A DUAL-TYPE PROBLEM TO CHRISTOFFEL FUNCTION

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Abstract. We study a dual-type problem to generalized Christoffel function. The solution is connected with other extremal problems in the H^p space of analytic functions on the unit circle considered by Macintyre, Rogosinski and Shapiro.

1. Introduction and main results

Let μ be a finite positive Borel measure on the unit circle $\mathbb{T}$ with infinitely many points in its support. The $L^p(\mu)$-*Christoffel function* is defined as

$$\lambda_{n,p}(\mu,\zeta) := \inf\{\|Q\|_p^p : Q \in \Pi_n, Q(\zeta) = 1\}, \quad 1 \leq p < \infty,$$

where ζ is a complex number, Π_n stands for the set of polynomials of degree at most n, and $\|Q\|_p = (\int |Q|^p \, d\mu)^{1/p}$. Christoffel functions have been useful for establishing Bernstein and Nikolskii inequalities, in estimating quadrature sums, and in studying convergence of Lagrange interpolation and orthogonal expansions [14, 17, 18]. These functions also play an important role in random matrix theory (see [5]). The extremal kernel for Christoffel functions in L^∞ is used for extensions of the conjugate gradient method for solving linear systems with certain symmetric form (see [9]). For the history of Christoffel functions, we refer the reader to [13, 17, 18, 22].

In this paper, we study the distances of certain varieties to the origin in $L^p(\mu)$, with $1 \leq p \leq \infty$. These are dual-type problems to Christoffel functions, and they also link with extremal problems in H^p. Our results are connected with extremal problems in H^p for $1 < p < \infty$ studied by Macintyre, Rogosinski and Shapiro (see [6, 16, 20], and the references therein).

From now on, we denote by q the conjugate exponent to $p \in [1,\infty]$ (i.e., $q = 1$ if $p = \infty$, $q = \infty$ if $p = 1$, and $1/p + 1/q = 1$ otherwise). Let

$$V_{n,q}(\mu,\zeta) := \Big\{ f \in L^q(\mu) : \int Q\bar{f} \, d\mu = Q(\zeta), \text{ for every } Q \in \Pi_n \Big\}.$$

It is a non-empty set, and has the structure of a variety in $L^q(\mu)$ (see the beginning of Section 2). Let

$$\Omega_{n,q}(\mu,\zeta) := \inf\{\|f\|_q^p : f \in V_{n,q}(\mu,\zeta)\}, \tag{1}$$

for $1 < q \leq \infty$. We define $\Omega_{n,1}(\mu,\zeta)$ as the infimum of the norms $\|\cdot\|_1$.

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Theorem 1.1. *If $1 \le p < \infty$ and $|\zeta| = 1$, then*

$$\Omega_{n,q}(\mu,\zeta) = \frac{1}{\lambda_{n,p}(\mu,\zeta)}.$$

Now, assume that μ is the normalized Lebesgue measure in the unit circle $\mathbb{T}$, which we will denote by $\mathfrak{m}$. In this case, we connect $\Omega_{n,q}(\mathfrak{m},\zeta)$ (and therefore $\lambda_{n,p}(\mathfrak{m},\zeta)$) with the following extremal problems:

$$\delta_{n,p} := \sup\left\{ \left| \int_{\mathbb{T}} \frac{1+z+\cdots+z^n}{z^{n+1}} f(z)\,\frac{dz}{2\pi} \right| \,:\, f \in H^p, \|f\|_p = 1 \right\},$$

$$\Delta_{n,q} := \inf\left\{ \left\| \frac{1+z+\cdots+z^n}{z^{n+1}} + g(z) \right\|_q \,:\, g \in H^q \right\},$$

where, as usual, H^p stands for the Hardy space of analytic function in the unit disc with bounded means of order p. If $f(z) = \sum_{k=0}^{\infty} a_k z^k \in H^p$, by the Cauchy integral formula we have

$$\int_{\mathbb{T}} \frac{1+z+\cdots+z^n}{z^{n+1}} f(z)\,\frac{dz}{2\pi i} = \sum_{k=0}^{n} a_k.$$

These extremal problems were considered by Landau, Schur, Macintyre, Rogosinski, Shapiro and other authors (see [20] and mainly [6, Chapter 8], and the references therein). For $p = \infty$, in $\delta_{n,\infty}$ the supremum is taken on $\mathcal{C} \cap H^\infty$, the algebra of analytic functions in the unit disc which are continuous on the closed unit disc. The extremal problem $\Delta_{1,2}$ in $H^2(\mu)$ is associated with the asymptotic behavior of certain Toeplitz determinants or the leading coefficients of orthonormal polynomials with respect to μ (see, for example, [21, Theorem 2.7.15]).

Theorem 1.2. *If $|\zeta| = 1$, then $\Omega_{n,1}(\mathfrak{m},\zeta) \le \delta_{n,\infty}$, and $\Omega_{n,q}(\mathfrak{m},\zeta) \le \delta_{n,p}^{p}$, for $1 < p < \infty$.*

It is easy to see that $\delta_{n,p}$ is a monotonically increasing sequence on n. Its asymptotic behavior has been studied for $p = 1$ and $p = \infty$. We refer the reader to Egerváry [7] for $p = 1$. This is a particular case of Carathéodory-Fejér problem (see [1, Appendix §D]). For $p = \infty$, see Landau [12, §2]. If $1 < p < \infty$, it is straightforward to verify (see Proposition 3.7) that

$$0 < \liminf_{n\to\infty} \frac{\delta_{n,p}}{n^{1/p}} \le \limsup_{n\to\infty} \frac{\delta_{n,p}}{n^{1/p}} < \infty.$$

The paper is organized as follows. In Section 2 we prove Theorem 1.1, and show that the variety $V_{n,q}(\mu,\zeta)$ has an element of minimal norm. This element is unique if $1 < q < \infty$. Section 3 contains the proof of Theorem 1.2 and remarks on other extremal problems for the Lebesgue measure.

2. Minimal norm in a variety

In this section we prove Theorem 1.1, and give some properties of $V_{n,q}(\mu,\zeta)$ and $\Omega_{n,q}(\mu,\zeta)$. Mainly, the infimum that defines $\Omega_{n,q}(\mu,\zeta)$ is indeed a minimum, and it is attained at a unique function in $V_{n,q}(\mu,\zeta)$ (see Proposition 2.2 below). We also obtain some characterizations of this minimal function.

Let us start by showing that the set $V_{n,q}(\mu,\zeta)$ has the structure of a variety. Denote by $\{\varphi_k : k = 0,1,\ldots\}$ the sequence of orthonormal polynomials with respect to μ with positive leading coefficients. The *reproducing kernel for* μ is the function $K_n(w,z) = \sum_{j=0}^{n} \overline{\varphi_j(w)}\varphi_j(z)$. Notice that $K_n(\zeta,\cdot)$ is an element in $V_{n,q}(\mu,\zeta)$. Let

$$W_{n,q}(\mu) := \left\{ f \in L^q(\mu) \,:\, \int Q\bar{f}\, d\mu = 0, \text{ for every } Q \in \Pi_n \right\}.$$

It is a subspace of $L^q(\mu)$. Therefore, $V_{n,p}(\mu,\zeta)$ is a variety in $L^q(\mu)$ that can be written as

$$V_{n,p}(\mu,\zeta) = K_n(\zeta,\cdot) + W_{n,q}(\mu).$$

For the proof of Theorem 1.1 we make use of the following lemma. Let

$$\widehat{\lambda}_{n,p}(\mu,\zeta) = \sup\{|Q(\zeta)|^p \,:\, Q \in \Pi_n,\ \|Q\|_p = 1\}.$$

Lemma 2.1. *If $1 \le p < \infty$, then there exists a function f_0 in $V_{n,q}(\mu,\zeta)$ such that $\|f_0\|_q^p = \widehat{\lambda}_{n,p}(\mu,\zeta)$.*

Proof. Let Λ be the point evaluation functional $\Lambda(Q) = Q(\zeta)$, acting on Π_n. If we consider the $L^p(\mu)-$norm in Π_n, then Λ is a continuous linear functional with $\|\Lambda\|^p = \widehat{\lambda}_{n,p}(\mu,\zeta)$. By the Hahn-Banach theorem, Λ can be extended to a linear functional $\widehat{\Lambda}$ on $H^p(\mu)$ with the same norm. Recall that $H^p(\mu)$ spaces are the $L^p(\mu)-$closure of the linear space of all polynomials (see [3] and [8, Sec. V.3]). By the Riesz duality theorem for $L^p(\mu)$, there is a unique $f_0 \in L^q(\mu)$ such that $\widehat{\Lambda}(h) = \int h\,\overline{f_0}\,d\mu$, for every $h \in H^p(\mu)$, and $\|\widehat{\Lambda}\| = \|f_0\|_q$. It is clear that $f_0 \in V_{n,q}(\mu,\zeta)$. Therefore the statement follows joining the equalities of norms obtained. □

It is well-known (and very easy to check from definitions) that

$$\lambda_{n,p}(\mu,\zeta) = \frac{1}{\widehat{\lambda}_{n,p}(\mu,\zeta)}, \quad 1 \le p < \infty. \tag{2}$$

Proof of Theorem 1.1. As a consequence of Lemma 2.1, notice that $\Omega_{n,q}(\mu,\zeta)$ is bounded above by $\widehat{\lambda}_{n,p}(\mu,\zeta)$. Hölder's inequality gives that $|Q(\zeta)| \le \|Q\|_p\|g\|_q$ for any g in $V_{n,q}(\mu,\zeta)$ and any Q in Π_n. Then $\Omega_{n,q}(\mu,\zeta)$ is bounded below by $\widehat{\lambda}_{n,p}(\mu,\zeta)$. Hence, using (2), the statement follows. □

Now we focus on the study of the variety $V_{n,q}(\mu,\zeta)$.

Proposition 2.2. *If $1 \le p < \infty$, then*

$$\Omega_{n,q}(\mu,\zeta) = \min\{\|f\|_q^p : f \in V_{n,q}(\mu,\zeta)\}.$$

Moreover, if $1 < p < \infty$, this minimum is attained at a unique function in $V_{n,q}(\mu,\zeta)$. If $p = 1$ and μ is absolutely continuous with respect to Lebesgue measure, then this minimum is also unique.

Proof. As an immediate consequence of Theorem 1.1 and Lemma 2.1, we obtain the existence of the minimum in Proposition 2.2. Indeed, it is attained for the function f_0 given in Lemma 2.1. If $1 < p < \infty$, the space $L^p(\mu)$ is a strictly convex. Since $V_{n,p}(\mu,\zeta)$ is a closed variety on it, the uniqueness in Proposition 2.2 follows for this case. Our proof of the uniqueness for $p = 1$ relies on the next fact, valid for any p in $[1,\infty)$.

Claim. *There exists a unique polynomial $Q_\#$ in Π_n such that*

$$\|Q_\#\|_p = 1, \qquad Q_\#(\zeta) > 0, \qquad \text{and} \qquad Q_\#(\zeta)^p = \widehat{\lambda}_{n,p}(\mu,\zeta).$$

Indeed, there exists a unique polynomial P in Π_n such that $P(\zeta) = 1$ and $\|P\|_p^p = \lambda_{n,p}(\mu,\zeta)$. The existence can be easily proved by compactness arguments (see [17, p. 106]). The uniqueness of this polynomial follows for $p > 1$ from the strict convexity of the space L^p; for $p = 1$, see, for example, [11]. Take $Q_\#(z) = \alpha P(z)/\|P\|_p$ for an appropriate α in the unit circle to complete the proof of the claim.

Now, let $p = 1$, and let g be any function in $V_{n,\infty}(\mu,\zeta)$ such that

$$\|g\|_\infty = \inf\{\|f\|_\infty : f \in V_{n,\infty}(\mu,\zeta)\} = \Omega_{n,\infty}(\mu,\zeta).$$

Take $Q_\#$ as in the claim. That is, $Q_\#$ is a polynomial in Π_n such that $\|Q_\#\|_1 = 1$, and $Q_\#(\zeta) = \|g\|_\infty$. Since $g \in V_{n,\infty}(\mu,\zeta)$ and $Q_\# \in \Pi_n$, we have

$$\int Q_\# \overline{g}\, d\mu = Q_\#(\zeta) = \|g\|_\infty \|Q_\#\|_1.$$

Thus, equality holds for Hölder's inequality in $L^1(\mu) - L^\infty(\mu)$−norm. Therefore, if μ is absolutely continuous, we obtain that

$$g = \|g\|_\infty \operatorname{sign}\left(\overline{Q_\#}\right), \quad \mathfrak{m} - \text{a.e.} \tag{3}$$

Recall that the sign function is defined as $\operatorname{sign}(z) = z/|z|$ if $z \neq 0$, and $\operatorname{sign}(0) = 0$. This shows the uniqueness of the function g (since the right hand side above is fixed), and the proof of Proposition 2.2 is complete. □

Remark 2.3. Notice that (3) gives an explicit formula, in terms of the polynomial $Q_\#$, for the unique element in $V_{n,\infty}(\mu,\zeta)$ with minimal norm.

In the following result, we connect $Q_\#$ with the unique element in $V_{n,q}(\mu,\zeta)$ with minimal norm, for $1 < q < \infty$.

Proposition 2.4. *Let $1 < p < \infty$ and let f_0 be the element of $V_{n,q}(\mu,\zeta)$ with minimal norm. Then*

$$\|f_0\|_q^q |Q_\#(t)|^p = |f_0(t)|^q, \quad \mu - a.e. \tag{4}$$

Moreover,

$$Q_\#(t)\overline{f_0(t)} \geq 0 \quad \mu - a.e. \tag{5}$$

Proof. By definitions of $Q_\#$ and f_0, and Lemma 2.1, we have $\int Q_\# \overline{f_0}\, d\mu = Q_\#(\zeta) = \|f_0\|_q$. Hölder's inequality gives that $\int |Q_\# f_0|\, d\mu \leq \|f_0\|_q$. Therefore $\int Q_\# \overline{f_0}\, d\mu = \int |Q_\# f_0|\, d\mu$. This implies (5). Moreover, since we have obtained the equality in Hölder's inequality, this means that there exist constants α and β, which depend on n and p but not on t, such that $\alpha|Q_\#(t)|^p = \beta|f_0(t)|^q$, μ-almost every t in the support of μ. Integrating both sides of this equality, we get that $\alpha = \beta\, \|f_0\|_q^q$. Hence (4) follows. □

In the next proposition, we give a characterization of the element of minimal norm in $V_{n,q}(\mu,\zeta)$, for $1 < q < \infty$.

Proposition 2.5. *Let f be a function in $V_{n,q}(\mu,\zeta)$, and let $1 < q < \infty$. The following statements are equivalent.*

(i) *f has minimal norm among the elements of $V_{n,q}(\mu,\zeta)$.*

(ii) $\|f\|_q = \inf\{\|K_n(\zeta,\cdot) - g\|_q : g \in W_{n,q}(\mu)\}$.
(iii) *For any* $g \in W_{n,q}(\mu)$,

$$\int_{\mathbb{T}} g(t)|f(t)|^{p-1}\,\mathrm{sign}\,(f(t))\; d\mu(t) = 0.$$

Proof. Since $V_{n,q}(\mu,\zeta) = K_n(\zeta,\cdot) + W_{n,q}(\mu)$, we have

$$\Omega_{n,q}(\mu,\zeta) = \inf\{\|K_n(\zeta,\cdot) - g\|_q : g \in W_{n,q}(\mu)\},$$

which implies the equivalence between (i) and (ii). Since $K_n(\zeta,\cdot)$ and f belong to $V_{n,q}(\mu,\zeta)$, we obtain that $K_n(\zeta,\cdot) - f$ belongs to $W_{n,q}(\mu)$. Therefore the equivalence between (ii) and (iii) follows using [23, Theorem 1.11]. $\square$

Propositions 2.4 and 2.5 suggest the study of the zeros of $Q_\#$. This is done in the next result. Given a polynomial P of degree n, we denote by P^* the polynomial $P^*(z) = z^n\,\overline{P(1/\bar{z})}$.

Proposition 2.6. *If* $|\zeta| = 1$*, then* $Q_\#^* = \zeta^n Q_\#$*. In particular, the zeros of* $Q_\#$ *are symmetric with respect to the unit circle.*

Moreover, if $\zeta = 1$*, then* $Q_\#$ *has real coefficients, and its zeros are further symmetric with respect to the real line.*

Proof. Obviously, $Q_\#^*$ has degree n and $\zeta^{-n}Q_\#^*(\zeta) = \overline{Q_\#(\zeta)} = Q_\#(\zeta)$, because $Q_\#(\zeta) > 0$. Since $|\zeta^{-n}Q_\#^*(z)| = |Q_\#(z)|$ for every z in the unit circle, $\|\zeta^{-n}Q_\#^*\|_p = \|Q_\#\|_p$. Using the uniqueness of $Q_\#$, we have $\zeta^{-n}Q_\#^* = Q_\#$.

Now consider the case $\zeta = 1$. Let P be the polynomial whose coefficients are the conjugate coefficients of $Q_\#$. Then P has degree n and $P(1) = Q_\#(1)$. Since $|P(z)| = |Q_\#(\bar{z})|$ for every z in the unit circle, $\|P\|_p = \|Q_\#\|_p$. Using the uniqueness of $Q_\#$ again, we have $P = Q_\#$. $\square$

3. Extremal problem in the unit circle

In this section, we consider μ as the normalized Lebesgue measure in the unit circle, and we denote it by $\mathfrak{m}$. In other words,

$$\int_{\mathbb{T}} f(z)\,d\mathfrak{m}(z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\,dt = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(z)}{z}\,dz.$$

We shall prove Theorem 1.2 and its consequences. To this end, consider the extremal problem

$$\Upsilon_{n,q} = \inf\left\{\left\|\frac{1+z+\cdots+z^n}{z^{n+1}} + f(z) + \frac{g(1/z)}{z^{n+2}}\right\|_q^p \;:\; f,g \in H^q\right\},$$

for $1 < q \le \infty$. If $q = 1$, we define $\Upsilon_{n,1}$ as the infimum of norms $\|\cdot\|_1$ as above. Notice that taking $g = 0$ in the definition of $\Upsilon_{n,q}$, the following result is trivial.

Proposition 3.1. *If* $1 < q \le \infty$*, then* $\Upsilon_{n,q}$ *is bounded above by* $\Delta_{n,q}^p$*. If* $q = 1$*, then* $\Upsilon_{n,1}$ *is bounded above by* $\Delta_{n,1}$*.*

Remark 3.2. Observe that $\Upsilon_{n,2} = \Delta_{n,2}^2 = n+1$, since for $p = 2$ the trigonometric system is orthonormal.

Lemma 3.3. *For* $1 \le q < \infty$ *we have* $\Omega_{n,q}(\mathfrak{m},\zeta) = \Upsilon_{n,q}$*.*

Proof. For the Lebesgue measure $\mathfrak{m}$ the reproducing kernel is given by $K_n(\zeta, z) = \sum_{j=0}^{n} (\overline{\zeta} z)^j$. Note that

$$V_{n,q}(\mathfrak{m}, \zeta) = K_n(\zeta, z) + \left\{ z^{n+1} g(z) + \frac{h(1/z)}{z} \,:\, g, h \in H^q \right\}.$$

Indeed, if $1 < q < \infty$, this follows from the convergence of Fourier series in $L^q(\mathfrak{m})$. If $q = 1$, the statement follows from the density of trigonometric polynomials in $L^1(\mathfrak{m})$.

Setting $w = \overline{\zeta} z$, we have

$$\begin{aligned} \left\| K_n(\zeta, z) + z^{n+1} g(z) + \frac{h(1/z)}{z} \right\|_q &= \left\| \sum_{j=0}^{n} w^j + (\zeta w)^{n+1} g(\zeta w) + \frac{h(1/\zeta w)}{\zeta w} \right\|_q \\ &= \left\| k_n(w) + \zeta^{n+1} g(\zeta w) + \frac{h(1/(\zeta w))}{\zeta w^{n+2}} \right\|_q, \end{aligned}$$

where $k_n(z) = (1 + z + \cdots + z^n)/z^{n+1}$. This equality immediately gives the desired result. ☐

Proof of Theorem 1.2. In [6, Theorem 8.1] is given the duality relation

$$\delta_{n,p} = \Delta_{n,q}. \tag{6}$$

Now Theorem 1.2 follows from Lemma 3.3, Proposition 3.1, and (6). ☐

For historical remarks on the extremal problems (6), we refer the reader to the notes at the end of Chapter 8 of [6].

Let us observe now a connection between $\Upsilon_{n,q}$ and the extremal problem

$$\Theta_{n,q} = \inf \left\{ \left\| \frac{1 + z + \cdots + z^n}{z^{n+1}} + f(z) + \frac{f(1/z)}{z^{n+2}} \right\|_q^p \,:\, f \in H^q \right\} \quad 1 < q \leq \infty.$$

If $q = 1$, we define $\Theta_{n,1}$ as the infimum of norms $\|\cdot\|_1$ as above.

Proposition 3.4. $\Upsilon_{n,q} = \Theta_{n,q}$.

Proof. Changing z by z^{-1} we obtain

$$\begin{aligned} \left\| k_n(z) + f(z) + \frac{g(1/z)}{z^{n+2}} \right\|_q &= \left\| z^{n+2} \left(k_n(z) + \frac{f(1/z)}{z^{n+2}} + g(z) \right) \right\|_q \\ &= \left\| k_n(z) + \frac{f(1/z)}{z^{n+2}} + g(z) \right\|_q. \end{aligned}$$

Therefore, $g(z) = f(z)$ for the optimal function ☐

Remark 3.5. We have shown the following:

$$\frac{1}{\lambda_{n,p}(\mathfrak{m}, \zeta)} = \Omega_{n,q}(\mathfrak{m}, \zeta) = \Upsilon_{n,q} = \Theta_{n,q} \leq \Delta_{n,q}^p = \delta_{n,p}^p.$$

The asymptotic behavior of the Chritoffel functions $\lambda_{n,p}$ is a cornerstone in many application as it was said in the introduction. Let us state a beautiful result of this type obtained by Levin and Lubinsky [15]. Let

$$\mathcal{E}_p := \inf \left\{ \int_{-\infty}^{\infty} |f(t)|^p \, dt \,:\, f \in L_\pi^p, f(0) = 1 \right\},$$

where the Paley-Wiener space L^p_π consists of all the entire functions f such that

$$\int_{-\infty}^{\infty} |f(t)|^p \, dt < \infty \qquad \text{and} \qquad |f(z)| \le Ce^{\pi|z|},$$

for some constant $C = C(f) > 0$, and for every $z \in \mathbb{C}$. Levin and Lubinsky proved that if μ is a regular measure in the sense of Stahl-Totik, and μ' is continuous at ζ (with $|\zeta| = 1$), then

$$\lim_{n\to\infty} n\lambda_{n,p}(\mu,\zeta) = 2\pi\mathcal{E}_p\mu'(\zeta). \tag{7}$$

Several authors have studied $\mathcal{E}_p$ (see [15] and the references therein, and [19]). While there are estimates of $\mathcal{E}_p$, it seems that there is no explicit formula for it.

Lemma 3.6 (see [2, Lemma 2.1]). *If $1 < q < \infty$ and $|\zeta| = 1$, then*

$$\|K_n(\zeta,\cdot)\|_q = \left(\int_0^{2\pi} \left|\frac{\sin\left((n+1)t/2\right)}{\sin(t/2)}\right|^q \frac{dt}{2\pi}\right)^{\frac{1}{q}} = C_q n^{\frac{q-1}{q}} + o_q(n^{\frac{q-1}{q}}),$$

as $n \to \infty$, where $C_q = \left(\frac{2}{\pi}\int_0^\infty \left|\frac{\sin t}{t}\right|^q dt\right)^{1/q}$.

Recall that the reproducing kernel for the Lebesgue measure can be written in terms of the Dirichlet kernel. Indeed, if $\zeta = e^{it_0}$ and $z = e^{it}$, with t_0 and t real numbers, then

$$K_n(\zeta,z) = \sum_{j=0}^{n} (\overline{\zeta}z)^j = \frac{(\overline{\zeta}z)^{n+1} - 1}{(\overline{\zeta}z) - 1} = e^{in(t_0+t)/2}\frac{\sin\left((n+1)(t_0+t)/2\right)}{\sin\left((t_0+t)/2\right)}.$$

Proposition 3.7. *If $1 < p < \infty$, then*

$$0 < 1/C_p \le \liminf_{n\to\infty} \frac{\delta_{n,p}}{n^{1/p}} \le \limsup_{n\to\infty} \frac{\delta_{n,p}}{n^{1/p}} \le C_q < \infty.$$

Proof. By the definition of $\delta_{n,p}$ and Hölder's inequality, it is immediate that $\delta_{n,p} \le \|K_n(1,\cdot)\|_q$. Hence, using Lemma 3.6 we obtain the desired upper bound. If we take $f(z) = K_n(1,z)/\|K_n(1,\cdot)\|_p$, by the definition of $\delta_{n,p}$,

$$\delta_{n,p} \ge \left|\int_{\mathbb{T}} \frac{1+z+\cdots+z^n}{z^{n+1}} f(z)\,\frac{dz}{2\pi}\right| = \frac{n+1}{\|K_n(1,\cdot)\|_p}.$$

Therefore, using Lemma 3.6 again, we obtain the desired lower bound. □

Remark 3.8. From (7), Remark 3.5, and Proposition 3.7, we deduce that $\mathcal{E}_p$ is bounded below by $1/(2\pi C_q^p)$, for $1 < p < \infty$. Moreover, using that $\delta_{n,1}/n \sim 2/\pi$ (see Egerváry [7]), it follows that $\mathcal{E}_1$ is bounded below by $1/4$. This result, however, is not relevant, since Korevaar [10] proved in 1949 (see also [4, Theorem 6.7.17]) that $\mathcal{E}_p \ge 1/p$ for $p \ge 1$. In particular, $\mathcal{E}_1 \ge 1$.

Remark 3.9. Some of the previous statements for $p = 2$ can be formulated for general measures, but this is out of the scope of this paper. Here we only give a result for the Jacobi measure in the interval $(-1,1)$ for the case $p = 2$. If $\alpha,\beta > -1$, and f is a function in $L^2((-1,1),(1-x)^\alpha(1+x)^\beta\,dx)$

such that $\int_{-1}^{1} Q(x)f(x)(1-x)^{\alpha}(1+x)^{\beta}\,dx = Q(1)$, for every $Q \in \Pi_n$, then

$$\int_{-1}^{1} |f(x)|^2(1-x)^{\alpha}(1+x)^{\beta}\,dx \geq 2^{-\alpha-\beta-1}\frac{\Gamma(n+\alpha+\beta+2)\Gamma(n+\alpha+2)}{\Gamma(\alpha+1)\Gamma(\alpha+2)\Gamma(n+1)\Gamma(n+\beta+1)}.$$

This follows immediately from Bessel's inequality and [24, formula (4.5.8)].

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