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An embedding theorem for proper n-types

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Abstract

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This paper is centred around an embedding theorem for the proper n-homotopy category at infinity of σ-compact simplicial complexes into the n-homotopy category of prospaces. There is a corresponding global version. This enables one to prove proper analogues of various classical results of Whitehead on n-types, J_n-complexes, etc.

Keywords: n-type, proper n-type, Edwards-Hastings embedding.

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Introduction

In his address to the 1950 Proceedings of the International Congress of Mathematicians, Whitehead described the program on which he has been working for some years [31]. He mentioned in particular the realization problem: that of deciding if a given set of homomorphisms defined on homotopy groups of complexes, K and K', have a geometric realization, $K \to K'$. His attempt to study this problem involved examining algebraic structures, containing more information than the homotopy groups, so that homomorphisms that preserved the extra structure would be realizable. In particular he divided up the problem of finding such models into simpler steps involving n-types as introduced earlier by Fox [13, 14] and gave algebraic models for 2-types in a paper with MacLane [24]. (*Warning*: their 3-types are our 2-types—see later.)

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The notion of n-type is given a clear geometric meaning and it is easy to see how to generalise this definition to the proper homotopy context of noncompact complexes and proper maps as has been done by Geoghegan [15]. The rest of Whitehead's program is less immediately easy to implement in the proper context as it involves algebraic models, etc. In this paper we show how to modify the Edwards–Hastings embedding theorem to embed the proper n-homotopy category (at infinity) into an n-homotopy procategory in which the "pro-analogues" of algebraic models of n-types can be applied more easily. Here we will apply this to questions involving the proper forms of Whitehead's theorems, proper analogues of J_n-complexes, to the classification of proper 2-types and an algebraic characterization of the pointed proper type of $\mathbb{R}^3$ among the connected open 3-manifolds that have only one Freudenthal end. As Whitehead's theory uses the cellular approximation theorem many times, we will restrict our attention to σ-compact simplicial complexes as in that setting there is a simplicial approximation theorem. In the larger category of σ-compact CW-complexes, no corresponding result holds.

1. The classical theory of n-types

The usual numbering of the approximating n-types has changed since Whitehead published his work, thus as was mentioned above the MacLane–Whitehead results on 3-types, [24] would now be phrased in terms of 2-types. Because of this, we have collected here a consistent set of definitions and results on the continuous as opposed to the proper case. The best sources for much of this material are Whitehead's original papers (see the reference list) and Hilton's monograph [23]. These sources use the old numbering system so their n-type would be called $(n-1)$-type here. We will phrase our version of these results either for CW-complexes or for simplicial sets.

Definition. If $f, g: X \to Y$ are two maps, we say f is *n-homotopic to g* (written $f \simeq_n g$) if for every map ϕ of an arbitrary CW-complex P, of dimension $\leq n$, into X, $f\phi$ is homotopic to $g\phi$.

Two CW-complexes X and Y are said to be of the same n-type if there are maps (assumed to be cellular),

$$\phi: X^{n+1} \to Y^{n+1},$$

$$\phi': Y^{n+1} \to X^{n+1},$$

such that $\phi'\phi \simeq_n 1$, $\phi\phi' \simeq_n 1$. We write $\phi: X^{n+1} \equiv_n Y^{n+1}$.

The inclusion of X^{n+1} into X gives $X^{n+1} \equiv_n X$.

Since there is a cellular approximation theorem, the n-type is a homotopy invariant.

- If X and Y are connected, then $\phi : X^{n+1} \equiv_n Y^{n+1}$ if and only if for $r = 1, \ldots, n$, $\phi_r : \pi_r(X) \to \pi_r(Y)$ is an isomorphism [27].

Whitehead's proof uses the long exact sequence of the mapping cylinder of ϕ and its "subcomplex" X.

A map $f: X \to Y$ will be called an n-equivalence if the homotopy fibre of f is n-connected. From the long exact sequence of the associated fibration, f is an n-equivalence if and only if it induces an isomorphism between $\pi_r(X)$ and $\pi_r(Y)$ for $r \leq n$ and an epimorphism for $r = n+1$.

In Spanier [26], we note (with shift of numbering):

- If $f: X \to Y$ is an n-equivalence, then for any CW-complex P, the induced map

$$f_{\#} : [P, X] \to [P, Y]$$

is surjective if $\dim P \leq n+1$ and injective if $\dim P \leq n$. In particular, if f is an n-equivalence, taking $P = Y^{n+1}$, we obtain a map $g: Y^{n+1} \to X$ such that $fg \simeq 1 : Y^{n+1} \to Y$, whilst $gf \simeq_n 1$, so X and Y have the same n-type.

We shall use the term weak n-equivalence for a map $f: X \to Y$ that induces isomorphisms on π_r for $r \leq n$ (no condition imposed on $f_{n+1} : \pi_{n+1}(X) \to \pi_{n+1}(Y)$). We showed in [22] how with this notion of weak n-equivalence, and with suitable notions of n-fibration and n-cofibration, one obtains a Quillen model category structure on the category of CW-complexes, simplicial sets, etc. The categories obtained by formally inverting the weak n-equivalences will be denoted $\mathrm{Ho}_n(\mathrm{CW})$, $\mathrm{Ho}_n(\mathrm{SS})$, etc.

For Baues [2, p. 364], the category **n-types** denotes the full subcategory of $\mathrm{Ho}(\mathrm{Top}_*)$ consisting of CW-spaces Y with $\pi_i Y = 0$ for $i > n$. The n-type of a CW-complex X is $P_n X$, the nth term in the Postnikov decomposition of X, obtained from X by killing homotopy groups. The nth Postnikov section $p_n : X \to P_n X$ is then an n-equivalence as its homotopy fibre is n-connected. This functor P_n with the natural transformation p_n provides a natural equivalence

$$\mathrm{Ho}_n(\mathrm{CW}) \xrightarrow{\simeq} \textbf{\textit{n}-types}.$$

A similar and even more elegant formulation occurs in the theory of simplicial sets where P_n is given by the coskeleton functor cosk_{n+1}. This theory will be briefly recalled shortly as we will need one or two lemmas that are "folklore" but do not seem to be in the literature. The basic reference is Artin and Mazur's lecture notes [1].

In the Postnikov formulation, two spaces X and Y have the same n-type if and only if $P_n X$ and $P_n Y$ have the same homotopy type. These results show that if one forms a category from CW by inverting the n-equivalences one gets a category isomorphic to $\mathrm{Ho}_n(\mathrm{CW})$. This may seem strange but as $\mathrm{Ho}_n(\mathrm{CW})$ and n-types are equivalent and p_n is naturally a (strong) n-equivalence, it is quite easy to check. Any weak n-equivalence $f: X \to Y$ can be factored

$$[f] = [p_n(Y)]^{-1}[P_n(f) p_n(X)]$$

where $P_n(f)$ is a homotopy equivalence, within the category $\mathrm{CW}(\Sigma^{-1})$ where Σ is the class of n-equivalences.

These results give a modern interpretation of Whitehead's theory of n-types and it is in this formulation that we will prove analogues in the proper context.

J_n-complexes. Whitehead's algebraic model for the 2-type of a complex involves crossed modules (cf. Hilton [23, p. 39]). These he generalised to what he called "homotopy systems" [28] which are special cases of the "crossed complexes" of Brown and Higgins ([7], [8], etc., see also [22]) and the "crossed chain complexes" of Baues [2]. These crossed complexes were used as an intermediate stage in the realization problem. For a reduced CW-complex X, that is one with only one 0-cell, the associated crossed complex consists of the relative homotopy groups $\pi_n(X^n, X^{n-1})$ for $n>1$, the group $\pi_1(X^1)$ and the boundary maps, actions, etc. between them. The class of J-complexes gives a class of complexes whose homotopy types are completely determined by their associated crossed complexes, similarly for the J_m-complexes, the crossed complex is an algebraic model for the m-type.

Definition. Given a CW-complex, X, we will say X is a *J_m-complex* (or X is J_m) if $\mathrm{im}(\pi_q(X^{q-1}) \to \pi_q(X^q))$ is zero for $q=2,\ldots,m$.

If X is J_m then clearly it is J_l for $l\leq m$. If X is J_m for all m then we say X is a J-complex.

- For any CW-complex X, let $\Pi_n=\pi_n(X)$, $H_n=H_n(\tilde{X})$, $\tilde{X}$ the universal cover of X, and $\Gamma_n=\mathrm{im}(\pi_n(X^{n-1})\to\pi_n(X^n))$ for $n\geq 2$, then there is a "certain exact sequence"

$$\cdots\to H_{n+1}\to\Gamma_n\to\Pi_n\xrightarrow{h_n}H_n\to\Gamma_{n-1}\to\cdots$$

where the homomorphism h_n is the Hurewicz homomorphism of the universal cover, thus if X is simply connected h_n *is* the usual Hurewicz homomorphism [30]. Note also (cf. [29]), X is J_m if and only if $\Gamma_q(X)=0$ for $q\leq m$. This implies:

- If X is a simply connected J_m-complex, then the Hurewicz homomorphisms $h_q:\pi_q(X)\to H_q(X)$ are isomorphisms if $q\leq m$ and h_{m+1} is an epimorphism.

If X is a J_{n+1}-complex and Y has the same n-type as X then Y is also J_{n+1} (see Whitehead [27]).

The connection between J_m-complexes and n-type is quite complex. Brown and Higgins [9] prove that if X is any CW-complex then there is a natural mapping $X\to B\pi X$ where πX is the crossed complex of X and B is the classifying space functor. They prove that the homotopy fibration sequence of this natural map is Whitehead's "certain exact sequence" and hence that X is a J_m-complex if and only if $X\to B\pi X$ is an m-equivalence.

In [22] the authors introduced a weakened version of the J_m condition that they denote $\mathcal{J}_m$. A CW-complex X is $\mathcal{J}_m$ if $X\to B\pi X$ is a *weak* m-equivalence. The corresponding condition on homotopy groups is that

$$\operatorname{im}(\pi_q(X^{q-1}) \to \pi_q(X^q)) \text{ is trivial for } q \leq n-1,$$

and

$$\operatorname{im}(\pi_n(X^{n-1}) \to \pi_n(X^{n+1})) \text{ is trivial.}$$

The classifying space functor $B: \mathrm{Crs} \to \mathrm{SS}$, where Crs is the category of crossed complexes, is right adjoint to πR where R is the geometric realization functor. This adjointness passes to the corresponding Ho_n-categories. We say a crossed complex Y is $\mathcal{J}_n$ if $\pi RBY \to Y$ is a weak n-equivalence and denote by $\mathrm{Ho}_n(\mathrm{Crs})|\mathcal{J}_n$ the full subcategory determined by $\mathcal{J}_n$-crossed complexes. We proved in [22] that

$$\pi R : \mathrm{Ho}_n(\mathrm{SS})|\mathcal{J}_n \to \mathrm{Ho}_n(\mathrm{Crs})|\mathcal{J}_n$$

is an equivalence of categories so if X, Y are simplicial sets, with Y, $\mathcal{J}_n$, and $\varphi: \pi RX \to \pi RY$ is a map of crossed complexes, then within $\mathrm{Ho}_n(\mathrm{SS})(X, Y)$ there is a unique map that is sent to φ by πR. This is a version of Whitehead's realization theorem 4 of [28].

Other results of Whitehead's involve realizability of chain maps on universal covers. Again the codomain complex is assumed to be J_m. We will not be handling this aspect here.

The above selection of results from the theory of n-types has been chosen to suggest the tools that are useful in this area, notably a Postnikov or coskeleton construction, a condition of J_m- or $\mathcal{J}_m$-type, the "certain exact sequence" and some elements of the theory of crossed complexes. Our aim will be to extend and adapt these ideas to the proper homotopy context. In particular note that Whitehead's realizability result interprets as an equivalence of categories or rather a result on hom-sets in two n-homotopy categories; in the proper homotopy context we will be working with embeddings into procategories rather than equivalences.

2. Terminology and notation

2.1. Categories of spaces and proper maps

Basic terminology will be derived from that of Edwards and Hastings [11]. As we sometimes need to use proper simplicial or cellular approximations, we restrict to the category SC_σ of σ-compact locally compact simplicial complexes and proper maps, the basic rayed version will be denoted $(\mathrm{SC}_\sigma)_*$ and the versions of these "at infinity", i.e., using only germs of proper maps "at infinity", by $(\mathrm{SC}_\sigma)_\infty$ and $((\mathrm{SC}_\sigma)_*)_\infty$ respectively. The corresponding proper homotopy categories are $\mathrm{Ho}(\mathrm{SC}_\sigma)$, $\mathrm{Ho}((\mathrm{SC}_\sigma)_*)_\infty$, etc. The Edwards–Hastings end functor $\varepsilon: (\mathrm{SC}_\sigma)_\infty \to \text{pro Top}$, and its variant $(\varepsilon, 1): \mathrm{SC}_\sigma \to (\text{pro Top}, \text{Top})$ provide embeddings (cf. [12]) $\mathrm{Ho}((\mathrm{SC}_\sigma)_\infty) \to \mathrm{Ho}(\text{pro Top})$ and $\mathrm{Ho}(\mathrm{SC}_\sigma) \to \mathrm{Ho}(\text{pro Top}, \text{Top})$. Similarly there are pointed/base rayed versions of these results.

2.2. Simplicial sets

We have already used the notation SS for the category of simplicial sets which we consider with the usual Quillen model category structure. We will also need the theory of skeleta and coskeleta as outlined in Artin and Mazur [1]. The category Δ consists of the ordered sets $[n]=\{0<1<\cdots<n\}$ with monotone maps between them so, as usual, a simplicial set is a functor $K:\Delta^{\mathrm{op}}\to \mathrm{Sets}$. Let $\Delta_{\leqslant k}$ denote the full subcategory of Δ determined by the objects $[q]$ for $q\leqslant k$ and let $\mathrm{SS}_{\leqslant k}$ denote the corresponding category of k-truncated simplicial sets, i.e., functors from $\Delta^{\mathrm{op}}_{\leqslant k}$ to Sets. The restriction functor

$$*/k:\mathrm{SS}\to \mathrm{SS}_{\leqslant k}$$

has, by the theory of Kan extensions, both left and right adjoints. We denote these by sk_k and cosk_k respectively, following Artin and Mazur [1]. This same notation is used for the composite functors defined on SS namely

$$(\mathrm{sk}_k:\mathrm{SS}\to \mathrm{SS})=\mathrm{sk}_k\circ */k,$$

$$(\mathrm{cosk}_k:\mathrm{SS}\to \mathrm{SS})=\mathrm{cosk}_k\circ */k.$$

The possibility of confusion is slight, as which functor is being used should be clear from the context. It is important to note that $*/k\circ \mathrm{sk}_k=\mathrm{identity}=*/k\circ \mathrm{cosk}_k$, which means that both sk_k and cosk_k are idempotent functors and that $\mathrm{sk}_k\mathrm{cosk}_k=\mathrm{sk}_k$, etc. We will use these identities freely later on. It is also easily checked that if $K, L\in \mathrm{SS}$,

$$\mathrm{SS}(\mathrm{sk}_kK, L)\cong \mathrm{SS}_{\leqslant k}(K/k, L/k)\cong \mathrm{SS}(K, \mathrm{cosk}_kL)$$

so sk_k is left adjoint to cosk_k.

We note that $\pi_i(\mathrm{cosk}_{n+1}K)=0$ if $i\geqslant n+1$, and that there is a canonical map $\eta(K):K\to \mathrm{cosk}_{n+1}K$ which is universal with respect to maps to objects, L, with $\pi_i(L)=0$ for $i\geqslant n+1$. This map is an n-equivalence as was mentioned earlier.

2.3. The singular complex and its relatives

Given a space X, SX will denote the singular simplicial set of X. The geometric realization functor, R, is left adjoint to S.

We set $(sX)_n=\{\alpha:\Delta^n\to X \mid \alpha\in SX,\ \alpha \text{ is cellular}\}$, where Δ^n is given its usual cell structure. The inclusion $k:sX\to SX$ is a homotopy equivalence and is natural provided we restrict to cellular maps. A simplicial variant $\underline{s}(X)$ of this can be defined if X is a simplicial complex. We note that $\underline{s}(X^n)=\mathrm{sk}_n\underline{s}(X)$.

2.4. The category, Crs, *of crossed complexes*

We have already mentioned the origins of this category in the work of Whitehead. The modern treatment is contained in a series of articles by Brown and Higgins, in particular [7, 8, 10]. The abstract homotopy theory of Crs can be found in Brown and Golasiński [6] and its n-homotopy theory in our own [22].

The classifying space functor $B: \mathrm{Crs} \to \mathrm{SS}$ is left adjoint to πR. This was introduced by Brown and Higgins in [9] in a cubical version and by us in [22] in a simplicial one. We also note the extension of all these structures to the procategories pro Crs and (pro Crs, Crs).

3. Proper n-homotopy

Suppose $f, g: X \to Y$ are two morphisms in SC_σ (respectively $(\mathrm{SC}_\sigma)_\infty$). We will say that f is properly n-homotopic to g (at infinity) written $f \simeq_n g$, if for every $\varphi: K \to X$ in SC_σ (respectively in $(\mathrm{SC}_\sigma)_\infty$) where K has dimension $\leq n$, $f\varphi$ and $g\varphi$ are equal in $\mathrm{Ho}(\mathrm{SC}_\sigma)$ (respectively in $\mathrm{Ho}((\mathrm{SC}_\sigma)_\infty)$).

As there are simplicial/cellular approximation theorems in both SC_σ and $(\mathrm{SC}_\sigma)_\infty$, it is easy to prove the following:

Proposition 3.1. *Let $f, g: X \to Y$ in SC_σ (respectively $(\mathrm{SC}_\sigma)_\infty$), then $f \simeq_n g$ if and only if $f|X^n = g|X^n$ in $\mathrm{Ho}(\mathrm{SC}_\sigma)$ (respectively $\mathrm{Ho}((\mathrm{SC}_\sigma)_\infty)$).*

The proper n-homotopy category, $\mathrm{Ho}_n(\mathrm{SC}_\sigma)$ (respectively at infinity, $\mathrm{Ho}_n((\mathrm{SC}_\sigma)_\infty)$) is defined to have as objects σ-compact simplicial complexes and as morphisms $\alpha: X \to Y$, the proper n-homotopy classes of proper maps (respectively germs of proper maps at infinity) $f: X^{n+1} \to Y^{n+1}$.

Remarks. (i) From now on we will not always repeat definitions and results in both global and "at the end" versions. There are also pointed/base rayed versions of the embedding theorems.

(ii) As all our spaces are σ-compact, the end of a space in SC_σ can be considered to be a tower of spaces. Replacing each space in the tower by its singular simplicial set takes us to tow SS, the category of towers in SS.

Theorem 3.2. *Let $f, g: X \to Y$ in $(\mathrm{SC}_\sigma)_\infty$, then $f \simeq_n g$ if and only if $\mathrm{cosk}_{n+1} S\varepsilon f = \mathrm{cosk}_{n+1} S\varepsilon g$ in $\mathrm{Ho}(\mathrm{tow\,SS})$.*

Proof. First some preliminary lemmas whose proofs use standard methods.

Lemma 3.3. *Let B be a Kan complex, then for any diagram*

$$\begin{array}{c}(\Delta[n+1]\times 0) \cup (\mathrm{sk}_n\Delta[n+1]\times\Delta[1]) \cup (\Delta[n+1]\times 1) \xrightarrow{h} B \\ \downarrow \\ \mathrm{sk}_{n+1}(\Delta[n+1]\times\Delta[1])\end{array}$$

h has an extension $h': \mathrm{sk}_{n+1}(\Delta[n+1]\times\Delta[1]) \to B$.

Lemma 3.4. *Let B be a fibrant object in $SS^{\mathbb{N}}$, $\mathbb{N}$ the ordered category of natural numbers. Suppose that for each $k \in \mathbb{N}$, $B(k)$ is $(n+1)$-coconnected, i.e., $\pi_i(B(k)) = 0$ if $i \geq n+1$ for any base point. If $f, g: A \to B$ are morphisms in $SS^{\mathbb{N}}$ and $F: \mathrm{sk}_n A \times \Delta[1] \to B$ is a homotopy from $f|\mathrm{sk}_n A$ to $g|\mathrm{sk}_n A$, then F extends to a homotopy from f to g.*

Lemma 3.5. *Let $i: A \to X$ be a morphism in $SS^{\mathbb{N}}$ such that*

(i) *the transition maps of A and X are inclusions*;

(ii) *for every $k \in \mathbb{N}$, $i_k: A_k \to X_k$ is an inclusion*;

(iii) $\dim X \backslash A = n+1$; *and*

(iv) $\lim X_k = \emptyset$.

Suppose that $p: E \to B$ is a fibration in $SS^{\mathbb{N}}$ such that for each $k \in \mathbb{N}$, $p_k: E_k \to B_k$ is an n-equivalence.

Then if the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & E \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[g]{} & B \end{array}$$

is commutative in $SS^{\mathbb{N}}$, there is a filler $h: X \to E$ in $SS^{\mathbb{N}}$.

This third lemma and the following proposition will later be applied to $\underline{s}\varepsilon X$ which clearly satisfies the condition $\lim \underline{s}\varepsilon X = \emptyset$.

These lead to the following.

Proposition 3.6. *Let $X = \{X_i\}$ be an object in* tow SS *such that the transition maps of X are inclusions and $\lim X_i = \emptyset$. Given $f, g: X \to Y$ morphisms in* tow SS, *then $f|\mathrm{sk}_n X = g|\mathrm{sk}_n X$ in* Ho(tow SS) *if and only if $\mathrm{cosk}_{n+1} f = \mathrm{cosk}_{n+1} g$ in* Ho(tow SS).

Proof. Since Y is weakly equivalent to a fibrant object, we may assume that Y is itself fibrant in $SS^{\mathbb{N}}$. We can also assume, by reindexing if necessary (cf. Artin and Mazur [1]), that f, g are morphisms in $SS^{\mathbb{N}}$. As $\mathrm{sk}_n \mathrm{cosk}_{n+1} = \mathrm{sk}_n$, if $f|\mathrm{sk}_n X = g|\mathrm{sk}_n X$ in Ho(tow SS), this is the same as saying $\mathrm{cosk}_{n+1} f|\mathrm{sk}_n \mathrm{cosk}_{n+1} X = \mathrm{cosk}_{n+1} g|\mathrm{sk}_n \mathrm{cosk}_{n+1} X$ in Ho(tow SS). Finally replacing $\mathrm{cosk}_{n+1} Y$ by a weakly equivalent fibrant object, $u: \mathrm{cosk}_{n+1} Y \to B$, the equality

$$u\mathrm{cosk}_{n+1} f|\mathrm{sk}_n \mathrm{cosk}_{n+1} X = \mathrm{cosk}_{n+1} g|\mathrm{sk}_n \mathrm{cosk}_{n+1} X$$

is realisable by a homotopy F between the two morphisms in $SS^{\mathbb{N}}$ and the chosen F can be extended by Lemma 3.4, to get a homotopy between $u\mathrm{cosk}_{n+1} f$ and $u\mathrm{cosk}_{n+1} g$. Since u is a weak equivalence, $\mathrm{cosk}_{n+1} f = \mathrm{cosk}_{n+1} g$.

Conversely suppose that $\operatorname{cosk}_{n+1} f = \operatorname{cosk}_{n+1} g$ in Ho(tow SS) and replace the map $Y \to \operatorname{cosk}_{n+1} Y$ by a fibration of fibrant objects:

$$\begin{array}{ccc} Y & \xrightarrow{v} & E \\ \downarrow & & \downarrow{\scriptstyle p} \\ \operatorname{cosk}_{n+1} Y & \xrightarrow{u} & B \end{array}$$

Thus here B is fibrant in tow SS, p is a fibration and u, v are weak equivalences.

As $u\operatorname{cosk}_{n+1} f = u\operatorname{cosk}_{n+1} g$ in Ho(tow SS) and B is fibrant, there is a homotopy $F: \operatorname{cosk}_{n+1} X \times \Delta[1] \to B$ from $u\operatorname{cosk}_{n+1} f$ to $u\operatorname{cosk}_{n+1} g$. Now the diagram

$$\begin{array}{ccc} X \times 0 \cup X \times 1 & \xrightarrow{vf \cup vg} & E \\ \downarrow & & \downarrow \\ X \times 0 \cup (\operatorname{sk}_n X \times \Delta[1]) \cup X \times 1 & \xrightarrow[pvf \cup \bar{F} \cup pvg]{} & B \end{array}$$

satisfies the conditions of Lemma 3.5, after reindexing if necessary. Here $\bar{F}$ is the restriction of F to $\operatorname{sk}_n X \times \Delta[1]$. By Lemma 3.5, we thus have a homotopy from $vf \mid \operatorname{sk}_n X$ to $vg \mid \operatorname{sk}_n X$.

Since v is a weak equivalence, this implies that

$$f \mid \operatorname{sk}_n X = g \mid \operatorname{sk}_n X$$

as required. □

Proof of Theorem 3.2 (continued). Consider the following diagram for f (and the analogous one for g):

$$\begin{array}{ccc} s\varepsilon X & \xrightarrow{s\varepsilon f} & s\varepsilon Y \\ \downarrow & & \downarrow \\ S\varepsilon X & \xrightarrow{S\varepsilon f} & S\varepsilon Y \end{array}$$

induced by the natural inclusion of s into S. The vertical arrows are weak equivalences and so $\operatorname{cosk}_{n+1} S\varepsilon f = \operatorname{cosk}_{n+1} S\varepsilon g$ if and only if $\operatorname{cosk}_{n+1} s\varepsilon f = \operatorname{cosk}_{n+1} s\varepsilon g$. Now assume that $f \mid X^n = g \mid X^n$ in $\mathrm{Ho}((\mathrm{SC}_\sigma)_\infty)$ then $\varepsilon f \mid \varepsilon X^n = \varepsilon g \mid \varepsilon X^n$ in Ho(tow Top). Applying the functor s, we have that $s(\varepsilon f \mid \varepsilon X^n) = s(\varepsilon g \mid \varepsilon X^n)$ in Ho(tow SS) and as $\operatorname{sk}_n s\varepsilon X^n = \operatorname{sk}_n s\varepsilon X$, we have

$$s(\varepsilon f \mid \varepsilon X^n) \mid \operatorname{sk}_n s(\varepsilon X^n) = s\varepsilon f \mid \operatorname{sk}_n s(\varepsilon X),$$

and similarly for g. Therefore $s\varepsilon f \mid \operatorname{sk}_n s(\varepsilon X) = s\varepsilon g \mid \operatorname{sk}_n s(\varepsilon X)$ and by Proposition 3.6, it follows that $\operatorname{cosk}_{n+1} s\varepsilon f = \operatorname{cosk}_{n+1} s\varepsilon g$ as required.

Conversely assume $\operatorname{cosk}_{n+1} s\varepsilon f = \operatorname{cosk}_{n+1} s\varepsilon g$. Using again Proposition 3.6, it follows that $s\varepsilon f | \operatorname{sk}_n s(\varepsilon X) = s\varepsilon g | \operatorname{sk}_n s(\varepsilon X)$ in Ho(tow SS).

Now the diagram

$$\begin{array}{ccccc} \operatorname{sk}_n \underline{s}(\varepsilon X^n) & \subseteq & \operatorname{sk}_n s(\varepsilon X^n) & = & \operatorname{sk}_n s(\varepsilon X) \\ \| & & \cap| & & \cap \\ \underline{s}(\varepsilon X^n) & \underset{u}{\subseteq} & s(\varepsilon X^n) & \subseteq & s\varepsilon(X) \end{array}$$

commutes, so we also have that

$$s(\varepsilon f | \varepsilon X^n) | \operatorname{sk}_n \underline{s}(\varepsilon X^n) = s(\varepsilon g | \varepsilon X^n) | \operatorname{sk}_n \underline{s}(\varepsilon X^n)$$

and as u is a weak equivalence, $s(\varepsilon f | \varepsilon X^n) = s(\varepsilon g | \varepsilon X^n)$ but this implies that $\varepsilon f | \varepsilon X^n = \varepsilon g | \varepsilon X^n$ in Ho(tow Top). Finally by the Edwards–Hastings embedding theorem, $f | X^n = g | X^n$ as required. □

A similar proof can be given for the global version:

Theorem 3.2′. *Let $f, g: X \to Y$ be morphisms in SC_σ, then f is proper n-homotopic to g if and only if*

$$(\operatorname{cosk}_{n+1} S\varepsilon f \to \operatorname{cosk}_{n+1} Sf) = (\operatorname{cosk}_{n+1} S\varepsilon g \to \operatorname{cosk}_{n+1} Sg)$$

in Ho(tow SS, SS).

Theorem 3.7. *The functor* $\operatorname{cosk}_{n+1} S\varepsilon$ *gives an embedding of* $\mathrm{Ho}_n((\mathrm{SC}_\sigma)_\infty)$ *into* Ho(pro SS).

Proof. Let K be a simplicial complex. If $i: K^{n+1} \to K$ is the inclusion of the $(n+1)$-skeleton of K, then $\operatorname{cosk}_{n+1} i: \operatorname{cosk}_{n+1} SK^{n+1} \to \operatorname{cosk}_{n+1} SK$ is a weak equivalence, so if X is in $(\mathrm{SC}_\sigma)_\infty$, the inclusion of X^{n+1} into X induces a weak equivalence

$$\operatorname{cosk}_{n+1} S\varepsilon i: \operatorname{cosk}_{n+1} S\varepsilon X^{n+1} \to \operatorname{cosk}_{n+1} S\varepsilon X.$$

Given a morphism $\alpha: X \to Y$ in $\mathrm{Ho}_n((\mathrm{SC}_\sigma)_\infty)$ represented by $f: X^{n+1} \to Y^{n+1}$, we set $\operatorname{cosk}_{n+1} S\varepsilon\alpha$ to be the composite $(\operatorname{cosk}_{n+1} S\varepsilon i_Y) \circ (\operatorname{cosk}_{n+1} S\varepsilon f) \circ (\operatorname{cosk}_{n+1} S\varepsilon i_X)^{-1}$ where i_X, i_Y are the corresponding inclusions. This does not depend on the choice of f and gives a functor

$$\operatorname{cosk}_{n+1} S\varepsilon: \mathrm{Ho}_n((\mathrm{SC}_\sigma)_\infty) \to \mathrm{Ho(pro\ SS)}.$$

Now let $\alpha, \beta: X \to Y$ in $\mathrm{Ho}((\mathrm{SC}_\sigma)_\infty)$ be represented by $f, g: X^{n+1} \to Y^{n+1}$. If $\operatorname{cosk}_{n+1} S\varepsilon\alpha = \operatorname{cosk}_{n+1} S\varepsilon\beta$, then $\operatorname{cosk}_{n+1} S\varepsilon f = \operatorname{cosk}_{n+1} S\varepsilon g$ in Ho(pro SS) or in Ho(tow SS) and so by Theorem 3.2, $f | X^n = g | X^n$ in $\mathrm{Ho}_n((\mathrm{SC}_\sigma)_\infty)$ and $\alpha = \beta$, i.e., the functor $\operatorname{cosk}_{n+1} S\varepsilon$ is faithful.

Now let $\varphi:\mathrm{cosk}_{n+1}S\varepsilon Y\to\mathrm{cosk}_{n+1}S\varepsilon Y$ in Ho(tow SS). Noting that $\underline{s}\varepsilon X\to S\varepsilon X$ and $\mathrm{cosk}_{n+1}\underline{s}\varepsilon X\to\mathrm{cosk}_{n+1}S\varepsilon X$ are isomorphisms in Ho(tow SS), we can assume that φ is a morphism of the form $\varphi:\mathrm{cosk}_{n+1}\underline{s}\varepsilon X\to\mathrm{cosk}_{n+1}\underline{s}\varepsilon Y$ and that $\dim X\leqslant n+1$. We thus have a commutative diagram

$$\begin{array}{ccccc} \emptyset & \longrightarrow & & & \underline{s}\varepsilon Y \\ \downarrow & & & & \downarrow{\scriptstyle\eta} \\ \underline{s}\varepsilon X=\mathrm{sk}_{n+1}\mathrm{cosk}_{n+1}\underline{s}\varepsilon X & \xrightarrow[\eta]{} & \mathrm{cosk}_{n+1}\underline{s}\varepsilon X & \xrightarrow[\varphi]{} & \mathrm{cosk}_{n+1}\underline{s}\varepsilon Y \end{array}$$

where the various η denote the natural morphisms. We can replace as before $\eta:\underline{s}\varepsilon Y\to\mathrm{cosk}_{n+1}\underline{s}\varepsilon Y$ by a fibration $p:E\to B$ of fibrant objects. As B is fibrant, the composite morphism from $\underline{s}\varepsilon X$ to B in Ho(tow SS) is represented by an actual morphism in tow SS. After reindexing if necessary we can apply Lemma 3.5 to obtain a filler from $\underline{s}\varepsilon X$ to E hence a morphism $\psi:\underline{s}\varepsilon X\to\underline{s}\varepsilon Y$ in Ho(tow SS) such that $\eta\psi=\varphi\eta$ in Ho(tow SS).

We note that as $\mathrm{sk}_n\mathrm{cosk}_{n+1}=\mathrm{sk}_n$,

$$\begin{aligned}\mathrm{cosk}_{n+1}\psi\,|\,\mathrm{sk}_n\mathrm{cosk}_{n+1}\underline{s}\varepsilon X &= (\mathrm{cosk}_{n+1}\psi)\eta\,|\,\mathrm{sk}_n\underline{s}\varepsilon X=\eta\psi\,|\,\mathrm{sk}_n\underline{s}\varepsilon X\\ &=\varphi\eta\,|\,\mathrm{sk}_n\underline{s}\varepsilon X=\varphi\,|\,\mathrm{sk}_n\mathrm{cosk}_{n+1}\underline{s}\varepsilon X.\end{aligned}$$

We can thus compose both morphisms with a weak equivalence, $u:\mathrm{cosk}_{n+1}\underline{s}\varepsilon Y\to B$ with B fibrant and $(n+1)$-coconnected and as

$$u\mathrm{cosk}_{n+1}\psi\,|\,\mathrm{sk}_n\mathrm{cosk}_{n+1}\underline{s}\varepsilon X=u\varphi\,|\,\mathrm{sk}_n\mathrm{cosk}_{n+1}\underline{s}\varepsilon X$$

in Ho(tow SS), we can find a homotopy between these maps in tow SS, then applying Lemma 3.4, we can find a homotopy between $u\mathrm{cosk}_{n+1}\psi$ and $u\varphi$, thus $\mathrm{cosk}_{n+1}\psi=\varphi$ in Ho(tow SS).

Since $S\varepsilon:\mathrm{Ho}((\mathrm{SC}_\sigma)_\infty)\to\mathrm{Ho}(\text{tow SS})$ is a full embedding, we can find a map $f:X\to Y$ such that $S\varepsilon f=\psi$. Thus $\mathrm{cosk}_{n+1}S\varepsilon f=\varphi$ and consequently $\mathrm{cosk}_{n+1}S\varepsilon$ is a full embedding. □

Corollary. *The functor* $S\varepsilon:\mathrm{Ho}_n((\mathrm{SC}_\sigma)_\infty)\to\mathrm{Ho}_n(\text{pro SS})$ *is a full embedding.*

Proof. This follows from Theorem 3.7 on noting that

$$S\varepsilon X\to\mathrm{cosk}_{n+1}S\varepsilon X$$

is naturally an n-equivalence in pro SS. □

There is, of course, a global version of Theorem 3.7, which we leave the reader to formulate.

4. Applications

4.1. A proper Whitehead theorem for n-types

Theorem 4.1 (at the end). *Let $f: X \to Y$ be a proper map of σ-compact simplicial complexes. Then f induces isomorphisms*

$$\text{pro-}\pi_i \varepsilon X \to \text{pro-}\pi_i \varepsilon Y$$

for $0 \leqslant i \leqslant n$, and for all choices of base ray in X, if and only if f is invertible in $\text{Ho}_n((\text{SC}_\sigma)_\infty)$.

Proof. Suppose f induces isomorphisms as claimed. By Theorem 3.7, it suffices to prove that $\text{cosk}_{n+1} S\varepsilon f$ is invertible in Ho(pro SS), but $\text{cosk}_{n+1} S\varepsilon f$ induces isomorphisms on prohomotopy groups in all dimensions hence is a ♮-isomorphism in the terminology of Artin and Mazur [1]. As $\text{cosk}_m \text{cosk}_{n+1}$ and cosk_{n+1} are the same if $m \geqslant n+1$, $\text{cosk}_{n+1} S\varepsilon f$ is an isomorphism in Ho(pro SS) by Edwards and Hastings' version of the Whitehead theorem in Ho(tow SS) ([12, p. 193], see also [12, Theorem 6.4.5, p. 226]).

The converse is easy. □

Theorem 4.1′ (Global version). *Let (X, α), (Y, β) be base rayed σ-compact simplicial complexes with only one end and let $f: (X, \alpha) \to (Y, \beta)$ be a proper map that preserves base rays. Then f is invertible in* $\text{Ho}_n((\text{SC}_\sigma)_*)$ *if and only if f induces isomorphisms*

$$\pi_i(X, \alpha(0)) \to \pi_i(Y, \beta(0))$$

and

$$\text{pro-}\pi_i(\varepsilon X, \alpha) \to \text{pro-}\pi_i(\varepsilon Y, \beta)$$

for $1 \leqslant i \leqslant n$.

Proof. Again one can reduce this to a statement about $(\text{cosk}_{n+1} S\varepsilon f, \text{cosk}_{n+1} Sf)$ within Ho(pro Top_*Top_*) using the global version of Theorem 3.7. This can then be attacked exactly as above, again using [12, p. 226]. □

4.2. n-equivalences and proper homotopy groups

There is no single, all embracing, definition of proper homotopy groups that will detect all the proper analogues of the geometric features detected by the homotopy groups. The two main definitions of proper homotopy group are the "Čerin" or "Steenrod" definition [11], and the Brown–Grossman definition [5]. There are close relationships between them (cf. [20, 25]), and relationships between the invariants at the end, globally and "compactly". Here we examine their effectiveness for detecting n-equivalences.

Let (X, α) be in $(\text{SC}_\sigma)_*$, or in $((\text{SC}_\sigma)_\infty)_*$. The group of proper homotopy classes $[(S^n \times [0,\infty), * \times [0,\infty)), (X, \alpha)]$ is isomorphic to $\pi_n(\text{holim}\, \varepsilon(X))$, the nth homotopy group of the homotopy limit of $\varepsilon(X)$ (cf. [11, 20, 25]). As X is σ-compact the Bousfield–Kan spectral sequence (cf. [4]) reduces to a short exact sequence, the form of which resembles that of the Milnor sequence, namely:

$$0 \to \lim{}^1 \pi_{n+1}(\varepsilon X) \to \pi_n(\text{holim } \varepsilon X) \to \lim(\pi_n \varepsilon X) \to 0.$$

Thus if $f:(X, \alpha) \to (Y, \beta)$ induces isomorphisms on pro-π_i for $i \leq n$, we immediately have that it induces isomorphisms on the "Čech" limiting groups $\lim \pi_i(\varepsilon\text{-})$, $i \leq n$ and on the "Čerin" or "Steenrod" groups, $\pi_i(\text{holim } \varepsilon\text{-})$, $i \leq n-1$. In general it does not seem possible to obtain an isomorphism at level n on these Čerin groups since they involve the $(n+1)$-dimensional space $S^n \times [0, \infty)$, but if both X and Y are end π_{n+1}-movable (i.e., the progroups pro-$\pi_{n+1}(\varepsilon X)$ and pro-$\pi_{n+1}(\varepsilon Y)$ are movable), then $\pi_n(\text{holim } \varepsilon X) \cong \lim(\pi_n \varepsilon X)$ and similarly for Y. In this case if f is an isomorphism in $\text{Ho}_n((\text{SC}_\sigma)_\infty)$, the corresponding maps on the $\pi_i(\text{holim } \varepsilon\text{-})$ groups for $i \leq n$ are isomorphisms.

The situation with the Brown–Grossman groups is simpler. These groups are defined (at the end) for (X, α) in $((\text{SC}_\sigma)_{\infty,*})$ by

$$\pi_q^\infty(X, \alpha) = \text{Ho}((\text{SC}_\sigma)_{\infty,*})((\underline{S}^q, *), (X, \alpha))$$

where $\underline{S}^q$ is the "infinite line of q-spheres", more precisely $\underline{S}^q$ consists of $[0, \infty)$ with a q-sphere attached at each integer. We denote by Σ^q the prospace with $\bigvee_{l=k} S^q = \Sigma^q(k)$ so $\Sigma^q \cong \varepsilon(\underline{S}^q)$ in Ho(pro Top). These groups were introduced by Brown in [5] and in [16], Grossman gave a reduced power construction I defined on sets, groups or Abelian groups such that $\pi_q^\infty(X, \alpha) = \varinjlim I(\pi_q(\varepsilon X))$. Thus if f induces an isomorphism on the prohomotopy groups pro-π_i, $i \leq n$, it induces an isomorphism on π_i^∞, $i \leq n$. (One can also use Brown's P-construction for this.) Grossman also proved that if $f: G \to H$ is a morphism of prosets (or progroups), it is an isomorphism if and only if $\lim If$ is an isomorphism of sets (or groups). This implies:

Corollary (Brown form of Whitehead theorem for n-types). *Let $f: X \to Y$ be a proper map of σ-compact simplicial complexes. Then f is invertible in* $\text{Ho}_n((\text{SC}_\sigma)_\infty)$ *if and only if f induces isomorphisms*

$$\pi_i^\infty(X, \alpha) \to \pi_i^\infty(Y, f\alpha)$$

for $0 \leq i \leq n$, and all choices of base ray α.

This is closely related to the proper Whitehead theorem proved by Brown [5]. There is a corresponding global version of this corollary using Theorem 4.1′ instead of Theorem 4.1.

Corollary. *Let $f:(X, \alpha) \to (Y, \beta)$ be a proper map of based rayed σ-compact simplicial complexes. Then f is invertible in* $\text{Ho}_n((\text{SC}_\sigma)_*)$ *if and only if f induces isomorphisms*

$$\pi_i^G(X, \alpha) \to \pi_i^G(Y, \beta)$$

and $\pi_i(X, \alpha(0)) \to \pi_i(Y, \beta(0))$ for all $0 \leq i \leq n$.

Proof. First the global version of the Brown–Grossman groups, π_q^G is defined by $\pi_q^G(X, \alpha) = \mathrm{Ho}((\mathrm{SC}_\sigma)_*)((\underline{S}^q, *), (X, \alpha))$ (cf. [21]). The global to "local at infinity" natural epimorphism introduced in [21] has kernel naturally isomorphic to $\bigoplus \pi_q(X, \alpha(0))$. Thus the conditions on π_i^G and π_i imply the corresponding condition at infinity. The result now follows by the same argument as before from Theorem 4.1′. □

4.3. Realization theorems for $\mathscr{J}_n$-spaces

In Section 1, we summarised the theory of J_m-complexes and the weaker $\mathscr{J}_n$-complexes. The reason for the introduction of the $\mathscr{J}_n$-condition was that it is easier to work with when considering weak n-equivalences. We will say X in SC_σ is J_n (respectively $\mathscr{J}_n$) at infinity if εX is a J_n-prospace (respectively $\mathscr{J}_n$-prospace), i.e., the natural map $S\varepsilon X \to B\pi RS\varepsilon X$ is an n-equivalence (respectively weak n-equivalence).

Given any X in SC_σ and a choice of base ray α in X, the pro-map $S\varepsilon X \to B\pi RS\varepsilon X$ yields functorially a "certain long exact sequence" of progroups

$$\cdots \to \text{pro-}H_{n+1}(\varepsilon\tilde{X}) \to \text{pro-}\Gamma_n(\varepsilon X) \to \text{pro-}\pi_n(\varepsilon X) \xrightarrow{h_n} \text{pro-}H_n(\varepsilon\tilde{X}) \to \cdots$$

where h_n is the Hurewicz homomorphism, where we have written εX for $S\varepsilon X$. If X is J_n at infinity then pro-$\Gamma_q(\varepsilon X) \cong 0$ for $q \leq n$ and conversely. If X is $\mathscr{J}_n$ at infinity, pro-$\Gamma_q(\varepsilon X) \cong 0$ for $q \leq n-1$ and the natural map from pro-$\Gamma_n(\varepsilon X)$ to pro-$\pi_n(\varepsilon X)$ is zero. As $\Gamma_n(X)$ can be identified with $\mathrm{im}(\pi_n(X^{n-1}) \to \pi_n(X^n))$, for X a CW-complex, this gives equivalent formulations of both J_n-space and $\mathscr{J}_n$-space:

Proposition 4.2. (a) *A space X in SC_σ with α a base ray in X, is a J_n-space at infinity if and only if*

$$\mathrm{im}(\text{pro-}\pi_q(X^{q-1}, \alpha) \to \text{pro-}\pi_q(X^q, \alpha)) \cong 0 \quad \textit{for } q \leq n.$$

(b) *A space X in SC_σ with α a base ray in X is a $\mathscr{J}_n$-space at infinity if and only if*

$$\mathrm{im}(\text{pro-}\pi_q(X^{q-1}, \alpha) \to \text{pro-}\pi_q(X^q, \alpha)) \cong 0 \quad \textit{for } q \leq n-1$$

and

$$\mathrm{im}(\text{pro-}\pi_n(X^{n-1}, \alpha) \to \text{pro-}\pi_n(X^{n+1}, \alpha)) \cong 0.$$

Proof. The proofs of (a) and (b) are similar. We identify $\mathrm{sk}_q \underline{s}\varepsilon X$ with $\underline{s}\varepsilon X^q$ and using the weak equivalence between $\underline{s}\varepsilon X$ and $S\varepsilon X$ for any X, we have that pro-$\Gamma_q(X)$ is defined up to isomorphism, by

$$\text{pro-}\Gamma_q(X) \cong \mathrm{im}(\text{pro-}\pi_q(\mathrm{sk}_{q-1}\underline{s}\varepsilon X) \to \text{pro-}\pi_q(\mathrm{sk}_q \underline{s}\varepsilon X)).$$

The result follows. □

Before we prove proper analogues of Whitehead's realization theorems for $\mathscr{J}_n$-spaces, let us note the following immediate implication of the above definitions.

Proposition 4.3. *If X is a J_n-space or a $\mathscr{J}_n$-space at infinity, then the Hurewicz homomorphism*

$$h_q : \text{pro-}\pi_q(\varepsilon X) \to \text{pro-}H_q(\varepsilon \tilde{X})$$

is an isomorphism for all $q \leq n$ and if X is J_n, h_{n+1} is an epimorphism.

Here pro-$H_q(\varepsilon\tilde{X})$ is interpreted as the cellular homology of a universal cover of each $\mathrm{cl}(X-C)$, C compact. Problems of functoriality are avoided by its actual definition as pro-$\pi_q B\pi Rs\varepsilon X$. If εX is cofinally simply connected then this is simply the prohomology of εX.

If X is both $\mathscr{J}_n$ at infinity and is an $\mathscr{J}_n$-space, we will say X is globally $\mathscr{J}_n$.

In the following, $\mathrm{Ho}((\mathrm{SC}_\sigma)_\infty)|(\mathscr{J}_n$ at infinity) denotes the full subcategory of $\mathrm{Ho}((\mathrm{SC}_\sigma)_\infty)$ determined by the spaces which are $\mathscr{J}_n$ at infinity. Similarly $\mathrm{Ho}_n(\mathrm{SC}_\sigma)|(\text{global } \mathscr{J}_n)$ is the restriction of $\mathrm{Ho}_n(\mathrm{SC}_\sigma)$ to those objects that are globally $\mathscr{J}_n$.

Proposition 4.4. *The functors*

$$\pi\varepsilon : \mathrm{Ho}_n((\mathrm{SC}_\sigma)_\infty)|(\mathscr{J}_n \text{ at infinity}) \to \mathrm{Ho}_n(\text{tow Crs}),$$

$$\pi\varepsilon : \mathrm{Ho}_n(\mathrm{SC}_\sigma)|(\text{global } \mathscr{J}_n) \to \mathrm{Ho}_n(\text{tow Crs}, \text{Crs})$$

are full embeddings.

Proof. This is a corollary of Theorems 4.1 and 4.1′ given the embedding results for pro-$\mathscr{J}_n$-complexes in [22]. □

Theorem 4.5 (realization). *Let X, Y be σ-compact simplicial complexes and suppose that Y is $\mathscr{J}_n$ at infinity, then*

$$\mathrm{Ho}_n((\mathrm{SC}_\sigma)_\infty)(X, Y) \cong \mathrm{Ho}_n(\text{tow Crs})(\pi RS\varepsilon X, \pi RS\varepsilon Y).$$

Proof. Using the model category structure given in [22], we can assume that $\pi\varepsilon Y$ is n-fibrant so any map in Ho_n(tow Crs) from $\pi\varepsilon X$ to $\pi\varepsilon Y$ is realizable by a map from $\pi RS\varepsilon X$ to $\pi RS\varepsilon Y$ in tow Crs. Using adjointness of π and B, this corresponds to a map in tow Top from $RS\varepsilon X$ to $B\pi RS\varepsilon Y$. As this latter tower is assumed to be linked to Y by an n-equivalence we check that

$$\begin{aligned}\mathrm{Ho}_n((\mathrm{SC}_\sigma)_\sigma)(X, Y) &\cong \mathrm{Ho}_n(\text{tow Top})(RS\varepsilon X, RS\varepsilon Y)\\ &\cong \mathrm{Ho}_n(\text{tow Crs})(\pi RS\varepsilon X, \pi RS\varepsilon Y). \qquad \square\end{aligned}$$

Remark. As Baues notes in [3], realization theorems are closely related to properties of functors and are similar to categorical embedding results. Here the result interprets as saying that any map between the towers of crossed complexes can be replaced, up to n-homotopy by a proper map between X^{n+1} and Y, i.e., it is a realization result.

As in [22], we denote by $\mathrm{Crs}|(\dim \leqslant n)$ the full subcategory of Crs determined by crossed complexes of dimension $\leqslant n$. Thus $\mathrm{Crs}|(\dim \leqslant 2)$ is the category of crossed modules. In [22] we proved that the category $\mathrm{Ho}_n(\mathrm{tow\,Crs})$ was equivalent to $\mathrm{Ho}(\mathrm{tow\,Crs}|(\dim \leqslant n))$ via a truncation functor, tr_n. We single out the case $n = 2$ of Proposition 4.4 for further comment.

Theorem 4.6. *The functors*

$$\mathrm{tr}_2\pi\varepsilon : \mathrm{Ho}_2((\mathrm{SC}_\sigma)_\infty) \to \mathrm{Ho}(\mathrm{tow\,Crs}|(\dim \leqslant 2)),$$

$$\mathrm{tr}_2\pi\varepsilon : \mathrm{Ho}_2((\mathrm{SC}_\sigma)) \to \mathrm{Ho}((\mathrm{tow\,Crs}, \mathrm{Crs})|(\dim \leqslant 2))$$

are full embeddings.

There are pointed versions of all these results. This result encorporates a version of the classification theorem of MacLane and Whitehead [24]. If X, Y are in $(\mathrm{SC}_\sigma)_\infty$ we will assume for simplicity they both have only one Freudenthal end. Consider the following objects in (tow Gps, Gps):

$$\underline{\pi}_1(X) = (\mathrm{tow}\ \pi_1, \pi_1)(\varepsilon X \to X) = \underline{G},$$

$$\underline{\pi}_2(X) = (\mathrm{tow}\ \pi_2, \pi_2)(\varepsilon X \to X) = \underline{H}.$$

(This also sets up the notation for future use.)

Assume $\underline{\pi}_1(Y) \cong \underline{G}$ and $\underline{\pi}_2(Y) \cong \underline{H}$ as $\underline{G}$-modules. Then X and Y have the same proper 2-type if and only if $\underline{\pi}_2(X, X^1) \to \underline{\pi}_1(X^1)$ and $\underline{\pi}_2(Y, Y^1) \to \underline{\pi}_1(Y^1)$ are isomorphic in $\mathrm{Ho}((\mathrm{tow\,Crs}, \mathrm{Crs})|(\dim \leqslant 2))$. Presumably those isomorphism classes for fixed $\underline{G}$, $\underline{H}$ form the cohomology group $H^3(\underline{G}, \underline{H})$ but we have not checked that this is so.

4.4. Open 3-manifolds of the proper homotopy type of $\mathbb{R}^3$

Using the same notation as above, we have $\underline{\pi}_1(\mathbb{R}^3) \cong (0 \to 0) = \underline{0}$, $\underline{\pi}_2(\mathbb{R}^3) \cong (\mathbb{Z} \to 0)$ and $\underline{\pi}_3(\mathbb{R}^3) \cong (\mathbb{Z} \to 0)$. Since $\varepsilon\mathbb{R}^3 \cong (S^2 \to *)$, the corresponding object of (tow Crs, Crs), $\underline{\pi}_2(\mathbb{R}^3, (\mathbb{R}^3)^1) \to \underline{\pi}_1((\mathbb{R}^3)^1)$ is isomorphic in $\mathrm{Ho}((\mathrm{tow\,Crs}, \mathrm{Crs})|(\dim \leqslant 2))$ to $\underline{\pi}_2(\mathbb{R}^3) \to \underline{0}$. In the following theorem we characterise using this sort of data, those open 3-manifolds that have the pointed proper homotopy type of $\mathbb{R}^3$.

Theorem 4.7. *Let M be an open 3-manifold with one Freudenthal end. Then M has the pointed proper homotopy type of $\mathbb{R}^3$ if and only if $\underline{\pi}_1(M) \cong (0 \to 0)$, $\underline{\pi}_2(M) \cong (\mathbb{Z} \to 0)$ and $\mathscr{H}^3(M; \underline{\pi}_3(M)) = 0$.*

Here $\mathscr{H}^3(M; \underline{\pi}_3(M))$ is the cohomology with (tow groups, groups)-coefficients introduced and studied in [17–19].

Proof. Because $\pi_1(M) \cong \underline{0}$, it follows that the "crossed module" $\pi_2(M, M^1) \to \pi_1(M^1)$ is onto $\pi_1(M^1)$. Consider the obvious map:

$$\begin{array}{ccccccc} & \nearrow & \pi_2(M, M^1) & \longrightarrow & \pi_1(M^1) & \searrow & \\ \pi_2(M) & & \uparrow & & \uparrow & & \pi_1(M)=0 \\ & \searrow & \pi_2(M) & \longrightarrow & \underline{0} & \nearrow & \end{array}$$

in ((tow Crs, Crs) | dim ≤ 2). As it induces an isomorphism on the two homotopy groups π_1 and π_2, it is a weak 2-equivalence, i.e., an isomorphism in Ho_2(tow Crs, Crs) or Ho((tow Crs, Crs) | dim ≤ 2). By Theorem 4.6, this implies that M has the same proper 2-type as $\mathbb{R}^3$ and thus that there exist proper maps $f : \mathbb{R}^3 \to M$, $g : M \to \mathbb{R}^3$ so that the two composites are properly 2-homotopic to the respective identities. It therefore remains to see if they are properly 3-homotopic to these identities as both spaces have dimension 3. The obstructions to being properly 3-homotopic give elements of $\mathcal{H}^3(\mathbb{R}^3; \pi_3(\mathbb{R}^3))$ and $\mathcal{H}^3(M; \pi_3, (M))$, see [17–19]. We have $\mathcal{H}^3(\mathbb{R}^3; \pi_3(\mathbb{R}^3)) \cong \mathcal{H}^3(S^2 \to *; \pi_3(\mathbb{R}^3)) = 0$ and we are given $\mathcal{H}^3(M; \pi_3(M)) = 0$, thus these obstructions must be trivial and both gf and fg are properly homotopic to the respective identities. The converse implication is easy. □

4.5. Computations of pointed proper homotopy classes

Proposition 4.8. *Let X and Y be σ-compact simplicial complexes with one Freudenthal end. Further suppose that X has finite dimension and the Y is a proper* 1*-type, i.e., $\pi_k(Y) \cong 0$ for $k \geq 2$. Then*

$$\mathrm{Ho}_n((\mathrm{SC}_\sigma)_*)(X, Y) \cong (\text{tow Gps, Gps})(\pi_1(X), \pi_1(Y)).$$

Proof. Let $f, g : X \to Y$ be morphisms in $(\mathrm{SC}_\sigma)_*$. If f is properly k-homotopic to g, the obstruction to it being $(k+1)$-homotopic is given by an element of $\mathcal{H}^{k+1}(X; \pi_{k+1}(Y))$ (cf. [17–19]) but this is trivial since $\pi_q(Y) \cong 0$ for $q \geq 2$, hence

$$\mathrm{Ho}((\mathrm{SC}_\sigma)_*)(X, Y) \cong \mathrm{Ho}_1((\mathrm{SC}_\sigma)_*)(X, Y).$$

Both X and Y are global $\mathcal{J}_1$-spaces, so

$$\mathrm{Ho}_1((\mathrm{SC}_\sigma)_*)(X, Y) \cong \mathrm{Ho}_1(\text{tow Crs}_*, \mathrm{Crs}_*)(\pi\underline{\varepsilon}X, \pi\underline{\varepsilon}Y)$$

where for convenience we have written $\underline{\varepsilon}X$ for $(\varepsilon X \to X)$. Since X and Y have only one Freudenthal end, we can replace $\pi\underline{\varepsilon}X$, $\pi\underline{\varepsilon}Y$ by reduced subcomplexes (cf. [22]) at each level and on using the fact that the 1-truncation map is a natural weak 1-equivalence, we conclude

$$\begin{aligned} \mathrm{Ho}_1(\text{tow Crs}_*, \mathrm{Crs}_*)(\pi\underline{\varepsilon}X, \pi\underline{\varepsilon}Y) &\cong \mathrm{Ho}_1(\text{tow Crs}_*, \mathrm{Crs}_*)(\mathrm{tr}_1\pi\underline{\varepsilon}X, \mathrm{tr}_1\pi\underline{\varepsilon}Y) \\ &\cong (\text{tow Gps, Gps})(\pi_1 X, \pi_1 Y) \end{aligned}$$

as claimed. □

Note that any connected open surface Y with one Freudenthal end satisfies the conditions of the above proposition.

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