On the zeros of orthogonal polynomials on the unit circle

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Abstract

Let $\{z_n\}$ be a sequence in the unit disk $\{z \in \mathbb{C} : |z| < 1\}$. It is known that there exists a unique positive Borel measure on the unit circle such that their orthogonal polynomials $\{\Phi_n\}$ satisfy

$$\Phi_n(z_n) = 0$$

for each $n = 1, 2, \ldots$ Characteristics of the orthogonality measure and asymptotic properties of the orthogonal polynomials are given in terms of the asymptotic behavior of the sequence $\{z_n\}$. Particular attention is paid to periodic sequences of zeros $\{z_n\}$ with period two and three.

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1. Introduction

In the last decade several papers on zeros of orthogonal polynomials on the unit circle (OPUC) have been published. For instance, we have [12], [13], [22], [23] or [24]. These articles joint to [5], [14], [15], [16], [25] and the seminal books of Simon, [20] and [21], bring us closer to a better understanding of the properties of the zeros of OPUC. However, there are several open questions about the zeros of OPUC, see for example pp. 97–98 of [20]. In [12] and [13] the properties of the zeros are studied in terms of analytic properties of the orthogonality measure, whereas in [22], [23] and [24] the information about the zeros is given in terms

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of Verblunsky coefficients. Other interesting problems deal with the description of properties of the zeros of OPUC in terms of other parameters which also characterize OPUC. We will study some of these questions in this paper.

We need to introduce some notation to state our results. Let μ be a nontrivial probability measure on $[0, 2\pi)$ and let $\varphi_n(z) = \varphi_n(z, \mu) = \kappa_n z^n + \dots, n = 0, 1, \dots$ be their orthonormal polynomials with positive leading coefficient. So, $\kappa_n > 0$ and

$$\langle \varphi_n, \varphi_m \rangle = \frac{1}{2\pi} \int \varphi_n(e^{i\theta}) \overline{\varphi_m(e^{i\theta})} \, d\mu(\theta) = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases}$$

We denote by $\Phi_n(z) = \dfrac{\varphi_n(z)}{\kappa_n}$ the corresponding monic orthogonal polynomials. It is well known that they satisfy the Szegő recurrence

$$\Phi_{n+1}(z) = z\Phi_n(z) + \Phi_{n+1}(0)\Phi_n^*(z), \quad n > 0, \quad \Phi_0(z) = 1, \tag{1}$$

$$\Phi_{n+1}^*(z) = \Phi_n^*(z) + \overline{\Phi_{n+1}(0)} z \Phi_n(z), \quad n \geq 0, \tag{2}$$

where we use the standard notation $\Phi_n^*(z) = z^n \overline{\Phi_n(\frac{1}{\bar{z}})}$. The parameters $\Phi_n(0)$ are called Verblunsky coefficients (also reflection coefficients or Schur parameters).

All the zeros of Φ_n lie in the unit disk $\mathbb{D} \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |z| < 1\}$ and, therefore, $\Phi_{n+1}(0) \in \mathbb{D}$. Moreover, Verblunsky's theorem (see Theorem 1.7.11, p. 97, of [20]) states that given a sequence $\{\alpha_n\}_{n \geq 1}$ in $\mathbb{D}$ there exists a unique probability measure μ on $[0, 2\pi)$ such that $\Phi_n(0, \mu) = \alpha_n$, $n = 1, 2, \dots$

If $\{z_n\}$ is a sequence in $\mathbb{D}$ with the property that

$$\Phi_n(z_n) = 0, \quad n = 1, 2, \dots, \tag{3}$$

then (1) yields

$$\Phi_{n+1}(0) = -z_{n+1} \frac{\Phi_n(z_{n+1})}{\Phi_n^*(z_{n+1})} \tag{4}$$

and Verblunsky's theorem states the existence of a unique orthogonality measure. So, the OPUC are uniquely determined by a sequence of their zeros, i.e., by a sequence $\{z_n\}$ such that (3) holds (see [2]).

In this paper we obtain properties of OPUC in terms of properties of a sequence of their zeros. Our first result is about OPUC determined by a periodic sequence $\{z_n\}$.

Theorem 1. *Suppose that, for n large enough, there exists a common zero for Φ_n and Φ_{n-3}. Let ζ_j $(j = 1, 2, 3)$ be such common zeros, i.e.,*

$$\Phi_n(\zeta_j) = 0, \quad n = j \mod 3, \quad n \geq n_0,$$

and let $r \stackrel{\text{def}}{=} \max\{|\zeta_j| : j = 1, 2, 3\}$. We claim:

(i) If $r \leq \frac{-1+\sqrt{5}}{2}$, then $\lim_n \Phi_n(0) = 0$.

(ii) If, in addition, $r < \frac{-1+\sqrt{5}}{2}$, then $\limsup_n |\Phi_n(0)|^{1/n} \leq \dfrac{r^2}{1-r} < 1$.

The numerical experiments show that if the three common zeros have modulus greater than $\frac{-1+\sqrt{5}}{2}$, then the zeros are uniformly distributed on three arcs of the unit circle like those corresponding to a measure supported on three arcs, see Figures 3 and 4. If the sequence $\{z_n\}$ is periodic with period two, then the Verblunsky coefficients are an asymptotic periodic sequence as states Lemma 3 below. This case has been studied in [5] and [23].

A completely different situation appears when the sequence $\{z_n\}$ is dense in $\mathbb{D}$ and so the zeros of the OPUC are also dense in $\mathbb{D}$. This last case was studied by Khrushchev in [11]. He showed that there exist OPUC with zeros dense in $\mathbb{D}$ with orthogonality measure in many classes of measures, including measures in the Szegő class ($\log \mu' \in L^1$) or measures such that the series of their Verblunsky coefficients is absolutely convergent (see also Example 1.7.18, p. 98, of [20]).

Our next theorem states that the zeros can not "approach the unit circle too fast" if and only if the orthogonality measure lies in the Nevai class. From now on $\{z_{n,j}\}_{j=1}^n$ will denote the zeros of $\Phi_n(z)$.

Theorem 2. *The following statements are equivalent:*

(a) $\lim_n \Phi_n(0) = 0$.

(b) $\lim_n \sum_{j=1}^{n} (1 - |z_{n,j}|) = \infty$.

The proof of this theorem is included in Section 5. This section also contains a study of the rate with which the zeros of OPUC for the Chebyshev weight on an arc of the unit circle approach to $\partial(\mathbb{D})$. In Section 2 we make some remarks about properties of OPUC in terms of the sequence of zeros given. In Section 3 we study orthogonal polynomials obtained from a two-periodic sequence $\{z_n\}$. The proof of Theorem 1 is contained in Section 4. Finally, we include several figures displaying the zeros of OPUC for some different sequences $\{z_n\}$.

2. Some estimates for the radius of the Mhaskar-Saff circle

It was proved in [14] that if $\lim_n \sum_{j=1}^n \Phi_j(0) = 0$ and Λ is an infinite subset of natural numbers such that

$$\lim_{n\in\Lambda} |\Phi_n(0)|^{1/n} = \limsup_n |\Phi_n(0)|^{1/n} \stackrel{\text{def}}{=} L,$$

then

$$\lim_{n\in\Lambda} \nu_{\Phi_n} = m_L,$$

where, as usual, we put $\nu_{\Phi_n} \stackrel{\text{def}}{=} \frac{1}{n}\sum_{j=1}^n \delta_{\{z_{n,j}\}}$ and m_L is the Lebesgue measure on the circle of radius L (the Mhaskar-Saff circle). Notice that throughout this paper the limit of a sequence of measures is always taken in the weak-* topology. In this section we give some estimates of L in terms of the behavior of a sequence of zeros of the OPUC.

Let $\{z_n\}$ be a sequence in $\mathbb{D}$ and let $\{\Phi_n\}$ be the unique sequence of monic orthogonal polynomials defined by (3). Then

$$\Phi_{n+1}(0) = -z_{n+1}\frac{\Phi_n(z_{n+1})}{\Phi_n^*(z_{n+1})} = -z_{n+1}\frac{z_{n+1}-z_n}{1-\overline{z_n}z_{n+1}}\prod_{j:z_{n,j}\neq z_n}\frac{z_{n+1}-z_{n,j}}{1-\overline{z_{n,j}}z_{n+1}}.$$

Lemma 1. *(i) If* $\lim_n \frac{z_{n+1}-z_n}{1-\overline{z_n}z_{n+1}} = 0$, *then* $\lim_n \Phi_n(0) = 0$. *This occurs, in particular, if* $\lim_n z_n = z_0$ *with* $|z_0| < 1$.

(ii) In this last case, putting $w_n = z_n - z_0$, *we have*

$$\limsup_n |\Phi_n(0)|^{1/n} \leq \limsup_n |w_n|^{1/n}. \tag{5}$$

(iii) Let Λ *denote any infinite subset of* $\mathbb{N}$ *such that there exists* $\lim_{n\in\Lambda}\nu_{\Phi_n} \stackrel{\text{def}}{=} \nu$. *If* $\lim_n z_n = z_0$ *and* $\nu(\{z_0\}) = 0$, *then*

$$\limsup_{n\in\Lambda} |\Phi_{n+1}(0)|^{1/(n+1)} \leq \exp\int \log\left|\frac{z_0-\zeta}{1-\bar{\zeta}z_0}\right| d\nu. \tag{6}$$

Proof. We only check (6) because the other statements are trivial since $\left|\frac{z_{n+1}-z_{n,j}}{1-\overline{z_{n,j}}z_{n+1}}\right| < 1$. Let $\delta > 0$ and $L = \limsup_{n\in\Lambda}|\Phi_{n+1}(0)|^{1/(n+1)}$. Then

$$\Phi_{n+1}(0) = -z_{n+1}\prod_{j:|z_{n,j}-z_0|<\delta}\frac{z_{n+1}-z_{n,j}}{1-\overline{z_{n,j}}z_{n+1}}\prod_{j:|z_{n,j}-z_0|\geq\delta}\frac{z_{n+1}-z_{n,j}}{1-\overline{z_{n,j}}z_{n+1}}$$

$$\Rightarrow \frac{1}{n+1}\log|\Phi_{n+1}(0)| \leq \frac{1}{n+1}\sum_{j:|z_{n,j}-z_0|\geq\delta}\log\left|\frac{z_{n+1}-z_{n,j}}{1-\overline{z_{n,j}}z_{n+1}}\right|$$
$$= \frac{n}{n+1}\int_{|\zeta-z_0|\geq\delta}\log\left|\frac{z_{n+1}-\zeta}{1-\bar{\zeta}z_{n+1}}\right| d\nu_{\Phi_n}(\zeta).$$

Since

$$\lim_n \log\left|\frac{z_{n+1}-\zeta}{1-\bar{\zeta}z_{n+1}}\right| = \log\left|\frac{z_0-\zeta}{1-\bar{\zeta}z_0}\right|$$

uniformly on $\mathrm{supp}(\nu)\cap\{\zeta : |\zeta-z_0|\geq\delta\}$ and $\lim_{n\in\Lambda}\nu_{\Phi_n} = \nu$, we deduce that

$$\log L \leq \int_{|\zeta-z_0|\geq\delta}\log\left|\frac{z_0-\zeta}{1-\bar{\zeta}z_0}\right| d\nu(\zeta).$$

Because $\delta > 0$ is arbitrary, $\left|\frac{z_0-\zeta}{1-\bar{\zeta}z_0}\right| \leq 1$ for $\zeta \in \overline{\mathbb{D}}$ and $\nu(\{z_0\}) = 0$, from the above inequality the proof is easily completed. $\square$

Remarks. 1. *From (5),* $\limsup_n |w_n|^{1/n}$ *is an upper bound of the radius* L *of the circle where the zeros of the polynomials of degree* $n \in \Lambda$ *are uniformly distributed.*

2. *Simon proved in [22] that if* $\limsup_n |\Phi_n(0)|^{1/n} < 1$, *then the rate of convergence of the zeros to the Nevai-Totik points*[2] *is geometric. So, according to (5), this geometric rate is smaller than the radius of the Mhaskar-Saff circle.*

3. *If* $L > 0$ *and* $|z_0| \le L$, *then*

$$\int \log \left| \frac{z_0 - \zeta}{1 - \bar{\zeta} z_0} \right| dm_L(\zeta) = \log L. \tag{7}$$

Thus, (6) and (7) imply that $\limsup_{n\in\Lambda} |\Phi_{n+1}(0)|^{1/(n+1)}$ *is less than or equal to the infimum of* $s > 0$ *such that* m_s *is the limit of any convergent subsequence of* $\{\nu_{\Phi_n}\}_{n\in\Lambda}$ *and*

$$\{z : |z| \le s\} \cap \left\{ z : \lim_n \operatorname{dist}(z, \{\zeta : \Phi_n(\zeta) = 0\}) = 0 \right\} \ne \emptyset.$$

Here $\operatorname{dist}(z, \{\zeta : \Phi_n(\zeta) = 0\}) \stackrel{\text{def}}{=} \inf_\zeta \{|z - \zeta| : \Phi_n(\zeta) = 0\}$ *and* m_s *denotes the Lebesgue measure on the circle* $\{z : |z| = s\}$.

Since $\Phi_{n+1}(0) = (-1)^{n+1} \prod_j z_{n+1,j}$, (4) implies

$$\Phi_{n+1}(0)^n = \prod_j \frac{\Phi_n(z_{n+1,j})}{\Phi_n^*(z_{n+1,j})}$$

and we get

$$|\Phi_{n+1}(0)|^{1/(n+1)} = \exp \iint \log \left| \frac{z - w}{1 - \overline{w} z} \right| d\nu_{\Phi_{n+1}}(z) d\nu_{\Phi_n}(w).$$

So, in addition to (6), we claim

Lemma 2. *Let* Λ *be any infinite subset of* $\mathbb{N}$. *If there exist* $\lim_{n\in\Lambda} \nu_{\Phi_n} \stackrel{\text{def}}{=} \nu_1$ *and* $\lim_{n\in\Lambda} \nu_{\Phi_{n+1}} \stackrel{\text{def}}{=} \nu_2$, *then*

$$\limsup_{n\in\Lambda} |\Phi_{n+1}(0)|^{1/(n+1)} \le \exp \iint \log \left| \frac{z - w}{1 - \overline{w} z} \right| d\nu_2(z) d\nu_1(w). \tag{8}$$

[2] The Nevai-Totik points are all the solutions of $D_{int}^{-1}(1/\overline{z}) = 0$ in $\{z : L < |z| < 1\}$, where D_{int} is the analytic extension of the *interior* Szegő function (see (17) below) and $L = \limsup_n |\Phi_n(0)|^{1/n}$.

Proof. The function $f(z,w) = \log\left|\dfrac{z-w}{1-\overline{w}z}\right|$ is non-positive upper semicontinuous in $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$. So, there is a monotone decreasing sequence of non-positive continuous functions $\{g_m\}$ such that $f(z,w) = \lim_m g_m(z,w)$ pointwise in $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$ (see Theorem 1.1, p.1, in [17]). Thus,

$$\begin{aligned}|\Phi_{n+1}(0)|^{1/(n+1)} = \exp \iint \log\left|\frac{z-w}{1-\overline{w}z}\right| d\nu_{\Phi_{n+1}}(z) d\nu_{\Phi_n}(w) \\ \leq \exp \iint g_m(z,w) d\nu_{\Phi_{n+1}}(z) d\nu_{\Phi_n}(w)\end{aligned}$$

and since $\lim_n (\nu_{\Phi_{n+1}} \times \nu_{\Phi_n}) = \nu_2 \times \nu_1$, the conclusion of the lemma follows immediately from the monotone convergence theorem. □

Remark. *Let* $\limsup_{n\in\Lambda} |\Phi_{n+1}(0)|^{1/(n+1)} = r > 0$ *and* $\nu_1 = \nu_2 = m_r$. *Then (8) becomes an equality as a consequence of (7).*

3. Two-periodic case

Lemma 3. *Suppose that the sequence* $\{z_n\}$ *is periodic with period two, i.e.,*

$$z_n = \begin{cases} \alpha_1, & n \text{ odd}, \\ \alpha_2, & n \text{ even}, \end{cases} \tag{9}$$

with $\{\alpha_1, \alpha_2\} \subset \mathbb{D}$. *Let* $\{\Phi_n\}$ *be the sequence of monic orthogonal polynomials on the unit circle such that (3) holds. Then the Verblunsky coefficients are*

$$\Phi_1(0) = -\alpha_1, \quad \Phi_2(0) = -\alpha_2\, C_{\alpha_1,\alpha_2} \tag{10}$$

and for all $n \geq 3$,

$$\Phi_n(0) = (-1)^{n-1} C_{\alpha_1,\alpha_2} \times \begin{cases} \alpha_1^{(n-1)/2} \alpha_2^{(n-1)/2}, & n \text{ odd}, \\ \alpha_1^{-1+n/2} \alpha_2^{n/2}, & n \text{ even}, \end{cases} \tag{11}$$

where $C_{\alpha_1,\alpha_2} = \dfrac{\alpha_2 - \alpha_1}{1 - \overline{\alpha_1}\alpha_2}$.

Proof. Iterating (1) and (2), we can write

$$\begin{aligned}\Phi_{n+1}(z) = z\left(z + \overline{\Phi_n(0)}\Phi_{n+1}(0)\right)\Phi_{n-1}(z) \\ + \left(\Phi_{n+1}(0) + z\Phi_n(0)\right)\Phi^*_{n-1}(z), \quad n \geq 2. \end{aligned}\tag{12}$$

Thus, if Φ_{n+1} and Φ_{n-1} have a common zero, ζ, setting $z = \zeta$ in (12) we get

$$\Phi_{n+1}(0) = -\zeta \Phi_n(0), \quad n \geq 2,$$

which proves the lemma. □

Remark. *If $\alpha_1 = \alpha_2 = \alpha$, then $\Phi_n(z) = z^{n-1}(z-\alpha)$ for all $n \geq 1$ and $\Phi_n(0) = 0$ for all $n \geq 2$. If $\alpha_1\alpha_2 = 0$, then $\Phi_n(0) = 0$ for all $n \geq 3$. Both cases are trivial, this is why we do not consider them through this section.*

Ortogonal polynomials on the unit circle were studied in [5] and [23] under the conditions

$$\lim_n \Phi_n(0) = 0; \quad \exists \lim_{\substack{n \\ n=j \bmod k}} \frac{\Phi_{n+1}(0)}{\Phi_n(0)} \quad j = 1, 2, \ldots, k. \tag{13}$$

More precisely, the Verblunsky coefficients considered in [23] satisfy the property

$$\Phi_n(0) = \sum_{j=1}^{l} C_j b_j^n + O(\lambda b)^n, \quad n \to \infty, \tag{14}$$

where $\lambda \in (0,1)$, $0 \notin \{C_j\}$, $\{b_j\}$ are all distinct and $|b_j| = |b| < 1$, $j = 1, \ldots, l$. If $\{z_n\}$ satisfies (9), then (14) holds for $l = 2$ with

$$C_1 = -\frac{C_{\alpha_1,\alpha_2}}{2}\left(\frac{1}{\alpha_1} - \frac{1}{\sqrt{\alpha_1\alpha_2}}\right), \quad C_2 = -\frac{C_{\alpha_1,\alpha_2}}{2}\left(\frac{1}{\alpha_1} + \frac{1}{\sqrt{\alpha_1\alpha_2}}\right) \tag{15}$$

and

$$b_1 = \sqrt{\alpha_1\alpha_2}, \quad b_2 = -\sqrt{\alpha_1\alpha_2}. \tag{16}$$

Therefore, all the results proved in [23] also hold for OPUC obtained from a two-periodic sequence $\{z_n\}$. Thus, according to Theorem 2.2 there, we state the following result.

Corollary 3. *If $\{z_n\}$ is defined by (9), then*

$$\lim_k \frac{\Phi_{2k}(z)}{\alpha_1^k \alpha_2^k C_{\alpha_1,\alpha_2}} = \frac{D(0)D(z)^{-1}}{(\alpha_1\alpha_2 - z^2)}(z - \alpha_2),$$

$$\lim_k \frac{\Phi_{2k+1}(z)}{\alpha_1^k \alpha_2^k C_{\alpha_1,\alpha_2}} = \frac{\alpha_2 D(0)D(z)^{-1}}{(\alpha_1\alpha_2 - z^2)}(\alpha_1 - z)$$

uniformly on each compact subset of $\{z : |z| < \sqrt{|\alpha_1\alpha_2|}\}$, where

$$D(z) = D(z,\mu) \stackrel{\text{def}}{=} \exp\left(\frac{1}{4\pi}\int \frac{e^{i\theta}+z}{e^{i\theta}-z}\log(\mu'(e^{i\theta}))d\theta\right) \tag{17}$$

is the Szegő function. Since

$$\kappa \stackrel{\text{def}}{=} \lim_n \kappa_n = D(0)^{-1} = \prod_{j=1}^{\infty}(1 - |\Phi_j(0)|^2)^{-1/2} < \infty,$$

the analogous result also holds for orthonormal polynomials.

As a consequence of Corollary 3, we have

$$\lim_n \frac{\Phi_{n+2}(z)}{\Phi_n(z)} = \alpha_1\alpha_2$$

uniformly on compact subsets of $\{z : |z| < \sqrt{|\alpha_1\alpha_2|}\} \setminus \{\alpha_1, \alpha_2\}$. This result is proved in [5] under the conditions (13). Actually, only the point α_1 or α_2 with smaller modulus lies in the circle $\{z : |z| < \sqrt{|\alpha_1\alpha_2|}\}$. For example, if $|\alpha_1| < |\alpha_2|$, we can be more precise and specify that the uniform convergence holds on compact subsets of $\{z : |z| < \sqrt{|\alpha_1\alpha_2|}\} \setminus \{\alpha_1\}$.

It is worthwhile studying the asymptotic behavior of OPUC in an annulus about the critical circle $\{z : |z| = \sqrt{|\alpha_1\alpha_2|}\}$. In this case some computations as those done in Sections 2 and 3 of [23], let us state:

Theorem 4. *If $\{z_n\}$ satisfies (9), then the function $D^{-1}(z)$ for $z \in \mathbb{D}$ admits a meromorphic extension, D_{int}^{-1}, to $\{z : |z| < \frac{1}{|\alpha_1\alpha_2|}\}$ which is analytic in $\{z : |z| < \frac{1}{\sqrt{|\alpha_1\alpha_2|}}\}$ and has only two poles at $\pm\frac{1}{\sqrt{\alpha_1\alpha_2}}$. Moreover,*

(i) We have

$$\lim_n \Phi_n^*(z) = D(0)D_{int}^{-1}(z)$$

uniformly on compact sets of $\{z : |z| < \frac{1}{\sqrt{|\alpha_1\alpha_2|}}\}$. Also,

$$\lim_n \frac{\Phi_n(z)}{z^n} = D(0)\overline{D_{int}(1/\overline{z})}^{-1}$$

uniformly on compact sets of $\{z : |z| > \sqrt{|\alpha_1\alpha_2|}\}$.

(ii) Let

$$F_n(z) \stackrel{\text{def}}{=} \Phi_{n+1}(0)\left(D(0)D^{-1}(z) - \Phi_n^*(z)\right)$$

and

$$s_n(z) \stackrel{\text{def}}{=} \sum_{j=0}^{\infty} z^{-j-1}F_{n+j}(z). \tag{18}$$

Then there exists $C > 0$ such that

$$\max_{|z|\leq 1} |F_n(z)| \leq C\,|\alpha_1\alpha_2|^n,$$

the series (18) converges in

$$\mathbb{A} = \{z : |\alpha_1\alpha_2| < |z| < 1\}$$

and

$$|s_n(z)| \leq C\frac{|\alpha_1\alpha_2|^n}{|z| - |\alpha_1\alpha_2|}, \quad z \in \mathbb{A}.$$

(iii) For $z \in \mathbb{A}$, we have

$$\Phi_{2k}(z) = -s_{2k}(z) + \frac{C_{\alpha_1,\alpha_2} D(0) D(z)^{-1} \alpha_1^k \alpha_2^k}{(\alpha_1\alpha_2 - z^2)} (z - \alpha_1) + z^{2k} D(0) \overline{D_{int}(1/\overline{z})}^{-1},$$

$$\Phi_{2k+1}(z) = -s_{2k+1}(z) + \frac{\beta C_{\alpha_1,\alpha_2} D(0) D(z)^{-1} \alpha_1^k \alpha_2^k}{(\alpha_1\alpha_2 - z^2)} (\alpha_1 - z) + z^{2k+1} D(0) \overline{D_{int}(1/\overline{z})}^{-1}.$$

Remark. *Using the above result, Simon also proved that if Verblunski coefficients satisfy (14), then the zeros of OPUC have "clock behavior" in the Mhaskar-Saff circle, see Theorem 5.1 in [23], and [22]. In our discussion this implies that the zeros approach to the circle $\{z : |z| = \sqrt{|\alpha_1\alpha_2|}\}$ with rate $O(\frac{\log n}{n})$, the quotient of magnitudes of consecutive zeros is $1 + O(\frac{1}{n \log n})$ and they are equally spaced with only larger gaps around $\pm\sqrt{\alpha_1\alpha_2}$ (see Figures 1 and 2 at the end). If $|\alpha_2| > |\alpha_1|$, see Figure 1, α_2 is a Nevai-Totik point. Notice that another Nevai-Totik point appears in this figure. The proof of its analytical existence will be the subject of a forthcoming paper.*

3.1. On the meromorphic extension of the Szegő function

In Theorem 4 we have used the meromorphic extension of the interior Szegő function to $\{z : |z| < \frac{1}{|\alpha_1\alpha_2|}\}$. This function has two poles at $\pm\frac{1}{\sqrt{\alpha_1\alpha_2}}$. Using a convergence result for Fourier-Padé approximants constructed from the expansion of the interior Szegő function in terms of orthogonal polynomials on the unit circle, we prove in this section that $\frac{1}{|\alpha_1\alpha_2|}$ is the larger radius of the circle with center at the origin where a such meromorphic extension with exactly two poles can be done. To obtain this result we will use a Lemma stated in [5] (Lemma 4 below).

Previously let us fix some notation and concepts. Let $f \in L^1(\mu)$. Its Fourier expansion with respect to the orthonormal system $\{\varphi_n\}$ is given by

$$\sum_{j=0}^{\infty} A_j \varphi_j(z),$$

where A_j denotes the jth Fourier coefficient of f, i.e.,

$$A_j \stackrel{\text{def}}{=} \langle f, \varphi_j \rangle.$$

The Fourier-Padé approximant of type (n, m), $n, m \in \{0, 1, \ldots\}$, of f is the ratio $\pi_{n,m}(f) = p_{n,m}/q_{n,m}$ of any two polynomials $p_{n,m}$ and $q_{n,m}$ such that

(i) $\deg(p_{n,m}) \le n$; $\deg(q_{n,m}) \le m$, $q_{n,m} \not\equiv 0$.

(ii) $q_{n,m}(z) f(z) - p_{n,m}(z) \sim A_{n,1}\varphi_{n+m+1}(z) + A_{n,2}\varphi_{n+m+2}(z) + \ldots$

The above condition (ii) means that

$$\langle q_{n,m} f - p_{n,m}, \varphi_j \rangle = 0, \quad j = 0, \ldots, n+m.$$

In the sequel, we take $q_{n,m}$ with leading coefficient equal to 1.

To prove the existence of such polynomials it suffices to solve a homogeneous linear system of m equations on the $m+1$ coefficients of $q_{n,m}$. Thus, a nontrivial solution is guaranteed. In general, the rational function $\pi_{n,m}$ is not uniquely determined but if for every solution $(p_{n,m}, q_{n,m})$ the polynomial $q_{n,m}$ has degree m, then $\pi_{n,m}$ is unique. For m fixed, a sequence of type $\{\pi_{n,m} : n \in \mathbb{N}\}$ is called an mth row of the Fourier-Padé approximants relative to f.

We write $\Delta_m(f)$ for the largest disk centered at $z = 0$ in which f can be extended to a meromorphic function with at most m poles and let $R_m(f)$ be the radius of a such circle. If $R_0(f) > 1$ and f has exactly m poles in $\Delta_m(f)$, then, for n large enough, $\pi_{n,m}$ is uniquely determined. This and other results for row sequences of Fourier-Padé approximants may be found in [18] and [19] for Fourier expansions with respect to measures supported on an interval of the real line whose absolutely continuous part with respect to Lebesgue measure is positive almost everywhere.

The following result is stated in [5].

Lemma 4. *Let μ be such that $R_0(D^{-1}) > 1$. The following assertions are equivalent:*

(a) *D^{-1} has exactly m poles in $\Delta_m = \Delta_m(D^{-1})$.*

(b) *The sequence $\{\pi_{n,m}(D^{-1}) : n = 0, 1, \ldots\}$, for all sufficiently large n, has exactly m finite poles and there exists a polynomial $w_m(z) = z^m + \ldots$ such that*

$$\limsup_n \|q_{n,m} - w_m\|^{1/n} = \delta < 1,$$

where $\|\cdot\|$ denotes any norm in the space of polynomials of degree at most m.

The poles of D^{-1} in Δ_m coincide with the zeros $z_1, \ldots, z_m$ of w_m, and

$$R_m(D^{-1}) = \frac{1}{\delta} \max_{1 \le j \le m} |z_j|. \tag{19}$$

Lemma 4 turns out to be definitive to prove the following result.

Theorem 5. *If the sequence of zeros $\{z_n\}$ satisfies (9), then*

$$R_2(D^{-1}) = \frac{1}{|\alpha_1 \alpha_2|}.$$

To carry out the proof we need some previous results (Lemma 5 and Lemma 6 below). The first of them is easily proved by using recurrence relation (1).

Lemma 5. *The orthonornal polynomials associated with $\{z_n\}$ satisfy the following relations:*

(i)

$$\langle z\varphi_j, 1\rangle = -\frac{\varphi_{j+1}(0)}{\kappa_j\kappa_{j+1}}, \quad j = 0, 1, \ldots,$$

$$\langle z\varphi_j, \varphi_m\rangle = \begin{cases} 0, & j < m-1, \\ \frac{\kappa_j}{\kappa_{j+1}}, & j = m-1, \\ -\frac{\varphi_{j+1}(0)\overline{\varphi_m(0)}}{\kappa_j\kappa_{j+1}}, & j > m-1. \end{cases}$$

(ii)

$$\langle z^2\varphi_j, \varphi_m\rangle = \begin{cases} 0, & j < m-2, \\ \frac{\kappa_{m-2}}{\kappa_m}, & j = m-2 \end{cases}$$

and

$$\langle z^2\varphi_{m-1}, \varphi_m\rangle = -\frac{\overline{\varphi_m(0)}}{\kappa_m^2}\left(\frac{\kappa_{m-1}}{\kappa_{m+1}}\varphi_{m+1}(0) + \frac{\varphi_m(0)\overline{\varphi_{m-1}(0)}}{\overline{\varphi_m(0)}}\right).$$

Moreover, if $j \geq m$, then

$$\langle z^2\varphi_j, \varphi_m\rangle = -\frac{\Phi_{j+1}(0)\overline{\varphi_m(0)}}{\kappa_j} \times \left(\frac{\overline{\Phi_{m-1}(0)}}{\overline{\Phi_m(0)}} + \frac{\Phi_{j+2}(0)}{\Phi_{j+1}(0)} - \sum_{l=m-1}^{j+1}\overline{\Phi_l(0)}\Phi_{l+1}(0)\right).$$

It is known that for measures in the Szegő class we have

$$D^{-1}(z) = \frac{1}{\kappa}\sum_{j=0}^{\infty}\overline{\varphi_j(0)}\varphi_j(z), \quad z \in \mathbb{D}$$

(see [9], p. 19; [15], Theorem 1; [14], Theorem 2.2 or [5], p. 174). When $\{z_n\}$ is defined by (9) the above expansion converges uniformly on compact subsets of $\{z : |z| < \frac{1}{\sqrt{|\alpha_1\alpha_2|}}\}$ and

$$D_{int}^{-1}(z) = \frac{1}{\kappa}\sum_{j=0}^{\infty}\overline{\varphi_j(0)}\varphi_j(z), \quad z \in \{z : |z| < \frac{1}{\sqrt{|\alpha_1\alpha_2|}}\}. \tag{20}$$

The next result is a consequence of (20) and Lemma 5.

Lemma 6. *The following equalities hold:*

$$\langle D_{int}^{-1}, \varphi_m\rangle = \frac{\overline{\varphi_m(0)}}{\kappa},$$

$$\langle zD_{int}^{-1}, \varphi_m\rangle = \frac{\overline{\varphi_m(0)}}{\kappa}\left(\frac{\overline{\Phi_{m-1}(0)}}{\overline{\Phi_m(0)}} - \sum_{j=m-1}^{\infty}\overline{\Phi_j(0)}\Phi_{j+1}(0)\right),$$

$$\langle z^2D_{int}^{-1}, \varphi_m\rangle = \frac{\overline{\varphi_m(0)}}{\kappa}\left(\frac{\overline{\Phi_{m-2}(0)}}{\overline{\Phi_m(0)}} + O(\varphi_{m-1}(0))\right).$$

Proof of Theorem 5. According to Theorem 4, D_{int}^{-1} is an analytic function on $\{z : |z| < \frac{1}{\sqrt{|\alpha_1\alpha_2|}}\}$ which has a meromorphic extension to $\{z : |z| < \frac{1}{|\alpha_1\alpha_2|}\}$ with only two poles at $\pm 1/\sqrt{\alpha_1\alpha_2}$. Thus, from Lemma 4, the denominators $q_{n,2}$ of the Fourier-Padé approximants of order $(n, 2)$ are exactly of degree 2 for n large enough and the zeros of $q_{n,2}$ converge to $\pm 1/\sqrt{|\alpha_1\alpha_2|}$. Therefore, according to (19) it only remains to find $\limsup_n \|q_{n,2}(z) - (z^2 - \dfrac{1}{\alpha_1\alpha_2})\|^{1/n}$.

Let $q_{n,2}(z) = (z - \beta_n)(z - \tau_n) = z^2 - (\beta_n + \tau_n)z + \beta_n\tau_n$. It satisfies

$$\langle q_{n,2} D_{int}^{-1}, \varphi_{n+1}\rangle = \langle q_{n,2} D_{int}^{-1}, \varphi_{n+2}\rangle = 0.$$

Thus,

$$(\beta_n + \tau_n)\langle zD_{int}^{-1}, \varphi_{n+1}\rangle - \beta_n\tau_n\langle D_{int}^{-1}, \varphi_{n+1}\rangle = \langle z^2 D_{int}^{-1}, \varphi_{n+1}\rangle,$$
$$(\beta_n + \tau_n)\langle zD_{int}^{-1}, \varphi_{n+2}\rangle - \beta_n\tau_n\langle D_{int}^{-1}, \varphi_{n+2}\rangle = \langle z^2 D_{int}^{-1}, \varphi_{n+2}\rangle.$$

Hence,

$$\beta_n + \tau_n = \frac{\begin{vmatrix} \langle z^2 D_{int}^{-1}, \varphi_{n+1}\rangle & -\langle D_{int}^{-1}, \varphi_{n+1}\rangle \\ \langle z^2 D_{int}^{-1}, \varphi_{n+2}\rangle & -\langle D_{int}^{-1}, \varphi_{n+2}\rangle \end{vmatrix}}{\begin{vmatrix} \langle zD_{int}^{-1}, \varphi_{n+1}\rangle & -\langle D_{int}^{-1}, \varphi_{n+1}\rangle \\ \langle zD_{int}^{-1}, \varphi_{n+2}\rangle & -\langle D_{int}^{-1}, \varphi_{n+2}\rangle \end{vmatrix}},$$

$$\beta_n\tau_n = \frac{\begin{vmatrix} \langle zD_{int}^{-1}, \varphi_{n+1}\rangle & \langle z^2 D_{int}^{-1}, \varphi_{n+1}\rangle \\ \langle zD_{int}^{-1}, \varphi_{n+2}\rangle & \langle z^2 D_{int}^{-1}, \varphi_{n+2}\rangle \end{vmatrix}}{\begin{vmatrix} \langle zD_{int}^{-1}, \varphi_{n+1}\rangle & -\langle D_{int}^{-1}, \varphi_{n+1}\rangle \\ \langle zD_{int}^{-1}, \varphi_{n+2}\rangle & -\langle D_{int}^{-1}, \varphi_{n+2}\rangle \end{vmatrix}}.$$

We have

$$\begin{vmatrix} \langle zD_{int}^{-1}, \varphi_{n+1}\rangle & -\langle D_{int}^{-1}, \varphi_{n+1}\rangle \\ \langle zD_{int}^{-1}, \varphi_{n+2}\rangle & -\langle D_{int}^{-1}, \varphi_{n+2}\rangle \end{vmatrix}$$
$$= \frac{\overline{\varphi_{n+1}(0)\varphi_{n+2}(0)}}{\kappa^2}\left(\frac{\overline{\Phi_{n+1}(0)}}{\overline{\Phi_{n+2}(0)}} - \frac{\overline{\Phi_n(0)}}{\overline{\Phi_{n+1}(0)}} + \overline{\Phi_n(0)}\Phi_{n+1}(0)\right). \tag{21}$$

Then, by Lemma 6, there exists $C \neq 0$ such that

$$\begin{vmatrix} \langle z^2 D_{int}^{-1}, \varphi_{n+1}\rangle & -\langle D_{int}^{-1}, \varphi_{n+1}\rangle \\ \langle z^2 D_{int}^{-1}, \varphi_{n+2}\rangle & -\langle D_{int}^{-1}, \varphi_{n+2}\rangle \end{vmatrix} = C\frac{\overline{\varphi_{n+1}(0)\varphi_{n+2}(0)}}{\kappa^2}\varphi_n(0). \tag{22}$$

Thus, from (21) and (22), we can find a constant $C' \neq 0$ such that

$$\beta_n + \tau_n = C'\varphi_n(0).$$

Doing the same calculations for $\beta_n\tau_n$, we deduce that

$$\beta_n\tau_n = \frac{\frac{\overline{\varphi_{n+1}(0)\varphi_{n+2}(0)}}{\kappa^2}\left(\frac{\overline{\Phi_n(0)}}{\overline{\Phi_{n+1}(0)}}\frac{\overline{\Phi_n(0)}}{\overline{\Phi_{n+2}(0)}} - \frac{\overline{\Phi_{n-1}(0)}}{\overline{\Phi_n(0)}}\frac{\overline{\Phi_{n-1}(0)}}{\overline{\Phi_{n+1}(0)}} + O(\varphi_n(0))\right)}{\frac{\overline{\varphi_{n+1}(0)\varphi_{n+2}(0)}}{\kappa^2}\left(\frac{\overline{\Phi_{n+1}(0)}}{\overline{\Phi_{n+2}(0)}} - \frac{\overline{\Phi_n(0)}}{\overline{\Phi_{n+1}(0)}} + \overline{\Phi_n(0)}\Phi_{n+1}(0)\right)}$$

and thus,

$$\lim_n \beta_n \tau_n = -\frac{1}{\overline{\alpha_1 \alpha_2}},$$

where the convergence is geometric with ratio $\sqrt{|\alpha_1\alpha_2|}$.

Therefore,

$$\lim_n \|q_{n,2}(z) - (z^2 - \frac{1}{\overline{\alpha_1\alpha_2}})\|^{1/n} = |\alpha_1\alpha_2|^{1/2}.$$

From Lemma 4,

$$R_2(D^{-1}) = \frac{|z_j|}{|\alpha_1\alpha_2|^{1/2}} = \frac{1}{|\alpha_1\alpha_2|}, \quad j = 1, 2,$$

where z_j are the roots of $z^2 - \dfrac{1}{\overline{\alpha_1\alpha_2}}$ (both with modulus $\dfrac{1}{|\alpha_1\alpha_2|^{1/2}}$). Hence, $\{z : |z| < \frac{1}{|\alpha_1\alpha_2|}\}$ is the largest disk centered at $z = 0$ in which $D_{int}^{-1}(z)$ can be extended to a meromorphic function with at most two poles.

$\square$

Remark. *It might seem that an alternative proof of Theorem 5 could be done using Hadamard-type formula for $R_m(D^{-1})$ given in [6]. Indeed, this formula is written in terms of derivatives $\varphi_n^{(j)}(0)$ which can be obtained from Corollary 3. It is also required to calculate*

$$c_k \stackrel{\text{def}}{=} \int e^{-ik\theta} \log w(e^{i\theta}) d\theta, \quad k = 0, 1, 2,$$

but the weight w is unknown. By the way, we can obtain c_0 and c_1 from the values $R_j(D^{-1})$, $j = 0, 1, 2$, given in Theorems 4 and 5.

Remark. *We can also prove that the Fourier-Padé approximants of type $(n, 1)$ of D^{-1} have exactly a pole at $\overline{\alpha_1}^{-1}$ or $\overline{\alpha_2}^{-1}$, depending on whether n is even or odd. Thus, they converge to D_{int}^{-1} in $\{z : |z| < \frac{1}{\sqrt{|\alpha_1\alpha_2|}}\}$.*

4. Three-periodic case

If the sequence of zeros is periodic with period three, then the Verblunsky coefficients are not a geometric progression as one might naively expect. This case is more complex: if the three periodic zeros have modulus at most $\frac{-1+\sqrt{5}}{2}$, then the measure is in the Nevai class, i.e., $\lim_n \Phi_n(0) = 0$, whereas if the periodic zeros have modulus greater than $\frac{-1+\sqrt{5}}{2}$, then some numerical experiments show that, for degree large enough, the zeros of OPUC are close to three arcs of the unit circle. So the orthogonality measure should be supported on these arcs (see Figures 3 and 4).

To prove Theorem 1, we need the following lemma whose easy proof is omitted.

Lemma 7. *Let $r \in (0,1)$ and let $\{a_k : k \geq 2\}$ be the sequence*

$$a_2 = a_3 = \frac{2r^2}{1+r^2}, \quad a_{n+1} = r\frac{ra_{n-1}+a_n}{1+ra_{n-1}a_n}, \quad n \geq 3.$$

The following statements hold:

(i) $a_n \in (0,r)$ *for all* $n \geq 2$.

(ii) The sequence $\{a_n\}$ *is monotone decreasing.*

(iii) If $r \in (0, \frac{-1+\sqrt{5}}{2}]$, *then* $\lim_n a_n = 0$, *whereas* $r \in (\frac{-1+\sqrt{5}}{2}, 1)$ *implies*

$$\lim_n a_n = \sqrt{r+1-\frac{1}{r}}.$$

(iv) When $r < \frac{-1+\sqrt{5}}{2}$, *we have*

$$\frac{a_n}{a_{n-1}} < \frac{a_{n+1}}{a_{n-1}} = r\frac{r+\frac{a_n}{a_{n-1}}}{1+ra_{n-1}a_n} < r(r+1) < 1.$$

(v) Let $H \stackrel{\text{def}}{=} \limsup_n \frac{a_n}{a_{n-1}}$. *Then*

$$H \leq r(r+H) \Leftrightarrow H \leq \frac{r^2}{1-r} < r(r+1) < 1.$$

Proof of Theorem 1. From (1), (2) and (12), it follows that

$$\begin{aligned}
&\Phi_{n+1}(z) \\
&\quad = z\left(z\left(z+\overline{\Phi_n(0)}\Phi_{n+1}(0)\right) + (\Phi_{n+1}(0)+z\Phi_n(0))\,\overline{\Phi_{n-1}(0)}\right)\Phi_{n-2}(z) \\
&\qquad + \left(z\left(z+\overline{\Phi_n(0)}\Phi_{n+1}(0)\right)\Phi_{n-1}(0)+\Phi_{n+1}(0)+z\Phi_n(0)\right)\Phi^*_{n-2}(z).
\end{aligned}$$

If Φ_{n+1} and Φ_{n-2} have a common zero ζ, then

$$\Phi_{n+1}(0) = -\zeta\frac{\zeta\Phi_{n-1}(0)+\Phi_n(0)}{1+\zeta\Phi_{n-1}(0)\overline{\Phi_n(0)}}.$$

Hence,

$$|\Phi_1(0)| = |\alpha|, \quad |\Phi_2(0)| \leq |\beta|\frac{|\beta|+|\alpha|}{1+|\alpha||\beta|} \leq \frac{2r^2}{1+r^2}, \quad |\Phi_3(0)| \leq \frac{2r^2}{1+r^2},$$

$$|\Phi_{n+1}(0)| = |\zeta|\left|\frac{\zeta\Phi_{n-1}(0)+\Phi_n(0)}{1+\zeta\Phi_{n-1}(0)\overline{\Phi_n(0)}}\right| \leq r\frac{r|\Phi_{n-1}(0)|+|\Phi_n(0)|}{1+r|\Phi_{n-1}(0)||\Phi_n(0)|}, \quad n \geq 3.$$

Let $\{a_n\}$ be as in Lemma 7. Then

$$|\Phi_n(0)| \leq a_n, \quad n \geq 2,$$

and, therefore, if $r \in (0, \frac{-1+\sqrt{5}}{2}]$, then $\lim_n a_n = 0$. Hence, $\lim_n \Phi_n(0) = 0$ holds. If $r \in (0, \frac{-1+\sqrt{5}}{2})$, then

$$\limsup_n |\Phi_n(0)|^{1/n} \leq \limsup_n |a_n(0)|^{1/n} \leq \limsup_n \left| \frac{a_n}{a_{n-1}} \right| < \frac{r^2}{1-r} < 1.$$

□

5. Distance from the zeros to the circle

The proof of Theorem 2 requires some auxiliary results.

Lemma 8. *Let Λ denote an infinite subset of the natural numbers. Let*

$$\{V_n(z) = \prod_{j=1}^{n} (z - v_{n,j}) : n \in \Lambda\}$$

be a sequence of monic polynomials with zeros in $\mathbb{D}$ such that

$$\lim_{n \in \Lambda} \frac{V_n(z)}{V_n^*(z)} = 0 \tag{23}$$

uniformly on compact subsets of $\mathbb{D}$. Suppose that there are $z_0 \in \mathbb{D}$ and $r > 0$ such that $V_n(z) \neq 0$ for all $z : |z - z_0| > r$ and $n \in \Lambda$. Then

$$\lim_{n \in \Lambda} \sum_{j=1}^{n} (1 - |v_{n,j}|) = \infty.$$

Proof. There is no loss of generality in assuming that $z_0 = 0$. Indeed, by the change of variables

$$z = \frac{\zeta + z_0}{1 + \overline{z_0}\zeta},$$

we get

$$\frac{W_n(\zeta)}{W_n^*(\zeta)} = \frac{V_n(\frac{\zeta + z_0}{1 + \overline{z_0}\zeta})}{V_n^*(\frac{\zeta + z_0}{1 + \overline{z_0}\zeta})},$$

where W_n is a monic polynomial whose zeros, $\zeta_{n,j}$, $j = 1, \ldots, n$, lie in $\mathbb{D}$. Besides, there exists $\delta > 0$ such that

$$|\zeta_{n,j}| > \delta \quad j = 1, \ldots, n.$$

Moreover, we have

$$\lim_{n \in \Lambda} \sum_{j=1}^{n} (1 - |v_{n,j}|) = \infty \quad \Leftrightarrow \quad \lim_{n \in \Lambda} \sum_{j=1}^{n} (1 - |\zeta_{n,j}|) = \infty$$

because there are $k_1 = k_1(z_0) > 0$, $k_2 = k_2(z_0) > 0$ such that

$$k_1(1 - |\zeta|) \leq 1 - \left| \frac{\zeta + z_0}{1 + \overline{z_0}\zeta} \right| \leq k_2(1 - |\zeta|), \quad \forall \zeta \in \mathbb{D}.$$

Thus, we can assume $z_0 = 0$. By hypothesis,

$$\lim_{n\in\Lambda} \prod_{j=1}^{n} v_{n,j} = \lim_{n\in\Lambda} \frac{V_n(0)}{V_n^*(0)} = 0 \Leftrightarrow \lim_{n\in\Lambda} \sum_{j=1}^{n} \log|v_{n,j}| = -\infty. \tag{24}$$

Since $|v_{n,j}| \geq r$ and there is $\alpha < -1$ such that $\alpha x < \log(1 - x)$, $\forall x \in (0, 1 - r)$, we deduce that

$$\alpha(1 - |v_{n,j}|) < \log(1 - (1 - |v_{n,j}|)) = \log|v_{n,j}|. \tag{25}$$

Therefore, the proof of the lemma follows from (24) and (25). □

Remark. *Sequences of monic polynomials V_n as those considered in Lemma 8 play an important role in rational approximation. Namely, condition (23) is equivalent to the set of rational functions*

$$\{\frac{p_n}{V_n^*} : p_n \text{ polynomial of degree} \leq n, n = 1, 2, \ldots\}$$

is dense in the space of analytic functions in $\overline{\mathbb{D}}$ with the uniform norm (see Corollary 2, p. 246, in [26]).

The next result is a generalization of Theorem 9, Chapter 9, in [26].

Lemma 9. *Let Λ denote an infinite subset of the natural numbers and let*

$$\{V_n(z) = \prod_{j=1}^{n}(z - v_{n,j}) : n \in \Lambda\}$$

be a sequence of monic polynomials whose zeros lie in $\mathbb{D}$. The following statements are equivalent:

(a) $\displaystyle\lim_{n\in\Lambda} \frac{V_n(z)}{V_n^*(z)} = 0$ *uniformly on compact subsets of $\mathbb{D}$.*

(b) $\displaystyle\lim_{n\in\Lambda} \sum_{j=1}^{n}(1 - |v_{n,j}|) = \infty.$

Proof. Assume that (b) holds. Let $T \in (0, 1)$ be fixed. We have the inequalities

$$\frac{1 - T}{T + 1}(1 - |v_{n,j}|) \leq \frac{(1 - T)(1 - |v_{n,j}|)}{1 + T|v_{n,j}|} \leq \frac{1 - T}{T}(1 - |v_{n,j}|).$$

Thus, (b) is equivalent to

$$\lim_{n\in\Lambda} \sum_{j=1}^{n} \frac{(1 - T)(1 - |v_{n,j}|)}{1 + T|v_{n,j}|} = \infty$$

for each $T \in (0,1)$. As $\dfrac{(1-T)(1-|v_{n,j}|)}{1+T|v_{n,j}|} < 1-T < 1$, there exists $\lambda < -1$ such that

$$\begin{aligned}\lambda\left(\frac{(1-T)(1-|v_{n,j}|)}{1+T|v_{n,j}|}\right) &\le \log\left(1-\frac{(1-T)(1-|v_{n,j}|)}{1+T|v_{n,j}|}\right)\\ &\le -\left(\frac{(1-T)(1-|v_{n,j}|)}{1+T|v_{n,j}|}\right)\end{aligned}$$

$$\Leftrightarrow \lambda\left(\frac{(1-T)(1-|v_{n,j}|)}{1+T|v_{n,j}|}\right) \le \log\left(\frac{T+|v_{n,j}|}{1+T|v_{n,j}|}\right) \le -\left(\frac{(1-T)(1-|v_{n,j}|)}{1+T|v_{n,j}|}\right).$$

Hence, (b) is equivalent to

$$\lim_{n\in\Lambda}\sum_{j=1}^{n}\log\left(\frac{T+|v_{n,j}|}{1+T|v_{n,j}|}\right) = -\infty.$$

If $|z| \le T$, using an inequality in [26], p. 229, we have

$$\left|\frac{V_n(z)}{V_n^*(z)}\right| \le \prod_{j=1}^{n}\frac{T+|v_{n,j}|}{1+T|v_{n,j}|}.$$

Therefore, (b) implies (a).

By Lemma 8, proving the other implication requires only to verify the following statement: *Assume that (a) holds and that for any infinite set $\Lambda_1 \subset \Lambda$, any $z_0 \in \mathbb{D}$ and any $\epsilon > 0$ there exists an infinite set $\Lambda_2 \subset \Lambda_1$ such that for any $n \in \Lambda_2$ there is $j \in \{1,\ldots,n\}$ such that $|v_{n,j} - z_0| < \epsilon$, i.e., V_n has a zero in $\{z : |z - z_0| < \epsilon\}$. Then*

$$\lim_{n\in\Lambda}\sum_{j=1}^{n}(1-|v_{n,j}|) = \infty.$$

To get a contradiction, we assume that there exist $M > 0$ and an infinite set $\Gamma \subset \Lambda$ such that

$$\sum_{j=1}^{n}(1-|v_{n,j}|) \le M, \quad \forall n \in \Gamma. \tag{26}$$

We can choose $w_1,\ldots,w_k$ on the circle $\{z : |z| = 1/2\}$ and $r > 0$ small enough, such that the disks $\{z : |z - w_j| < r\}$, $j = 1,\ldots,k$, are all disjoint and

$$k(1/2 - r) > M.$$

By hypothesis, we can choose an infinite set $\Gamma_1 \subset \Gamma$ such that

$$V_n(z) \text{ has a zero in } \{z : |z - w_1| < r\} \text{ for all } n \in \Gamma_1.$$

Given Γ_1, we can choose $\Gamma_2 \subset \Gamma_1 \subset \Gamma$ such that

$$V_n(z) \text{ has a zero in } \{z : |z - w_2| < r\} \text{ for all } n \in \Gamma_2$$

and so, V_n, $n \in \Gamma_2$, has a zero in $\{z : |z-w_2| < r\}$ and a zero in $\{z : |z-w_1| < r\}$. In this way, there exists an infinite set of natural numbers $\Gamma_k \subset \Gamma$ such that

$$V_n(z) \text{ has a zero in each } \{z : |z - w_j| < r\} \text{ for all } n \in \Gamma_k \text{ and } j = 1, \dots, k.$$

According to the choice of $w_1, \cdots, w_k$, r and Γ_k, for $n \in \Gamma_k$, we get the contradiction

$$M \geq \sum_{j=1}^{n}(1 - |v_{n,j}|) \geq \sum_{\substack{j : |v_{n,j} - w_l| < r \\ l = 1, \dots, k}} (1 - |v_{n,j}|) > k(1/2 - r) > M.$$

□

Proof of Theorem 2. It is very well known that $\lim_n \Phi_n(0) = 0$ is equivalent to

$$\lim_n \frac{\Phi_n(z)}{\Phi_n^*(z)} = 0$$

uniformly on compact subsets of $\mathbb{D}$ (see, for example, Theorem 1.7.4, p. 91 in [20]). Therefore, Theorem 2 follows immediately from Lemma 9.

□

5.1. Distance from the zeros to an arc of the circle

Polynomials orthogonal with respect to a weight in the form

$$W(z) = w(z) \prod_{k=1}^{m} |z - a_k|^{2\beta_k}, \quad |z| < 1,$$

where $|a_k| = 1$, $\beta_k > -1/2$, $k = 1, \dots, m$, and $w(z) > 0$ for $|z| = 1$ are been studied in [13]. It is proved there that if $w(z)$ can be extended as a holomorfic and nonvanishing function to an annulus around the unit circle, then

$$|z_{n,j}| = 1 - \frac{\log n}{n} + O(\frac{1}{n}).$$

In the case of polynomials orthogonal with respect to any weight on an arc Δ of the unit circle which is positive almost everywhere on Δ, the behavior of the zeros is known (see [4] and [7]): They approach to Δ as the degree of the polynomials increases. Moreover, (see [8]) for n large enough, there exist $O(n)$ zeros of Φ_n on every neighborhood of each arc $\Delta' \subset \Delta$.

Next, we obtain the rate of approach to Δ for the zeros of OPUC of Chebyshev weight on an arc of the unit circle[3]. The property just above-mentioned justifies the existence of sequences of zeros as the one used in Theorem 6.

[3]These polynomials were already studied by Akhiezer in [1].

Let consider the weight

$$w(\theta) \stackrel{\text{def}}{=} \begin{cases} \dfrac{\sin(\alpha/2)}{2\sin(\theta/2)\sqrt{\cos^2\alpha/2 - \cos^2\theta/2}}, & \theta \in [\alpha, 2\pi - \alpha], \\ 0, & \theta \notin [\alpha, 2\pi - \alpha]. \end{cases}$$

Theorem 6. *Let z_{n,j_n} be any zero of the polynomial $\Phi_n(z)$. Provided that*

$$\lim_n z_{n,j_n} = e^{i\theta_0}, \quad \theta_0 \in [\alpha, 2\pi - \alpha],$$

it follows that

$$|z_{n,j_n}| = 1 - \frac{f(\theta_0)}{n} + O(1/n^2),$$

where f is a positive continuous function in $[\alpha, 2\pi - \alpha]$ which is nonzero in $(\alpha, 2\pi - \alpha)$.

Proof. For the nth-orthonormal polynomial, $\varphi_n(z)$, the following expression appears in [10]:

$$\varphi_n(z) = K_n \left\{ \frac{w^n(v)}{1 - \beta v} + \frac{v\, w^n(1/v)}{v - \beta} \right\}, \quad z = h(v), \tag{27}$$

where $\beta = i \tan \frac{\pi - \alpha}{4}$,

$$w(v) = i\frac{1 - \beta v}{v + \beta}, \quad w(1/v) = i\frac{v - \beta}{1 + \beta v}, \quad z = h(v) = \frac{(v - \beta)(\beta v - 1)}{(v + \beta)(\beta v + 1)}$$

and K_n is a nonzero complex number.

The function $w = w(v)$ is an invertible analytic homeomorphism from $\mathbb{D}$ to $\mathbb{D}$ and $z = h(v)$ is analytic in $\mathbb{C} \setminus \{-\beta, -\frac{1}{\beta}\}$ and a homeomorphism from $\mathbb{D} \setminus \{-\beta\}$ to $\mathbb{C} \setminus \Delta_\alpha$, where $\Delta_\alpha = \{e^{i\theta} : \theta \in [\alpha, 2\pi - \alpha\}$.

For simplicity, from now on, we will write $z_{n,j}$ instead of $z_{n.j_n}$. Also we will use the standard notation $x_n \sim y_n$ in place of $\lim_n \frac{x_n}{y_n} = 1$ and $x_n \underset{n}{\to} x$ in place of $\lim_n x_n = x$.

To every $z_{n,j}$ there corresponds a unique $v_{n,j}$ such that $h(v_{n,j}) = z_{n,j}$ and $|v_{n,j}| < 1$. Since $z_{n,j}$ approach to Δ_α as $n \to \infty$, we have $|v_{n,j}| \underset{n}{\to} 1$. Moreover, we know that

$$\frac{w^n(v_{n,j})}{1 - \beta v_{n,j}} + \frac{v_{n,j}\, w^n(1/v_{n,j})}{v_{j,n} - \beta} = 0 \Leftrightarrow \frac{w^n(1/v_{n,j})}{w^n(v_{n,j})} = -\frac{v_{n,j} - \beta}{v_{n,j}(1 - \beta v_{n,j})}. \tag{28}$$

Thus,

$$\lim_n v_{n,j} = e^{i\omega_0}, \quad e^{i\theta_0} = h(e^{i\omega_0}), \quad \omega_0 \in (0, \pi),$$

$$\lim_n \Im(v_{n,j}) = \sin\omega_0,$$

and

$$\lim_n \frac{1}{|v_{n,j}|^2} \frac{|v_{n,j}|^2 - 2\Re(\overline{v_{n,j}}\beta) + |\beta|^2}{1 - 2\Re(v_{n,j}\beta) + |v_{n,j}\beta|^2} = \frac{1 - 2\tan\eta\sin\omega_0 + |\beta|^2}{1 + 2\tan\eta\sin\omega_0 + |\beta|^2} \in (0, 1),$$

where $\eta = \frac{\pi-\alpha}{4}$.

On the other hand,

$$\left|\frac{v_{n,j}-\beta}{v_{n,j}(1-\beta v_{n,j})}\right|^2 = \frac{1}{|v_{n,j}|^2}\frac{(v_{n,j}-\beta)(\overline{v_{n,j}}-\overline{\beta})}{(1-\beta v_{n,j})(1-\overline{\beta v_{n,j}})}$$
$$= \frac{1}{|v_{n,j}|^2}\frac{|v_{n,j}|^2-2\Re(\overline{v_{n,j}}\beta)+|\beta|^2}{1-2\Re(v_{n,j}\beta)+|v_{n,j}\beta|^2}$$

and $\Re(v_{n,j}\beta) = -\Im(v_{n,j})\tan\eta$, $\Re(\overline{v_{n,j}}\beta) = \Im(v_{n,j})\tan\eta$. Moreover,

$$|w(1/v_{n,j})|^2 = \left|\frac{1-\beta v_{n,j}}{v_{n,j}+\beta}\right|^2 = \frac{1-2\Re(v_{n,j}\beta)+|v_{n,j}\beta|^2}{|v_{n,j}|^2+2\Re(\overline{v_{n,j}}\beta)+|\beta|^2} \underset{n}{\to} 1,$$

$$|w(1/v_{n,j})|^2-1$$
$$= \frac{(1-|v_{n,j}|^2)(1-|\beta|^2)}{1+2\Re(\overline{v_{n,j}}\beta)+|\beta|^2} \sim 2(1-|v_{n,j}|)\frac{(1-|\beta|^2)}{1+2\tan\eta\sin\omega_0+\tan^2\eta}.$$

Also,

$$|w(v_{n,j})|^2 = \left|\frac{v_{n,j}-\beta}{1+\beta v_{n,j}}\right|^2 = \frac{|v_{n,j}|^2-2\Re(\overline{v_{n,j}}\beta)+|\beta|^2}{1+2\Re(v_{n,j}\beta)+|v_{n,j}\beta|^2} \underset{n}{\to} 1,$$

$$|w(v_{n,j})|^2-1 = \frac{(|v_{n,j}|^2-1)(1-|\beta|^2)}{1+2\Re(v_{n,j}\beta)+|v_{n,j}\beta|^2} \sim -2(1-|v_{n,j}|)\frac{(1-|\beta|^2)}{1-2\tan\eta\sin\omega_0+|\beta|^2}.$$

Hence,

$$\left|\frac{w(1/v_{n,j})}{w(v_{n,j})}\right|^2 - 1 = \frac{|w(1/v_{n,j})|^2-|w(v_{n,j})|^2}{|w(v_{n,j})|^2} = \frac{-2(1-|\beta|^2)(1-|v_{n,j}|)}{|w(v_{n,j})|^2}$$
$$\underset{n}{\to} \frac{4\tan\eta\sin\omega_0}{(1+2\tan\eta\sin\omega_0+|\beta|^2)(1-2\tan\eta\sin\omega_0+|\beta|^2)}$$
$$= \frac{-8(1-|\beta|^2)(1-|v_{n,j}|)\tan\eta\sin\omega_0}{(1-2\tan\eta\sin\omega_0+|\beta|^2)(1+2\tan\eta\sin\omega_0+|\beta|^2)}.$$

From (28), we deduce that

$$\left|\frac{w(1/v_{n,j})}{w(v_{n,j})}\right|^n \underset{n}{\to} \frac{1-2\tan\eta\sin\omega_0+|\beta|^2}{1+2\tan\eta\sin\omega_0+|\beta|^2}$$

and so,

$$n\left(\left|\frac{w(1/v_{n,j})}{w(v_{n,j})}\right|-1\right) \underset{n}{\to} \log\left(\frac{1-2\tan\eta\sin\omega_0+|\beta|^2}{1+2\tan\eta\sin\omega_0+|\beta|^2}\right).$$

Thus,

$$\lim_n n\,(1-|v_{n,j}|) = f(\omega_0),$$

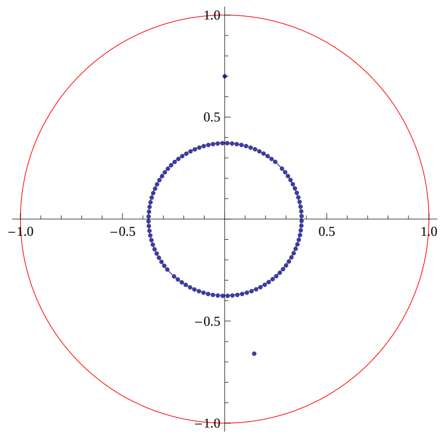

Figure 1: Zeros of Φ_{100} for two period zeros: 0.2 and $0.7i$.

where

$$f(\omega_0) = \frac{(1 - 2\tan\eta\sin\omega_0 + |\beta|^2)(1 + 2\tan\eta\sin\omega_0 + |\beta|^2)}{8(1-|\beta|^2)\tan\eta\sin\omega_0} \times \log\left(\frac{1 + 2\tan\eta\sin\omega_0 + |\beta|^2}{1 - 2\tan\eta\sin\omega_0 + |\beta|^2}\right).$$

Therefore,

$$\begin{aligned} |z_{n,j}| &= |h(v_{n,j})| = |w(v_{n,j})||w(\frac{1}{v_{n,j}})| \\ &= (1 + |w(v_{n,j})| - 1)(1 + |w(\frac{1}{v_{n,j}})| - 1) \\ &\sim \left(1 + \frac{2(1-|v_{n,j}|)(1-|\beta|^2)}{1 + 2\tan\eta\sin\omega_0 + |\beta|^2}\right)\left(1 + \frac{2(|v_{n,j}|-1)(1-|\beta|^2)}{1 - 2\tan\eta\sin\omega_0 + \tan^2\eta}\right) \\ &\sim 1 - \frac{1}{n}\widetilde{f}(\omega_0), \end{aligned}$$

where

$$\widetilde{f}(\omega_0) = \log\left(\frac{1 + 2\tan\eta\sin\omega_0 + \tan^2\eta}{1 - 2\tan\eta\sin\omega_0 + \tan^2\eta}\right).$$

□

Remark. *The Figures 1–4 were generated in Mathematica 6.*

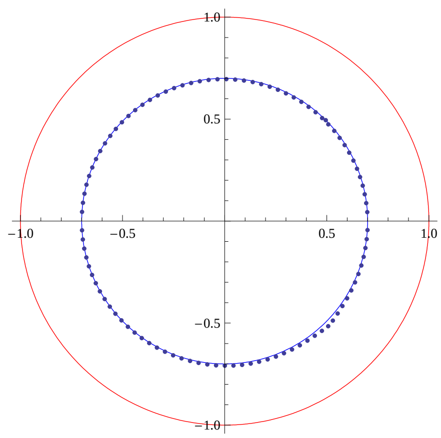

Figure 2: Zeros of Φ_{100} for two period zeros: $0.7e^{-i\frac{\pi}{4}}$ and $0.7e^{i\frac{\pi}{4}}$.

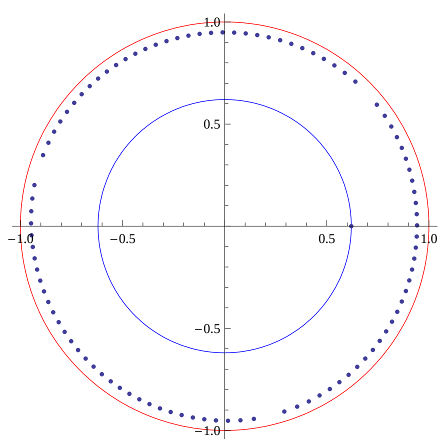

Figure 3: Zeros of Φ_{100} for three period zeros: 0.62, $0.62e^{i\frac{2\pi}{3}}$ and $0.62e^{-i\frac{2\pi}{3}}$. Observe $\frac{-1+\sqrt{5}}{2} = 0.618034\ldots < 0.62$

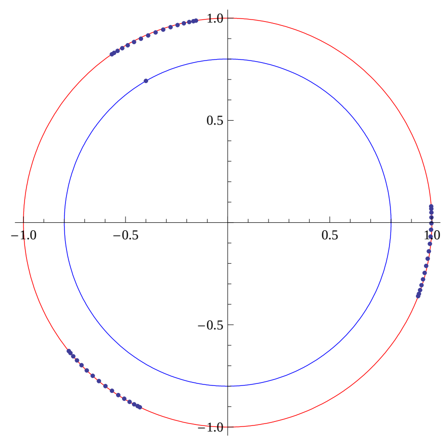

Figure 4: Zeros of Φ_{50} for three period zeros: 0.8, $0.8e^{i\frac{2\pi}{3}}$ and $0.8e^{-i\frac{2\pi}{3}}$. When the degree of the OPUC is larger than 50 calculating the zeros appear numerical instability in Mathematica

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