

Iterated integrals and Borwein-Chen-Dilcher polynomials

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Abstract

We study the zero location and the asymptotic behavior of iterated integrals of polynomials. Borwein-Chen-Dilcher's polynomials play an important role in this issue. For these polynomials we find their strong asymptotics and give the limit measure of their zero distribution. We apply these results to describe the zero asymptotic distribution of iterated integrals of ultraspherical polynomials with parameters $(2\alpha + 1)/2$, $\alpha \in \mathbb{Z}_+$.

1 Introduction and main results

Several problems require the location of the zeros of a polynomial in areas such as numerical analysis, approximation theory, differential equations, and complex dynamics. The zeros of a polynomial can represent equilibrium points in a certain force field, geometric points of certain curves, critical points, and so on (see [6, 9, 10, 14], and the references therein). The objective of this paper is the study of some algebraic and asymptotic properties of the zeros of iterated integrals of polynomials.

Given a monic polynomial p_n of degree n , $\lambda \in \mathbb{C}$, and $m \in \mathbb{Z}_+$, its m -fold integral

$$I_{m,\lambda}(p_n)(z) := \frac{(n+m)!}{n!} \int_{\lambda}^z \int_{\lambda}^{t^{m-1}} \dots \int_{\lambda}^{t^1} p_n(t_0) dt_0 \dots dt_{m-2} dt_{m-1} \quad (1)$$

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defines a monic polynomial of degree $n + m$ for which the derivatives of order j , $0 \leq j \leq m - 1$, at λ are zero and

$$(I_{m,\lambda}(p_n))^{(m)}(z) = \frac{(n+m)!}{n!} p_n(z), \quad (2)$$

where $m \in \mathbb{Z}_+$ and $\lambda \in \mathbb{C}$. Of course, $I_{0,\lambda}(p_n) := p_n$. When $\lambda = 0$, for simplicity of notation, let $I_m(p_n) := I_{m,0}(p_n)$. The interchange of the order of integration or integration by parts yields

$$\begin{aligned} I_{m,\lambda}(p_n)(z) &= \frac{(n+m)!}{(m-1)!n!} \int_{\lambda}^z (z-s)^{m-1} p_n(s) ds \\ &= \frac{(n+m)!z^m}{(m-1)!n!} \int_{\lambda/z}^1 (1-t)^{m-1} p_n(zt) dt. \end{aligned}$$

Let $Q_{n,m}$ be the polynomials of degree n given by

$$\begin{aligned} Q_{n,m}(z) &:= \frac{(n+m)!}{n!} \sum_{k=0}^n \binom{n}{k} \frac{k!}{(m+k)!} z^k = \sum_{k=0}^n \binom{n+m}{k+m} z^k \\ &= \frac{(n+m)!}{(m-1)!n!} \int_0^1 (1-t)^{m-1} (1+zt)^n dt = \frac{(n+m)!}{n!z^m} I_m((1+z)^n). \end{aligned} \quad (3)$$

Borwein, Chen and Dilcher ([4, Th. 1]) prove that the zeros of $Q_{n,n+1}$ are dense in the curve

$$\Gamma := \left\{ z \in \mathbb{C} : \left| \frac{(z+1)^2}{4z} \right| = 1 \text{ and } |z| \geq 1 \right\}.$$

and these are the only limit points of the zeros. Using this result, they give estimations for the radius of a disc containing the zeros of $I_n(p_n)$ for every n . Moreover, they also obtain the curve to which the zeros of the n -fold integral of the n -th Legendre polynomial converge, as n goes to infinity. We call $Q_{n,m}$ *Borwein-Chen-Dilcher polynomials*.

We give strong asymptotics for $Q_{n,n}$. This allows us to characterize the measure which describes their zero distribution. The steepest descent method is used to obtain this estimation. For this description, let us consider two regions induced by Γ :

$$\begin{aligned} \mathcal{E}_1 &:= \left\{ z \in \mathbb{C} : \left| \frac{(z+1)^2}{4z} \right| > 1, |z| > 1 \right\}, \\ \mathcal{E}_2 &:= \left\{ z \in \mathbb{C} : \left| \frac{(z+1)^2}{4z} \right| < 1 \right\} \cup \left\{ z \in \mathbb{C} : \left| \frac{(z+1)^2}{4z} \right| > 1, |z| < 1 \right\}. \end{aligned}$$

Theorem 1. *We have*

$$Q_{n,n}(z) = \begin{cases} \frac{(z+1)^{2n}}{z^n} (1 + o(1)) & \text{if } z \in \mathcal{E}_1, \\ \frac{2^{2n}}{\sqrt{\pi n(1-z)}} (1 + o(1)) & \text{if } z \in \mathcal{E}_2, \end{cases} \quad (4)$$

uniformly, as $n \rightarrow \infty$, on compact subsets of each stated domain¹.

¹Here we state results for $m = n$ but analogous statements hold for $m = n + j$ with j a fixed integer.

A consequence of the above result is the zero distribution of $Q_{n,n}$. Define

$$v[Q_{n,n}] := \frac{1}{n} \sum_{Q_{n,n}(z)=0} \delta_z,$$

its weak-* limit is given in terms of the equilibrium measure of Γ . For each Borel set $B \subset \mathbb{C}$, $\mu_\Gamma(B) = m(\varphi(B \cap \Gamma))$, where $dm = d\theta/(2\pi)$, the normalized arc-length on $\partial\mathbf{D}_1$, and $\varphi(z) = \frac{(z+1)^2}{4z}$.

Corollary 1. *We have $\text{w-lim}_{n \rightarrow \infty} v[Q_{n,n}] = \mu_\Gamma$, where μ_Γ is the equilibrium measure on Γ . The support of μ_Γ is Γ . The measure μ_Γ is the pre-image of the normalized arc-length on $\partial\mathbf{D}_1$ under the mapping $\varphi(z) = \frac{(z+1)^2}{4z}$ from Γ to $\partial\mathbf{D}_1$. We have $\text{cap}(\Gamma) = 4$. Moreover, for n large enough and all $m \in \mathbb{Z}_+$ the zeros of $Q_{n,m}$ are inside Γ .*

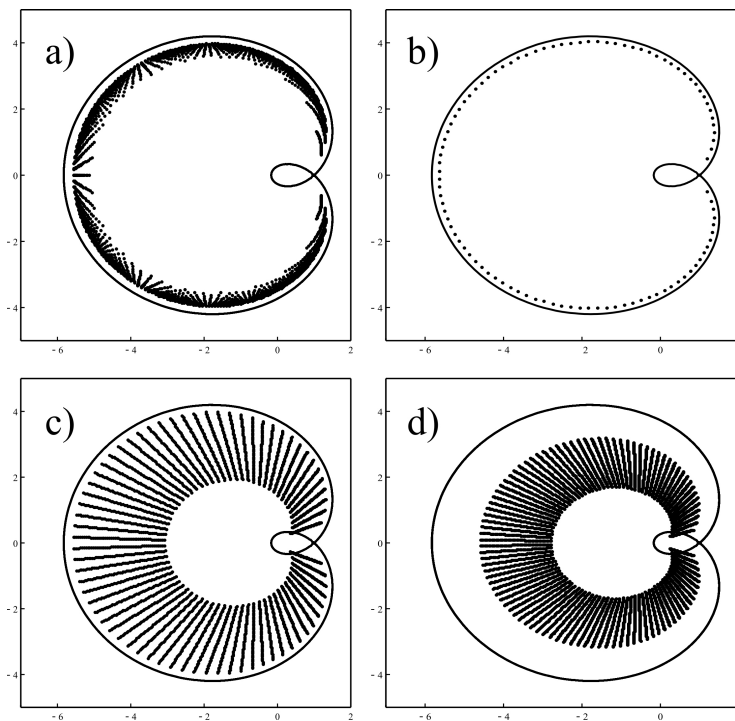


Figure 1: a) Zeros of $Q_{n,n}$ for $n = 20, \dots, 70$. b) Zeros of $Q_{100,100}$. c) Zeros of $Q_{n,m}$ for $n = 70$ and $m = 20, \dots, 70$. d) Zeros of $Q_{n,m}$ for $n = 100$ and $m = 20, \dots, 70$. In all cases the curve is $\left| \frac{(z+1)^2}{4z} \right| = 1$.

In Figures 1 we can see the zeros of $Q_{n,m}$ for several values of n and m . Another consequence of Theorem 1 is the strong asymptotics and zeros distribution of iterated integrals of ultraspherical polynomials. Let $\alpha \in \mathbb{Z}_+$ be a positive integer and let

$\widehat{P}_n^{(\alpha+1/2)}$ be monic ultraspherical polynomials with parameter $\alpha + 1/2$. The polynomials $\widehat{P}_n^{(1/2)}$ are monic Legendre polynomials.

Theorem 2. *Let $\alpha \in \mathbb{Z}_+$ be a nonnegative integer. Then we have*

$$I_{2n} \left(\widehat{P}_{2n}^{(\alpha+1/2)} \right) (z) = \begin{cases} (z^2 - 1)^{2n-\alpha} z^{2\alpha} (1 + o(1)) & \text{if } z \in \mathcal{F}_1, \\ \frac{(-1)^n 2^{2n-\alpha} z^{2n}}{\sqrt{\pi n} (1+z^2)} (1 + o(1)) & \text{if } z \in \mathcal{F}_2, \end{cases}$$

$$I_{2n+1} \left(\widehat{P}_{2n+1}^{(\alpha+1/2)} \right) (z) = \begin{cases} (z^2 - 1)^{2n+1-\alpha} z^{2\alpha} (1 + o(1)) & \text{if } z \in \mathcal{F}_1, \\ \frac{(-1)^n 2^{2n+1-\alpha} z^{2n+2}}{\sqrt{\pi n} (1+z^2)} (1 + o(1)) & \text{if } z \in \mathcal{F}_2, \end{cases}$$

as $n \rightarrow \infty$, where \mathcal{F}_j are the pre-image of \mathcal{E}_j under the transformation $\varphi_1(z) = -z^2$. Moreover, its zero distribution² β is supported on $\varphi_1(\Gamma)$ and for each Borel set E this measure satisfies

$$\beta(E) = \mu_\Gamma(\varphi_1(E)).$$

Ultimately, the behavior of $Q_{n,n}$ is the same as that obtained by integrating polynomials a number of times which does not change with the integrand (see Section 2). In order to set our result in this context, let us fix some notations. Let \mathbf{A} be a Jordan rectifiable arc in \mathbb{C} and $\Omega := \overline{\mathbb{C}} \setminus \mathbf{A}$. Given $r \in (0, \infty)$, $\mathbf{D}_r := \{z \in \mathbb{C} : |z| < r\}$, $\partial\mathbf{D}_r := \{z \in \mathbb{C} : |z| = r\}$, $\overline{\mathbf{D}}_r^c := \{z \in \mathbb{C} : |z| > r\}$, and $\mathbf{D}_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\}$ for $z_0 \in \mathbb{C}$. Let τ denote the conformal mapping of Ω onto $\overline{\mathbf{D}}_1^c$ such that $\tau'(\infty) := \text{cap}(\mathbf{A}) := \lim_{z \rightarrow \infty} \frac{\tau(z)}{z} > 0$. It is well known that τ can be extended continuously to \mathbf{A} and $\lim_{z \rightarrow \zeta} |\tau(z)| = 1$, $\zeta \in \mathbf{A}$. If $\lambda \in \Omega$, let $\Lambda_\lambda := \{z \in \Omega : |\tau(z)| < |\tau(\lambda)|\}$, $\partial\Lambda_\lambda := \{z \in \Omega : |\tau(z)| = |\tau(\lambda)|\}$, $\overline{\Lambda}_\lambda^c := \{z \in \Omega : |\tau(z)| > |\tau(\lambda)|\}$, when $\lambda \in \mathbf{A}$, we consider $\partial\Lambda_\lambda := \mathbf{A}$ and $\overline{\Lambda}_\lambda^c := \Omega$. Let (ϕ_n) denote a sequence of monic polynomials such that $\deg \phi_n = n$ for all n and

$$\lim_{n \rightarrow \infty} \frac{\phi_n(z)}{\text{cap}(\mathbf{A})^n \tau^n(z)} = \mathfrak{F}(z), \quad (5)$$

uniformly on compact subsets of Ω , in which the analytic function \mathfrak{F} has no zeros. There are several sequences of monic polynomials which satisfy condition (5) such as extremal polynomials with respect to a measure whose weight satisfies the Szegő condition (see [16]). If (ϕ_n) is a sequence of monic orthogonal polynomials with respect to a measure μ , $(I_{m,\lambda}(\phi_n))_{n \in \mathbb{Z}_+}$ is a sequence of polynomials orthogonal in a non-standard sense, i.e., they satisfy Sobolev-type orthogonality with respect to the inner product

$$\langle f, g \rangle := \sum_{k=0}^m \int f^{(k)} g^{(k)} d\mu_k,$$

where $\mu_k = \delta_\lambda$, $0 \leq k \leq m-1$, and $\mu_m = \mu$. There are several papers on this issue (e.g. [1, 2, 7, 8, 11, 13]). These polynomials are useful in Fourier analysis ([13]), numerical analysis ([5]), and so on (see [8] and references therein).

²The zeros of $I_n(\widehat{P}_n)$ different from 0.

Theorem 3. Let (ϕ_n) be a sequence of monic polynomials which satisfies (5).

(i) If $\lambda \in \mathbf{A}$, then

$$\lim_{n \rightarrow \infty} \frac{\phi_n(z)}{I_{m,\lambda}(\phi_n)(z)} = \psi^m(z),$$

uniformly on compact subsets of Ω , where the function $\psi(z) := \frac{\tau'(z)}{\tau(z)}$. Also, we have $\text{w-lim}_{n \rightarrow \infty} \nu^*[I_{m,\lambda}(\phi_n)] = \mu_{\mathbf{A}}$, where $\mu_{\mathbf{A}}$ is the equilibrium measure on the arc \mathbf{A} and

$$\nu^*[I_{m,\lambda}(\phi_n)] = \sum_{\substack{I_{m,\lambda}(\phi_n)(\zeta)=0 \\ \zeta \neq 0}} \delta_{\zeta}.$$

(ii) If $\lambda \in \Omega$, then

$$\lim_{n \rightarrow \infty} \frac{n^{m-1} \phi_n(z)}{I_{m,\lambda}(\phi_n)(z)} = -\frac{(m-1)! \psi^m(z)}{(z-\lambda)^{m-1}}$$

uniformly on compact subsets of Λ_λ and

$$\lim_{n \rightarrow \infty} \frac{\phi_n(z)}{I_{m,\lambda}(\phi_n)(z)} = \psi^m(z),$$

uniformly on compact subsets in $\overline{\Lambda_\lambda}^c$. Moreover,

$$\text{w-lim}_{n \rightarrow \infty} \nu^*[I_{m,\lambda}(\phi_n)] = \mu_{\Lambda_\lambda},$$

where μ_{Λ_λ} is the equilibrium measure on the arc $\partial \Lambda_\lambda$.

Section 5 is devoted to the proof of the above theorem. Theorem 2 is proved in Section 3. In Section 4 we state some results about the location of the zeros of iterated integrals of polynomials and the next section includes the proof of Theorem 1 as well as some of its consequences.

2 Proof of Theorem 1 and some consequences

Let

$$P_{n,m}(z) := \int_0^1 (1-t)^{m-1} (1+zt)^n dt.$$

The relations in (4) are equivalent to

$$P_{n,n}(z) = \begin{cases} \frac{\sqrt{\pi}}{2^{2n} \sqrt{n}} \frac{(z+1)^{2n}}{z^n} (1+o(1)) & \text{if } z \in \mathcal{E}_1, \\ \frac{1}{n(1-z)} (1+o(1)) & \text{if } z \in \mathcal{E}_2, \end{cases}$$

uniformly, as $n \rightarrow \infty$, on compact subsets of each stated domain.

Changing $t = \frac{1}{2}(1 - \frac{1}{z}) + \frac{1}{2}(1 + \frac{1}{z})x$, a straightforward computation gives us

$$P_{n,n}(z) := \int_0^1 (1-t)^{n-1} (1+zt)^n dt = \frac{(z+1)^{2n}}{2^{2n} z^n} \int_w^1 (1-x^2)^{n-1} (1+x) dx, \quad (6)$$

where $w = \frac{1-z}{1+z}$. Observe that the Möbius transformation $w = \frac{1-z}{1+z}$ transforms $\overline{\mathbf{D}}_1^c$ onto $\Re(w) < 0$ and $1-w^2 = \frac{4z}{(1+z)^2}$. Define

$$G_n(w) := \int_w^1 (1-x^2)^n dx. \quad (7)$$

We have

$$\int_w^1 (1-x^2)^{n-1} (1+x) dx = G_{n-1}(w) + \frac{(1-w^2)^n}{2n},$$

so, the theorem 1 is equivalent to the following statements:

- (i) Uniformly on compact subsets of $\mathcal{G}_2 := \{w \in \mathbb{C} : |1-w^2| > 1\} \cup \{w \in \mathbb{C} : |1-w^2| < 1 \text{ and } \Re(w) > 0\}$,

$$G_n(w) = \frac{(1-w^2)^{n+1}}{2nw} (1+o(1)). \quad (8)$$

- (ii) Uniformly on compact subsets of $\mathcal{G}_1 := \{w \in \mathbb{C} : |1-w^2| < 1 \text{ and } \Re(w) < 0\}$,

$$G_n(w) = \frac{\sqrt{\pi}}{\sqrt{n}} (1+o(1)), \quad (9)$$

The integral (7) does not depend on the contour of integration. Thus, we deform the integration interval $[w, 1]$ as we need in each case.

First, we obtain (8). Let $K \subset (\mathcal{G}_2 \cap \{w \in \mathbb{C} : \Re(w) \geq 0\})$ be a compact set and $w \in K$. From the change of variable $\frac{t^2}{n} = \frac{x^2-w^2}{1-w^2}$ and the dominated converge theorem, we get

$$\begin{aligned} \int_w^1 (1-x^2)^n dx &= \frac{(1-w^2)^{n+1}}{n} \int_0^{\sqrt{n}} \left(1 - \frac{t^2}{n}\right)^n \frac{t}{\sqrt{(1-w^2)\frac{t^2}{n} + w^2}} dt \\ &= \frac{(1-w^2)^{n+1}}{2nw} (1+o(1)). \end{aligned}$$

If $K \subset (\mathcal{G}_2 \cap \{w \in \mathbb{C} : \Re(w) \leq 0\})$ is a compact set and $w \in K$, the proof is reduced to the former case because

$$G_n(w) = -G_n(-w) + 2G_n(0),$$

and

$$G_n(0) = \frac{\Gamma(n+1)\Gamma(1/2)}{2\Gamma(n+3/2)} = \frac{\sqrt{\pi}}{2\sqrt{n}} (1+o(1)). \quad (10)$$

Second, we get (9). Let $K \subset \mathcal{G}_1$ and $w \in K$; we can choose $w_1 \in (-1, 0)$ such that $K \subset \{u : \Re(u) < w_1\}$. Then

$$G_n(w) = \int_w^{w_1} (1-x^2)^n dx + \int_{w_1}^0 (1-x^2)^n dx + G_n(0) = 2G_n(0)(1+o(1)).$$

□

The following result is a consequence of Theorem 1.

Corollary 2. *We have*

$$\lim_n |\mathcal{Q}_{n,n}(z)|^{1/n} = \left| \frac{(z+1)^2}{z} \right|, \quad z \in \mathcal{E}_1,$$

$$\lim_n |\mathcal{Q}_{n,n}(z)|^{1/n} = 4, \quad z \in \mathcal{E}_2,$$

uniformly on compact subsets of each of the two regions. Moreover,

$$\lim_n \left(\max_{z \in \Gamma} |\mathcal{Q}_{n,n}(z)| \right)^{1/n} = 4. \quad (11)$$

Proof. We will only check (11). From (6), it is equivalent to

$$\lim_{n \rightarrow \infty} \left(\max_{w \in \partial \mathcal{G}_1} |G_n(w)| \right)^{1/n} = 1.$$

By the maximum principle,

$$G_n(0) \leq \max_{w \in \partial \mathcal{G}_1} |G_n(w)| \leq 1.$$

From (10), the conclusion of the proof is straightforward. □

Remark 1. If $z \in \overline{\mathcal{E}}_1 := \{z \in \mathbb{C} : \left| \frac{(z+1)^2}{4z} \right| \geq 1, |z| \geq 1\}$, then there exists $K_1(z) > 0$ ($K_1(z)$ only depends on z) such that

$$\frac{K_1(z)}{\sqrt{n}} \leq |\mathcal{Q}_{n,n}(z)|$$

for n large enough. In fact, from (3) and (6), the above relation is equivalent to

$$\frac{K_2(w)}{\sqrt{n}} \leq |G_n(w)|$$

for $w \in \overline{\mathcal{G}}_1 := \{w \in \mathbb{C} : |1-w^2| \leq 1 \text{ and } \Re(w) \leq 0\}$, where $K_2(z)$ is a positive function. The case $w = 0$ is straightforward. If $w \neq 0$,

$$G_n(w) = 2G_n(0) - \int_{-w}^1 (1-x^2)^n dx$$

and there exists $C(w) > 0$ such that

$$\left| \int_{-w}^1 (1-x^2)^n dx \right| \leq \frac{C(w)}{n}.$$

Actually, if z is outside a disc with center at 1 (w is outside of a disc with center at 0) with small radius, then we can take the positive constant $C(w)$ independent of w .

With the same arguments we obtain

$$Q_{n,n+1}(z) = \begin{cases} \frac{(z+1)^{2n+1}}{z^{n+1}}(1+o(1)) & \text{if } z \in \mathcal{E}_1, \\ \frac{2^{2n+1}}{\sqrt{\pi n}(1-z)}(1+o(1)) & \text{if } z \in \mathcal{E}_2, \end{cases} \quad (12)$$

as $n \rightarrow \infty$, uniformly on compact subsets of the two regions.

2.1 Proof of Corollary 1

Given that $w = \frac{(z+1)^2}{z}$ is a conformal transform from \mathcal{E}_1 onto $\overline{\mathbf{D}}_4^c$, and $\text{cap}(\mathbf{D}_4) = 4$ (see [10, Th. 5.2.3]), we have $\text{cap}(\Gamma) = 4$. Moreover, by the subordination principle ([10, Th. 4.3.8]), for each Borel set $B \subset \mathbb{C}$, $\mu_\Gamma(B) = m(\varphi(B \cap \Gamma))$, where $dm = d\theta/(2\pi)$ on $\partial\mathbf{D}_1$.

Let μ be a weak-* limit of $\nu[Q_{n,n}] := \frac{1}{n} \sum_{z: Q_{n,n}(z)=0} \delta_z$, i.e., there exists a subsequence $(\nu[Q_{n_k, n_k}])$ such that $\lim_{k \rightarrow \infty} \int f(\zeta) d\nu[Q_{n_k, n_k}](\zeta) = \int f(\zeta) d\mu(\zeta)$ for all continuous function f in \mathbb{C} with compact support. To simplify notation, we write n instead of n_k . By Theorem 1, $\text{supp}(\mu) \subset \Gamma$. It is well known that³

$$\lim_{n \rightarrow \infty} V(\nu[Q_{n,n}], z) = V(\mu, z)$$

uniformly on compact subsets of $\mathbb{C} \setminus \Gamma$. Then, by (3), and Corollary 2, we obtain

$$V(\nu[Q_{n,n}], z) = \frac{1}{n} \log \left(\frac{(n-1)!n!}{(2n)!} \right) + V(\nu[P_{n,n}], z)$$

and

$$V(\mu, z) = \log \left(\frac{1}{4} \right) + \begin{cases} 0 & \text{if } z \in \mathcal{E}_2, \\ \log \left| \frac{4z}{(z+1)^2} \right| & \text{if } z \in \mathcal{E}_1. \end{cases}$$

Then, from the identity principle for harmonic functions (see [10, Th. 1.1.7]) we get $\text{supp}(\mu_\Gamma) = \Gamma$ and from the principle of descent, if $z \in \Gamma$ and $(z_n) \subset \mathcal{E}_2$ such that $\lim_{n \rightarrow \infty} z_n = z$,

$$V(\mu, z) \leq \liminf_{n \rightarrow \infty} V(\nu[Q_{n,n}], z_n) = \log \left(\frac{1}{4} \right).$$

³Hereafter, if ν is a positive Borel measure with compact support in the complex plane, its logarithmic potential and energy are respectively $V(\nu, z) := \int \frac{1}{|z-x|} d\nu(x)$ and $I(\nu) := \int V(\nu, z) d\nu(z)$.

Since μ is a probability measure, $I(\mu) = \int V(\mu, z) d\mu(z) \leq \log(\frac{1}{4})$, but μ_Γ is the unique measure which minimizes the energy between the probability measures with support on Γ , and $I(\mu_\Gamma) = \log \frac{1}{\text{cap}(\Gamma)} = \log(\frac{1}{4})$. Therefore, $\mu = \mu_\Gamma$,

$$\text{w-lim}_{n \rightarrow \infty} \nu[Q_{n,n}] = \mu_\Gamma.$$

□

3 Asymptotic analysis for the integral of ultraspherical polynomials

Proof of Theorem 2.

We shall proceed by induction on the parameter α . For $\alpha = 0$, we have Legendre polynomials. By Rodrigues' formula (see [15, (4.3.1)]) the monic Legendre polynomials $\widehat{P}_n^{(1/2)}$ are given by

$$\widehat{P}_n^{(1/2)}(z) := (-1)^n \frac{n!}{(2n)!} \partial_z^n ((1-z^2)^n).$$

So,

$$\begin{aligned} I_{2n}(\widehat{P}_{2n}^{(1/2)})(z) &= (-z^2)^n Q_{n,n}(-z^2), \\ I_{2n+1}(\widehat{P}_{2n+1}^{(1/2)})(z) &= -(-z^2)^{n+1} Q_{n,n+1}(-z^2). \end{aligned}$$

Let $\mathcal{F}_1 := \{z \in \mathbb{C} : \left| \frac{z^2-1}{2z} \right| > 1, |z| > 1\}$, $\mathcal{F}_2 := (\{z \in \mathbb{C} : \left| \frac{z^2-1}{2z} \right| < 1\}) \cup (\{z \in \mathbb{C} : \left| \frac{z^2-1}{2z} \right| > 1, |z| < 1\})$. Observe that the coefficient of z^{2n} in $I_{2n}(\widehat{P}_{2n}^{(1/2)})$ is $\frac{P_{2n}^{(1/2)}(0)}{\kappa_{2n}} \sim \frac{(-1)^n 2^{2n}}{\sqrt{\pi n}}$ as $n \rightarrow \infty$, where κ_{2n} is the leading coefficient of $P_{2n}^{(1/2)}$. From (12), Theorem 1, and Corollary 1, we obtain

$$\begin{aligned} I_{2n}(\widehat{P}_{2n}^{(1/2)})(z) &= \begin{cases} (z^2 - 1)^{2n} (1 + o(1)) & \text{if } z \in \mathcal{F}_1, \\ \frac{(-1)^n 2^{2n}}{\sqrt{\pi n}} \frac{z^{2n}}{1+z^2} (1 + o(1)) & \text{if } z \in \mathcal{F}_2, \end{cases} \\ I_{2n+1}(\widehat{P}_{2n+1}^{(1/2)})(z) &= \begin{cases} (z^2 - 1)^{2n+1} (1 + o(1)) & \text{if } z \in \mathcal{F}_1, \\ \frac{(-1)^n 2^{2n+1}}{\sqrt{\pi n}} \frac{z^{2(n+1)}}{1+z^2} (1 + o(1)) & \text{if } z \in \mathcal{F}_2, \end{cases} \end{aligned}$$

as $n \rightarrow \infty$, where \mathcal{F}_j are the pre-image of \mathcal{E}_j under the transformation $\varphi_1(z) = -z^2$. Moreover, its zero distribution⁴ β is supported on $\varphi_1(\Gamma)$ and for each Borel set E this measure satisfies $\beta(E) = \mu_\Gamma(\varphi_1(E))$.

Next, we assume that the statement holds for α . According to [15, (4.21.7)], we have

$$\partial_z \left(P_n^{(\alpha+1/2)}(z) \right) = \frac{1}{2} (n + 2\alpha + 1) P_{n-1}^{(\alpha+3/2)}(z).$$

⁴The zeros of $I_n(\widehat{P}_n^{(1/2)})$ different from 0.

Remember that $I_m(z)$ are monic polynomials and $\partial_z^j(I_m(z)) = 0$, $j = 0, 1, \dots, m-1$. The factor $\frac{(2n-2)!}{(n-2)!n!}$ is to guarantee that $\partial_z^{n-2}(I_{n-1}(\widehat{P}_{n-1}^{(\alpha+3/2)})) = 0$. So,

$$I_{n-1}(\widehat{P}_{n-1}^{(\alpha+3/2)})(z) = \left(I_{n-2}(\widehat{P}_n^{(\alpha+1/2)})(z) - \widehat{P}_n^{(\alpha+1/2)}(0) \frac{(2n-2)!}{(n-2)!n!} z^{n-2} \right).$$

Moreover, we have

$$I_{n-2}(\widehat{P}_n^{(\alpha+1/2)})(z) = \frac{1}{2n(2n-1)} \partial_z^2 \left(I_n(\widehat{P}_n^{(\alpha+1/2)})(z) \right).$$

Next, we consider $n = 2k + 2$ even. By the Cauchy integral formula for the derivative and by induction hypothesis, we get

$$\begin{aligned} I_{2k}(\widehat{P}_{2k+2}^{(\alpha+1/2)})(z) &= \frac{1}{(4k+2)(4k+1)} \partial_z^2 \left(I_{2k+2}(\widehat{P}_{2k}^{(\alpha+1/2)})(z) \right) \\ &= (z^2 - 1)^{2k-\alpha} z^{2\alpha} (1 + o(1)) \end{aligned}$$

uniformly on compact subsets of \mathcal{F}_1 . Observe that on \mathcal{F}_1 ,

$$\left| \frac{z^2 - 1}{2z} \right| > 1,$$

and by [15, (4.7.31)],

$$\widehat{P}_{2k+2}^{(\alpha+1/2)}(0) = \frac{(-1)^{k+1} \Gamma(k + \alpha + 3/2) \Gamma(2k + 3)}{2^{2k+2} \Gamma(k + 2) \Gamma(2k + \alpha + 5/2)}.$$

Moreover, by Stirling's formula,

$$\frac{\Gamma(k + \alpha + 3/2)}{\Gamma(k + 2)} = k^{\alpha-1/2} (1 + o(1)), \quad \frac{(4k+2)!}{(2k)!(2k+2)!} = \frac{2^{4k+2}}{\sqrt{2\pi k}} (1 + o(1)).$$

Thus,

$$\begin{aligned} I_{2k}(\widehat{P}_{2k+2}^{(\alpha+1/2)})(z) - \widehat{P}_{2k+2}^{(\alpha+1/2)}(0) \frac{(2k)!}{k!(k+2)!} z^{2k-2} &= I_{2k}(\widehat{P}_{2k+2}^{(\alpha/2)})(z) (1 + o(1)) \\ &= (z^2 - 1)^{2k+2-\alpha} z^{2\alpha} (1 + o(1)) \end{aligned}$$

uniformly on compact subsets of \mathcal{F}_1 . Therefore,

$$I_{2k+1}(\widehat{P}_{2k+1}^{(\alpha+3/2)})(z) = (z^2 - 1)^{2k+1-(\alpha+1)} z^{2(\alpha+1)} (1 + o(1)),$$

uniformly on compact subsets of \mathcal{F}_1 .

Consider $z \in \mathcal{F}_2$; we have

$$\frac{1}{(4k+4)(4k+3)} \partial_z^2 \left(I_{2k+2}(\widehat{P}_{2k+2}^{(\alpha+1/2)})(z) \right) = \frac{(-1)^{k+1} 2^{2k+2-\alpha} z^{2k}}{2^2 \sqrt{\pi k} (1+z^2)} (1 + o(1)).$$

In fact, it is immediately checked that

$$\begin{aligned} \partial_z^2 \left(\frac{(-1)^{k+1} 2^{2k+2-\alpha} z^{2k+2}}{\sqrt{\pi k} (1+z^2)} \right) &= \frac{(-1)^{k+1} 2^{2k+2-\alpha}}{\sqrt{\pi k}} \partial_z^2 \left(\frac{z^{2k+2}}{1+z^2} \right) \\ &= \frac{(-1)^{k+1} 2^{2k+2-\alpha}}{\sqrt{\pi k}} \\ &\quad \times \frac{2z^{2n}(1-3z^2+n(1+z^2)(3+2n+(-1+2n)z^2))}{(1+z^2)^3}, \end{aligned}$$

and $\frac{z^{2k}}{1+z^2} - z^{2k} = \frac{-z^{2k+2}}{1+z^2}$. Therefore,

$$I_{2k+1} \left(\widehat{P}_{2k+1}^{(\alpha+3/2)} \right) (z) = \frac{(-1)^k 2^{2k+1-(\alpha+1)}}{\sqrt{\pi k}} \frac{z^{2k+2}}{1+z^2} (1+o(1))$$

uniformly on compact subsets of \mathcal{F}_2 .

If n is odd, then $\widehat{P}_n^{(\alpha+1/2)}(0) = 0$,

$$I_{2n} \left(\widehat{P}_{2n}^{(\alpha+3/2)} \right) (z) = \frac{1}{(4n+2)(4n+1)} \partial_z^2 \left(I_{2n+1} \left(\widehat{P}_{2n+1}^{(\alpha+1/2)} \right) (z) \right),$$

and the proof is concluded as in the first case.

The conclusion about the limiting distribution of the zeros of $I_n \left(\widehat{P}_n^{(\alpha+1/2)} \right)$ follows in a straightforward manner from their asymptotic behavior and the unicity theorem for potentials (see [12, Theorem 2.1, p. 97]).

□

In Figures 2 and 3 we can see the zeros of iterated integral of ultraspherical polynomials, in particular, of Legendre polynomials, for different values of n .

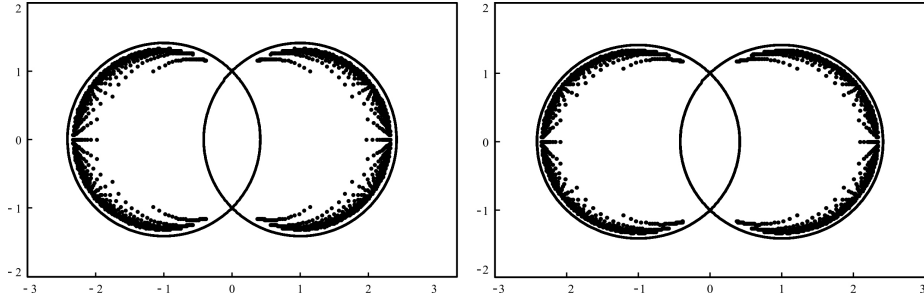


Figure 2: Zeros of the iterated integral of Legendre polynomials $I_n(P_n^{(1/2)})$: to the left, for n even $n = 10$ to $n = 80$. To the right, for n odd $n = 11$ to $n = 81$. In both the curve $\left| \frac{1-z^2}{2z} \right| = 1$.

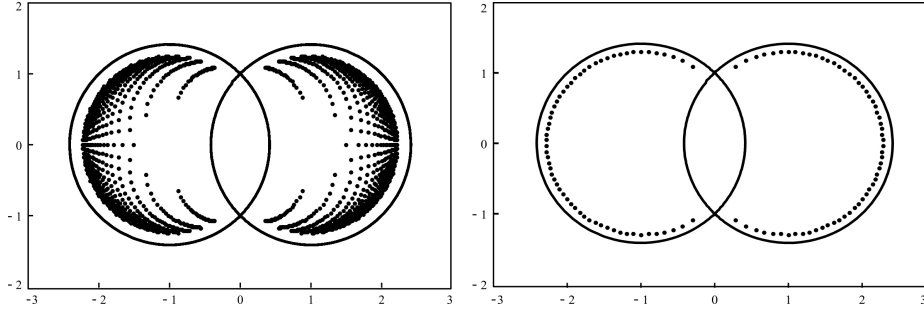


Figure 3: Zeros of the integral of ultraspherical polynomials $I_n(P_n^{(13/2)})$: to the left, for n even, $n = 20, \dots, 160$. To the right, for $n = 120$. In both the curve $\left| \frac{1-z^2}{2z} \right| = 1$.

4 Zeros of polynomials with critical points on a disc

It is clear that if we know the location of the critical points of a polynomial and one of its zeros, the remaining zeros are uniquely determined. Nonetheless, there are only a few general results about the zero locations of polynomials in terms of their critical points and a given zero, most of them contained in [9, §4.5] and [4].

Since

$$Q_{n,m}(z) = \frac{(n+m)!}{m!} \frac{1}{z^m} I_m[(1+z)^n] = \frac{1}{z^m} \left((1+z)^{n+m} - \sum_{k=0}^{m-1} \binom{n+m}{k} z^k \right),$$

we have the following lemma.

Lemma 1.

$$Q_{n,m}(z) = \begin{cases} \frac{(z+1)^{n+m}}{n^{m-1} z^m} (1 + o(1)) & \text{if } |z+1| > 1, \\ -\frac{1}{z^{(m-1)!}} (1 + o(1)) & \text{if } |z+1| < 1. \end{cases}$$

as $n \rightarrow \infty$. Moreover, $w\text{-}\lim_{n \rightarrow \infty} \nu[Q_{n,m}] = \mu_{\mathbf{D}_1(-1)}$, where $\mu_{\mathbf{D}_1(-1)}$ is the equilibrium measure on $\mathbf{D}_1(-1)$. So it is given by the normalized arc length on $\mathbf{D}_1(-1)$.

If p_n is a polynomial of degree n ,

$$p_n(z) = \sum_{k=0}^n \binom{n}{k} a_k z^k,$$

$$I_{m,0}(p_n)(z) = \frac{(n+m)!}{n!} \sum_{k=0}^n \binom{n}{k} \frac{k!}{(m+k)!} a_k z^{k+m} = z^m \sum_{k=0}^n \binom{n}{k} c_k z^k,$$

then $c_k = a_k b_k$, where $\sum_{k=0}^n \binom{n}{k} b_k z^k = Q_{n,m}(z)$.

This means that, as Borwein, Chen and Dilcher [4] observed, $I_{m,0}$ is a Hadamard product of p_n and $Q_{n,m}$. Then, by a theorem of Szegő and Schur (as quoted, for instance, in [4]), for n large enough, we have:

Corollary 3. *If the zeros of p_n lie in the disc \mathbf{D}_r , then the zeros of $I_{m,0}(p_n)$ lie in $\mathbf{D}_{(2+\varepsilon(n))r}$, where $(\varepsilon(n))$ is a decreasing sequence of positive number with $\lim_{n \rightarrow \infty} \varepsilon_n = 0$.*

The next result also helps to locate the zeros of the iterated integral of a polynomial. This is an extension of [14, Th. 5.7.8] and the proof is carried out with analogous arguments.

Lemma 2. *Let P be a polynomial of degree $n \geq 2$ with all its critical points in the closed disc $\overline{\mathbf{D}}_r$, where $r \in \mathbb{R}_+$ is fixed. If $P(\lambda) = P(z) = 0$, with $\lambda, z \in \mathbb{C}$, then*

(i) *there exists $w \in \overline{\mathbf{D}}_1$ such that*

$$z = F_r(w) := 2rw - \overline{\lambda}w^2. \quad (13)$$

(ii) $\|z\| - \|\lambda\| \leq 2r$.

(iii) F_r is univalent on $\overline{\mathbf{D}}_1$ if and only if $|\lambda| \leq r$.

Proof. As z and λ are zeros of P , from the bisector lemma (see [9, Th. 4.3.1]), if we draw a straight line ℓ which cuts perpendicularly the segment joining the two zeros at its middle point, then P' has at least one zero in each of the closed half planes in which ℓ divides the complex plane. But, we have assumed that all the zeros of P' lie in $\overline{\mathbf{D}}_r$ and therefore ℓ must intersect $\partial\mathbf{D}_r$. Hence, there exists $u \in \partial\mathbf{D}_1$ such that $|z - ru| = |ru - \lambda| = |r - \overline{\lambda}u|$. It follows that there exists $v \in \partial\mathbf{D}_1$ such that $z - ru = v(r - \overline{\lambda}u)$, where we have $z = r(u + v) - \overline{\lambda}uv$. This expresses z as a value of a symmetric linear form in the variables u and v taking their values on $\partial\mathbf{D}_1$, and therefore in $\overline{\mathbf{D}}_1$. It follows from Walsh's coincidence lemma [9, Th. 3.4.1b] that z is a value of the polynomial obtained by putting $w = u = v$ with $w \in \overline{\mathbf{D}}_1$, which establishes (13) and the inequality in statement (ii) as an immediate consequence.

If $\lambda = 0$ then obviously F_r is univalent. Assume that $\lambda \neq 0$, if there exist $w_1, w_2 \in \overline{\mathbf{D}}_1$ such that $w_1 \neq w_2$ and $F_r(w_1) = F_r(w_2)$, we get that $w_1 + w_2 = 2(r/\overline{\lambda})$. Therefore, F_r is univalent on $\overline{\mathbf{D}}_1$ if and only if $|\lambda| \leq r$ and we get the third statement of the lemma. \square

Remark 2. Under the above assumptions, as a consequence of Lemma 2, the possible region of zeros of P is the set $F_r(\overline{\mathbf{D}}_1)$ and if $|\lambda| \leq r$ then F_r maps $\partial\mathbf{D}_1$ onto a Jordan curve (for $r = 1$ see Figure 4).

Corollary 4. *Given two integers $n, m > 0$ and $\lambda \in \mathbb{C}$, let $\rho = 2^m(|\lambda| + r) - |\lambda|$. If all the zeros of ϕ_n are in $\overline{\mathbf{D}}_r$, then all the zeros of the polynomials $I_{m,\lambda}(\phi_n)$ lie in the closed disc $\overline{\mathbf{D}}_\rho$.*

Proof. For $m = 1$, as all the zeros of ϕ_n lie in \mathbf{D}_r (i.e., the critical points of $I_{1,\lambda}(\phi_n)$), the assertion follows from Lemma 2. The rest of the proof runs by induction. \square

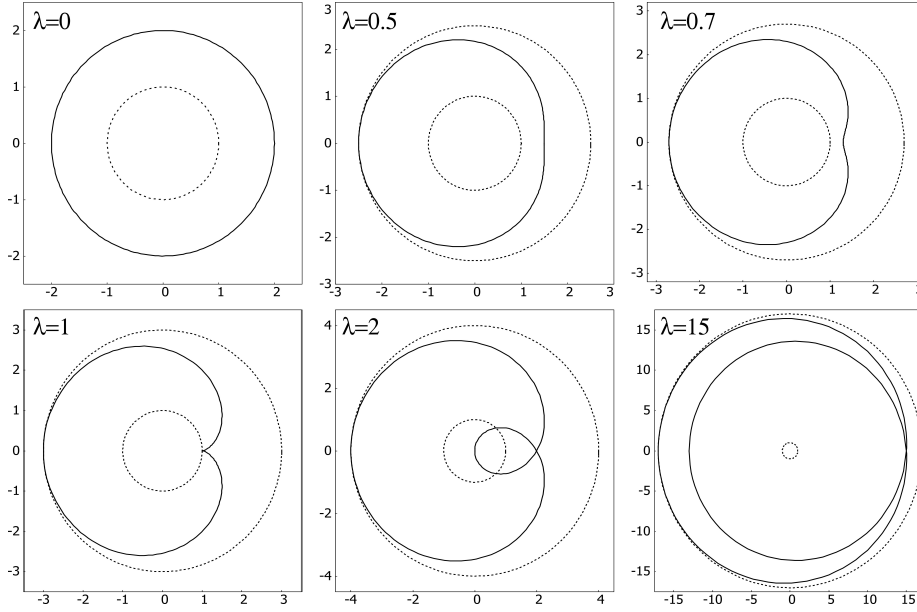


Figure 4: The cardioidal curve $F_r(\partial\mathbf{D}_1)$, for $r = 1$ and several values of λ . The interior circle is $\partial\mathbf{D}_1$ and the exterior one is given by $|z| = 2 + |\lambda|$.

5 Proof of Theorem 3

The proof of Theorem 3 is divided into three subsections: first, two auxiliary lemmas, second, the case $\lambda \in \mathbf{A}$, and third, the case $\lambda \notin \mathbf{A}$.

5.1 Auxiliary lemmas

The following result plays a main role in obtaining the strong asymptotic behavior of the m th iterated integrals $I_{m,\lambda}(\phi_n)$.

Lemma 3. *Set $\lambda \in \Omega$, and let K_1, K_2 be two compact sets with $K_1 \subset (\Lambda_\lambda \cup \mathbf{A})$ and $K_2 \subset \bar{\Lambda}_\lambda^c$. Let $z_1 \in K_1, z_2 \in K_2$ and $\{\phi_n\}_{n \in \mathbb{Z}_+}$ be a sequence of polynomials which satisfies (5). Then*

$$\lim_{n \rightarrow \infty} \left| \int_{z_1}^{z_2} \phi_n(s) ds \right|^{1/n} = \text{cap}(\mathbf{A}) |\tau(z_2)| \quad \text{uniformly on } K_1 \text{ and } K_2. \quad (14)$$

Proof. For $z_1 \in K_1$ and $z_2 \in K_2$, denote $J_n := \int_{z_1}^{z_2} \phi_n(s) ds$. This integral is independent of the contour of integration from z_1 to z_2 . Then, by the maximum principle for holomorphic functions and (5) we have

$$\limsup_{n \rightarrow \infty} |J_n|^{1/n} \leq \left(\int_{z_1}^{z_2} |ds| \max_{\{z \in \mathbb{C}: |\tau(z)| = |\tau(z_2)|\}} |\phi_n(z)| \right)^{1/n} = \text{cap}(\mathbf{A}) |\tau(z_2)|.$$

Since the convergence in (5) is uniform on compact subsets of Ω , the above relation also holds uniformly on K_1 and K_2 .

The proof of the inequality

$$\liminf_{n \rightarrow \infty} |J_n|^{1/n} \geq \text{cap}(\mathbf{A}) |\tau(z_2)| \quad (15)$$

requires a more detailed analysis. We chose ζ_0 near z_2 such that $[\zeta_0, z_2] \cap \mathbf{A} = \emptyset$,

$$|\tau(\zeta_0)| > |\tau(\lambda)| \text{ and } \text{Arg}(\tau(\zeta_0)) = \text{Arg}(\tau(z_2)). \quad (16)$$

Let Δ_1 be a Jordan rectifiable arc from z_1 to ζ_0 in $\{z \in \mathbb{C} : |\tau(z_1)| \leq |\tau(z)| \leq |\tau(\zeta_0)|\}$ and let Δ_2 be the arc given by $r(t) = \tau^{-1}(t\tau(z_2) + (1-t)\tau(\zeta_0))$, $t \in [0, 1]$, which satisfies

$$\text{Arg}(\tau(r(t))) \text{ is constant and } r'(t) = \frac{\tau(z_2) - \tau(\zeta_0)}{\tau'(t\tau(z_2) + (1-t)\tau(\zeta_0))} \neq 0, \quad t \in [0, 1].$$

We take as integration path in J_n the curve $\Delta := \Delta_1 + \Delta_2$. Then,

$$|J_n| = \left| \int_{\Delta} \phi_n(s) ds \right| \geq \left| \int_{\Delta_2} \phi_n(s) ds \right| - \int_{\Delta_1} |\phi_n(s)| |ds|.$$

By the maximum modulus principle and (5), it follows that

$$\limsup_n \left(\int_{\Delta_1} |\phi_n(s)| |ds| \right)^{1/n} \leq \text{cap}(\mathbf{A}) |\tau(\zeta_0)|. \quad (17)$$

Since $\Im(r(1))\overline{\tau'(r(1))} \neq 0$, we have that its real or its imaginary part is different from zero. We assume that

$$\Re \left(\Im(r(1))\overline{\tau'(r(1))} \right) \neq 0.$$

The case where the imaginary part is nonzero is reduced to this one by multiplying ϕ_n by i . So, by (5), we can chose $\Delta_2^1 \subset \Delta_2$ a piece of arc of Δ_2 containing z_2 such that

$$\lim_{n \rightarrow \infty} \Re \left(\frac{\phi_n(s)\overline{\tau'(s)}}{\text{cap}(\mathbf{A})^n |\tau^n(s)| |\tau'(s)|^2} \right) = \Re \left(\frac{\Im(s)\overline{\tau'(s)}}{|\tau'(s)|^2} \right) \neq 0$$

uniformly on Δ_2^1 and $H_n(s) := \Re \left(\frac{\phi_n(s)\overline{\tau'(s)}}{\text{cap}(\mathbf{A})^n |\tau^n(s)| |\tau'(s)|^2} \right)$ has constant sign for $s \in \Delta_2^1$ and n large enough. From (16), we have

$$\begin{aligned} \left| \int_{\Delta_2} \phi_n(s) ds \right| &\geq \left| \int_{\Delta_2^1} \frac{\phi_n(s)}{(\text{cap}(\mathbf{A}))^n |\tau^n(s)|} (\text{cap}(\mathbf{A}))^n |\tau^n(s)| ds \right| \\ &\quad - \int_{\Delta_2 \setminus \Delta_2^1} |\phi_n(s)| |ds| \\ &\geq \left| \int_J H_n(r(t)) (\text{cap}(\mathbf{A}))^n |\tau^n(r(t))| dt \right| - \int_{\Delta_2 \setminus \Delta_2^1} |\phi_n(s)| |ds| \end{aligned} \quad (18)$$

where J denote the parameter interval for the Δ_2^1 path. By the choice of Γ_2^1 , it follows that

$$\limsup_{n \rightarrow \infty} \left(\int_{\Delta_2 \setminus \Delta_2^1} |\phi_n(s)| |ds| \right)^{1/n} < |\text{cap}(\mathbf{A}) \tau(z_2)|.$$

Now, we deduce

$$\lim_{n \rightarrow \infty} \left| \int_J H_n(r(t)) (\text{cap}(\mathbf{A}))^n |\tau^n(r(t))| dt \right|^{1/n} = |\text{cap}(\mathbf{A}) \tau(z_2)|. \quad (19)$$

In fact, since $|\tau(r(t))| \leq |\tau(r(1))| = |\ell(z_2)|$ for all $t \in J$, we have

$$\limsup_{n \rightarrow \infty} \left| \int_J H_n(r(t)) (\text{cap}(\mathbf{A}))^n |\tau^n(r(t))| dt \right|^{1/n} \leq |\text{cap}(\mathbf{A}) \tau(z_2)|.$$

On the other hand, given $\varepsilon \in (0, |\tau(z_2)|)$, we choose $\delta > 0$ such that $[1 - \delta, 1] \subset J$ and $|\tau(r(t))| > |\tau(z_2)| - \varepsilon, \forall t \in J$. Then,

$$\left| \int_J H_n(r(t)) (\text{cap}(\mathbf{A}))^n |\tau^n(r(t))| dt \right| \geq \left| \int_{[1-\delta, 1]} H_n(r(t)) dt \right| (\text{cap}(\mathbf{A}))^n (|\tau(z_2)| - \varepsilon)^n$$

and we get

$$\liminf_{n \rightarrow \infty} \left| \int_J H_n(r(t)) (\text{cap}(\mathbf{A}))^n |\tau^n(r(t))| dt \right|^{1/n} \geq |\text{cap}(\mathbf{A}) \tau(z_2)|.$$

Once we have obtained (19), from (16)–(17) and (18)–(19) the relations (14) and (15) follow readily. Observe that the limit in (15) is uniform on z_1 and z_2 . By the continuity of $\mathfrak{F}(s)\overline{\tau'(s)}/|\tau'(s)|^2$, we can consider that the relations (16)–(19) hold for all z'_1 and z'_2 in a closed disc around z_1 and z_2 . \square

The following lemma is a well-known result, but we include a proof for easy reference.

Lemma 4. *We have*

$$\psi(z) = \int \frac{d\mu_{\mathbf{A}}(w)}{z-w}, \quad z \in \Omega,$$

where $\psi(z) = \tau'(z)/\tau(z)$, as it was defined in Theorem 3.

Proof. Let $\{\phi_n\}$ be a sequence of monic polynomials which satisfy (5). So, their zeros tend to \mathbf{A} and

$$\text{w-lim}_{n \rightarrow \infty} \nu[\phi_n] = \mu_{\mathbf{A}}.$$

(See arguments to check $\text{w-lim}_{n \rightarrow \infty} \nu[I_{1,\lambda}(\phi_n)] = \mu_{\mathbf{A}}$ in the proof of statement (i) of Theorem 3 in the next section.) Let (g_n) be the sequence of analytic functions $g_n(z) := (\text{cap}(\mathbf{A}))^n \mathfrak{F}(z) (\tau(z))^n, z \in \Omega, n \in \mathbb{N}$. Then, $\lim_{n \rightarrow \infty} \frac{g'_n(z)}{ng_n(z)} = \psi(z)$, uniformly on compact subsets of Ω . Since

$$\left(\frac{\phi_n(z)}{g_n(z)} \right)' = \frac{\phi_n(z)}{g_n(z)} \left(\frac{\phi'_n(z)}{\phi_n(z)} - \frac{g'_n(z)}{g_n(z)} \right),$$

we get $\lim_{n \rightarrow \infty} \frac{\phi_n'(z)}{n\phi_n(z)} = \lim_{n \rightarrow \infty} \frac{g_n'(z)}{ng_n(z)} = \psi(z)$, uniformly on compact subsets of Ω and

$$\psi(z) = \lim_{n \rightarrow \infty} \frac{\phi_n'(z)}{n\phi_n(z)} = \lim_{n \rightarrow \infty} \int \frac{d\nu[\phi_n](w)}{z-w} = \int \frac{d\mu_{\mathbf{A}}(w)}{z-w}, \quad z \in \Omega.$$

□

5.2 Proof of Theorem 3 (i): $\lambda \in \mathbf{A}$

We only consider the case $m = 1$ because the next step of induction on m follows with the same argument. It is well known that, if P_n is a monic polynomial of degree $n \geq 1$, we have $\|P_n\|_{\mathbf{A}} \geq (\text{cap}(E))^n$. (See [10, Th. 5.5.4].) Combining this relation with the definition of $I_{1,\lambda}(\phi_n)$ and the condition (5) on ϕ_n , it plainly follows that

$$\lim_{n \rightarrow \infty} \|I_{1,\lambda}(\phi_n)\|_{\mathbf{A}}^{1/n} = \text{cap}(\mathbf{A}).$$

Thus, since \mathbf{A} has empty interior and a connected complement (see [3]), we have

$$\text{w-lim}_{n \rightarrow \infty} \nu[I_{1,\lambda}(\phi_n)] = \mu_{\mathbf{A}}. \quad (20)$$

As τ has no zeros in Ω , by Lemma 3, we know that the zeros of $(I_{1,\lambda}(\phi_n))$ converge to \mathbf{A} . So, we obtain

$$\frac{\phi_n(z)}{I_{1,\lambda}(\phi_n)(z)} = \frac{(I_{1,\lambda}(\phi_n))'(z)}{(n+1)I_{1,\lambda}(\phi_n)(z)} = \int \frac{d\nu[I_{1,\lambda}(\phi_n)](w)}{z-w},$$

and by (20),

$$\lim_{n \rightarrow \infty} \frac{\phi_n(z)}{I_{1,\lambda}(\phi_n)(z)} = \int \frac{d\mu_{\mathbf{A}}(w)}{z-w} = \psi(z), \quad z \in \Omega.$$

Therefore, the statement (i) of Theorem 3 follows immediately from the hypothesis (5) on (ϕ_n) . □

5.3 Proof of Theorem 3 (ii): $\lambda \notin \mathbf{A}$

We choose $\lambda_0 \in \mathbf{A}$ and $\lambda \in \Omega$. From (1), $I_{m,\lambda}(\phi_n)$ can be written alternatively as

$$I_{m,\lambda}(\phi_n)(z) = I_{m,\lambda_0}(\phi_n)(z) - P_{m-1}(z),$$

where $P_{m-1}(z)$ is the $m - 1$ -th Taylor polynomial of $I_{m,\lambda_0}(\phi_n)$ in powers of $(z - \lambda)$. From (2), we have that

$$(I_{m,\lambda_0}(\phi_n))^{(k)}(\lambda)k! = \binom{n+m}{k} I_{m-k,\lambda_0}(\phi_n)(\lambda),$$

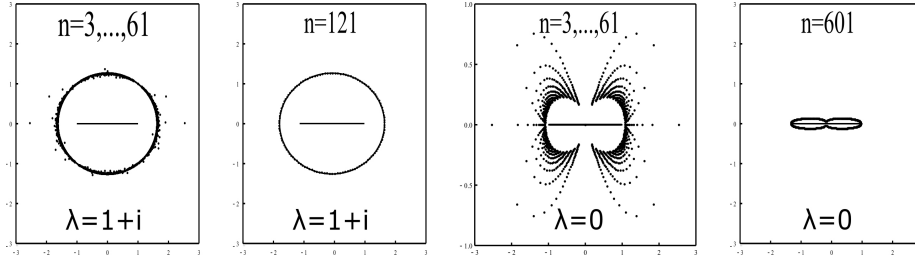


Figure 5: Here we illustrate Theorem 3 for ultraspherical polynomials on $[-1, 1]$: $I_6(P_n^{(3/2)})$.

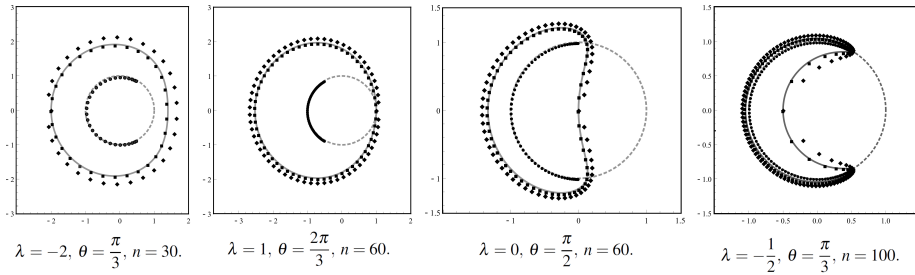


Figure 6: We illustrate Theorem 3 for orthogonal polynomials on the arc $\{e^{it} : \theta \leq t \leq 2\pi - \theta\}$ of the unit circle. In each case we take the Chebyshev measure on the arc. The zeros of ϕ_n are marked by bullets, the zeros of $I_{1,\lambda}(\phi_n)$ by squares, and the zeros of $I_{2,\lambda}(\phi_n)$ by diamonds. In each case, the dashed circle is $\partial\mathbf{D}_1$ and the gray path is the curve $\partial\Lambda_\lambda$ for the corresponding values of λ .

$0 \leq k \leq m - 1$. Therefore,

$$P_{m-1}(z) = \sum_{k=0}^{m-1} \binom{n+m}{k} I_{m-k,\lambda_0}(\phi_n)(\lambda)(z-\lambda)^k.$$

The proof is completed using (i) of Theorem 3 for $I_{m-k,\lambda_0}(\phi_n)$. \square

By way of illustrating Theorem 3, Figures 5 and 6 include several examples of the curve $\partial\Lambda_\lambda$, the zeros of the ϕ_n , and the zeros of their iterated integrals for different values of λ, m .

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