Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **14** (2020), 729–753.
https://doi.org/10.2298/AADM190218050B

EFFICIENT OPTIMAL FAMILIES OF HIGHER-ORDER ITERATIVE METHODS WITH LOCAL CONVERGENCE

Dedicated to Academician Professor Gradimir Milovanović on the occasion of his 70th birthday.

Ramandeep Behl, J. M. Gutiérrez, I. K. Argyros and A. S. Alshomrani*

The main aim of this manuscript is to propose two new schemes having three and four substeps of order eight and sixteen, respectively. Both families are optimal in the sense to Kung-Traub conjecture. The derivation of them are based on the weight function approach. In addition, theoretical and computational properties are fully investigated along with two main theorems describing the order of convergence. Further, we also provide the local convergence of them in Banach space setting under weak conditions. From the numerical experiments, we find that they perform better than the existing ones when we checked the performance of them on a concrete variety of nonlinear scalar equations. Finally, we analyze the complex dynamical behavior of them which also provide a great extent to this.

1. INTRODUCTION

The conceptualization and construction of root finding techniques have always been a paramount importance in the field of numerical analysis that provide accurate and efficient approximate solution of a nonlinear scalar equation. Newton's method is one of the most basic and popular iterative method for such type of

*Corresponding author. Ramandeep Behl
2010 Mathematics Subject Classification. 65H05, 65D99, 65D10.
Keywords and Phrases. Nonlinear equations, Newton's method, Basin of attraction, Convergence analysis, Banach space.

problems. But, it is a one-point iterative method and one-point methods have several major drawbacks. Some of them are related to order of convergence, efficiency index and consumption of higher-order derivatives (for details please see [**27**]).

It is important to note that the computation of higher-order derivatives is not an easy task. Therefore, scholars from the worldwide turned towards multi-point iterative methods. The main advantage of them is that they do not use higher-order derivatives. In addition, they have a great practical importance since they overcome from the theoretical limitations of one-point methods (for details please see the Traub's book [**27**]).

Due to the advancement of advanced computer arithmetic and symbolic computation, the construction of higher-order finding techniques become more vital and popular because they provide more accurate and efficient approximated root with in a small number of iterations. In the last two decades, a variety of three-point optimal eighth-order methods have been proposed by scholars in [**4, 6, 7, 17, 21, 22, 23, 24, 25, 26, 30**]. Most of them are the extension of Newton's method or Newton like method at the expense of additional functional evaluations or substeps.

In 1974, Kung and Traub [**15**] proposed two general classes of n-point optimal iterative methods with and without derivatives. After few years, Neta [**19**], gave an optimal sixteenth-order family of iterative methods. Recently, Guem and Kim [**9, 10**], Sharma et al. [**21**], Ullah et al. [**28**], have also proposed optimal sixteen order extension of Newton's method. Nowadays, obtaining new optimal methods of order eight and sixteen are very important and interesting task from the computational point of view because they have faster convergence towards the required root/s and better efficiency indices than the classical Newton's method.

First of all, we present an optimal family of eighth-order methods based on the weight function approach. Then, we will extend this scheme to optimal sixteenth-order. Notice also that local convergence results are useful because they give us the degree of difficulty for choosing initial point. Therefore, we analysis the local convergence of our methods in a Banach space setting. The efficiency of the proposed methods is tested on a concrete variety of numerical examples. It is found that our proposed methods perform better than existing optimal methods of same order. The basin of attraction also validate this to a great extent.

The outline of the paper is as follows. In Section , we propose a new three-point scheme and we also discuss the convergence analysis which confirms the optimal eighth-order convergence. In Section 2.1, we extend this family to optimal sixteenth-order. Section 2.1 contains the local convergence for Banach space valued operators under weak conditions. Section 2.1 is devoted to demonstrate the theoretical results which are proposed in sections and 2.1. Finally, some numerical experiments and a first approach to the dynamic study of the methods in the family are also performed.

2. DEVELOPMENT OF EIGHTH-ORDER ITERATIVE SCHEMES

In this section, we develop an optimal eighth-order family of iterative methods. Therefore, we consider the following three substeps scheme

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{f(y_n)}{f'(x_n)} M(h_n,\ k_n), \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(x_n)} Q(h_n,\ t_n), \end{aligned} \tag{1}$$

where the weight functions $M : \mathbb{C}^2 \to C$ and $Q : \mathbb{C}^2 \to C$ are analytic in the neighborhood of $(0,\ 0)$ with

$$h_n = \frac{f(y_n)}{f(x_n)},\ k_n = \frac{f(x_n)}{f'(x_n)},\ t_n = \frac{f(z_n)}{f(y_n)}. \tag{2}$$

Note that our scheme uses just four functional evaluations. The next results establish the conditions needed for (1) in order to reach an optimal eighth-order of convergence.

Lemma 2.1. *Let us assume that ξ be a simple zero of $f : \mathbb{C} \to \mathbb{C}$, an analytic function defined in a region containing ξ. Then the quotients defined in (2) satisfies the following expressions*

$$\begin{aligned} h_n &= \frac{f(y_n)}{f(x_n)} = c_2 e_n + O(e_n^2), \\ k_n &= \frac{f(x_n)}{f'(x_n)} = e_n + O(e_n^2), \\ t_n &= \frac{f(z_n)}{f(y_n)} = \frac{1}{2}\left(-2c_2 M_{11} + c_2^2(-(M_{20} - 10)) - 2c_3 - M_{02}\right) e_n^2 + O(e_n^3), \end{aligned} \tag{3}$$

where $e_n = x_n - \xi$ is the error in n^{th} iteration, $M_{ij} = \frac{\partial^{i+j}}{\partial u^i \partial k^j} M_f(h,\ k)|_{(h=0,\ k=0)}$ for $i, j = 0, 1, 2, 3$ and $c_m = \frac{f^{(m)}(\xi)}{m! f'(\xi)}$ for $m = 1,\ 2,\ \ldots,\ 16$.

Proof. Let us expand the function $f(x_n)$ and its first order derivative $f'(x_n)$ around $x = \xi$ by using Taylor's series with the assumption $f'(\xi) \neq 0$, which leads us to:

$$f(x_n) = f'(\xi)\left[\sum_{j=1}^{16} c_j e_n^j + O(e_n^{17})\right], \tag{4}$$

and

$$f'(x_n) = f'(\xi)\left[\sum_{j=1}^{17} j c_j e_n^{j-1} + O(e_n^{17})\right], \tag{5}$$

respectively.
By using the equations (4) and (5) in the first substep of scheme (1), we get

$$y_n - \xi = c_2 e_n^2 + \sum_{j=1}^{14} G_j e_n^{j+2} + O(e_n^{17}), \tag{6}$$

where $G_j = G_j(c_2, c_3, \ldots, c_{16})$ are given in the term of $c_2, c_3, \ldots, c_{16}$ and some of them are $G_1 = 2c_3 - 2c_2^2$, $G_2 = 4c_2^3 - 7c_3c_2 + 3c_4$, $G_3 = -8c_2^4 + 20c_3c_2^2 - 10c_4c_2 - 6c_3^2 + 4c_5$ and $G_4 = 16c_2^5 - 52c_3c_2^3 + 28c_4c_2^2 + (33c_3^2 - 13c_5)c_2 - 17c_3c_4 + 5c_6$, etc.
With the help of Taylor's series and expression (6), we obtain

$$f(y_n) = f'(\xi)\left[c_2 e_n^2 + (2c_3 - 2c_2^2)e_n^3 + \sum_{j=1}^{13} \bar{G}_j e_n^{j+3} + O(e_n^{17})\right], \tag{7}$$

where $\bar{G}_1 = (5c_2^3 - 7c_3c_2 + 3c_4)$, $\bar{G}_2 = -2(6c_2^4 - 12c_3c_2^2 + 5c_4c_2 + 3c_3^2 - 2c_5)$ and $\bar{G}_3 = 28c_2^5 - 73c_3c_2^3 + 34c_4c_2^2 + (37c_3^2 - 13c_5)c_2 - 17c_3c_4 + 5c_6$, etc.
Using equations (4), (5) and (7), we further yield

$$\begin{aligned} h_n = \frac{f(y_n)}{f(x_n)} &= c_2e_n + (2c_3 - 3c_2^2)e_n^2 + (8c_2^3 - 10c_3c_2 + 3c_4)e_n^3 + (37c_3c_2^2 - 20c_2^4 \\ &- 14c_4c_2 - 8c_3^2 + 4c_5)e_n^4 + \big\{48c_2^5 - 118c_3c_2^3 + 51c_4c_2^2 + (55c_3^2 - 18c_5)c_2 \\ &- 22c_3c_4 + 5c_6\big\}e_n^5 + \big\{344c_3c_2^4 - 163c_4c_2^3 - 112c_2^6 + (65c_5 - 252c_3^2)c_2^2 \\ &+ 2(75c_3c_4 - 11c_6)c_2 + 26c_3^3 - 15c_4^2 - 28c_3c_5 + 6c_7\big\}e_n^6 + O(e_n^7). \end{aligned} \tag{8}$$

and

$$\begin{aligned} k_n = \frac{f(x_n)}{f'(x_n)} &= e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + (7c_3c_2 - 3c_4 - 4c_2^3)e_n^4 + (8c_2^4 - 20c_3c_2^2 \\ &+ 10c_4c_2 + 6c_3^2 - 4c_5)e_n^5 + \{52c_3c_2^3 - 16c_2^5 - 28c_4c_2^2 + (13c_5 - 33c_3^2)c_2 \\ &+ 17c_3c_4 - 5c_6\}e_n^6 + O(e_n^7). \end{aligned} \tag{9}$$

Since, it is clear from the expressions (8) and (9), that h_n and k_n both have linear order$\Big(O(e_n)\Big)$. Therefore, we can expand the weight function $M(h_n,\ k_n)$ in the neighborhood of $(0,\ 0)$ by Taylor series expansion up to third-order terms as follows:

$$\begin{aligned} M(h_n,\ k_n) &= M_{00} + M_{10}h_n + M_{01}k_n + \frac{1}{2!}\big(M_{20}h_n^2 + 2M_{11}h_nk_n + M_{02}k_n^2\big) \\ &+ \frac{1}{3!}\big(M_{30}h_n^3 + 3M_{21}h_n^2k_n + 3M_{12}h_nk_n^2 + M_{03}k_n^3\big). \end{aligned} \tag{10}$$

Now, by using the equations (4), (5), (7)–(10) in the second substep of (1), we obtain

$$z_n - \xi = -(M_{00} - 1)c_2e_n^2 + \sum_{j=1}^{14} P_j e_n^{j+2} + O(e_n^{17}), \tag{11}$$

where $P_j = P_j(M_{ij}, c_2, c_3, \ldots, c_{16})$ are given in the term of $M_{ij}, c_2, c_3, \ldots, c_{16}$, e.g. $P_1 = c_2^2(4M_{00} - M_{10} - 2) - 2c_3(M_{00} - 1) - c_2M_{01}$, etc.
In order to obtain at least third-order convergence, we have to choose

$$M_{00} = 1. \tag{12}$$

By using the above equation (12) in $P_1 = 0$, we obtain

$$M_{01} = 0, \ M_{10} = 2. \tag{13}$$

Substituting the equations (12) and (13) in (11), we get

$$z_n - \xi = \sum_{j=2}^{14} P_j e_n^{j+2} + O(e_n^{17}), \tag{14}$$

where $P_2 = -\frac{c_2}{2}\left(2c_2M_{11} + c_2^2(M_{20} - 10) + 2c_3 + M_{02}\right)$, etc.
With the help of expression (14),we obtain

$$f(z_n) = f'(\xi)\left[\sum_{j=2}^{14} \bar{P}_j e_n^{j+2} + O(e_n^{17})\right], \tag{15}$$

where $\bar{P}_2 = P_2$ and $\bar{P}_3 = \frac{c_2^2}{2}\Big(c_3(64 - 6M_{20}) + 6M_{02} - M_{12}\Big) - c_3(2c_3 + M_{02}) - \frac{1}{6}c_2(24c_3M_{11} + 12c_4 + M_{03}) + c_2^3\left(8M_{11} - \frac{M_{21}}{2}\right) + c_2^4\left(5M_{20} - \frac{M_{30}}{6} - 36\right)$, etc.
By using the expressions (7) and (14), we yield

$$t_n = \frac{f(z_n)}{f(y_n)} = \sum_{j=2}^{16} R_j e_n^j + O(e_n^{17}), \tag{16}$$

where $R_2 = \frac{1}{2}\left(-2c_2M_{11} + c_2^2(-(M_{20} - 10)) - 2c_3 - M_{02}\right)$, etc. □

Theorem 2.2. *Let us consider that an initial guess $x = x_0$ is sufficiently close to ξ for the guaranteed convergence. Then, the iterative scheme defined by (1) has eighth-order convergence when*

$$\begin{aligned} &Q_{00} = 1, \ Q_{10} = 2, \ Q_{01} = 1, \ M_{02} = 0, \ M_{11} = 0, \ Q_{11} = 4, \ M_{12} = 0, \\ &M_{20} = Q_{20} - 2, \ M_{03} = 0, \ M_{21} = 0, \ Q_{30} = 6Q_{20} + M_{30} - 36, \end{aligned} \tag{17}$$

where $Q_{ij} = \frac{\partial^{i+j}}{\partial h^i \partial k^j} Q(h, \ t)|_{(h=0, \ t=0)}$ for $i, j = 0, 1, 2, 3$.

Proof. We expand the weight function $Q(h_n, \ k_n)$ in the neighborhood of $(0, \ 0)$ by Taylor series expansion, which leads us to:

$$\begin{aligned} Q(h_n, \ t_n) = Q_{00} + Q_{10}h_n + Q_{01}t_n + \frac{1}{2!}\left(Q_{20}h_n^2 + 2Q_{11}h_nt_n + Q_{02}t_n^2\right) \\ + \frac{1}{3!}\left(Q_{30}h_n^3 + 3Q_{21}h_n^2t_n + 3Q_{12}h_nt_n^2 + Q_{03}t_n^3\right). \end{aligned} \tag{18}$$

By using (18), in the third substep of (1), we yield

$$e_{n+1} = \frac{1}{2}c_2(Q_{00}-1)\left(2c_2M_{11}+c_2^2(M_{20}-10)+2c_3+M_{02}\right)e_n^4+\sum_{j=1}^{12}H_je_n^{(j+4)}+O(e_n^{17}), \tag{19}$$

where $H_j = H_j(M_{ij}, Q_{i,j}c_2, c_3, \dots, c_{16})$ are given in the term of $M_{ij}, Q_{ij}c_2, c_3, \dots, c_{16}$. We can easily attain minimum fifth-order convergence by choosing $Q_{00} = 1$. Now, we obtain the following independent relation by substituting $Q_{00} = 1$ in $H_1 = 0$

$$\frac{1}{2}c_2^2(Q_{10}-2)\left(2c_2M_{11}+c_2^2(M_{20}-10)+2c_3+M_{02}\right)=0,$$

which further yields

$$Q_{10} = 2. \tag{20}$$

Again, we use $Q_{00} = 1$ and $Q_{10} = 2$, in $H_2 = 0$, we have

$$\begin{aligned}
&-\frac{c_2^3}{4}(Q_{01}(M_{20}-10)-Q_{20}+12)\big(2c_2M_{11}+c_2^2(M_{20}-10)+2c_3+M_{02}\big)=0,\\
&-\frac{c_2^2}{2}Q_{01}M_{11}\big(2c_2M_{11}+c_2^2(M_{20}-10)+2c_3+M_{02}\big)=0,\\
&-\frac{c_2c_3}{2}(Q_{01}-1)\big(2c_2M_{11}+c_2^2(M_{20}-10)+2c_3+M_{02}\big)=0,\\
&-\frac{c_2}{4}Q_{01}M_{02}\big(2c_2M_{11}+c_2^2(M_{20}-10)+2c_3+M_{02}\big)=0,
\end{aligned}$$

which further provide

$$M_{20} = Q_{20}-2,\ Q_{01}=1,\ M_{02}=0,\ M_{11}=0. \tag{21}$$

Using $Q_{00} = 1$, expressions (20) and (21) in $H_3 = 0$, we get

$$\begin{aligned}
&-\frac{c_2^4}{12}\Big(3Q_{11}(Q_{20}-12)-6Q_{20}-Q_{30}+M_{30}+108\Big)\Big(c_2^2(Q_{20}-12)+2c_3\Big)=0,\\
&-\frac{c_2c_3}{4}\Big(2c_3(Q_{11}-4)+M_{12}\Big)\Big(c_2^2(Q_{20}-12)+2c_3\Big)=0,\\
&-\frac{c_2^3}{4}M_{21}\Big(c_2^2(Q_{20}-12)+2c_3\Big)=0,\\
&-\frac{c_2}{12}M_{03}\Big(c_2^2(Q_{20}-12)+2c_3\Big)=0,
\end{aligned}$$

which further obtain

$$Q_{11}=4,\ M_{12}=0,\ Q_{30}=6Q_{20}+M_{30}-36,\ M_{21}=0,\ M_{03}=0. \tag{22}$$

Finally, by inserting $Q_{00} = 1$ and expressions (20)–(22) in (19), we have

$$\begin{aligned}
e_{n+1} = {}& \frac{c_2\left(c_2^2(Q_{20}-12)+2c_3\right)}{48}\Big[c_2^4\Big\{3Q_{02}(Q_{20}-12)^2-2\Big(3Q_{20}(Q_{21}-18)+4\big(M_{30}-9Q_{21}+162\big)\Big)\Big\}\\
&+12c_3c_2^2\Big(Q_{02}(Q_{20}-12)-Q_{20}-Q_{21}+40\Big)+12c_3^2(Q_{02}-2)-24c_4c_2\Big]e_n^8+\sum_{j=5}^{12}H_je_n^{(j+4)}+O(e_n^{17}),
\end{aligned} \tag{23}$$

The above error equation reveals that the scheme (1) reaches at optimal eighth-order convergence by using only four functional evaluations (viz. $f(x_n)$ $f'(x_n)$ $f(y_n)$ and $f(z_n)$) per iteration. □

2.1 Special cases

Here, we discuss some special cases of our eighth-order scheme (1) by considering some different weight functions. The beauty of the weight function is that we can choose any arbitrary function which employs the conditions define in Theorem 2.2. In this way, we can easily obtain several new optimal eighth-order methods. Some of them are define as follows:
(i) Let us consider the following weight function in scheme (1)

$$\begin{cases} M(h,k) = \dfrac{1}{1-2h} \\ Q(h,t) = 1+2h+t+5h^2+4ht+t^2+12h^3+13h^2t. \end{cases} \tag{24}$$

By using the above weight functions in the scheme (1), we will obtain a new optimal eight-order modification of Ostrowski's method, denoted by (M_1).
(ii) Again, we have selected the following weight function

$$\begin{cases} M(h,k) = 1+2h+5h^2, \\ Q(h,t) = 1+2h+t+6h^2+4ht+t^2+6h^3+14h^2t. \end{cases} \tag{25}$$

By using this weight function in (1), we get another optimal eighth-order iterative method, called by (M_2).
(iii) We obtain another optimal eighth-order iterative method, known as (M_3), by considering the following weight function

$$\begin{cases} M(h,k) = 1+2h, \\ Q(h,t) = 1+2h+t+h^2+4ht-4h^3. \end{cases} \tag{26}$$

(iv) Finally, we obtain an interesting optimal eighth-order iterative method, denoted by (M_4) that does not involve cubic terms. It is given by the weight function

$$\begin{cases} M(h,k) = 1+2h+2h^2, \\ Q(h,t) = 1+2h+t+3h^2+4ht. \end{cases} \tag{27}$$

3. CONSTRUCTION OF SIXTEENTH-ORDER SCHEMES

In this section, we propose a new optimal sixteenth-order family of iterative methods. In this regards, we consider one additional step to the family (1), in the following way

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)},\\
z_n &= y_n - \frac{f(y_n)}{f'(x_n)} M(h_n,\ k_n),\\
s_n &= z_n - \frac{f(z_n)}{f'(x_n)} Q(h_n,\ t_n),\\
x_{n+1} &= x_n - \frac{a_n b_n\big[u_1 f(x_n)^3 f(y_n) + u_2 f(x_n) f'(x_n) f(s_n) f(z_n)\big]}{v_1 f(x_n)^3 + v_2 f'(x_n) f(s_n) f(z_n)},
\end{aligned}
\tag{28}
$$

where

$$
\begin{aligned}
u_1 &= f(s_n)\big(b_n^2 f'(x_n) + b_n f(x_n) - c_n f(z_n)\big) + a_n\big(f(x_n) - a_n f'(x_n)\big) f(z_n),\\
u_2 &= a_n b_n c_n f'(x_n)\big(f(y_n) - f(x_n)\big) + c_n f(y_n) f(x_n)(a_n - b_n),\\
v_1 &= f(y_n)\big[b_n f(s_n)\ \big(b_n^2 f'(x_n) + b_n f(x_n) - c_n f(z_n)\big) +\\
&\qquad \big(a_n^3 f'(x_n) + c_n a_n f(s_n) - a_n^2 f(x_n)\big) f(z_n)\big],\\
v_2 &= a_n^2 b_n^2 c_n f'(x_n)^2\big(2f(y_n) - f(x_n)\big) + a_n b_n c_n (2a_n - c_n) f'(x_n) f(y_n) f(x_n)\\
&\qquad + c_n\big(a_n b_n - a_n c_n - b_n^2\big) f(y_n) f(x_n)^2,\\
a_n &= x_n - z_n,\ b_n = s_n - x_n,\ c_n = s_n - z_n.
\end{aligned}
\tag{29}
$$

Note that method (28) needs four evaluations of the function f and one of its first-order derivative. Then, the optimal order of convergence for (28) would be sixteen according Kung-Traub conjecture. In fact, we demonstrate that the optimal sixteenth-order of convergence is attained in the next Theorem 3.3.

Theorem 3.3. *Under the assumptions of Theorem 2.2, the iterative scheme defined by* (28) *has sixteenth-order convergence.*

Proof. From the expression (23) in the third substep of (28), we can easily obtain

$$
s_n - \xi = H_4 e_n^8 + \sum_{j=5}^{12} H_j e_n^{(j+4)} + O(e_n^{17}),
\tag{30}
$$

which further provides

(31)

$$
\begin{aligned}
f(s_n) =& f'(\xi)\Big(H_4 e_n^8 + H_5 e_n^9 + H_6 e_n^{10} + H_7 e_n^{11} + H_8 e_n^{12} + H_9 e_n^{13} + H_{10} e_n^{14} + H_{11} e_n^{15}\\
&+ e_n^{16}(c_2 H_8^2 + H_{16}) + O(e_n^{17})\Big),
\end{aligned}
$$

where H_4 defined previously.

Using equations (4)–(23) and (31), we yield

$$\begin{aligned}
&\frac{a_n b_n\big[u_1 f(x_n)^3 f(y_n) + u_2 f(x_n) f'(x_n) f(s_n) f(z_n)\big]}{v_1 f(x_n)^3 + v_2 f'(x_n) f(s_n) f(z_n)}\\
&= \frac{c_2}{96}\Bigg[\big\{3(Q_{20}-12)(Q_{02}\big(Q_{20}-12\big)-2Q_{21}+36\big)\\
&- 8M_{30}\big\}c_2^4 + 12c_3^2(Q_{02}-2) - 24c_4c_2 + 12(Q_{02}(Q_{20}-12) - Q_{20} - Q_{21} + 40)c_3c_2^2\Bigg]\\
&\times \Big((Q_{20}-12)c_2^3 + 2c_3c_2\Big)^2 (c_2^4 - 3c_3c_2^2 + 2c_4c_2 + c_3^2 - c_5)e_n^{16} + O(e_n^{17}).
\end{aligned} \tag{32}$$

Finally, we use the above expression (32) in the last substep of scheme (28), we obtain

$$\begin{aligned}
e_{n+1} &= -\frac{c_2}{96}\Bigg[\big\{3(Q_{20}-12)(Q_{02}\big(Q_{20}-12\big)-2Q_{21}+36\big) - 8M_{30}\big\}c_2^4 + 12c_3^2(Q_{02}-2)\\
&- 24c_4c_2 + 12(Q_{02}(Q_{20}-12) - Q_{20} - Q_{21} + 40)c_3c_2^2\Bigg]\big((Q_{20}-12)c_2^3 + 2c_3c_2\big)^2\\
&\times (c_2^4 - 3c_3c_2^2 + 2c_4c_2 + c_3^2 - c_5)e_n^{16} + O(e_n^{17}).
\end{aligned} \tag{33}$$

This reveals that the proposed scheme (28) reaches an optimal sixteen-order convergence. □

In order to develop our numerical experiments, we have considered the same weight functions, (24), (25), (26) and (27), which are given in the eight-order schemes M_i, $i = 1, 2, 3, 4$, to obtain the corresponding sixteenth-order iterative methods given by (28). We have denoted the new iterative methods as S_1, S_2, S_3 and S_4.

4. CONVERGENCE IN BANACH SPACES

Let $\mathbb{X}$, $\mathbb{Y}$ denote Banach spaces and $\mathbb{D} \subset \mathbb{X}$ a nonempty, open and convex set. In addition, we assume that $U(u, a)$ and $\bar{U}(u, a)$ are open and closed balls in $\mathbb{X}$, respectively with center $u \in \mathbb{X}$ and of radius $a > 0$. Moreover, we consider that $\mathcal{L}(\mathbb{X}, \mathbb{Y})$ to be the space of bounded linear operators from $\mathbb{X}$ in to $\mathbb{Y}$. Consider the problem of finding a locally unique solution ξ of equation

$$F(x) = 0, \tag{34}$$

where $F : \mathbb{D} \to \mathbb{Y}$ is continuously differentiable in the sense of Fréchet. Many problems can be reduced to solving equations like (34) using mathematical modeling [**18**]. Most of the solution techniques are iterative, since analytical or closed form solutions can be found only in rare cases.

We consider the method defined for each $n = 0, 1, 2, \ldots$ by

$$
\begin{aligned}
y_n &= x_n - F'(x_n)^{-1}F(x_n),\\
z_n &= y_n - L_1(x_n, y_n)F'(x_n)^{-1}F(y_n),\\
s_n &= z_n - L_2(x_n, y_n, z_n)F'(x_n)^{-1}F(z_n),\\
x_{n+1} &= x_n - L_3(x_n, y_n, z_n, s_n)F(x_n),
\end{aligned}
\tag{35}
$$

where $x_0 \in \mathbb{D}$ is an initial point, $L_1(\cdot,\cdot) : \mathbb{D}\times\mathbb{D} \to \mathcal{L}(\mathbb{Y},\mathbb{X})$, $L_2(\cdot,\cdot,\cdot) : \mathbb{D}\times\mathbb{D}\times\mathbb{D} \to \mathcal{L}(\mathbb{Y},\mathbb{X})$ and $L_3(\cdot,\cdot,\cdot,\cdot) : \mathbb{D}\times\mathbb{D}\times\mathbb{D}\times\mathbb{D} \to \mathcal{L}(\mathbb{Y},\mathbb{X})$.

The proceeding method are clearly special cases of method for (35), $\mathbb{X} = \mathbb{Y} = \mathbb{C}$. Moreover, many Newton-type single or multi-step methods [**18, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 16, 17, 28, 19, 20, 21, 22, 24, 25, 26, 27, 29, 30, 14, 15, 23**] are the special cases of method (35). Therefore, examining the convergence of multi-step methods in a unified way is important. This is our first motivation for introducing the method (35). As it was shown in the proceeding sections, Taylor expansions and bounded derivatives reaching up to the order of sixteen are required to show the convergence of those methods. This fact limits the applicability of the methods (see examples in [1], where higher-order derivatives do not exist even for the scalar equations).

In this section, we only use hypotheses on the first Fréchet derivative. Hence, the applicability of these methods is extended. This is our second motivation for introducing method (35). Notice that Taylor expansions are not used, since the order of convergence can be found by using computational order of convergence (COC) or approximate computational order of convergence ($ACOC$), which do not require the computation of higher than two derivatives [**18**].

To continue, we need to define some scalar functions, sets and parameters. Set $\mathbb{I} = [0, \infty)$. Let $w_0 : \mathbb{I} \to \mathbb{I}$ be a non-decreasing and continuous function with $w_0(0) = 0$. Suppose that equation

$$
w_0(t) = 1, \tag{36}
$$

has at least one positive root. Denote by r_0 the smallest such root. Set $\mathbb{I}_0 = [0, r_0)$. Let $w : \mathbb{I}_0 \to \mathbb{I}$, $v : \mathbb{I}_0 \to \mathbb{I}$, $l_1 : \mathbb{I}_0\times\mathbb{I}_0 \to \mathbb{I}$, $l_2 : \mathbb{I}_0\times\mathbb{I}_0 \to \mathbb{I}$ and $l_3 : \mathbb{I}_0\times\mathbb{I}_0\times\mathbb{I}_0 \to \mathbb{I}$ be continuous and non-decreasing functions in each of their arguments with $w(0) = 0$. Define functions $\phi_i, \psi_i, i = 1, 2, 3, 4$ on the interval $\mathbb{I}_0$ by

$$\begin{aligned}
\phi_1(t) =& \frac{\int_0^1 w\big((1-\tau)t\big)d\tau}{1-w_0(t)},\\
\phi_2(t) =& \left(1+\frac{l_1(t,\phi_1(t)t)\int_0^1 v(\tau\phi_1(t))d\tau}{1-w_0(t)}\right)\phi_1(t),\\
\phi_3(t) =& \left(1+\frac{l_2(t,\phi_1(t)t,\phi_2(t)t)\int_0^1 v(\tau\phi_2(t))d\tau}{1-w_0(t)}\right)\phi_2(t),\\
\phi_4(t) =& \phi_1(t)+\frac{l_3(t,\phi_1(t)t,\phi_2(t)t,\phi_3(t)t)\int_0^1 v(\tau t)d\tau}{1-w_0(t)},
\end{aligned}$$

and

$$\psi_i(t) = \phi_i(t) - 1.$$

We have $\psi_j(0) = -1$ and $\psi_j(t) \to +\infty$ as $t \to r_0^-, j = 1,2,3$. The intermediate value theorem assumes that equations $\psi_j(t) = 0$ have at least one positive root. We denote by r_j the smallest such roots, respectively.

Suppose that

$$l_3(0,0,0,0) < 1. \tag{37}$$

Then, we also obtain from expression (37) that $\psi_4(0) = -1$ and $\psi_4(t) \to +\infty$ as $t \to r_0^-$. Denote by r_4 the smallest such root. Finally, define the radius of convergence r by

$$r = \min\{r_i\}, i = 1,2,3,4. \tag{38}$$

Clearly, we have that for each $t \in [0, r)$

$$0 \le w_0(t) < 1 \tag{39}$$

and

$$0 \le \phi_i(t) < 1. \tag{40}$$

We base the local convergence of iterative method (35) on aforementioned notation and the conditions (A):

(a_1) There exists $\xi \in \mathbb{D}$ such that
$F(\xi) = 0$ and $F'(\xi)^{-1} \in \mathcal{L}(\mathbb{Y}, \mathbb{X})$.

(a_2) There exists function $w_0 : \mathbb{I} \to \mathbb{I}$ continuous and non-decreasing with $w_0(0) = 0$ such that for each $x \in \mathbb{D}$
$\|F'(\xi)^{-1}\big(F'(x) - F'(\xi)\big)\| \le w_0(\|x - \xi\|)$.
Condition (36) holds. Set $\mathbb{D}_0 = \mathbb{D} \cap U(\xi, r_0)$.

(a_3) There exist functions $w : \mathbb{I}_0 \to \mathbb{I}, w(0) = 0$, $l_1 : \mathbb{I}_0 \times \mathbb{I}_0 \to \mathbb{I}$, $l_2 : \mathbb{I}_0 \times \mathbb{I}_0 \times \mathbb{I}_0 \to \mathbb{I}$ and $l_3 : \mathbb{I}_0 \times \mathbb{I}_0 \times \mathbb{I}_0 \times \mathbb{I}_0 \to \mathbb{I}$ continuous and non-decreasing in each argument such that for each $x, y, z, u \in \mathbb{D}_0$

$$\|F'(\xi)^{-1}\big(F'(x) - F'(y)\big)\| \le w(\|x - y\|),$$
$$\|F'(\xi)^{-1}F'(x)\| \le v(\|x - \xi\|),$$
$$\|L_1(x, y)\| \le l_1(\|x - \xi\|, \|y - \xi\|),$$
$$\|L_2(x, y, z)\| \le l_2(\|x - \xi\|, \|y - \xi\|, \|z - \xi\|),$$

and

$$\left\|F'(\xi)^{-1}\Big(I - F'(x)L_3(x, y, z, u)\Big)F'(\xi)\right\| \le l_3(\|x - \xi\|, \|y - \xi\|, \|z - \xi\|, \|u - \xi\|),$$

where linear operators L_j are given previously.

(a_4) $\bar{U}(\xi, r) \subseteq \mathbb{D}$ and (37) holds, where the radius r is defined in (38).

(a_5) There exist $r^* \ge r$ such that

$$\int_0^1 w_0(\tau r^*) d\tau < 1.$$

Set $D_1 = D \cap \bar{U}(\xi, r^*)$.

Theorem 4.4. *Suppose that the conditions (A) hold. Then, the sequence $\{x_n\}$ initiated at $x_0 \in U(\xi, r) - \{\xi\}$ and produced by method (35) is well defined in $U(\xi, r)$ remains in $U(\xi, r)$ for each $n = 0, 1, 2, 3, \dots$ and converges to ξ. Moreover, the upper error bound estimates hold*

$$\|y_n - \xi\| \le \phi_1(\|x_n - \xi\|)\|x_n - \xi\| \le \|x_n - \xi\| < r, \tag{41}$$

$$\|z_n - \xi\| \le \phi_2(\|x_n - \xi\|)\|x_n - \xi\| \le \|x_n - \xi\|, \tag{42}$$

$$\|s_n - \xi\| \le \phi_3(\|x_n - \xi\|)\|x_n - \xi\| \le \|x_n - \xi\|, \tag{43}$$

and

$$\|x_{n+1} - \xi\| \le \phi_4(\|x_n - \xi\|)\|x_n - \xi\| \le \|x_n - \xi\|, \tag{44}$$

where the functions "ϕ_i" are defined previously. Furthermore, the point ξ is the only solution of the equation $F(x) = 0$ in the set $\mathbb{D}_1$.

Proof. The techniques of proof is induction based. Let $x \in U(\xi, r)$. Using (a_1) and (a_2), we have

$$\|F'(x)^{-1}\Big(F'(x) - F'(\xi)\Big)\| \le w_0(\|x_n - \xi\|) \le w_0(r) < 1, \tag{45}$$

which together with the Banach perturbation lemma [**18**] imply that

$$F'(x)^{-1} \in \mathcal{L}(\mathbb{Y}, \mathbb{X}) \ \ and \tag{46}$$
$$\|F'(x)^{-1}F'(\xi)\| \leq \frac{1}{1 - w_0(\|x-\xi\|)}.$$

We also have y_0, z_0, s_0 and x_1 are well defined by method (35) for $n = 0$. Next, we shall prove consecutively estimates (41) – (44). Firstly, by using the expressions (38), (40) (for $i = 1$), the first condition in (a_3), the first substep in method (35) and (46), we get in turn that

(47)

$$\begin{aligned}\|y_0 - \xi\| &= \|x_0 - \xi - F'(x_0)^{-1}F(x_0)\| \\ &\leq \|F'(x_0)^{-1}F'(\xi)\| \left\| \int_0^1 F'(\xi)^{-1}\Big[F'\big(\xi + \tau(x_0-\xi) - F'(x_0)\big)\Big](x_0-\xi)d\tau \right\| \\ &\leq \frac{\int_0^1 w\Big((1-\tau)\|x_0-\xi\|\Big)d\tau}{1 - w_0(\|x_0-\xi\|)}\|x_0-\xi\| \\ &\leq \phi_1(\|x_0-\xi\|)\|x_0-\xi\| \leq \|x_0-\xi\| < r,\end{aligned}$$

so, estimate (41) holds for $n = 0$ and $y_0 \in U(\xi, r)$. Then, similarly but using the second, third and fourth conditions in (a_3), (38), (40) (for $i = 2, 3$), (46), (47) and the second and third substep in method (35), we obtain in turn that

$$\begin{aligned}\|z_0 - \xi\| &= \|y_0 - \xi - L_1(x_0, y_0)F'(x_0)^{-1}F(y_0)\| \\ &\leq \|y_0-\xi\| + \|L_1(x_0,y_0)\|\|F'(x_0)^{-1}F'(\xi)\|\|F'(\xi)^{-1}F'(y_0)\| \\ &\leq \left(1 + \frac{l_1(\|x_0-\xi\|, \|y_0-\xi\|)\int_0^1 v(\tau\|y_0-\xi\|)d\tau}{1 - w_0(\|x_0-\xi\|)}\right)\|y_0-\xi\| \\ &\leq \phi_2(\|x_0-\xi\|)\|x_0-\xi\| \leq \|x_0-\xi\|,\end{aligned} \tag{48}$$

so, estimate (37) holds for $n = 0$ and $z_0 \in U(\xi, r)$, and

(49)

$$\begin{aligned}\|s_0 - \xi\| &= \|z_0 - \xi + L_2(x_0, y_0, z_0)F'(x_0)^{-1}F(z_0)\| \\ &\leq \|z_0-\xi\| + \|L_2((x_0,y_0,z_0)\|\|F'(x_0)^{-1}F'(\xi)\|\|F'(\xi)^{-1}F(z_0)\| \\ &\leq \left(1 + \frac{l_2\Big(\|x_0-\xi\|, \|y_0-\xi\|, \|z_0-\xi\|\Big)\int_0^1 v(\tau\|z_0-\xi\|)d\tau}{1 - w_0(\|x_0-\xi\|)}\right)\|z_0-\xi\| \\ &\leq \left(1 + \frac{l_2\Big(\|x_0-\xi\|, \phi_1(\|x_0-\xi\|)\|x_0-\xi\|, \phi_2(\|x_0-\xi\|)\|x_0-\xi\|\Big)\int_0^1 v\Big(\tau\phi_2(\|x_0-\xi\|)\|x_0-\xi\|\Big)d\tau}{1 - w_0(\|x_0-\xi\|)}\right) \\ &\quad \times \phi_2(\|x_0-\xi\|)\|x_0-\xi\| \\ &\leq \phi_3(\|x_0-\xi\|)\|x_0-\xi\| \leq \|x_0-\xi\|,\end{aligned}$$

so, estimate (43) holds for $n = 0$ and $s_0 \in U(\xi, r)$. Moreover, by using the expressions (38), (40) (for $i = 3$), (46)(for $x = x_0$), (47)–(49), the last condition in (a_3)

for $n = 0$ and the identity

$$x_1 - \xi = \Big(x_0 - \xi - F'(x_0)^{-1}F(x_0)\Big) + F'(x_0)^{-1}F'(\xi)F'(\xi)^{-1}\Big(I - F'(x_0)L_3(x_0, y_0, z_0, s_0)\Big)F'(\xi)F'(\xi)^{-1}F(x_0)$$

which further yields

(50)

$$\begin{aligned}\|x_1 - \xi\| &= \|y_0 - \xi\| + \Big\|F'(x_0)^{-1}F(\xi)\|\|F'(\xi)^{-1}\Big(I - F'(x_0)L_3(x_0, y_0, z_0, s_0)\Big)F'(\xi)\Big\| \|F'(\xi)^{-1}F(x_0)\| \\ &\le \phi_1(\|x_0 - \xi\|)\|x_0 - \xi\| + \frac{l_3(\|x_0 - \xi\|, \|y_0 - \xi\|, \|z_0 - \xi\|, \|s_0 - \xi\|)\int_0^1 v(\tau\|x_0 - \xi\|)d\tau}{1 - w_0(\|x_0 - \xi\|)}\|x_0 - \xi\| \\ &\le \left(1 + \frac{l_2\Big(\|x_0 - \xi\|, \|y_0 - \xi\|, \|z_0 - \xi\|\Big)\int_0^1 v(\tau\|z_0 - \xi\|)d\tau}{1 - w_0(\|x_0 - \xi\|)}\right)\|x_0 - \xi\| \\ &\le \phi_4(\|x_0 - \xi\|)\|x_0 - \xi\| \le \|x_0 - \xi\|,\end{aligned}$$

so, estimate (44) holds for $n = 0$ and $x_1 \in U(\xi, r)$.

The induction for estimates (41)–(44) is completed if we substitute x_0, y_0, z_0, s_0, x_1 by $x_m, y_m, z_m, s_m, x_{m+1}$ in the proceeding estimates, respectively. Then, it follows from the estimate

$$\|x_{m+1} - \xi\| \le \|x_m - \xi\| < r, \tag{51}$$

that $\lim_{m\to+\infty} x_m = \xi$ and $x_{m+1} \in U(\xi,\ r)$, where $\lambda = \phi_4(\|x_0 - \xi\|) \in [0, 1]$. Furthermore, the uniqueness is proved as follows. Let $\xi^* \in \mathbb{D}_1$ with $F(\xi^*) = 0$. Define linear operator $T = \int_0^1 F'(\xi + \tau(\xi - \xi^*))d\tau$. Then, in view (a_2) and (a_5), we get that

$$\left|F'(\xi)^{-1}(T - F'(\xi))\right| \le \int_0^1 w_0(\tau\|\xi - x^*\|)d\tau \quad \le \int_0^1 w_0(\tau r^*)d\tau < 1, \tag{52}$$

So, $T^{-1} \in \mathcal{L}(\mathbb{Y}, \mathbb{X})$. Finally, from the identity $0 = F(\xi) - F(\xi^*) = T(\xi - \xi^*)$, we deduce that $\xi = \xi^*$. □

5. NUMERICAL EXPERIMENTS

Here, we verify the theoretical convergence properties of our methods on two standard academic examples 5.1–5.2. Therefore, we described the number of iteration indexes (n), approximated zeros (x_n), absolute residual error of the corresponding function ($|f(x_n)|$), errors $|e_n|$ (where $e_n = x_n - \xi$), $\left|\frac{e_n}{e_{n-1}^p}\right|$ and the asymptotic error constant $\eta = \lim_{n\to\infty}\left|\frac{e_n}{e_{n-1}^p}\right|$ (where p is the order of convergence which is either 8 or 16), in the Tables 1 – 2. In order to calculate the computational order of convergence (ρ) , we use the formula proposed by Grau-Sánchez et al. in [**11**], which is defined as follows

$$\rho = \frac{\ln \frac{|x_{n+2}-\xi|}{|x_{n+1}-\xi|}}{\ln \frac{|x_{n+1}-\xi|}{|x_n-\xi|}}, \quad \text{for each } n = 0, 1, 2, \dots$$

or the approximate computational order of convergence (ACOC) [**11**]

$$\rho^* = \frac{\ln \frac{|x_{n+2}-x_{n+1}|}{|x_{n+1}-x_n|}}{\ln \frac{|x_{n+1}-x_n|}{|x_n-x_{n-1}|}}, \quad \text{for each } n = 1, 2, \dots$$

We calculate the values of all constants up to several number of significant digits(minimum 1000 significant digits to minimize the round off error). Because of limited paper space, we display the value of x_n up to 15 significant digits. In addition, ρ, $\left(\left|\frac{e_n}{e^p_{n-1}}\right| \ \& \ \eta\right)$, $|e_n|$ and absolute residual error in the function $|f(x_n)|$, are presented up to 5 and 10, 2 and 2 significant digits, respectively. Further, the approximated zeros up to 35 significant digits and many more academic problems are also displayed in the Tables 3.

We shall also compute the convergence radii for the examples 5.1–5.2, when $\mathbb{X} = \mathbb{Y} = \mathbb{R}$, $F = f$, $L_1 = L_2 = I$, the identity operator and $L_3 = F'(x)^{-1}$. Then, we can choose $l_1(t,s) = l_2(t,s,q) = 1$ and $l_3(t,s,q,p) = 0$.

Example 5.1. *We consider the following nonlinear scalar function from [**2**]*

$$f(x) = \tan^{-1}\left(x^2 - x\right). \tag{53}$$

We can choose by (a_1)–(a_3) *that* $\xi = 1$, $\mathbb{D} = [0,1]$, $w_0(t) = w(t) = 2.25t$ *and* $v(t) = 1$. *Then, using* (38), *we obtain*

$$r = 0.11748.$$

Example 5.2. *We choose a standard scalar function from [**25**], which is defined as follows:*

$$f(x) = e^{-x}\sin x + \log(1 + x^2). \tag{54}$$

In this case of this example, we can set by (a_1)–(a_3) *that* $\xi = 0$, $\mathbb{D} = \left[-\frac{1}{2}, \frac{1}{2}\right]$, $w_0(t) = w(t) = \frac{1}{2}t$ *and* $v(t) = \frac{3}{2}$. *By using* (38), *we get*

$$r = 0.391756.$$

First, we will compare our eighth-order methods with existing optimal methods of same order proposed by Thukral in [**25**] and Cordero et al. in [**7**], out of these families we shall choose the expression (36-37) and expression (9) (for $\beta_1 = \beta_2 = 1$ and $\beta_3 = 2$), called by (TM) and (CM), respectively. In addition, we also compare our methods with some other optimal eight-order methods which were proposed by Li and Wang in [**17**] and Soleymani et al. in [**23**], we will choose the expression (20) (for $\beta_1 = 0$ and $\beta_2 = 0$) and method (16) out of these methods, denoted by (LM) and (SM), respectively.

Then, we will also compare our sixteenth-order methods with optimal sixteen-order family of iterative methods presented by Ullah et al. [**28**] and Neta [**19**], we consider the methods namely, method (10) and method (5), from their methods, denoted by (UM) and (NM), respectively. Finally, we also compare them with the optimal families of sixteenth-order methods which were proposed by Geum and Kim in [**10, 9**], out of these families we shall choose the expression (Y1) (defined in Table 1 of Geum and Kim [**10**]) and expression (K2) (for details of this method please see Table 1 of Geum and Kim [**9**]), respectively called by $(G1)$ and $(G2)$.

For better comparisons of our proposed methods with the other existing ones, we have given absolute error between the exact root and approximated root ($|e_n| = |x_n - \xi|$) and also calculate the computational order of convergence in the Table 4 – 8. Further, we consider the approximated zero of test functions when the exact zero is not available, which is corrected up to 1000 significant digits to calculate $|x_n - \xi|$. For the computer programming, all computations have been performed using the programming package $Mathematica$ 9 with multiple precision arithmetic.

Now, we finish our numerical experiments with a brief incursion into the dynamical properties of the iterative methods M_i and S_i, $i = 1, 2, 3, 4$ defined in the previous sections. This study allows us to obtain a first approach to the global behavior of the considered iterative methods. First we have plotted, by using the package *Mathematica* and following the patterns given by Varona [**29**], the basins of attraction of the eight considered methods applied to the quadratic complex polynomial $p(z) = z^2 - 1$. We plot in magenta the points $z_0 = x_0 + iy_0$, with $x_0, y_0 \in [-2, 2]$ whose orbits converge to the root $z = 1$, whereas yellow corresponds to initial seeds whose orbits converge to the root $z = -1$. Black zones represent points whose orbits does not converge to any root with the considered number of iterations (in our numerical experiments we have considered 30 iterations and a fixed precision $|z_n - \text{root}| < 10^{-3}$). As we can see (Figure 1) the basins of the eighth-order methods present a more intricate fractal structure than the sixteenth-order methods. The black zones that appears for the cases M_2, M_3 and M_4 reveals the existence of initial seeds whose orbits have a very slow convergence to the roots. This fact can produce numerical instabilities.

In the case of the sixteenth order methods, we can see that the locus of the two roots, that is, the imaginary axis belongs to the Julia set. As a first glance, the problem looks like Cayley's problem, where the right plane is the basin of attraction of the root $z = 1$ and the left plane is the basin of attraction of the root $z = -1$. In Figure-2 we have shown a general picture of the basins of attraction of method S_1 applied to $z^2 - 1$, but the situation is very similar for S_2, S_3 and S_4. But a more detailed analysis reveals that there are regions in the right plane with points whose orbit converges to -1 and vice versa. So in this case, the Fatou set is not connected and there are "islands" of the basin of attraction of the root $z = -1$ in the right plane and vice versa. See again Figure 2 to distinguish some of these islands. Note that there are many other islands, in both half-planes. This is a first step in the study of the Cayley Quadratic Test that can be used as a test to check the efficiency iterative methods for solving nonlinear equations [**1**].

It is not our target to make here a deeper study of other dynamical properties of these methods, that can be undertaken in further research. For instance, if we consider the dynamics of the methods defined in Sections and 2.1 applied to polynomials, we must deal with rational functions whose degrees increase with the degree of the polynomials as shown in Table 9. As a particular case, we plot the basins of attraction of the methods introduced in this paper applied to the polynomial $p(z) = z^3 - 1$ (see figures 3–4). Rational functions of degrees 81, 171, 87 and 123 are obtained for methods M_1, M_2, M_3 and M_4, whereas rational functions of degrees 189, 393, 213 and 297 are obtained for methods S_1, S_2, S_3 and S_4. The basins of the roots are colored in cyan, magenta and yellow. The black zones are again initial points whose orbits does not converge to any root with the considered number of iterations (30 iterations) and the given precision ($|z_n - \text{root}| < 10^{-3}$). As in the quadratic case, the stability of the 16th order methods seems better than the 8th order methods.

Acknowledgments. This work was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, under grant No. (130 - 3 - D1441). The authors, therefore, acknowledge with thanks DSR technical and financial support.

REFERENCES

1. D.K.R. Babajee, A. Cordero, J.R. Torregrosa: *Study of iterative methods through the Cayley Quadratic Test.* J. Comput. Appl. Math. **291** (2016) 358–369.
2. R. Behl, V. Kanwar, K.K. Sharma: *Optimal equi-scaled families of Jarratt's method.* Int. J. Comput. Math. **90** (2013) 408–422.
3. R. Behl, R., A. Cordero, S.S. Motsa, J.R. Torregrosa: *Construction of fourth-order optimal families of iterative methods and their dynamics.* Appl. Math. Comput. 5 **271** (2015) 89–10.
4. R. Behl, S.S. Motsa: *Geometric Construction of Eighth-Order Optimal Families of Ostrowski's Method.* T. Sci. W. J. 5 **2015** (2015) article ID 614612, 11 pages.
5. W. Bi, H. Ren, Q. Wu: *Three-step iterative methods with eighth-order convergence for solving nonlinear equations.* J. Comput. Appl. Math. **255** (2009) 105–112.
6. A. Cordero, J.L. Hueso, E. Martínez, J.R. Torregrosa: *New modifications of Potra-Pták's method with optimal fourth and eighth order of convergence.* J. Comput. Appl. Math. **234** (2010) 2969–2976.
7. A. Cordero, J.R. Torregrosa, M.P. Vassileva: *Three-step iterative methods with optimal eighth-order convergence.* J. Comput. Appl. Math. **235** (201 1) 3189-3194.
8. Y.H. Geum, Y.I. Kim, Y.I., B. Neta: *On developing a higher-order family of double-Newton methods with a bivariate weighting function.* Appl. Math. Comput. **254** (2015) 277–290.

9. Y.H. Geum, Y.I. Kim: *A family of optimal sixteenth-order multipoint methods with a linear fraction plus a trivariate polynomial as the fourth-step weighting function.* Comput. Math. Appli. **61** (2011) 3278–3287.

10. Y.H. Geum, Y.I. Kim: *A biparametric family of optimally convergent sixteenth-order multipoint methods with their fourth-step weighting function as a sum of a rational and a generic two-variable function.* J. Comput. Appl. Math. **235** (201) 3178–3188.

11. M. Grau-Sánchez, M. Noguera, J.M. Gutiérrez: *On some computational orders of convergence.* Appl. Math. Lett. **23** (201 0) 472–478.

12. P. Jarratt: *Some fourth-order multipoint iterative methods for solving equations.* Math. Comput. **20** (1966) 434–437.

13. P. Jarratt: *Some efficient fourth-order multipoint methods for solving equations.* BIT. **9** (1969) 119–124.

14. R.F. King: *A family of fourth order methods for nonlinear equations.* SIAM J. Numer. Anal. 10 (1973) 876–879.

15. H.T. Kung, J.F. Traub: *Optimal order of one-point and multipoint iteration.* J. ACM. **21** (1974) 643–651.

16. Y..I. Kim, R. Behl, S.S. Motsa, *Higher-order efficient class of Chebyshev-Halley type methods.* Appl. Math. Comput. **273** (2016) 1148–1159.

17. L. Liu, X. Wang: *Eighth-order methods with high efficiency index for solving nonlinear equations.* J. Comput. Appl. Math. **215** (2010) 3449–3454.

18. Á.A. Magreñán, I.K. Argyros: *A contemporary study of iterative methods: convergence, dynamics and applications.* Elsevier 2018

19. B. Neta: *On a family of multipoint methods for non-linear equations.* Int. J. Comput. Math. **9** (1981) 353–361.

20. M.S. Petković, B. Neta, L.D. Petković, J. Džunić: *Multipoint methods for solving nonlinear equations.* Academic Press, 2012.

21. J.R. Sharma, R.K. Guha, P. Gupta: *Improved King's methods with optimal order of convergence based on rational approximations.* Appl. Math. Lett. 26 (2013) 473–480.

22. J.R. Sharma, R. Sharma: *A new family of modified Ostrowski's methods with accelerated eighth order convergence.* Numer. Algor. **54** (201 0) 445–458.

23. F. Soleymani, S.K. Vanani, M. Khan, M. Sharifi: *Some modifications of King's family with optimal eighth-order of convergence.* Math. Comput. Model. **55** (2012) 1373–1380.

24. F. Soleymani, M. Sharifi, B.S. Mousavi: *An improvement of Ostrowski's and King's techniques with optimal convergence order eight.* J. Opt. The. Appl. **153(1)** (201 2) 225–236.

25. R. Thukral: *new eighth-order iterative method for solving nonlinear equations.* Appl. Math. Comput. **217** (2010) 222–229.

26. R. Thukral, M.S. Petkovíc: *A family of three point methods of optimal order for solving nonlinear equations.* J. Comput. Appl. Math. **233** (2010) 2278–2284.

27. J.F. Traub: *Iterative methods for the solution of equations.* Prentice-Hall, Englewood Cliffs, 1964.

28. M.Z. Ullah, M.Z., A.S. Al-Fhaid, F. Ahmad: *Four-Point Optimal Sixteenth-Order Iterative Method for Solving Nonlinear Equations.* J. Appl. Math. **2013** (2013) Article ID 850365, 5 pages.

29. J.L. Varona: *Graphic and numerical comparison between iterative methods.* Math. Intelligencer, **24** (2002) 37–46.

30. X. Wang, L. Liu: *Modified Ostrowski's method with eighth-order convergence and high efficiency index.* Appl. Math. Lett. **23** (2010) 549–554.

(Received 18.02.2019)
(Revised 04.09.2019)

Ramandeep Behl
Department of Mathematics,
King Abdulaziz University,
Jeddah 21589, Saudi Arabia
E-mail: *ramanbehl87@yahoo.in*

J. M. Gutiérrez
Department of Mathematics and Computer Sciences,
University of La Rioja, Spain
E-mail: *jmguti@unirioja.es*

I. K. Argyros
Department of Mathematics Sciences Lawton, OK 73505, USA
Cameron University,
E-mail: *ioannisa@cameron.edu*

A. S. Alshomrani
Department of Mathematics,
King Abdulaziz University,
Jeddah 21589, Saudi Arabia
E-mail: *aszalshomrani@kau.edu.sa*

Table 1: Convergence behavior of our methods on example 5.1

Cases	n	x_n	$\|f(x_n)\|$	$\|e_n\|$	$\frac{e_n}{e_{n-1}^p}$	η
M_1	0	1.09	$9.8e(-2)$	$9.0e(-2)$		
	1	0.999999976061565	$2.4e(-8)$	$2.4e(-8)$	5.561023808	13.33333591
	2	1.00000000000000	$1.4e(-60)$	$1.4e(-60)$	13.33333591	
M_2	0	1.09	$9.8e(-2)$	$9.0e(-2)$		
	1	1.00000000127424	$1.3e(-9)$	$1.3e(-9)$	0.2960124755	0.2222222448
	2	1.00000000000000	$1.5e(-72)$	$1.5e(-72)$	0.2222222448	
S_1	0	1.09	$9.8e(-2)$	$9.0e(-2)$		
	1	1.0000000000000	$5.7e(-14)$	$5.7e(-14)$	2.892583102	16.19753086
	2	1.00000000000000	$7.5e(-260)$	$7.5e(-260)$	16.19753086	
S_2	0	1.09	$9.8e(-2)$	$9.0e(-2)$		
	1	1.00000000000000	$2.4e(-18)$	$2.4e(-18)$	0.1307020275	0.06748971193
	2	1.00000000000000	$9.5e(-284)$	$9.5e(-284)$	0.06748971193	

Table 2: Convergence behavior of our methods on example 5.2

Cases	n	x_n	$\|f(x_n)\|$	$\|e_n\|$	$\frac{e_n}{e_{n-1}^p}$	η
M_1	0	0.37	$3.8e(-1)$	$3.7e(-1)$		
	1	$1.5550306436457e(-9)$	$1.6e(-9)$	$1.6e(-9)$	$4.427159550e(-6)$	$1.791200166e(-19)$
	2	$6.12428981700460e(-90)$	$6.1e(-90)$	$6.1e(-90)$	$1.791200166e(-19)$	
M_2	0	0.37	$3.8e(-1)$	$3.7e(-1)$		
	1	$2.98761085780169e(-9)$	$3.0e(-9)$	$3.0e(-9)$	$8.505703991(-6)$	$6.611717331e(-19)$
	2	$4.19668633660919e(-87)$	$4.2e(-87)$	$4.2e(-87)$	$6.611717331e(-19)$	
S_1	0	0.37	$3.8e(-1)$	$3.7e(-1)$		
	1	$1.24978392159601e(-18)$	$1.2e(-18)$	$1.2e(-18)$	$1.012995108e(-11)$	$3.867257001e(-75)$
	2	$-1.37013059863481e(-361)$	$1.4e(-361)$	$1.4e(-361)$	$3.867257001e(-75)$	
S_2	0	0.37	$3.8e(-1)$	$3.7e(-1)$		
	1	$4.49648896558032e(-18)$	$4.5e(-18)$	$4.5e(-18)$	$3.644567071e(-11)$	$6.479737753e(-73)$
	2	$-1.80939731413284e(-350)$	$3.8e(-331)$	$3.8e(-331)$	$6.479737753e(-73)$	

Table 3: Test problems

$f(x)$	x_0	$Root(r)$
$f_1(x) = e^{2x} + \sin^{-1}\left(x^2-1\right) - 7$; [**17**]	1.2	0.97629186887861075372580403259043572
$f_2(x) = (x-2)^2 - 33x - \log(x)$;[**4**]	39	36.989473582944669865344734734912736
$f_3(x) = x^3 + \log\left(x^2-2\right)\cos\left(\frac{\pi}{2x^2+1}\right) - 3\sqrt{3}$; [**10**]	1.6	$\sqrt{3}$
$f_4(x) = \log\left(x^2+x+1\right) - x + 1$; [**20**]	3.9	0
$f_5(x) = (x-2)\left(x^{10}+x+1\right)\exp(-x-1)$;[**28**]	2.1	2

Table 4: Performance of different methods for test function $f_1(x)$

Methods	$\|x_1 - \xi\|$	$\|x_2 - \xi\|$	$\|x_3 - \xi\|$	ρ
TM	3.2e−5	1.4e−35	1.6e−278	8.000
CM	7.4e−7	1.6e−50	6.0e−400	8.000
LM	1.4e−6	7.9e−48	6.7e−378	8.000
SM	8.8e−6	2.1e−42	2.5e−335	8.000
M_1	6.4e−6	9.4e−42	1.9e−328	8.000
M_2	8.3e−7	5.2e−50	1.2e−395	8.000
UM	3.3e−11	3.1e−166	1.5e−2646	16.00
NM	7.0e−8	3.7e−104	1.8e−1644	16.00
$G1$	1.7e−10	4.9e−156	1.5e−2484	16.00
$G2$	7.8e−10	4.2e−143	2.5e−2275	16.00
S_1	1.3e−12	6.4e−192	4.4e−3061	16.00
S_2	3.3e−13	1.6e−202	2.8e−3231	16.00

Table 5: Performance of the methods for test function $f_2(x)$

Methods	$\|x_1 - \xi\|$	$\|x_2 - \xi\|$	$\|x_3 - \xi\|$	ρ
TM	9.2e−8	3.5e−66	1.4e−533	8.000
CM	8.1e−9	9.0e−76	2.1e−611	8.000
LM	2.1e−8	5.9e−72	1.9e−580	8.000
SM	1.6e−7	1.2e−66	1.7e−539	8.000
M_1	1.6e−10	3.6e−92	1.9e−745	8.000
M_2	3.8e−10	2.0e−94	1.3e−768	8.000
UM	1.5e−19	2.4e−332	4.5e−5337	16.00
NM	4.0e−17	8.8e−284	2.4e−4550	16.00
$G1$	3.2e−17	1.2e−285	1.5e−4580	16.00
$G2$	1.2e−15	4.6e−258	5.6e−4137	16.00
S_1	1.9e−18	1.9e−306	1.8e−4914	16.00
S_2	1.1e−20	1.1e−354	2.2e−5698	16.00

Table 6: Performance of different methods for test function $f_3(x)$

Methods	$\|x_1-\xi\|$	$\|x_2-\xi\|$	$\|x_3-\xi\|$	ρ
TM	6.3e−8	3.6e−60	4.3e−478	8.000
CM	5.8e−8	8.3e−60	1.5e−474	8.000
LM	7.1e−8	1.1e−59	2.9e−474	8.000
SM	2.5e−8	2.5e−63	1.9e−503	8.000
M_1	8.5e−8	1.5e−59	1.7e−473	8.000
M_2	3.4e−8	2.7e−62	3.3e−495	8.000
UM	2.7e−13	5.2e−203	2.0e−3238	16.00
NM	4.9e−16	6.3e−249	2.8e−3975	16.00
$G1$	1.6e−15	3.7e−240	1.6e−3834	16.00
$G2$	2.8e−15	1.3e−236	3.8e−3778	16.00
S_1	1.8e−15	3.8e−240	7.5e−3835	16.00
S_2	1.8e−15	4.9e−240	5.0e−3833	16.00

Table 7: Performance of different methods for test function $f_4(x)$

Methods	$\|x_1-\xi\|$	$\|x_2-\xi\|$	$\|x_3-\xi\|$	ρ
TM	4.9e−12	1.2e−96	1.1e−773	8.000
CM	5.8e−13	5.4e−105	2.9e−841	8.000
LM	2.2e−12	7.9e−100	2.3e−799	8.000
SM	5.5e−12	3.5e−96	8.8e−770	8.000
M_1	2.3e−12	1.2e−99	5.9e−798	8.000
M_2	6.3e−14	1.7e−113	4.5e−910	8.000
UM	7.4e−25	2.9e−399	9.2e−6390	16.00
NM	7.7e−25	5.8e−399	6.5e−6385	16.00
$G1$	7.2e−25	1.1e−399	4.7e−6397	16.00
$G2$	2.4e−23	7.6e−374	9.5e−5982	16.00
S_1	1.1e−25	4.1e−413	3.8e−6612	16.00
S_2	2.5e−27	6.2e−441	1.3e−7058	16.00

Table 8: Performance of different methods for test function $f_5(x)$

Methods	$\|x_1 - \xi\|$	$\|x_2 - \xi\|$	$\|x_3 - \xi\|$	ρ
TM	2.9e−4	2.7e−23	1.4e−175	8.000
CM	3.6e−5	6.7e−32	9.8e−246	8.000
LM	6.8e−5	3.0e−29	4.7e−224	8.000
SM	2.3e−5	3.2e−34	5.7e−265	8.000
M_1	6.4e−5	3.1e−29	9.2e−224	8.000
M_2	5.6e−5	5.7e−32	6.9e−248	7.998
UM	9.0e−9	6.6e−119	4.5e−1881	16.00
NM	7.0e−8	3.7e−104	1.8e−1644	16.00
$G1$	5.8e−8	4.2e−106	2.5e−1676	16.00
$G2$	1.9e−7	1.2e−95	2.3e−1506	16.00
S_1	9.6e−10	4.1e−136	5.9e−2158	16.00
S_2	7.5e−10	1.6e−140	2.2e−2231	16.00

Table 9: Degrees of the rational maps related to methods M_i and S_i, $i = 1, 2, 3, 4$ applied to polynomials of degree n.

Method	Degree	Method	Degree
M_1	$3n^3 + n^2 - 3n$	S_1	$3n^4 - n^3 - 4n^2 + 3n$
M_2	$9n^3 - 8n^2$	S_2	$9n^4 - 14n^3 + 4n^2 + 2n$
M_3	$4n^3 - 2n^2 - n$	S_3	$4n^4 - 4n^3 - n^2 + 2n$
M_4	$6n^3 - 4n^2 - n$	S_4	$6n^4 - 7n^3 - n^2 + 3n$

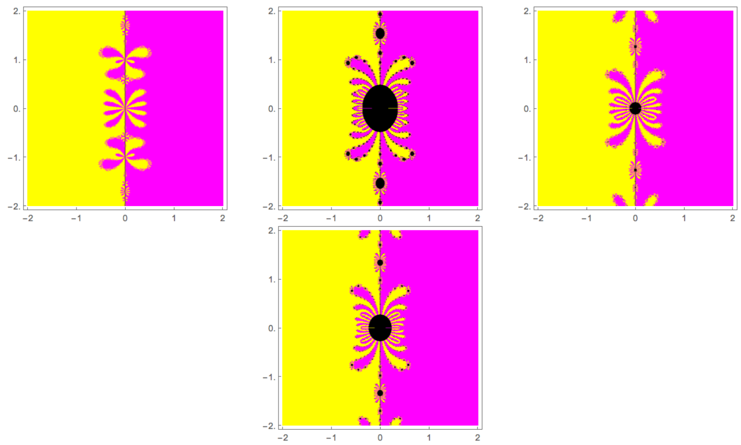

Figure 1: Basins of attraction of the eighth-order iterative methods M_i, $i = 1, \ldots, 4$ applied to the quadratic complex polynomial $p(z) = z^2 - 1$.

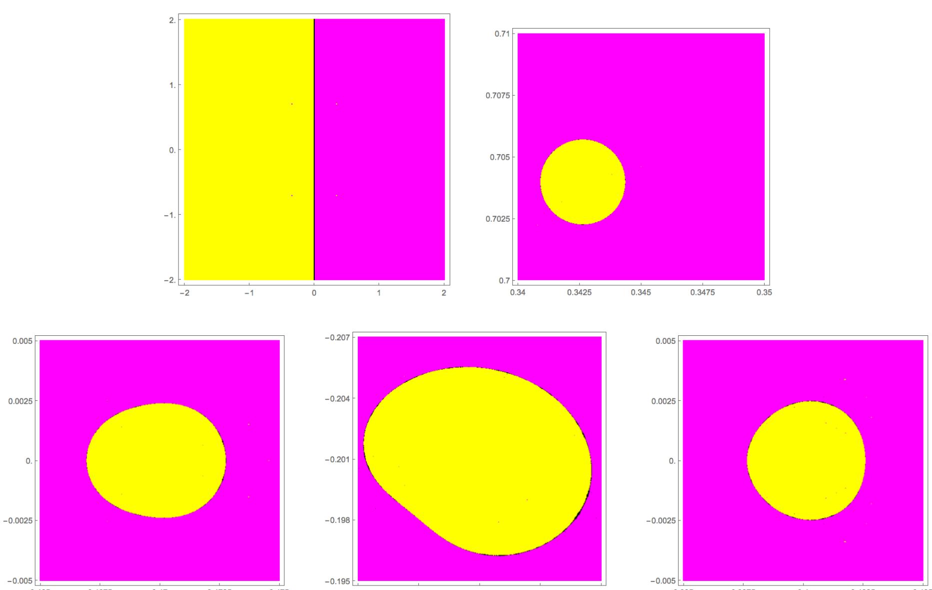

Figure 2: On the top left, basins of attraction of the sixteenth-order iterative method S_1 applied to the quadratic complex polynomial $p(z) = z^2 - 1$. On the top right, a magnification showing a region of initial points whose orbits converge to the root $z = -1$ that is included in the right plane. On the bottom, details of the basins of attraction of the sixteenth-order iterative method S_i, $i = 2, 3, 4$ applied to the quadratic complex polynomial $p(z) = z^2 - 1$ showing regions of initial points whose orbits converge to the root $z = -1$ but that are included in the right plane.

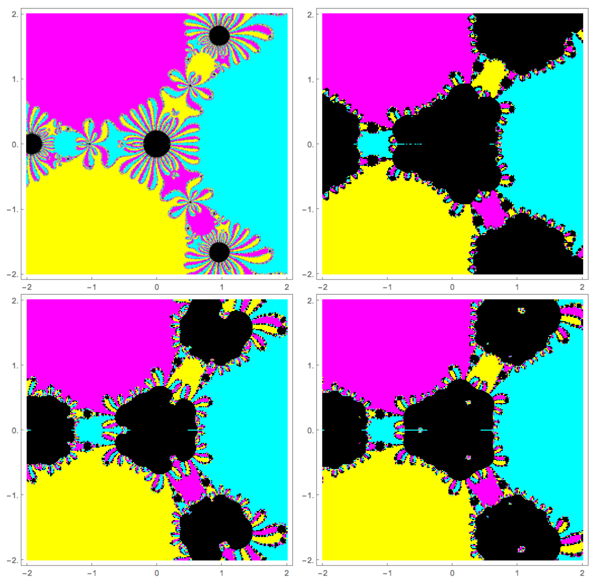

Figure 3: Basins of attraction of the eighth-order iterative methods M_i, $i = 1, \ldots, 4$ applied to the cubic complex polynomial $p(z) = z^3 - 1$.

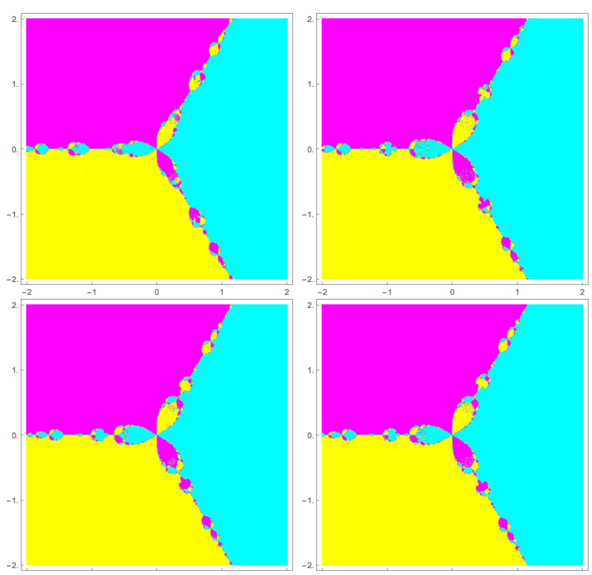

Figure 4: Basins of attraction of the sixteenth-order iterative methods S_i, $i = 1, \ldots, 4$ applied to the cubic complex polynomial $p(z) = z^3 - 1$.