accessibility of the modified Newton's method and Steffensen's method when they are applied to solve the complex equation $F(z) = z^3 - 1 = 0$. Observe again the big difference between both regions, that indicates that the domain of starting points for the modified Newton's method is much bigger than that of Steffensen's method.

4. An efficient iterative predictor–corrector method

The main aim of this paper is to construct a modification of Steffensen's method with a better region of accessibility than that of Steffensen's method. For this, we rely on a predictor method.

As we can observe in Fig. 9, the domain of parameters associated with Steffensen's method is included in that of the modified Newton's method. Therefore, the convergence conditions required to Steffensen's method, that guarantees its semilocal convergence, are more restrictive than those required to the modified Newton's method.

Our immediate aim is, for a initial pair (α_0, β_0) that satisfies condition (12), namely, this pair is within the domain of parameters associated with the modified Newton's method, to obtain a pair $(\tilde{\alpha}, \tilde{\beta})$ that satisfies the two conditions given in (8), after making a certain number N_0 of iterations with the modified Newton's method, so that we can guarantee the semilocal convergence of Steffensen's method when it starts in the iteration N_0 of the modified Newton's method. So, we can consider the pair $(\alpha_{N_0}, \beta_{N_0})$, obtained from the modified Newton's method, as the initial pair $(\tilde{\alpha}, \tilde{\beta})$ for Steffensen's method.

In other words, we construct a simple modification of Steffensen's method, so that this method is convergent from the same starting points from which the modified Newton's method is. We then consider the following predictor–corrector iterative method:

$$\begin{cases} \begin{cases} x_0 \in \Omega, \\ x_{j+1} = x_j - [F(x_0)]^{-1}F(x_j), \quad j = 0, 1, \ldots, N_0, \end{cases} \\ \begin{cases} z_0 = x_{N_0}, \\ y_n = z_n + F(z_n), \\ z_{n+1} = z_n - [z_n, y_n; F]^{-1}F(z_n), \quad n \geq 0, \end{cases} \end{cases} \tag{13}$$

where x_0 satisfies condition (12), while $z_0 = x_{N_0}$ satisfies the two conditions given in (8). For method (13) is convergent, we must do the following:

1. Find x_0, so that predictor method (5) is convergent.
2. From the convergence of predictor method (5), calculate the value N_0 such that x_{N_0} is a good starting point from which the convergence of corrector method (4) is guaranteed.

In short, we use the modified Newton's method (5) for a finite number of steps N_0, provided that the starting point x_0 satisfies condition (12), until $z_{N_0} = x_0$ satisfy the two conditions given in (8), and we then use Steffensen's method (4) instead of the modified Newton's method (5). The key to the problem is therefore to guarantee the existence of N_0.

Now, we study the semilocal convergence of method (13). From predictor method (5) we consider the following situation. Given the initial approximation x_0, we consider the initial pair (α_0, β_0) which is defined from conditions $(C1)$ and $(C2)$:

$$\|F(x_0)\| \leq \alpha_0, \qquad \|[F'(x_0)]^{-1}\| \leq \beta_0.$$

According to (12) and Theorem 3, the sequence $\{x_n\}$, defined by method (5), is convergent if

$$k\alpha_0\beta_0^2 \leq \frac{1}{2}.$$

From the next approximations x_n obtained with method (5), we define the pairs (α_n, β_n) from the corresponding conditions $(C1)$ and $(C2)$, so that

$$\|F(x_n)\| \leq p(t_n) = \alpha_n, \qquad \frac{k}{2}a_nb_n = \alpha_n, \qquad \frac{k^2}{2}a_nb_n = k\alpha_n,$$

where $a_n = t^* - t_n$ and $b_n = t^{**} - t_n$. From Theorem 4 we have

$$k\alpha_n = kp(t_n) < \frac{kt^*t^{**}(t^*k\beta_0)^n}{\beta_0(t^{**} - t^*(t^*k\beta_0)^n)}(1 - t^*k\beta_0), \tag{14}$$

where t^* and t^{**} $(t^* \leq t^{**})$ are the two real positive zeros of polynomial (9).

Now, we write β_n based on β_0, consider

$$\|I - \Gamma_0 F'(x_n)\| \leq \|\Gamma_0\| \, \|F'(x_0) - F'(x_n)\| \leq k\beta_0 t^*$$

and obtain

$$\|\Gamma_n\| \leq \tilde{\beta}, \quad \text{where } \tilde{\beta} = \beta_n = \frac{\beta_0}{1 - k\beta_0 t^*}.$$

So, we obtain the pair $(\alpha_n, \tilde{\beta})$ with $\tilde{\beta} = \beta_n$ for some $n \in \mathbb{N}$.

After that, the pair $(\alpha_n, \tilde{\beta})$ must satisfy the two corresponding convergence conditions of Steffensen's method given in (8), namely $k\tilde{\alpha}\tilde{\beta} \leq 2$ and $M\tilde{\alpha}b^2 \leq \frac{1}{2}$, where $\tilde{\alpha} = \alpha_n$, $b = \frac{2\tilde{\beta}}{2-k\tilde{\alpha}\tilde{\beta}}$ and $M = k\left(1 + \frac{1}{b}\right)$. Thus, the pair $(\alpha_n, \tilde{\beta})$, obtained previously from predictor method (5), must satisfy the following:

$$k\alpha_n\tilde{\beta} \leq 2 \quad \text{and} \quad M\alpha_n b^2 \leq \frac{1}{2}.$$

Moreover, the second inequality is equivalent to

$$5\tilde{\beta}^2(k\alpha_n)^2 - 4\tilde{\beta}(3 + 2\tilde{\beta})(k\alpha_n) + 4 \geq 0,$$

which is satisfied if

$$k\alpha_n \le P, \quad \text{where } P = \frac{2(3+2\tilde{\beta})\tilde{\beta} - 4\sqrt{\tilde{\beta}^2 + 3\tilde{\beta} + 1}}{5\tilde{\beta}^2}.$$

Taking into account the two inequalities, we infer that

$$k\alpha_n \le \min\left\{\frac{2}{\tilde{\beta}}, P\right\} = \begin{cases} P \text{ si } \tilde{\beta} \le 2.5303\ldots \\ \dfrac{2}{\tilde{\beta}} \text{ si } \tilde{\beta} \ge 2.5303\ldots. \end{cases} \tag{15}$$

The following is then to find N_0, so that this value indicates when hybrid method (13) jumps from the predictor method to the corrector method.

Notice that the sequence $\{\alpha_n\}$ is decreasing. Then, we look for the first value of n in (14) that satisfied (15).

First, if $\tilde{\beta} \le 2.5303\ldots$, then

$$k\alpha_n < \frac{kt^* t^{**}(k\beta_0)^n t^*}{\beta_0(t^{**} - t^*(t^* k\beta_0)^n)}(1 - k\beta_0 t^*) < P.$$

Now taking logarithms, we find a value $N_0 \in \mathbb{N}$ such that the pair $(\alpha_{N_0}, \beta_{N_0})$ satisfies the two convergence conditions of the corrector method and given in (8). So,

$$N_0 \ge \frac{\log\left(\frac{P\beta_0 t^{**}}{t^*(P\beta_0 + kt^{**}(1 - t^* k\beta_0))}\right)}{\log(t^* k\beta_0)},$$

so that

$$N_0 = 1 + \left[\frac{\log\left(\frac{P\beta_0 t^{**}}{t^*(P\beta_0 + kt^{**}(1 - k\beta_0 t^*))}\right)}{\log(k\beta_0 t^*)}\right],$$

where $[t]$ denotes the integer part of any real number t.

Second, if $\tilde{\beta} \ge 2.5303\ldots$, then

$$k\alpha_n < \frac{kt^* t^{**}(t^* k\beta_0)^n}{\beta_0(t^{**} - t^*(t^* k\beta_0)^n)}(1 - t^* k\beta_0) < \frac{2}{\tilde{\beta}}$$

and, following the same procedure as in the previous case, we have

$$N_0 = 1 + \left[\frac{\log\left(\frac{2\beta_0 t^{**}}{t^*(2\beta_0 + k\tilde{\beta} t^{**}(1 - k\beta_0 t^*))}\right)}{\log(k\beta_0 t^*)}\right].$$

Finally, once the value of N_0 is a priori estimated, we summarize all the above in the following result, which guarantees the semilocal convergence of hybrid method (13).

Theorem 5. *Let X be a Banach spaces, $F : \Omega \subseteq X \to X$ be a once continuously differentiable operator defined on a non-empty open convex domain Ω and $x_0 \in \Omega$. We suppose that the operator $\Gamma_0 = [F'(x_0)]^{-1} \in \mathcal{L}(X, X)$ exists, conditions $(C1)$–$(C3)$ are satisfied and $B(x_0, t^*) \subseteq \Omega$, where t^* is the smallest positive zero of polynomial (9). If $k\alpha_0\beta_0^2 \le \frac{1}{2}$, then the sequence $\{x_n\}$, given by method (13), is well defined and converges to a solution x^* of the equation $F(x) = 0$ with*

$$N_0 = \begin{cases} 1 + \left[\dfrac{\log\left(\frac{P\beta_0 t^{**}}{t^*(P\beta_0 + kt^{**}(1 - k\beta_0 t^*))}\right)}{\log(k\beta_0 t^*)}\right] & \text{if } \tilde{\beta} \le 2.5303\ldots, \\ 1 + \left[\dfrac{\log\left(\frac{2\beta_0 t^{**}}{t^*(2\beta_0 + k\tilde{\beta} t^{**}(1 - k\beta_0 t^*))}\right)}{\log(k\beta_0 t^*)}\right] & \text{if } \tilde{\beta} \ge 2.5303\ldots, \end{cases} \tag{16}$$

and $\tilde{\beta} = \frac{\beta_0}{1 - k\beta_0 t^*}$.

Table 1
Numerical solution $\mathbf{x}^*$ of (18).

i	x_i^*	i	x_i^*	i	x_i^*	i	x_i^*
1	1.012239 . . .	3	1.118079 . . .	5	1.159804 . . .	7	1.058428 . . .
2	1.058428 . . .	4	1.159804 . . .	6	1.118079 . . .	8	1.012239 . . .

5. Application

With this application we show that we cannot apply Steffensen's method in principle to approximate a solution of the discrete problem corresponding to a nonlinear integral equation of Hammerstein type, since the convergence conditions of Theorem 2 are not satisfied. However, we can approximate such a solution by Steffensen's method from a certain approximation N_0 which is estimated by Theorem 5 and obtained before by the modified Newton's method.

It seems that the application of Steffensen's method is more restrictive than the modified Newton's method. Let us look through the following example that this is true.

Consider the following nonlinear integral equation of mixed Hammerstein type

$$x(s) = 1 + \int_0^1 G(s,t)\,x(t)^2\,dt, \quad s \in [0,1], \tag{17}$$

where $x \in C[0,1]$, $t \in [0,1]$ and the kernel G is $G(s,t) = \begin{cases} (1-s)t, & t \le s, \\ s(1-t), & s \le t. \end{cases}$

To solve (17), we transform it into a finite dimensional problem by using a process of discretization. For this, we approximate the integral that appears in (17) by the Gauss–Legendre formula

$$\int_0^1 h(t)\,dt \simeq \sum_{i=1}^{8} w_i h(t_i),$$

where the nodes t_i and the weights w_i are known.

If we denote the approximation of $x(t_i)$ by x_i $(i = 1, 2, \dots, 8)$, then (17) is equivalent to the following nonlinear system of equations

$$x_i = 1 + \sum_{j=1}^{8} a_{ij}\,x_j^2, \quad j = 1, 2, \dots, 8, \tag{18}$$

where

$$a_{ij} = \begin{cases} w_j t_j (1-t_i) & \text{if } j \le i, \\ w_j t_i (1-t_j) & \text{if } j > i. \end{cases}$$

System (18) is now written as

$$F(\mathbf{x}) \equiv \mathbf{x} - \mathbf{1} - A\mathbf{v_x} = 0, \quad F : \mathbb{R}^8 \longrightarrow \mathbb{R}^8,$$

where

$$\mathbf{x} = (x_1, x_2, \dots, x_8)^T, \qquad \mathbf{1} = (1, 1, \dots, 1)^T, \qquad A = (a_{ij})_{i,j=1}^8, \qquad \mathbf{v_x} = (x_1^2, x_2^2, \dots, x_8^2)^T.$$

Moreover, $F'(\mathbf{x}) = I - 2AD(\mathbf{x})$, where $D(\mathbf{x}) = \operatorname{diag}\{x_1, x_2, \dots, x_8\}$.

Choosing as starting point $\mathbf{x_0} = (1.7, 1.7, \dots, 1.7)^T$ and the max-norm, we obtain $\alpha_0 = 0.6713\dots$, $\beta_0 = 1.6549\dots$, $k = 0.2471\dots$, $k\alpha_0\beta_0^2 = 0.4543\dots < \frac{1}{2}$. In consequence, we can apply the modified Newton's method to solve system (18), since condition (12) is satisfied, but we cannot apply Steffensen's method, since the second condition of (8) is not satisfied:

$$M\tilde{\alpha}b^2 = 0.9286\dots > \frac{1}{2},$$

where $\tilde{\alpha} = \alpha_0 = 0.6713\dots$, $b = 1.9182\dots$ and $M = 0.3759\dots$.

As the modified Newton's method is convergent by Theorem 3, then we use it to approximate the numerical solution $\mathbf{x}^* = (x_1^*, x_2^*, \dots, x_8^*)^T$ of (18), which is shown in Table 1, after 23 iterations and using the stopping criterion $\|\mathbf{x_n} - \mathbf{x_{n-1}}\| < 10^{-32}$. In Table 2 we show the errors $\|\mathbf{x_n} - \mathbf{x}^*\|$ obtained with the same stopping criterion.

On the other hand, we apply predictor–corrector iterative method (13) to approximate the solution of (18) given in Table 1. For this, we only need to calculate the value N_0 which is fixed by Theorem 5 taking into account that $\tilde{\beta} = 5.4777\dots$. According to formula (16), we obtain $N_0 = 1$, so that after one iteration by the modified Newton's method, we apply Steffensen's method and approximate the solution given in Table 1 after four iterations more. In Table 3 we show the errors $\|\mathbf{x_n} - \mathbf{x}^*\|$ when the stopping criterion $\|\mathbf{x_n} - \mathbf{x_{n-1}}\| < 10^{-32}$ is used.

Table 2
Absolute errors for method (5).

i	$\|\mathbf{x_n} - \mathbf{x}^*\|$	i	$\|\mathbf{x_n} - \mathbf{x}^*\|$	i	$\|\mathbf{x_n} - \mathbf{x}^*\|$
0	0.687760 . . .	8	$3.814614\ldots \times 10^{-7}$	16	$3.956494\ldots \times 10^{-13}$
1	0.061560 . . .	9	$6.814479\ldots \times 10^{-8}$	17	$7.067778\ldots \times 10^{-14}$
2	0.011811 . . .	10	$1.217348\ldots \times 10^{-8}$	18	$1.262849\ldots \times 10^{-14}$
3	0.002093 . . .	11	$2.174688\ldots \times 10^{-9}$	19	$2.253264\ldots \times 10^{-15}$
4	0.000374 . . .	12	$3.884895\ldots \times 10^{-10}$	20	$4.052338\ldots \times 10^{-16}$
5	0.000066 . . .	13	$6.940031\ldots \times 10^{-11}$	21	$6.968408\ldots \times 10^{-17}$
6	0.000011 . . .	14	$1.239777\ldots \times 10^{-11}$	22	$1.515594\ldots \times 10^{-17}$
7	$2.135342\ldots \times 10^{-6}$	15	$2.214753\ldots \times 10^{-12}$		

Table 3
Absolute errors for method (13).

n	$\|\mathbf{x_n} - \mathbf{x}^*\|$
0	0.687760 . . .
1	0.061560 . . .
2	0.000827 . . .
3	$1.458275\ldots \times 10^{-7}$
4	$4.498237\ldots \times 10^{-15}$

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