THE CONVERGENCE OF DISCRETE FOURIER-JACOBI SERIES

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Abstract. The discrete counterpart of the problem related to the convergence of the Fourier-Jacobi series is studied. To this end, given a sequence, we consider the analogue of the partial sum operator related to Jacobi polynomials and characterize its convergence in the $\ell^p(\mathbb{N})$-norm.

1. Introduction

By using Rodrigues' formula (see [17, p. 67, eq. (4.3.1)]), the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, $n \geq 0$, are defined as

$$(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n\, n!}\frac{d^n}{dx^n}\left((1-x)^{\alpha+n}(1+x)^{\beta+n}\right).$$

For $\alpha, \beta > -1$, they are orthogonal on the interval $[-1,1]$ with respect to the measure

$$d\mu_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta\, dx.$$

The family $\{p_n^{(\alpha,\beta)}(x)\}_{n\geq 0}$, given by $p_n^{(\alpha,\beta)}(x) = w_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(x)$, where

$$\begin{aligned} w_n^{(\alpha,\beta)} &= \frac{1}{\|P_n^{(\alpha,\beta)}\|_{L^2((-1,1),d\mu_{\alpha,\beta})}} \\ &= \sqrt{\frac{(2n+\alpha+\beta+1)\, n!\, \Gamma(n+\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}}, \quad n \geq 1, \end{aligned}$$

and

$$w_0^{(\alpha,\beta)} = \frac{1}{\|P_0^{(\alpha,\beta)}\|_{L^2((-1,1),d\mu_{\alpha,\beta})}} = \sqrt{\frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}},$$

is a complete orthonormal system in the space $L^2([-1,1], d\mu_{\alpha,\beta})$. Given a function $f \in L^2([-1,1], d\mu_{\alpha,\beta})$ its Fourier-Jacobi coefficients are defined by

$$c_n^{(\alpha,\beta)}(f) = \int_{-1}^{1} f(x) p_n^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x).$$

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The application

$$\begin{array}{ccc} L^2([-1,1],d\mu_{\alpha,\beta}) & \longrightarrow & \ell^2(\mathbb{N}) \\ f & \longmapsto & \{c_n^{(\alpha,\beta)}(f)\}_{n\geq 0} \end{array}$$

is an isometry and Parseval's identity

$$\|f\|_{L^2([-1,1],d\mu_{\alpha,\beta})} = \|c_n^{(\alpha,\beta)}(f)\|_{\ell^2(\mathbb{N})}$$

holds. For functions $f \in L^p([-1,1],d\mu_{\alpha,\beta})$, we define the n-th partial sum operator by

$$S_n^{(\alpha,\beta)}f(x) = \sum_{k=0}^{n} c_k^{(\alpha,\beta)}(f)p_k^{(\alpha,\beta)}(x).$$

It is well known (see [16] and [14]) that the mean convergence of $S_n^{(\alpha,\beta)}$, i.e.,

$$S_n^{(\alpha,\beta)}f \longrightarrow f \qquad \text{in } L^p([-1,1],d\mu_{\alpha,\beta}), \tag{1}$$

holds for $\alpha,\beta \geq -1/2$ if and only if

$$\max\left\{\frac{4(\alpha+1)}{2\alpha+3},\frac{4(\beta+1)}{2\beta+3}\right\} < p < \min\left\{\frac{4(\alpha+1)}{2\alpha+1},\frac{4(\beta+1)}{2\beta+1}\right\}.$$

This partial sum operator has been extensively analysed. In [12] some weighted inequalities were studied for $\alpha,\beta > -1$. The weak behaviour of $S_n^{(\alpha,\beta)}$ (weak (p,p)-type and restricted weak (p,p)-type inequalities) was treated in [7] for the case $\alpha = \beta = 0$ and in [9] for the general case. Weighted weak type inequalities were analysed in [10].

In this paper, we focus on the analysis of discrete Fourier-Jacobi expansions. More precisely, given an appropriate sequence $\{f(n)\}_{n\geq 0}$, its (α,β)-transform $\mathcal{F}_{\alpha,\beta}$ is given by the identity

$$\mathcal{F}_{\alpha,\beta}f(x) = \sum_{k=0}^{\infty} f(k)p_k^{(\alpha,\beta)}(x)$$

and its inverse by

$$\mathcal{F}_{\alpha,\beta}^{-1}F(n) = c_n^{(\alpha,\beta)}(F).$$

We are interested in recovering the given sequence by means of the multiplier of an interval for $\mathcal{F}_{\alpha,\beta}$. In a more concrete way, we define the multiplier of an interval $[a,b] \subset [-1,1]$, denoted by $T_{[a,b]}$ and simply by $\mathcal{T}_r$ when $[a,b] = [-r,r]$, with $0 < r < 1$, by the relation

$$T_{[a,b]}f = \mathcal{F}_{\alpha,\beta}^{-1}(\chi_{[a,b]}\mathcal{F}_{\alpha,\beta}f).$$

where $\chi_{[a,b]}$ is the characteristic function of the interval $[a,b]$. We want to study the conditions under

$$\lim_{r\to 1^-} \|\mathcal{T}_r f - f\|_{\ell^p(\mathbb{N})} = 0. \tag{2}$$

This problem is the discrete counterpart of (1) and it belongs to the study of the discrete harmonic analysis for Jacobi series developed in [1, 2, 3] by the authors. In those papers, the starting point is a discrete Laplacian defined by the three-term recurrence relation for the Jacobi polynomials. Recently, some classical operators in harmonic analysis have been treated in other discrete settings. For example, in [8] a complete study of the operators associated with the discrete Laplacian

$$\Delta_d u(u) = u(n+1) - 2u(n) + u(n-1), \qquad n \in \mathbb{Z},$$

was carried out. On its behalf, the same analysis was done in [6] for a discrete Laplacian defined in terms of the three-term recurrence relation for the ultraspherical polynomials.

In order to study (2), we give a complete characterization of the uniform boundedness of the operator $T_{[a,b]}$ on the spaces $\ell^p(\mathbb{N})$. This result will be a consequence of a more general one about the boundedness with discrete weights of $T_{[a,b]}$. Therefore, the convergence in (2) will follow from this characterization.

To state our result containing the weighted inequalities for the operator $T_{[a,b]}$, we need some preliminaries. A weight on $\mathbb{N} = \{0, 1, 2, \dots\}$ will be a strictly positive sequence $w = \{w(n)\}_{n\geq 0}$. We consider the weighted ℓ^p-spaces

$$\ell^p(\mathbb{N}, w) = \left\{ f = \{f(n)\}_{n\geq 0} : \|f\|_{\ell^p(\mathbb{N},w)} := \left(\sum_{m=0}^{\infty} |f(m)|^p w(m) \right)^{1/p} < \infty \right\},$$

$1 \leq p < \infty$, and the weak weighted ℓ^1-space

$$\ell^{1,\infty}(\mathbb{N}, w) = \left\{ f = \{f(n)\}_{n\geq 0} : \|f\|_{\ell^{1,\infty}(\mathbb{N},w)} := \sup_{t>0} t \sum_{\{m\in\mathbb{N}:|f(m)|>t\}} w(m) < \infty \right\},$$

and we simply write $\ell^p(\mathbb{N})$ and $\ell^{1,\infty}(\mathbb{N})$ when $w(n) = 1$ for all $n \in \mathbb{N}$.

Furthermore, we say that a weight $w(n)$ belongs to the discrete Muckenhoupt $A_p(\mathbb{N})$ (see, for instance, [11]) if

$$[w]_{A_p(\mathbb{N})} := \sup_{\substack{0\leq n\leq m \\ n,m\in\mathbb{N}}} \frac{1}{(m-n+1)^p} \left(\sum_{k=n}^{m} w(k) \right) \left(\sum_{k=n}^{m} w(k)^{-1/(p-1)} \right)^{p-1} < \infty,$$

for $1 < p < \infty$,

$$[w]_{A_1(\mathbb{N})} := \sup_{\substack{0\leq n\leq m \\ n,m\in\mathbb{N}}} \frac{1}{m-n+1} \left(\sum_{k=n}^{m} w(k) \right) \max_{n\leq k\leq m} w(k)^{-1} < \infty,$$

for $p = 1$. The value $[w]_{A_p(\mathbb{N})}$ is called the $A_p(\mathbb{N})$ constant of w.

Now we are in position to state the following result.

Theorem 1.1. *Let $\alpha, \beta \geq -1/2$, $[a,b] \subset [-1,1]$, $1 \leq p < \infty$, and $w \in A_p(\mathbb{N})$. Then,*

$$T_{[a,b]}f(n) = \sum_{m=0}^{\infty} f(m) K_{[a,b]}(m,n), \qquad f \in \ell^p(\mathbb{N}, w),$$

where

$$K_{[a,b]}(m,n) = \int_a^b p_m^{(\alpha,\beta)}(x) p_n^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x).$$

Moreover, for $1 < p < \infty$

$$\|T_{[a,b]}f\|_{\ell^p(\mathbb{N},w)} \leq C \|f\|_{\ell^p(\mathbb{N},w)}, \tag{3}$$

and for $p = 1$

$$\|T_{[a,b]}f\|_{\ell^{1,\infty}(\mathbb{N},w)} \leq C \|f\|_{\ell^1(\mathbb{N},w)}, \tag{4}$$

where C is a constant independent of f and $[a,b]$ in both inequalities.

As a consequence of the previous theorem, we can characterize the uniform boundedness of $T_{[a,b]}$ on the spaces $\ell^p(\mathbb{N})$.

Theorem 1.2. *Let $\alpha, \beta \geq -1/2$ and $1 \leq p < \infty$. Then,*

$$\|T_{[a,b]} f\|_{\ell^p(\mathbb{N})} \leq C \|f\|_{\ell^p(\mathbb{N})}, \tag{5}$$

where C is a constant independent of f and $[a,b] \subset [-1,1]$, if and only if $1 < p < \infty$.

Finally, from Theorem 1.2, we deduce that

Theorem 1.3. *Let $\alpha, \beta \geq -1/2$ and $1 \leq p < \infty$. Then (2) holds if and only if $1 < p < \infty$.*

Of course, from Theorem 1.3, the pointwise convergence

$$\lim_{r \to 1^-} \mathcal{T}_r f(n) = f(n), \qquad n \in \mathbb{N},$$

follows immediately.

The paper is organised as follows. Section 2 contains the proof of Theorem 1.1. To prove it we obtain a proper expression for the kernel of $T_{[a,b]}$ to write it in terms of some classical operators. The mapping properties of such operators will be used to complete the result. The proofs of Theorem 1.2 and Theorem 1.3 are contained in Section 3 where some technical lemmas are also included.

2. Proof of Theorem 1.1

From the identity

$$\chi_{[a,b]}(x) = \chi_{[-1,b]}(x) - \chi_{[-1,a)}(x),$$

we can focus on analysing the operator $T_{[-1,b]}$, denoted by T_b. For sequences $f \in \ell^2(\mathbb{N}) \cap \ell^p(\mathbb{N}, w)$, by using the identity

$$\int_{-1}^{1} \mathcal{F}_{\alpha,\beta} g(x) \mathcal{F}_{\alpha,\beta} h(x) \, d\mu_{\alpha,\beta}(x) = \sum_{m=0}^{\infty} g(m) h(m), \qquad g, h \in \ell^2(\mathbb{N}),$$

we have

$$T_b f(n) = \sum_{m=0}^{\infty} f(m) K_b(m,n), \tag{6}$$

where

$$K_b(m,n) = \int_{-1}^{b} p_m^{(\alpha,\beta)}(x) p_n^{(\alpha,\beta)}(x) \, d\mu_{\alpha,\beta}(x).$$

Our first step to prove Theorem 1.1 is to obtain an explicit expression for the kernel K_b.

Lemma 2.1. *Let $\alpha, \beta > -1$. Then, for $n \neq m$ we have the identity*

$$K_b(m,n) = \frac{(1-b)^{\alpha+1}(1+b)^{\beta+1}}{\lambda_n^{(\alpha,\beta)} - \lambda_m^{(\alpha,\beta)}} \Big(p_n^{(\alpha,\beta)}(b) \big(p_m^{(\alpha,\beta)}\big)'(b) - \big(p_n^{(\alpha,\beta)}\big)'(b) p_m^{(\alpha,\beta)}(b) \Big),$$

where $\lambda_j^{(\alpha,\beta)} = j(j+\alpha+\beta+1)$.

Proof. First, we note that (see [17, p. 60, eq. (4.2.1)])

$$L^{\alpha,\beta} p_n^{(\alpha,\beta)}(x) = \lambda_n^{(\alpha,\beta)} p_n^{(\alpha,\beta)}(x),$$

with

$$L^{\alpha,\beta} = -(1-x^2)\frac{d^2}{dx^2} - (\beta - \alpha - (\alpha+\beta+2)x)\frac{d}{dx}.$$

It is well known that $L^{\alpha,\beta}$ is a symmetric operator in $L^2([-1,1], d\mu_{\alpha,\beta})$, but for every interval $[r,s] \subset [-1,1]$, $r < s$, it is verified that

$$\int_r^s f(x) L^{\alpha,\beta} g(x)\, d\mu_{\alpha,\beta}(x) = U_{\alpha,\beta}(f,g)(x)\Big|_{x=r}^{x=s} + \int_r^s g(x) L^{\alpha,\beta} f(x)\, d\mu_{\alpha,\beta}(x),$$

with

$$U_{\alpha,\beta}(f,g)(x) = (1-x)^{\alpha+1}(1+x)^{\beta+1}\Big(g(x)\frac{df}{dx}(x) - f(x)\frac{dg}{dx}(x)\Big).$$

Then,

$$\begin{aligned}
\lambda_n^{(\alpha,\beta)} K_b(m,n) &= \int_{-1}^b p_m^{(\alpha,\beta)}(x) L^{\alpha,\beta} p_n^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x) \\
&= U_{\alpha,\beta}(p_m^{(\alpha,\beta)}, p_n^{(\alpha,\beta)})(x)\Big|_{x=-1}^{x=b} + \int_{-1}^b L^{\alpha,\beta} p_m^{(\alpha,\beta)}(x) p_n^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x) \\
&= U_{\alpha,\beta}(p_m^{(\alpha,\beta)}, p_n^{(\alpha,\beta)})(x)\Big|_{x=-1}^{x=b} + \lambda_m^{(\alpha,\beta)} K_b(m,n).
\end{aligned}$$

and

$$K_b(m,n) = \frac{1}{\lambda_n^{(\alpha,\beta)} - \lambda_m^{(\alpha,\beta)}} \left(U_{\alpha,\beta}(p_m^{(\alpha,\beta)}, p_n^{(\alpha,\beta)})(x)\Big|_{x=-1}^{x=b} \right).$$

Now the result follows immediately. □

The proof of Theorem 1.1 will be obtained by using the mapping properties of some classical operators. We consider

$$Hf(n) = \sum_{\substack{m=0\\ m\neq n}}^{\infty} \frac{f(m)}{n-m} \qquad \text{and} \qquad Q_a f(n) = \sum_{\substack{m=0\\ m\neq n}}^{\infty} \frac{f(m)}{n+m+a},$$

for some non-negative constant a. In the definition of Q_a we have considered $m \neq n$ because it is more convenient for us, but that value can be included without any problem.

The operator H is the well known discrete Hilbert transform and its boundedness with weights was treated in [11, Theorem 10]. There, it was proved that

$$\|Hf\|_{\ell^p(\mathbb{N},w)} \le C\|f\|_{\ell^p(\mathbb{N},w)} \Longleftrightarrow w \in A_p(\mathbb{N}), \tag{7}$$

for $1 < p < \infty$, and

$$\|Hf\|_{\ell^{1,\infty}(\mathbb{N},w)} \le C\|f\|_{\ell^1(\mathbb{N},w)} \Longleftrightarrow w \in A_1(\mathbb{N}). \tag{8}$$

Moreover, the constant C in (7) and (8) only depends on the $A_p(\mathbb{N})$ constant of the weight w.

In the case of the operator Q_a, we have

$$|Q_a f(n)| \le C\left(\frac{1}{n+1}\sum_{m=0}^{n} |f(m)| + \sum_{m=n}^{\infty} \frac{|f(m)|}{m+1}\right) =: C\left(O_1 f(n) + O_2 f(n)\right).$$

The operator O_1 is the discrete Hardy operator and it can be controlled by the discrete maximal operator, so it is bounded from $\ell^p(\mathbb{N},w)$ into itself when $1 < p < \infty$ and $w \in A_p(\mathbb{N})$, and from $\ell^1(\mathbb{N},w)$ into $\ell^{1,\infty}(\mathbb{N},w)$ for $w \in A_1(\mathbb{N})$. From the identity

$$\sum_{m=0}^{\infty} f(m) O_1 g(m) = \sum_{m=0}^{\infty} g(m) O_2 f(m),$$

we have that O_2 is the adjoint operator of O_1 (in fact, it is the adjoint Hardy operator), and we conclude that

$$\|Q_a f\|_{\ell^p(\mathbb{N},w)} \le C\|f\|_{\ell^p(\mathbb{N},w)}, \tag{9}$$

for $1 < p < \infty$ and $w \in A_p(\mathbb{N})$. Moreover, for O_2, using Fubini's theorem and the definition of $A_1(\mathbb{N})$, we can deduce that it is a bounded operator from $\ell^1(\mathbb{N}, w)$ into itself and, finally, we have

$$\|Q_a f\|_{\ell^{1,\infty}(\mathbb{N},w)} \le C\|f\|_{\ell^1(\mathbb{N},w)}, \tag{10}$$

when $w \in A_1(\mathbb{N})$. The constant appearing in the boundedness of the discrete maximal operator also depends on the value $[w]_{A_p(\mathbb{N})}$ and, then, so it occurs for the constant C in (9) and (10).

Proof of Theorem 1.1. Set

$$r_b(n) = (1-b)^{\alpha/2+1/4}(1+b)^{\beta/2+1/4} p_n^{(\alpha,\beta)}(b)$$

and

$$R_b(n) = \frac{(1-b)^{\alpha/2+3/4}(1+b)^{\beta/2+3/4}}{2n+\alpha+\beta+1}(p_n^{(\alpha,\beta)})'(b).$$

By Lemma 2.1 and the identities

$$\frac{1}{\lambda_n^{(\alpha,\beta)} - \lambda_m^{(\alpha,\beta)}} = \frac{1}{2m+\alpha+\beta+1}\left(\frac{1}{n-m} - \frac{1}{n+m+\alpha+\beta+1}\right) \tag{11}$$

$$= \frac{1}{2n+\alpha+\beta+1}\left(\frac{1}{n-m} + \frac{1}{n+m+\alpha+\beta+1}\right) \tag{12}$$

we have

$$\begin{aligned} T_b f(n) = r_b(n)H(R_b f)(n) - R_b(n)H(r_b f)(n) - r_b(n)Q_{\alpha+\beta+1}(R_b f)(n) \\ - R_b(n)Q_{\alpha+\beta+1}(r_b f)(n) + f(n)K_b(n,n). \end{aligned} \tag{13}$$

To estimate the weights r_b and R_b we need some bounds for the Jacobi polynomials. For $a, b > -1$, the estimate (see [13, eq. (2.6) and (2.7)])

$$\begin{aligned} &|p_n^{(a,b)}(x)| \\ &\le C \begin{cases} (n+1)^{a+1/2}, & 1 - 1/(n+1)^2 < x < 1, \\ (1-x)^{-a/2-1/4}(1+x)^{-b/2-1/4}, & -1 + 1/(n+1)^2 \le x \le 1 - 1/(n+1)^2, \\ (n+1)^{b+1/2}, & -1 < x < -1 + 1/(n+1)^2, \end{cases} \end{aligned} \tag{14}$$

holds, where C is a constant independent of n and x. When $a, b \ge -1/2$ the previous bound can be replaced by the simpler one

$$|p_n^{(a,b)}(x)| \le C(1-x)^{-a/2-1/4}(1+x)^{-b/2-1/4}. \tag{15}$$

In this way, using the identity (see [15, eq. 18.9.15])

$$\frac{dP_n^{(a,b)}}{dx}(x) = \frac{n+a+b+1}{2}P_{n-1}^{(a+1,b+1)}(x)$$

and (15), we obtain the bounds

$$|r_b(n)| \le C \qquad \text{and} \qquad |R_b(n)| \le C. \tag{16}$$

Then, by (13), (16), (7), (9) and the estimate $K_b(n,n) \le 1$, we deduce that

$$\|T_b f\|_{\ell^p(\mathbb{N},w)} \le C\left(\|H(R_b f)\|_{\ell^p(\mathbb{N},w)} + \|H(r_b f)\|_{\ell^p(\mathbb{N},w)} + \|Q_{\alpha+\beta+1}(R_b f)\|_{\ell^p(\mathbb{N},w)}\right.$$

$$+\|Q_{\alpha+\beta+1}(r_b f)\|_{\ell^p(\mathbb{N},w)} + \|f\|_{\ell^p(\mathbb{N},w)}\big) \le C\|f\|_{\ell^p(\mathbb{N},w)}$$

and the proof of (3) is completed when $f \in \ell^2(\mathbb{N}) \cap \ell^p(\mathbb{N}, w)$. To prove (4) we proceed in the same way but using (8) and (10) instead of (7) and (9).

At this point, we know that the operator T_b, which is given by (6) for sequences $f \in \ell^2(\mathbb{N}) \cap \ell^p(\mathbb{N}, w)$, admits an extension, that we denote by $\mathbb{T}_b$, bounded from $\ell^p(\mathbb{N}, w)$ into itself when $1 < p < \infty$, and from $\ell^1(\mathbb{N}, w)$ into $\ell^{1,\infty}(\mathbb{N}, w)$. Let us see that

$$\mathbb{T}_b f(n) = \sum_{m=0}^{\infty} f(m) K_b^{(\alpha,\beta)}(m, n), \qquad f \in \ell^p(\mathbb{N}, w),$$

to complete the proof of our result. We provide the details for $1 < p < \infty$ and we omit them for $p = 1$ (see [6]).

First, let us consider the functional

$$\mathfrak{T}_{b,n} : \ell^p(\mathbb{N}, w) \longrightarrow \mathbb{R}$$

$$f \longmapsto \mathfrak{T}_{b,n} f := \sum_{m=0}^{\infty} f(m) K_b^{(\alpha,\beta)}(m, n).$$

For $1 < p < \infty$, it is easy to check that

$$|\mathfrak{T}_{b,n} f| \le C \frac{\|f\|_{\ell^p(\mathbb{N},w)}}{w^{1/p}(n)}, \tag{17}$$

for sequences $f \in \ell^2(\mathbb{N}) \cap \ell^p(\mathbb{N}, w)$. Then we can prove that $K_b(\cdot, n)$ is a sequence in $\ell^q(\mathbb{N}, w^{-1/(p-1)})$, where q is the conjugate exponent of p; i.e., $p^{-1} + q^{-1} = 1$. Then, the operator $\mathfrak{T}_{b,n}^{(\alpha,\beta)}$ is bounded and it verifies (17) for $f \in \ell^p(\mathbb{N}, w)$ with $1 < p < \infty$.

Now, given $f \in \ell^p(\mathbb{N}, w)$ and $\{f_k\}_{k\ge 0} \subset \ell^2(\mathbb{N}) \cap \ell^p(\mathbb{N}, w)$ such that $f_k \longrightarrow f$ in $\ell^p(\mathbb{N}, w)$, we have

$$\mathbb{T}_b f_k = T_b f_k \longrightarrow \mathbb{T}_b f, \qquad \text{in } \ell^p(\mathbb{N}, w)$$

and

$$\mathcal{T}_b f_k(n) = \mathfrak{T}_{b,n} f_k \longrightarrow \mathbb{T}_b f(n), \qquad \text{in } \mathbb{R}.$$

In this way, by the boundedness of $\mathfrak{T}_{b,n}$, we conclude that

$$\mathbb{T}_b f(n) = \mathfrak{T}_{b,n} f = \sum_{m=0}^{\infty} f(m) K_b^{(\alpha,\beta)}(m, n), \qquad n \in \mathbb{N},$$

finishing the proof □

Remark 1. For the complete range $\alpha, \beta > -1$, it is also possible to obtain (3) and (4) for T_b but with more involved conditions on the weight w than the simple one $w \in A_p(\mathbb{N})$. Indeed, from (14) it is clear that, for $a, b > -1$,

$$|p_n^{(a,b)}(x)| \le C \left(1 - x + \frac{1}{(n+1)^2}\right)^{-a/2-1/4} \left(1 + x + \frac{1}{(n+1)^2}\right)^{-b/2-1/4}.$$

Then, taking

$$M_b^{r,s}(n) = \left(\frac{1-b}{1-b+\frac{1}{(n+1)^2}}\right)^r \left(\frac{1+b}{1+b+\frac{1}{(n+1)^2}}\right)^s,$$

we have

$$|r_b(n)| \le C M_b^{\alpha/2+1/4,\beta/2+1/4}(n) \qquad \text{and} \qquad |R_b(n)| \le C M_b^{\alpha/2+3/4,\beta/2+3/4}(n).$$

In this way, provided that

$$w(n)M_b^{p(\alpha/2+1/4),p(\beta/2+1/4)}(n) \qquad \text{and} \qquad w(n)M_b^{p(\alpha/2+3/4),p(\beta/2+3/4)}(n)$$

are uniform weights in $A_p(\mathbb{N})$ (uniform in the sense that the $A_p(\mathbb{N})$ constant of such weights does not depend on b) and using that

$$M_b^{\alpha/2+1/4,\beta/2+1/4}(n)M_b^{\alpha/2+3/4,\beta/2+3/4}(n) = M_b^{\alpha+1,\beta+1}(n) \le C,$$

it is possible to prove (3) and (4). This fact is so because the constants in the boundedness of the discrete Hilbert transform and the discrete maximal function in $\ell^p(\mathbb{N}, w)$ only depend on the $A_p(\mathbb{N})$ constant of the weight w.

3. Proofs of Theorem 1.2 and Theorem 1.3

The main tool to prove Theorem 1.2 and Theorem 1.3 is the following lemma in which we analyse if $\{K_b(m,n)\}_{n\ge 0}$ is an element of $\ell^p(\mathbb{N})$.

Lemma 3.1. *Let $\alpha, \beta \ge -1/2$ and $m \in \mathbb{N}$. Then*

$$|K_b(m,n)| \le C|m-n|^{-1}, \qquad n \ne m, \tag{18}$$

and $K_b(m,\cdot) \in \ell^p(\mathbb{N})$ for $1 < p < \infty$. Moreover,

$$\sum_{n=m+1}^{2m} \left| \int_0^{1-1/m^2} p_m^{(\alpha,\beta)}(x) p_n^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x) \right| \simeq \log m. \tag{19}$$

Proof. From Lemma 2.1, applying the identities in (11) and the bounds for r_b and R_b in (16), we have the estimate $|K_b(m,n)| \le C|m-n|^{-1}$ for $n \ne m$. This estimate it is enough to show that $K_b(m,\cdot) \in \ell^p(\mathbb{N})$ for $1 < p < \infty$ (note that $K_b(m,m) \le 1$).

Denoting by $I(m,n)$ the integral appearing in (19), to obtain the result it is enough to prove that

$$I(m,n) = A\left(\frac{1}{n-m} + \frac{1}{N}\log\left(\frac{N}{n-m}\right) + \frac{1}{M}\log\left(\frac{M}{n-m}\right) + O(M^{-1})\right) \tag{20}$$

for $m+1 \le n \le 2m$, with A a positive constant, $N = n + (\alpha+\beta+1)/2$, and $M = m + (\alpha+\beta+1)/2$. To attain this, we consider the expansion (deduce from known asymptotics for Jacobi polynomials in [4, formula (9)])

$$\begin{aligned} 2^{(\alpha+\beta+1)/2} p_n^{(\alpha,\beta)}(\cos\theta) &= (\sin\theta/2)^{-(\alpha+1/2)}(\cos\theta/2)^{-(\beta+1/2)} \\ &\times \left(A\cos(N\theta - \phi_\alpha) + A\frac{\sin(N\theta-\phi_\alpha)}{N\theta} + O(N^{-1}) + O((N\theta)^{-2}) \right), \end{aligned} \tag{21}$$

for $\delta/n < \theta \le \pi/2$, with $\delta > 0$, and $\phi_\alpha = (2\alpha+1)\pi/4$. Then, using the change of variable $x = \cos\theta$, taking $B = \arccos(1 - 1/m^2)$ (observe that $B \simeq 1/m \simeq 1/n$ for $m+1 \le n \le 2m$), and applying (21) for $p_n^{(\alpha,\beta)}$ and $p_m^{(\alpha,\beta)}$, we have

$$I(m,n) = J_1(m,n) + J_2(m,n) + J_3(m,n) + O(M^{-1}),$$

with

$$J_1(m,n) = A\int_{1/m}^{\pi/2} \cos(M\theta - \phi_\alpha)\cos(N\theta-\phi_\alpha)\, d\theta,$$

$$J_2(m,n) = \frac{A}{N}\int_{1/m}^{\pi/2} \cos(M\theta-\phi_\alpha)\sin(N\theta-\phi_\alpha)\,\frac{d\theta}{\theta}$$

and $J_3(m,n) = J_2(n,m)$. Following [4] (see [5] for some technical details), we obtain that

$$J_1(m,n) = \frac{A}{n-m} + O(M^{-1}),$$

$$J_2(m,n) = \frac{A}{N}\log\left(\frac{N}{n-m}\right) + O(M^{-1})$$

and the similar estimates for $J_3(m,n)$ by changing the roles of m and n. Now, the proof of (20) is completed. In this way, (19) follows immediately because

$$\sum_{n=m+1}^{2m}\left(\frac{1}{N}\log\left(\frac{N}{n-m}\right) + \frac{1}{M}\log\left(\frac{M}{n-m}\right)\right) \le C$$

and

$$\sum_{n=m+1}^{2m}\frac{1}{n-m} \simeq \log m. \qquad \square$$

Now, let us proceed with the proofs of Theorem 1.2 and Theorem 1.3.

Proof of Theorem 1.2. By Theorem 1.1 (note that $w(n) = 1$ is a weight in $A_p(\mathbb{N})$ for $1 < p < \infty$), it is enough to show the existence of a sequence $f \in \ell^1(\mathbb{N})$ such that the inequality

$$\|T_{[a,b]}f\|_{\ell^1(\mathbb{N})} \le C\|f\|_{\ell^1(\mathbb{N})} \tag{22}$$

does not hold for some interval $[a,b]$.

In this way, we take $m \ge 1$ and consider the interval $[a,b] = [0, 1-1/m^2]$ and the sequence $f_m(n) = \delta_{nm}$, where δ_{nm} denotes the usual Kronecker delta function. Note that

$$\mathcal{F}_{\alpha,\beta}f_m(x) = p_m^{(\alpha,\beta)}(x).$$

Now, if (22) was be true, it would imply

$$1 = \|f_m\|_{\ell^1(\mathbb{N})} \ge \sum_{n=m+1}^{2m}\left|\int_0^{1-1/m^2} p_n^{(\alpha,\beta)}(x)p_m^{(\alpha,\beta)}(x)\,d\mu_{\alpha,\beta}(x)\right|,$$

but this inequality is not possible because the right hand side is greater than $C\log m$ by (19). $\square$

Remark 2. By using the identity $p_n^{(a,b)}(-z) = (-1)^n p_n^{(b,a)}(z)$, for $-1 < z < 1$, proceeding as in the proof of Lemma 2.1, it is possible to prove that

$$\int_{-1+1/m^2}^{1-1/m^2} p_m^{(\alpha,\beta)}(x)p_n^{(\alpha,\beta)}(x)\,d\mu_{\alpha,\beta}(x)$$
$$= A\left(\frac{1}{n-m} + \frac{1}{N}\log\left(\frac{N}{n-m}\right) + \frac{1}{M}\log\left(\frac{M}{n-m}\right) + O(M^{-1})\right).$$

Then, in particular, we can deduce that the operators $\mathcal{T}_r$ are not bounded from $\ell^1(\mathbb{N})$ into itself.

To prove Theorem 1.3, first we have to check the convergence of $\mathcal{T}_r$ for sequence in c_{00}, the space of sequences having a finite number of non-null terms, and this is done in the following lemma.

Lemma 3.2. *Let $\alpha,\beta \geq -1/2$, $1<p<\infty$, and $f\in c_{00}$. Then*

$$\lim_{r\to 1^-} \|\mathcal{T}_r f - f\|_{\ell^p(\mathbb{N})} = 0.$$

Proof. Since each $f\in c_{00}$ could be stated by a finite linear combination of $f_m(n) = \delta_{nm}$, we prove the result for the latter sequences. Then, using that

$$f_m(n) = \int_{-1}^{1} p_m^{(\alpha,\beta)}(x) p_n^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x),$$

we have

$$\mathcal{T}_r f_m(n) - f_m(n) = -\int_{-1}^{-r} p_m^{(\alpha,\beta)}(x) p_n^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x) - \int_{r}^{1} p_m^{(\alpha,\beta)}(x) p_n^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x). \tag{23}$$

Owing to the orthogonality of the Jacobi polynomials, for $m\neq n$, it is verified that

$$\int_{r}^{1} p_m^{(\alpha,\beta)}(x) p_n^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x) = -\int_{-1}^{r} p_m^{(\alpha,\beta)}(x) p_n^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x)$$

and, from (23), we deduce that

$$\mathcal{T}_r f_m(n) - f_m(n) = K_r(m,n) - K_{-r}(m,n), \qquad m\neq n.$$

When $n=m$, applying (15) in (23), the estimate

$$|\mathcal{T}_r f_m(m) - f_m(m)| \leq C(1-r)^{1/2}$$

is attained. Then

$$\lim_{r\to 1^-} \|\mathcal{T}_r f_m - f_m\|^p_{\ell^p(\mathbb{N})} = \lim_{r\to 1^-} \sum_{\substack{n=0\\ n\neq m}}^{\infty} |K_{-r}(m,n) - K_r(m,n)|^p . \tag{24}$$

From (18), we have $|K_{-r}(m,n) - K_r(m,n)| \leq C|m-n|^{-1}$, when $m\neq n$. Then applying the dominated convergence theorem the result follows (note that $|m-n|^{-1}$, for $n\neq m$, is p-summable for $1<p<\infty$) from (24) because $\lim_{r\to 1^-}(K_{-r}(m,n) - K_r(m,n)) = 0$. □

Proof of Theorem 1.3. To prove (2) for $1<p<\infty$ and sequences $f\in \ell^p(\mathbb{N})$, it is enough to approximate them by sequences in c_{00} and use Theorem 1.2 and Lemma 3.2. Indeed, given $\varepsilon>0$, we consider a sequence $g\in c_{00}$ such that $\|f-g\|_{\ell^p(\mathbb{N})}<\varepsilon$, then, applying (5), we have

$$\begin{aligned}\|\mathcal{T}_r f - f\|_{\ell^p(\mathbb{N})} &\leq \|\mathcal{T}_r f - \mathcal{T}_r g\|_{\ell^p(\mathbb{N})} + \|\mathcal{T}_r g - g\|_{\ell^p(\mathbb{N})} + \|g-f\|_{\ell^p(\mathbb{N})}\\ &\leq C\|g-f\|_{\ell^p(\mathbb{N})} + \|\mathcal{T}_r g - g\|_{\ell^p(\mathbb{N})} \leq C\varepsilon,\end{aligned}$$

where in the last step we have used Lemma 3.2.

The convergence in $\ell^1(\mathbb{N})$ is not possible because in such case the uniform boundedness principle would imply the uniform boundedness of the $\mathcal{T}_r$ in $\ell^1(\mathbb{N})$ and that is impossible (see Remark 2). □

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